Auxiliary material for:

Constraints on the lake volume required for hydro-fracture through ice sheets

M J Krawczynski, M D Behn, S B Das, I Joughin

1 Numerical Approach

The first question we address when discussing supraglacial lake drainage is whether there are favorable conditions for crack formation. We use the stress intensity factor at the crack tip to determine whether critical crack propagation will occur for a given set of parameters, and calculate the crack length if there is propagation. The stress intensity factor, K_{tot}, at the tip of the crack is a linear combination of the individual stress intensities:

$$K_{tot} = K_T + K_I + K_W \tag{1}$$

where the terms on the right hand side of the equation are expressed as:

$$K_{T,I,W} = \sigma_{t,i,w}\sqrt{\pi \cdot z} \tag{2}$$

Thus, K_{tot} is a function of the deviatoric (longitudinal) stress σ_t (assumed to be independent of z), the ice overburden pressure $\sigma_i = -\rho_i g z$, as well as the added stress of the water filling a crack $\sigma_w = \rho_w g[z - d_w]$. Here, z is the crack depth, $\rho$is density, and $d_w$is the depth to the top of the water in the crack. When $K_{tot}$is greater than the fracture toughness of the ice, the crack will critically propagate. For simplicity we use a single value for the fracture toughness of 0.1 MPa [*Hooke*, 2005], which strictly speaking is a function of the temperature. However, we feel this is a reasonable approximation, as the fracture toughness within the bulk of the ice sheet will not vary greatly.

Because of the dependence of equation (1) on z, critical crack propagation requires an initial (small) crack to be present in order for K_{tot} to exceed the fracture toughness. We have calculated the depth an initial crack must reach before it starts to critically propagate for the water-filled condition as a function of the differential stress which varies widely in the ice sheet from tensile to compressive. We find this initial depth is typically less tha 1 m, but under cases of neutral differential stress, or slight compression, initial cracks of only 4-7 m are necessary (Figure S1). This initial flaw length is frequently observed in many regions of the Greenland Ice Sheet in the form of dry crevasses. These dry crevasses may be created far from supraglacial lakes in areas of overall tension, and once advected into the lake basin may serve as the initial cracks needed for water-filled propagation.

The depth of a crack can then be determined as a function of the fracture toughness, water content, and the longitudinal stress. This crack length is unbounded in the case of water-filled cracks because K_W and K_I are of opposite signs, and K_W is always greater than K_I (Figure S2). The length of the crack (z) is then used to calculate the opening geometry of an edge crack after *Weertman* [1996]:

$$D(y) = \frac{2\alpha\sigma}{\mu}\sqrt{z^2-y^2} + \frac{2\alpha\rho_i g}{\pi\mu} z\sqrt{z^2-y^2} - \frac{2\alpha\rho_w g}{\pi\mu}\sqrt{z^2-d_w^2}\sqrt{z^2-y^2} - 2\alpha\rho_i g\frac{y^2}{2\pi\mu}\ln\left(\frac{z+\sqrt{z^2-y^2}}{z-\sqrt{z^2-y^2}}\right) + 2\alpha\rho_w g\frac{y^2-d_w^2}{2\pi\mu}\ln\left|\frac{\sqrt{z^2-d_w^2}+\sqrt{z^2-y^2}}{\sqrt{z^2-d_w^2}-\sqrt{z^2-y^2}}\right| - 2\alpha\rho_w g\frac{d_w y}{\pi\mu}\ln\left|\frac{d_w\sqrt{z^2-y^2}+y\sqrt{z^2-d_w^2}}{d_w\sqrt{z^2-y^2}-y\sqrt{z^2-d_w^2}}\right| + 2\alpha\rho_w g\frac{d_w^2}{\pi\mu}\ln\left|\frac{\sqrt{z^2-y^2}+\sqrt{z^2-d_w^2}}{\sqrt{z^2-y^2}-\sqrt{z^2-d_w^2}}\right| \quad (3)$$

where stress, σ, is expressed as:

$$\sigma = \sigma_t - \frac{2\rho_i g z}{\pi} - \rho_w g d_w + \frac{2}{\pi}\rho_w g d_w \arcsin\left(\frac{d_w}{z}\right) + \frac{2\rho_w g}{\pi}\sqrt{z^2 - d_w^2} \quad (4)$$

