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Summer Study Program
in
GEOPHYSICAL FLUID DYNAMICS
at
The WOODS HOLE OCEANOGRAPHIC INSTITUTION

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Contents of the Volumes

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Volume II  Fellowship Lectures
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Editor's Preface

The principal lectures of this twelfth Summer Program were given by Joseph Pedlosky of the University of Chicago. On the following page one sees Dr. Pedlosky demonstrating advanced effects caused by rotation and stratification. Only in his last few lectures do these novel phenomena emerge from the analysis. This volume contains a restatement by the Fellows of these introductory lectures.

In the first weeks of the program additional afternoon and evening lectures were given by the staff with the purpose of suggesting classes of unsolved problems and techniques to attack them. Abstracts of these lectures follow the Pedlosky notes, attesting to the enthusiasms of the staff and the blunt edges of our formal tools.

Abstracts of the summer research seminars are also recorded here. These include the contributions to a micro-symposium on Turbulence organized by Robert Kraichnan. As in past years it was concluded that turbulence theory is on the verge of resolving pressing problems in the real world.

Mrs. Mary Thayer again has done all the work in assembling and reproducing the lectures. Our debt to her skill and perseverance defies repayment.

We are also indebted to the National Science Foundation for financial support and to the Woods Hole Oceanographic Institution for encouragement and shelter.

Willem V.R. Malkus
Our principal lectures, Joseph Pedlosky - rotating and stratified.
## CONTENTS OF VOLUME I

Course Lectures and Seminars

Course Lectures

by

Dr. Joseph Pedlosky
University of Chicago, Chicago, Illinois

**FLOW IN ROTATING STRATIFIED SYSTEMS**

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FLOW IN ROTATING STRATIFIED SYSTEMS

Lecture #1  
Joseph Pedlosky

1. The interest in rotating stratified flows stems from the fact that those are the flows encountered in the oceans and in the atmosphere. To study those problems, one fruitful line of attack is to consider some idealized models - realizable or not in the laboratory. One of the aims of these lectures is to give through theoretical consideration an insight into some of the mechanisms involved. Eventually consideration will be given to a particular topic, to be studied experimentally soon in Chicago.

2. Review of the basic equations

We consider a frame of reference rotating with constant angular velocity $\Omega = |\Omega| \hat{k}$. We take the $\Omega \hat{z}$ axis parallel to $\hat{k}$, and, without much loss of generality, consider situations where gravity is acting along the negative $\Omega \hat{z}$ direction: $g = -|g| \hat{k}$.

Then, denoting by $q$ the velocity observed in the rotating frame, and with standard notation for the other quantities, we have:

a) Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0$$

or

$$\frac{D \rho}{D t} + \rho \nabla \cdot \mathbf{q} = 0$$

(1.a) (1.b)
b) Momentum:

\[ \rho \left( \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} \right) + 2 \Omega \times \mathbf{q} = -\nabla \mathbf{p} + \rho \nabla \phi + \mu \nabla^2 \mathbf{q} - \frac{\alpha}{\rho} \mathbf{u} \nabla \cdot \mathbf{q} \quad (2) \]

\( \phi \) is the force potential, including centrifugal potential:

\[ \phi = -g z + \frac{1}{2} (\Omega \times \mathbf{n})^2 \]

c) Energy:

\[ \rho \frac{\partial e}{\partial t} = \kappa \nabla^2 T + p \frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) + \mathcal{D} \quad (3) \]

d) State relation: usually \( \rho = \rho(T, p) \); in our case \( \rho = \rho_0 \left[ 1 - \alpha (T - T_0) \right] \)

where \( \mathcal{D} = \frac{1}{2} \sum_{i,j} (\mathbf{q}_i \cdot \nabla \mathbf{q}_j) \) is the viscous dissipation.

Although Eqs. (1), (2) and (3) constitute a closed system, it is convenient to introduce the following equation for the entropy \( S \)

\[ \rho T \frac{\partial S}{\partial t} = \rho \frac{\partial e}{\partial t} - p \frac{\partial (1/\rho)}{\partial t} \quad (4) \]

This, together with the thermodynamic relation

\[ \delta S = \frac{\partial A}{\partial P} \right|_T \delta P + \frac{\partial A}{\partial T} \right|_P \delta T = \frac{C_p}{T} \delta T - \frac{\partial (1/\rho)}{\partial T} \right|_P \delta P \quad (5) \]

enables us to write (3) in the more convenient form:

\[ \rho \left[ C_p \frac{\partial T}{\partial t} - \frac{\alpha}{\rho} \frac{\partial p}{\partial t} \right] = \kappa \nabla^2 T + \mathcal{D} \quad (6) \]

where \( \alpha = -\left( \frac{\rho}{T} \right) \frac{\partial \rho}{\partial T} \)

Our basic set of equations is now (1), (2) and (6); they are non-linear and represent a coupling between the temperature, velocity, pressure and density fields,

3. **Sealing**

In order to reduce this system to a more tractable one it is necessary to do some approximations. For this purpose we non-dimensionalize
the equations by a scaling process. We thus assume that there is a scale that characterizes the fields over most of the domain of interest (there might be some singular regions where this fails).

Let $U$ be the velocity scale, $L$ be the length scale (we assume the system has only one length scale). We also write

$$T = T_0 + \sqrt{\Delta T_T} T_S(x) + \sqrt{\Delta T_T} T'(x,y,z,t),$$

where $T_S$ is the static pressure:

$$\nabla T_S = \Delta T_T T_0 T_S(x) - \rho_S q_S + \rho_S \frac{\alpha^2 R}{R^2} \left( \frac{R}{R} \frac{R}{R} = 0 \right)$$

Now we write $q = U q'$

$\alpha = L \alpha'$

$\tau = \frac{L}{U} t'$

Since the main feature of rotating flows is the presence of the Coriolis force and we shall consider systems where this force is of the same order of magnitude as the pressure gradient, we choose

$$\mathcal{P} = \rho S \sum U L$$

We want to relate $\Delta T$ to the external parameters, like $U$; we also insist that in most regions of the flow

Coriolis force $= 0$ (horizontal pressure gradient)

Buoyancy force $= 0$ (vertical pressure gradient)$= \rho S \sum U \rho g = \Delta T$

$$\Rightarrow \Delta T = \frac{\Omega \frac{U}{g}}{g} = \frac{\alpha^2 L}{g} \varepsilon$$

where $\varepsilon = \frac{U}{\Omega L}$ is the Rossby number.
If $\Delta T$ is given, one has: $E = \frac{g \alpha \Delta T}{\Omega^2 L}$ and $\varepsilon$ is thought of as the thermal Rossby number. We can now rewrite the equations with the non-dimensional quantities; for convenience we drop the primes. With $\Theta = (\Delta T_v + \Delta T) \frac{T_v}{T}$, we have

$$
(1-\alpha \Theta)[E \frac{Dq}{Dt} + k \times q] = -\nabla p + T \hat{R} + \frac{2 \Omega^2 L}{g} T \nabla \Theta + \frac{\alpha \Delta T_v}{\varepsilon} R \frac{T_v}{T} + E \nabla^2 q - \frac{1}{2} E \nabla ( \nabla \cdot q )
$$

(11)

where $E = \frac{\nu}{\Omega L}$ is the Eckman number, $\gamma = \frac{\alpha \Delta T_v}{\varepsilon}$. We shall discuss only problems where $E \ll 1, \varepsilon \ll 1$.

$$
\nabla \cdot q = (\alpha \Delta T) \frac{Dq}{Dt} + \omega (\alpha \Delta T_v) \frac{dT_v}{Dt} \\
(1-\alpha \Theta)(\omega \frac{Dq}{Dt} + \omega S \frac{dT_v}{Dt}) = \frac{\alpha \Delta T_v}{\varepsilon} \frac{d}{c_p \alpha T_v - \frac{dT_v}{c_p \alpha T}} + E \frac{\alpha \Delta T_v}{\varepsilon} \frac{d^2 \Theta}{c_p \alpha T_v} + E \frac{\alpha \Delta T_v}{\varepsilon} \frac{d^2 \Theta}{c_p \alpha T_v}
$$

(12)

(13)

where $T_v$ is the total (dimensional) temperature

$\mathcal{S} = \frac{g \alpha \Delta T_v}{\Omega^2 L}$ is the stratification parameter

$\sigma = \frac{\gamma}{\chi}$ is the Prandtl number

and $\frac{d^2 \Theta}{c_p \alpha T_v} = 0$

For a fluid like water $c_p$ turns out to be $\sim 10^{-11}$ and dissipation is unimportant for temperature changes. Also $\alpha \Delta T \ll 1$, and $\alpha \Delta T_v \ll 1$.

Then we can simplify the equations and get the Boussinesq equations:

$$
\nabla \cdot q = 0
$$

(14)

$$
E \frac{Dq}{Dt} + k \times q = -\nabla p + T \hat{R} + E \nabla^2 q
$$

(15)

$$
E \frac{DT}{Dt} + \omega (\alpha \Delta T) \frac{dT_v}{Dt} = \frac{E}{\sigma} \nabla^2 T
$$

(16)

We have also neglected terms in $\frac{\alpha \Delta T_v}{\varepsilon}$ (centrifugal effect) and in $\frac{\alpha \Delta T_v}{\varepsilon}$ (slow circulation due to vertical temperature gradient). This is equivalent to assuming total gravitational force is anti-parallel to $\Omega$. 


We also note that under the assumption $\frac{\Omega^2 L}{2} \ll 1$, $T_s(s)$ is solution of the static equilibrium equations only if $T_s(s) = \mathbf{z}$. (This neglects a very slow steady convection.) Then (16) becomes:

$$\mathcal{E} \frac{dT}{dt} + \mathbf{w} \cdot \nabla T = \frac{E}{\sigma} \nabla^2 T$$  \hspace{1cm} (16a)

4. Potential vorticity equation

From the preceding equations we can derive a useful relation for the vorticity $\omega \equiv \nabla \times \mathbf{q}$. Taking the curl of (15):

$$\mathcal{E} \frac{D\mathbf{w}}{Dt} = \mathcal{E} (\omega \cdot \nabla)q + 2 \mathbf{\hat{R}} \cdot \nabla q + \nabla T \times \mathbf{\hat{R}} + \mathcal{E} \nabla^2 \omega$$  \hspace{1cm} (17)

The vertical component of this is (with $\mathbf{\hat{R}} = \nabla \times \mathbf{\hat{R}}$)

$$\mathcal{E} \frac{DT}{Dt} = \mathcal{E} (\omega \cdot \nabla)w + 2 \frac{\partial w}{\partial z} + \mathcal{E} \nabla^2 \mathbf{\hat{R}}$$  \hspace{1cm} (18)

Combining this with the $z$ derivative of (16a):

$$\mathcal{E} \frac{D}{Dt} \left[ y + \frac{2}{5} \frac{\partial T}{\partial z} \right] - \mathcal{E} \left\{ \nabla \left[ (1 + \frac{2}{5} \frac{\partial T}{\partial z}) \right] \right\} = \mathcal{E} (\omega \cdot \nabla)w - \frac{2E}{3} \frac{\partial w}{\partial z} \cdot \nabla T$$  \hspace{1cm} (19)

Evaluation of (a) by taking $\mathbf{w}$ from the thermal Eq.16 and of (b) by taking $\frac{\partial w}{\partial z} = \mathbf{\hat{R}} \cdot \nabla q$ from the vorticity Eq.17, shows that the right-hand side of (19) contains only terms of order $\mathcal{E}^2$ or $\mathcal{E}E$ and thus can be neglected; therefore we write:

$$\mathcal{E} \frac{D}{Dt} \left[ y + \frac{2}{5} \frac{\partial T}{\partial z} \right] = \mathcal{E} \nabla^2 \left( y + \frac{1}{\sigma^2} \frac{\partial T}{\partial a} \right)$$  \hspace{1cm} (20)

The quantity $\xi + \frac{2}{3} \frac{\partial T}{\partial a}$ is called the potential vorticity. We see that if $\mathcal{E} \ll E$, it is conserved for each fluid element. On the other hand, if $\mathcal{E} \ll E$ we have

$$\nabla^2 \left( \mathbf{\xi} + \frac{2}{\sigma^2} \frac{\partial T}{\partial z} \right) = 0$$  \hspace{1cm} (21)

This would be a conduction equation for the potential vorticity if $\sigma = \mathbf{1}$. If $\sigma \neq \mathbf{1}$ we consider (21) as a quasi-diffusion equation for the potential
vorticity, which will be useful in the study of very slow steady motions.

Even if $E$ is not small with respect to $E$, the conduction equation may hold if the circulation is constrained by conditions of symmetry. For example, if (as in cases we shall study) the motion is symmetric about the rotation axis, and steady,

$$\frac{3}{Dt} = u \frac{a}{\partial n} + w \frac{\partial}{\partial n}.$$

It turns out (as will be shown later) that over most of the field in which ease:

$$u = O(E), \quad w = O(E/\sigma s) \quad \text{if } \sigma s > E.$$

Hence the conduction equation will still hold under the weaker condition $\sigma s > E$ instead of $E > E$. This is an important consideration for the conduction problem is linear: we may thus say that we are dealing with two distinct classes of flow, viz:

1) Over that part of the fluid where our scaling parameters are accurate, flows with strong symmetries are often governed by the law of conduction of (essentially) potential vorticity,

2) In the absence of symmetry $E > E$ is enough to upset this result and the principle of conservation of potential vorticity applies.

Notes submitted by
Yves J. Desaubies
EKMAN LAYER IN A STRATIFIED, ROTATING FLUID

Lecture #2 Joseph Pedlosky

            " " " " " J.F.M. 1967 29: 609-621
            " " " " " J.F.M. 1967 29: 673-690
            Veronis, G. Tellus 1967 XIX(2): 326
            " " Tellus 1967 XIX(4): 620

Consider a cylindrical annulus filled with fluid and rotating with angular velocity $\Omega$. The top lid of the annulus rotates differentially with respect to the sides and the bottom. The top is heated to produce stable stratification before the system is set in rotation. The side walls are insulated. Let $r, \theta, z$ be cylindrical coordinates and let $u, v, w$ be the radial, swirl, and vertical velocity components in the rotating frame respectively.

The boundary conditions on the flow are:

\[(u,v,w) = 0 \text{ on all walls except the top} \]
\[(u,v,w) = (0, v_t(r), 0) \text{ for } z = L \]

where $v_t(r)$ is the velocity of the top lid in the rotating frame.

The thermal condition for the insulating side walls is

\[\frac{\partial T}{\partial r} = 0 \text{ for } r = r_i \text{ and } r_o\]

There are two alternatives for thermal conditions at the top and bottom. One can either impose a temperature at the top and bottom or require that
there, The former condition is easily achieved experimentally but leads to a complicated flow. The latter can be realized experimentally with some difficulty. The classical rotating annulus experiments were performed with the temperature fixed at the inner and outer walls.

Recall the equations of motion

\[
\nabla \cdot \mathbf{q} = 0
\]

\[
\epsilon \frac{Dq}{Dt} + 2 \hat{\kappa} \times \mathbf{q} = -\nabla p + \hat{\mathbf{T}} + E \nabla^{2} \mathbf{q}
\]

\[
\epsilon \frac{DT}{Dt} + \omega \mathbf{S} = \frac{E}{\epsilon} \nabla^{2} T
\]

The length and velocity scales are \( L \) and \( v_T(r_o) \) respectively.

For the problem we wish to study, \( E \ll 1 \) and \( \epsilon \ll 1 \). In regions where the length scale is accurately represented by \( L \) the viscous effects are negligible to \( O(1) \) as far as the force balances are concerned. We anticipate that a singular region, a boundary layer, will develop in the regions where viscous forces are important. The first such region to be studied is the Ekman layer on a horizontal plate. In particular, the effect of stratification on these layers will be considered.

Let the velocity of the lower plate, \( z = 0 \), be \( \mathbf{q}_L = \mathbf{q}_L(z, \mathbf{y}) \). We impose the condition \( \nabla \cdot \mathbf{q}_L = 0 \).

The approach adopted here assumes that for small \( \epsilon \) the length scale in the boundary layer can be obtained from linear theory. The linearized equations of motion in cartesian coordinates are:

\[
2u = -p_y + E v_{zz}
\]

\[
-2v = -p_x + E u_{zz}
\]

\[
0 = -p_z + T + E [w_{zz}]
\]
In these equations $\nabla^a$ has been replaced by $\frac{\partial}{\partial z}$, the dominant part in the boundary layer. A "tracer bracket" has been placed around $w_{zz}$ to enable us to follow the contribution of this term.

Eliminate $p$ from the first two of these equations to obtain

$$2(u_x + v_y) = E \frac{\partial}{\partial z} (v_x - u_y)$$

Also, apply $\frac{\partial}{\partial y}$ to the first equation, $\frac{\partial}{\partial x}$ to the second to obtain

$$2(v_x - u_y) = \nabla^a p - E \frac{\partial}{\partial z} (u_x + v_y)$$

Using the equation of continuity and the definition of the $z$-component of vorticity

$$\zeta = v_x - u_y$$

we obtain the following four equations in $p$, $\zeta$, $T$, $w$:

$$-2 \frac{\partial w}{\partial z} = E \frac{\partial \zeta}{\partial z}$$

$$2 \zeta = \nabla^a p + E \frac{\partial w}{\partial z}$$

$$\frac{\partial p}{\partial z} = T + E \left[ \frac{\partial w}{\partial z} \right]$$

$$\sigma - S w = E \frac{\partial T}{\partial z}$$

Now successively eliminate $T$, $p$, and $\zeta$ from these equations to obtain an equation in $w$

$$E^2 \left\{ \frac{\partial^3 w}{\partial z^3} + \nabla^a \left[ \frac{\partial w}{\partial z} \right] \right\} + \sigma - S \nabla^2 w + 4 \frac{\partial^2 w}{\partial z^2} = 0$$

The second term in the curly brackets is always negligible with respect to the first because of the large vertical gradients in the boundary layer. This term arises from the $w_{zz}$ term being traced, hence this term could be neglected.
If $\sigma S = 0$ then the equation is balanced with a length scale of $\ell = E^{\frac{1}{2}}$. As long as $\sigma S < E^{-1}$ the proper length scale remains $\ell = E^{\frac{1}{2}}$. (Physically, $\ell$ may be interpreted as the distance over which the fluid appears homogeneous.) If the first and Past terms of the equation were to be balanced the appropriate length scale would be

$$
\ell = \left( \frac{E^2}{\sigma S} \right)^{\frac{1}{2}}
= \left( \frac{g \alpha \Delta T \beta}{k} \right)^{\frac{1}{2}}
= R_a^{-\frac{1}{4}}
$$

where $R_a$ is the Rayleigh number.

The above calculations were undertaken to suggest that $E^{\frac{1}{2}}$ is the proper length scale in the boundary layer for the full problem. Now introduce the stretched variable

$$\xi = E^{\frac{1}{2}} z$$

It is assumed that the velocity $q = q(x, y, \xi)$ for the steady problem depends explicitly on $\xi$. It is convenient to scale the vertical velocity according to

$$w = E^\frac{1}{2} q(x, y, \xi)$$

The boundary layer equations of motion can now be expressed in the form

$$\begin{align*}
\epsilon \left[ u_x + v_x + w_x \right] + 2 \frac{\partial}{\partial z} \left( \rho + \frac{1}{\sigma} \right) &= - \frac{\partial}{\partial x} \left( \frac{1}{\sigma} \right) + \frac{1}{\sigma} \frac{\partial}{\partial x} \left( \frac{1}{\sigma} \right) + E \frac{\partial^2}{\partial \xi^2} \frac{\partial}{\partial z} \\
\epsilon \left[ u_x + v_x + w_x \right] + 2 \frac{\partial}{\partial z} \left( \rho + \frac{1}{\sigma} \right) &= - \frac{\partial}{\partial y} \left( \frac{1}{\sigma} \right) + \frac{1}{\sigma} \frac{\partial}{\partial y} \left( \frac{1}{\sigma} \right) + E \frac{\partial^2}{\partial \xi^2} \frac{\partial}{\partial z} \\
E \left[ u_x + w_x + w_x \right] - E^\frac{1}{8} \frac{\partial}{\partial z} &= - \frac{\partial}{\partial \xi} \left( \frac{1}{\sigma} \right) + \frac{1}{\sigma} \frac{\partial}{\partial \xi} \left( \frac{1}{\sigma} \right) + E \frac{\partial^2}{\partial \xi^2} \frac{\partial}{\partial z} \\
E \left[ u_x + w_x + w_x \right] - E^\frac{1}{8} \frac{\partial}{\partial z} &= - \frac{\partial}{\partial \xi} \left( \frac{1}{\sigma} \right) + \frac{1}{\sigma} \frac{\partial}{\partial \xi} \left( \frac{1}{\sigma} \right) + E \frac{\partial^2}{\partial \xi^2} \frac{\partial}{\partial z} \\
E \left[ u_x + w_x + w_x \right] - E^\frac{1}{8} \frac{\partial}{\partial z} &= - \frac{\partial}{\partial \xi} \left( \frac{1}{\sigma} \right) + \frac{1}{\sigma} \frac{\partial}{\partial \xi} \left( \frac{1}{\sigma} \right) + E \frac{\partial^2}{\partial \xi^2} \frac{\partial}{\partial z}
\end{align*}$$

As long as $\epsilon \ll 1$ the $O(1)$ balance in the momentum equation is
Equation (3) implies that \( \hat{p}_x \) and \( \hat{p}_y \) are independent of \( \hat{c} \). Thus, by differentiating Eqs. (2.1) and (2.3) with respect to \( \hat{c} \) we obtain the following differential equations for \( \hat{u} \) and \( \hat{v} \):

\[
-2 \hat{u} = -\hat{p}_x + \hat{u}_{\hat{c}} \tag{2.1}
\]

\[
2 \hat{v} = -\hat{p}_y + \hat{v}_{\hat{c}} \tag{2.2}
\]

\[
0 = \hat{p}_z \tag{2.3}
\]

If we introduce the function

\[
\mathcal{Y} = u + \nu v
\]

then upon integration in \( \hat{c} \) we find

\[
\mathcal{Y} = 2 \mathcal{V} + \mathcal{I}_0
\]

The solution for \( \mathcal{Y} \) which decays into the interior is

\[
\mathcal{Y} = A e^{-(1+\nu)\mathcal{V}} - \frac{1}{2} \mathcal{V} \mathcal{I}_0
\]

where the integration constants \( A \) and \( \mathcal{I}_0 \) are obtained from matching conditions. As \( \mathcal{V} \to 0 \) the flow must approach the interior, geostrophic flow so

\[
-\frac{1}{2} \nu \mathcal{I}_0 = -\frac{1}{2} \left( \mathcal{P}_{2y} - \nu \mathcal{P}_{2x} \right)
\]

At \( \mathcal{V} = 0 \) the flow must match the velocity of the lower plate so

\[
u_L + \nu v_L = A - \frac{1}{2} \left( \mathcal{P}_{2y} - \nu \mathcal{P}_{2x} \right)
\]

Thus, the solution for \( \mathcal{V} \) becomes

\[
\mathcal{V} = \frac{1}{2} \left( \mathcal{P}_{2y} + \nu \mathcal{P}_{2x} \right) + e^{-\frac{1}{2} \mathcal{V}} \left[ \nu_L + \frac{1}{2} \mathcal{P}_{2x} - \nu \mathcal{P}_{2y} \right] + e^{\frac{1}{2} \mathcal{V}} \left[ \left( \nu_L - \frac{1}{2} \mathcal{P}_{2x} \right) - \nu \mathcal{P}_{2y} \right]
\]

Expressed in vector notation this is

\[
\mathbf{q}_I = \frac{1}{2} \mathbf{k} \times \mathbf{v}_L + e^{-\frac{1}{2} \mathcal{V}} \left[ \mathbf{q}_L - \frac{1}{2} \mathbf{k} \times \mathbf{v}_L \right] - e^{\frac{1}{2} \mathcal{V}} \left[ \mathbf{q}_L - \frac{1}{2} \mathbf{k} \times \mathbf{v}_L \right]
\]

where \( \mathbf{q}_I \) is the horizontal velocity.
Since \( \hat{p} \xi = 0 \) we have

\[
\hat{p} (x, y, \xi) = \hat{p}_I (x, y, 0)
\]

From the equation of continuity

\[
\hat{w} = u_x + v_y = \nabla_n \cdot q = e^{-\xi \tan \beta} [k \cdot \nabla_q + \frac{1}{2} \nabla_n^2 q]
\]

The boundary condition \( \hat{w} = 0 \) for \( \xi = 0 \) enables us to evaluate \( \hat{w}_0 \). Thus

\[
\hat{w} = \frac{1}{2} [k \cdot \nabla_q + \frac{1}{2} \nabla_n^2 q] e^{-\xi (\cos \xi + \sin \xi)} + \hat{w}_0
\]

(But as \( \xi \rightarrow \infty \), \( \hat{w} \) approaches the interior vertical velocity, thus)

\[
\hat{w} = \frac{1}{2} [k \cdot \nabla_q + \frac{1}{2} \nabla_n^2 q] e^{-\xi (\cos \xi + \sin \xi) - 1}
\]

and \( \hat{w} = E^{\xi} \hat{w} \)

Now suppose \( q \) and \( T \) are partitioned into interior and boundary layer parts, namely

\[
q = q_I + q_B (x, y, \xi)
\]

\[
T = T_I + T_B (x, y, \xi)
\]

If \( \sigma S = O(1) \) then

\[
T_B = 0 \left[ E^{\xi} \max (E, E^{\xi}) \right]
\]

Hence, to the lowest order the boundary conditions are

i) \( w_I = 0 \) for \( z = 0, 1 \)

ii) \( T_I \) satisfies the thermal conditions.

At \( z = 1 \) the same situation prevails. The analogous solution for the vertical velocity in the upper Ekman Payer gives

\[
E^{\xi} \hat{w}(x, y, \xi) = w_I(x, y, \xi) = \frac{1}{2} E^{\xi} \left[ k \cdot \nabla_q (q_I - q_B) \right].
\]

Notes submitted by
William B. Heard
FLOW IN ROTATING STRATIFIED SYSTEMS

Lecture #3
Joseph Pedlosky

The dynamics of axially symmetric motions: the interior,

The equations governing axially symmetric, steady motions are, in the nondimensional form

\[ r - \text{momentum: } \varepsilon \left[ u u_r + \omega u_z - \frac{v^2}{r} \right] - 2v = -p_r + E \left[ \nabla^2 u - \frac{u}{r} \right] \]  

(3.1a)

\[ \theta - \text{momentum: } \varepsilon \left[ u v_r + \omega v_z + \frac{w v}{r} \right] + 2u = E \left[ \nabla^2 \theta - \frac{\theta}{r} \right] \]  

(3.1b)

\[ z - \text{momentum: } \varepsilon \left[ u w_r + \omega w_z \right] - T = -p_z + E \nabla^2 \omega \]  

(3.1c)

\[ \text{continuity: } \frac{r}{\varepsilon} (ru)_r + \omega_z = 0 \]  

(3.1d)

\[ \text{temperature: } \varepsilon \left[ u T_r + \omega T_z \right] + S \omega = \frac{E}{\varepsilon} \nabla^2 T \]  

(3.1e)

in which the nondimensional parameters are

- the Ekman number \( E = \frac{\nu}{\varepsilon L} \ll 1 \)
- the Rossby number \( \varepsilon = \frac{U}{\Omega L} \ll 1 \)  

(3.2)

- the stratification measure \( S = \frac{g \alpha \sigma T}{\varepsilon L} \)

It will be assumed that in the interior the scaling of the swirl velocity, the pressure, the temperature, and the lengths are correct, i.e., the interior is not a singular region.

First it is necessary to estimate the divergence of the vertical flow in the interior. From the Ekman boundary layers studied in the previous lecture, (Lecture #2, