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A Modified Leapfrog Scheme for Shallow Water Equations

by

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Abstract

In the 1D linearized shallow water equations, the Courant number should be < 0.5 for stability in the original Leapfrog (LF) scheme. Here, we propose using the time-averaged heights in the pressure gradient force in the momentum equations. The stability analysis shows that the new scheme is neutral when Courant number <1. The scheme is 2^{nd} order accurate in both time and space. It does not require iterations and can be easily applied in 2D or 3D wave equations. The numerical simulations for 2-D linearized shallow water equations are consistent with those obtained from a 2-time-step semi-implicit scheme.

Keywords: Shallow Water Equations, leapfrog scheme, Courant number, eigenvalue, semi-implicit, finite-volume, stability

1. Introduction

The LF scheme has been applied to the shallow water equations by Mesinger and Arakawa (1976), Haltiner and Williams (1980), and many others. The stability criterion is Courant number (Co) < 0.5 in the 1D Leapfrog scheme in the staggered C-grids. A new semi-implicit approach is introduced which shows that the new scheme is neutral when Co<1. The number of calculations is almost the same as the original LF scheme and can be easily applied in the nonlinear 2D and 3D wave equations. Since both pressure and momentum fields are calculated at the every time step, they can easily be applied in the nonlinear terms. On the other hand, the time average may be required in order to calculate the nonlinear terms in the alternative Leapfrog scheme (Zhou 2002), which is also stable when Co<1.

2. Numerical Schemes and Eigenvalues

2a. Original LF scheme

The 1D linearized shallow water equations are

$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x} \tag{1}$$

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \tag{2}$$

where *H* is the mean depth, *g* is gravity, *h* and *u* are the depth perturbation and velocity.

In the staggered C-grids, the finite difference equations for the LF scheme become

$$\begin{aligned}\frac{(h_p^{n+1} - h_p^{n-1})}{2\Delta t} &= -H\frac{u_{p+1/2}^n - u_{p-1/2}^n}{\Delta x} \\ \frac{(u_q^{n+1} - u_q^{n-1})}{2\Delta t} &= -g\frac{h_{q+1/2}^n - h_{q-1/2}^n}{\Delta x}\end{aligned} \tag{3}$$

where $q=p\pm 1/2$, Δx is the spatial interval between *p* and *p+1* grids. If a wave-type solution at the n^{th} time step is assumed:

$$h_p^n = \hat{h}^n \exp[ikp\Delta x] = \hat{h}^0 \lambda^n \exp[ikp\Delta x] \tag{4a}$$

and

$$u_q^n = \hat{u}^n \exp[ikq\Delta x] = \hat{u}^0 \lambda^n \exp[ikq\Delta x] \tag{4b}$$

where $i = \sqrt{-1}$, $\hat{h}^n$ and $\hat{u}^n$ are the amplitude of *h* and *u* at the n^{th} time step; *k* is the wave number. The difference equations become

$$\frac{(\hat{h}^{n+1} - \hat{h}^{n-1})}{2\Delta t} = -H\hat{u}^n \frac{\exp(ik\Delta x/2) - \exp(-ik\Delta x/2)}{\Delta x} = -iH\hat{u}^n X \tag{5a}$$

$$\frac{(\hat{u}^{n+1} - \hat{u}^{n-1})}{2\Delta t} = -g\hat{h}^n \frac{\exp(ik\Delta x/2) - \exp(-ik\Delta x/2)}{\Delta x} = -ig\hat{h}^n X \tag{5b}$$

The eigenvalue of (5) is

$$\lambda^2 = \{1-2R^2\} \pm \sqrt{\{1-2R^2\}^2 - 1} \tag{7}$$

where $X = \frac{\sin[k\Delta x/2]}{\Delta x/2}$, $R^2 = gH\Delta t^2 X^2 = (CX\Delta t)^2 = [2\sin(k\Delta x/2)Co]^2$, $C = \sqrt{gH}$,

and Courant number $Co = \frac{C\Delta t}{\Delta x}$. The original LF is neutrally stable when Co<0.5, according to (7) (Mesinger and Arakawa 1976, Haltiner and Williams, 1980). It becomes weakly unstable because of repeated eigenvalues at Co=0.5 (Sun 2010a). Eq. (7) includes two physical modes and two computational modes. The latter are introduced by three-time-step used in the LF. The phase speed of the physical mode can be obtained by

$$\begin{aligned} h_p^n &= \hat{h}^n \exp[ikp\Delta x] = \hat{h}^0 \lambda^n \exp[ikp\Delta x] = \hat{h}^0 \left[|\lambda| \exp(\pm i\alpha)\right]^n \exp[ikp\Delta x] \\ &= \hat{h}^0 |\lambda|^n \exp[ik(p\Delta x \pm \frac{n\Delta t\alpha}{k\Delta t})] = \hat{h}^0 \exp[ik(x \pm cp_{LF} t)] \end{aligned} \tag{8a}$$

where

$$cp_{LF} = \frac{\alpha}{k\Delta t} = \frac{\cos^{-1}(1-2R^2)}{2k\Delta t} \tag{8b}$$

The value of cp_{LF} is shown in Table 1. The LF scheme requires two initial conditions (i.e., n=0 and 1) at the beginning. We can use the center-in-space and forward-in-time, or upstream method to obtain the value at n=1. The computational modes can be minimized if the values are very close to the real solution at n=1(Haltiner and Williams 1980).

We can represent (1) and (2) in the following finite difference forms:

$$\frac{(h_p^{n+1} - h_p^{n-1})}{2\Delta t} = -H \frac{u_{p+1/2}^n - u_{p-1/2}^n}{\Delta x} \tag{9a}$$

$$\frac{(u_q^{n+1} - u_q^{n-1})}{2\Delta t} = -g \frac{\bar{h}_{q+1/2}^n - \bar{h}_{q-1/2}^n}{\Delta x} \tag{9b}$$

where $\overline{h}^{n}_{q+1/2} = \dfrac{h^{n+1}_{q+1/2} + 2h^{n}_{q+1/2} + h^{n-1}_{q+1/2}}{4}$ (10)

Since $h^{n+1}_{q+1/2}$ in (10) can be obtained from (9a), no iteration is required in (9b). Hence, this is a semi-implicit scheme. It is a second order accuracy in both space and time, the same accurate as the original LF scheme. The eigenvalue can be obtained from

$$\frac{(\lambda - \lambda^{-1})\hat{h}}{2\Delta t} = -H\hat{u}\left[\frac{\exp(ik\Delta x/2) - \exp(-ik\Delta x/2)}{\Delta x}\right] = -iH\hat{u}X \quad (11a)$$

$$\frac{(\lambda - \lambda^{-1})}{2\Delta t}\hat{u} = -g\hat{h}\left(\frac{\lambda + 2 + \lambda^{-1}}{4}\right)\left[\frac{\exp(ik\Delta x/2) - \exp(-ik\Delta x/2)}{\Delta x}\right] = -ig\hat{h}X\left(\frac{\lambda + 2 + \lambda^{-1}}{4}\right) \quad (11b)$$

and

$$\lambda = \left[\left\{(2 - R^2\right\} \pm \sqrt{[2 - R^2]^2 - 4}\right]/2 \quad (12)$$

$|\lambda|$=1, as long as Co<1 which is twice allowed in the original LF scheme. It is also noted that eigenvalue of (12) is identical to the forward-backward scheme (Sun, 1984; 2010a), which becomes weakly unstable because of repeated eigenvalue at *Co=1* (Sun 2010a). In addition to the physical modes of (12). There are two identical computational models: $(\lambda+1)^2$=0. The phase speed of (12) is given by

$$cp_{MLF} = \frac{\alpha}{k\Delta t} = \frac{\cos^{-1}\left(\dfrac{2 - R^2}{2}\right)}{k\Delta t} \quad (13)$$

The results are shown in Table 2, in which the values at Co =0.2, 0. 4, 0.6, 0.8 and 1.0 are identical those at Co=0.1, 0.2, 0.3, 0.4 and 0.5 in Table 1.

3. Numerical Simulations

The scheme is applied to simulate a 2D dam break in the linearized equations. The initial water height is 1.0001 m inside a cylinder of radius = 11m; and 1 m outside in a domain of 200x200 m^2, shown as the inner circle in Fig. 1a. A small initial height perturbation ensures an accurate phase speed of surface gravity wave, C_p= 3.13ms^{-1}(with gravity=9.8ms^{-2}). The spatial intervals $\Delta x=\Delta y=1$ m, and $\Delta t=0.2$s. The Courant Number Co= $C_p*\Delta t /\Delta x$ =0.626 and the critical Courant number is 0.707 for a 2D problem. A forth order Shuman smoothing is applied each time step with coefficient of 0.5 (Sun 2010a). The result at t=24 s is shown as the outer rings in Fig. 1a, which are in good agreement with Fig. 1b simulated from a new 2-time-step semi-implicit finite volume scheme (Sun 2010b) with same Courant number and smoothing. The time sequences of vertical cross section at y=0 for both schemes at t=0, 8, 16, and 24 s are shown in Fig. 2. The results show that the critical Courant number in the modified leapfrog scheme is the same as that in the forward-backward scheme (Sun 2010a). They are twice as that in the conventional leapfrog scheme in the staggered grids.

It is noted that Zhou’s (2002) alternative leapfrog scheme using a staggered time grid system to solve surface gravity waves is also twice as efficient as the standard leap scheme. However, his height and velocity are not calculated at the same time step. On the other hand, the height and velocity are calculated at each time step here. The nonlinear terms can be easily calculated (without time-average) using the finite-difference schemes (Haltiner and Williams 1980) or semi-Lagrangian schemes (Sun et al. 1996, Sun and Yeh 1997, Sun and Sun 2004), etc.

4. Summary

In the linearized shallow water equations, the Courant number should be less than 0.5 for stability in the original Leapfrog (LF) scheme in the staggered grids. Here, we propose a simple modification by using the time-averaged heights in the pressure gradient force in the momentum equation. The stability analysis shows that the new scheme is neutral as long as Courant number is less than 1 in the 1D in the C-grids. The new scheme is semi-implicit and does not require iterations. Hence, it is easy to calculate. The scheme is 2^{nd} order accurate in both time and space, same as the original LF scheme. The numerical simulations for 2-D linearized shallow water equations are consistent with those obtained from a 2-time-step semi-implicit scheme.

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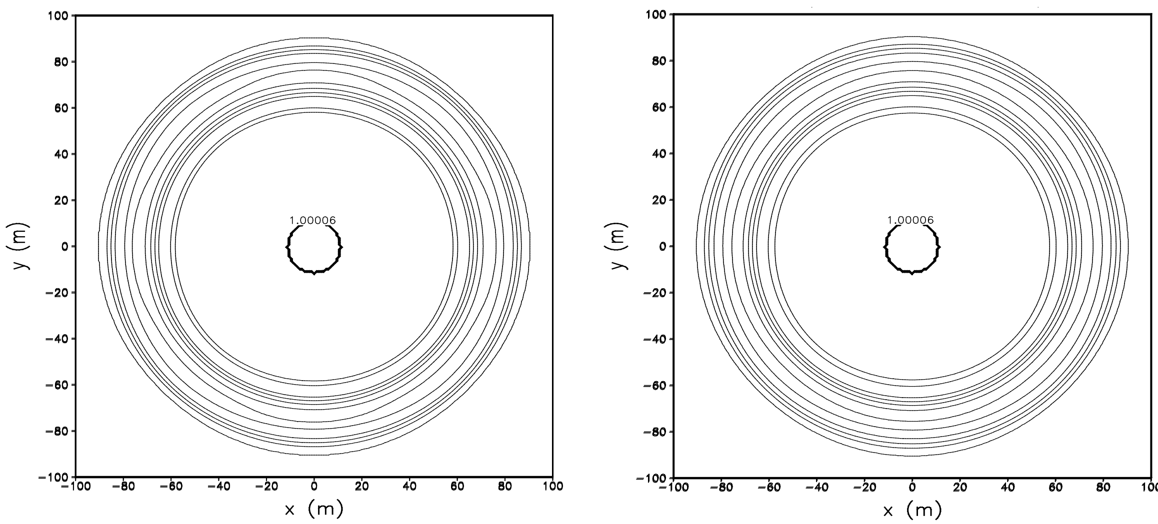
Caption of Figures

Fig.1: (a) Horizontal plane of initial condion (inner circle) and numerical simulation at t=24 s from modified leapfrog scheme, (b) same as (a) except from a two-time step semi-implicit scheme (Sun 2010b).

Fig. 2: Vertical cross section at y=0 of numerical simulations at t=0, 8, 16, and 24s from modified leapfrog scheme (solid line) and a two-time step semi-implicit scheme (x).



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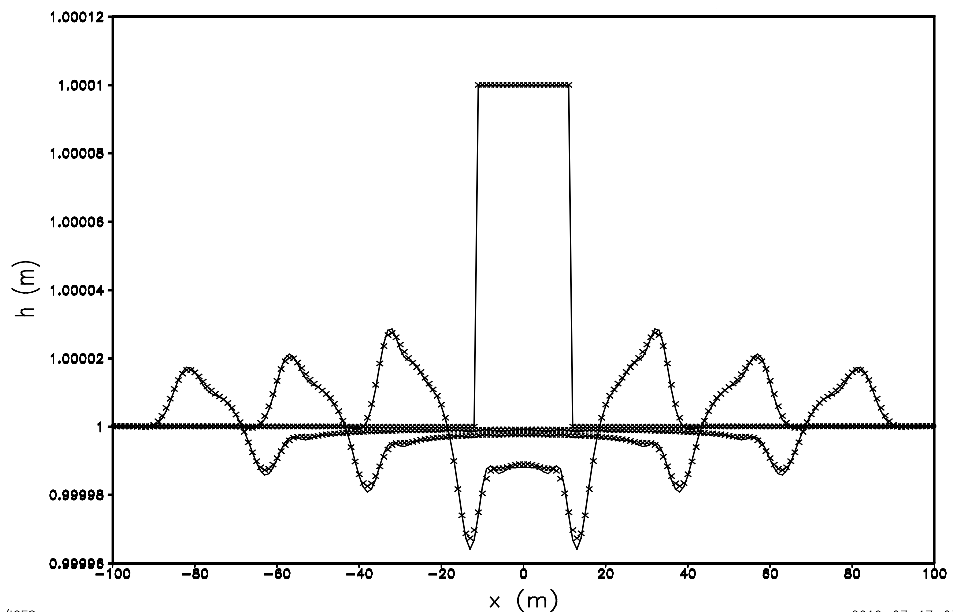

Fig. 2: Vertical cross section of numerical simulations at y=0 in Fig. 1 at t=0, 8, 16, and 24s from modified leapfrog scheme (solid line) and a two-time step semi-implicit scheme (x).

Table 1: The ratio of phase speed from LF to analytical solution as function of Courant number Co and wavelength L=mΔx

	mΔx	2	4	6	8	10	12	14	16	18	20
	0.1	.6409	.9033	.9565	.9754	.9843	.9891	.9920	.9938	.9951	.9961
	0.2	.6549	.9128	.9614	.9783	.9862	.9904	.9929	.9946	.9957	.9965
Co	0.3	.6828	.9298	.9699	.9833	.9894	.9926	.9946	.9959	.9967	.9974
	0.4	.7379	.9569	.9824	.9904	.9939	.9958	.9969	.9977	.9982	.9985
	0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 2: The ratio of phase speed from modified LF to analytical solution as function of Courant number Co and wavelength L=mΔx

	mΔx	2	4	6	8	10	12	14	16	18	20
	0.1	.6377	.9011	.9553	.9747	.9838	.9887	.9917	.9936	.9950	.9959
	0.2	.6409	.9033	.9565	.9754	.9843	.9891	.9920	.9938	.9951	.9961
	0.3	.6466	.9072	.9585	.9766	.9850	.9896	.9924	.9942	.9954	.9963
	0.4	.6549	.9128	.9614	.9783	.9862	.9904	.9929	.9946	.9957	.9965
CO	0.5	.6667	.9202	.9652	.9805	.9876	.9914	.9937	.9952	.9962	.9969
	0.6	.6828	.9298	.9699	.9833	.9894	.9926	.9946	.9959	.9967	.9974
	0.7	.7052	.9418	.9756	.9865	.9915	.9941	.9957	.9967	.9974	.9979
	0.8	.7379	.9569	.9824	.9904	.9939	.9958	.9969	.9977	.9982	.9985
	0.9	.7921	.9759	.9905	.9949	.9968	.9978	.9984	.9988	.9990	.9992
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000