The derivation of these equations is outlined by *Weertman* [1973, 1996]. $D(y)$ is the displacement of one side of the crack with respect to the center line as a function of the depth, y. α is $(1-\nu)$, where ν is Poisson's ratio and a value of 0.3 is used for ice. The depth (d_w) to the top of the water in the crack as measured from the surface can be varied, and $d_w = 0$ for water filled cracks. The shear modulus for ice (μ) has been found to vary based on loading rates, grain size, and temperature, none of which are constrained by our model. We use 3 different values across the known range for the shear moduli (3.9, 1.5, and 0.32 GPa) to determine the sensitivity in our model to this variable [*Vaughn*, 1995]. The density of water (ρ_w) and ice (ρ_i) are 1000 and 920 kg/m^3, respectively. g is the gravitational constant taken here at 9.78 m/s^2. We ignore uncompacted snow or firn at the surface of the ice sheet as it is likely a negligable component for the lake bottom environment in the Greenland ablation zone. Two example calculations are shown in Figure S3, one for the dry case ($z = d_w$) and one for a completely water filled case ($d_w = 0$).

Because the general shape of a water-filled crack does not vary significantly with depth, we approximated it as a channel with parallel sides in the calculations of lake drainage time. The mean opening of the crack is used for this calculation, and is determined by dividing twice the integral of D from 0 to z by z. The Reynolds number for this system is calculated to be on the order of 10^6 and thus falls in the turbulent regime. We follow the approach of *White* [1974] to estimate the turbulent flux of water through a channel with a mean opening determined as above (Figure S4). Using the flux the drainage time can be calculated for any lake of a known volume. If a crack is positioned underneath a supraglacial lake and is to remain water-filled throughout its propagation, then the cross sectional area of a conical lake must be at least equal to the cross-sectional area of the crack. There is a simple geometric relationship between the area of the crack and the minimum mean lake diameter necessary to keep the crack water-filled:

$$lake\ diameter \geq \sqrt{400 \cdot A} \tag{5}$$

Where A is the cross-sectional area of the crack, which can be calculated by integrating equation (3) over the entire length of the crack and multiplying it by two, to account for both sides of the crack. Using the above equation a minimum lake size can be estimated for a given ice sheet thickness (as shown in Figure 4). We also calculated drainage times for turbulent pipe flow, used to similate a moulin, which is a possible drainage mechanism for supraglacial lakes. The moulin drainage (pipe flow) represents a volume per time calculation, while the calculation for a crack drainage (channel flow) is a 2-D flux. Thus, in this case we use lake volume rather than lake area to estimate drainage time. We note that channel flow is always faster, due to the larger frictional forces associated with the pipe walls.

2 Satellite Imagery

Daily MODIS images collected throughout the 2006 melt season were used to determine the maximum summer lake extent of 1300 lakes across our study area. Lake boundaries were estimated by thresholding the ratio of the blue to the other visible channels. Lake locations were tracked through time and transient features were discarded (e.g., single-day false detections caused by clouds). Daily lake extent was then tabulated for each lake throughout each summer allowing us to determine the maximum lake surface area. Following our discussion in the main text, we prescribe an aspect ratio (mean surface diameter:maximum depth) of 100:1 to then calculate the water volume.

3 Field Observations

Although it is impossible to measure the shape and depth of many of these crevasses at depth in the field, the observations we do have support the validity of the Weertman model for opening geometry, as well as our calculations of the crack volume and our conclusions about modeling lake drainage through channel flow. Areas of the ice sheet that had previously been submerged beneath a supraglacial lake display a distinctive ‘egg-carton’ like appearance on the surface, due to the melting processes that occur at the lake bottom. We have observed many crevasses that are now frozen closed (healed) running across the ice sheet through areas displaying these egg-carton textures (Figure S5), indicating that they were once at the bottom of a lake. It is easy to see these crevasses because the ice filling them has a distinctly bluer color than surrounding lake bottom material. It is common for these crevasses to be longer than 1 km in their horizontal dimension, and they maintain a constant opening width across their entire length. Healed crevasses also typically have thin lines of bubbles indicating that they re-froze from the outside in.

Ice canyons, which are erosional features formed from surface meltwater flow, can be tens of meters deep, and provide 3-D observations of these healed crevasses (Figure S6). In these cases we see that the dilation of the crevasses remains remarkably constant in the vertical as well as horizontal dimension, consistent with a model in which the crack was water filled during its formation (Figure S3). The ‘healing’ process for these crevasses is ideal because it prevents them from being subjected to viscous flow and deformation processes, and ‘fossilizes’ the shape the crevasse has when it initially forms (i.e., parallel sides, constant opening width, etc.).

References

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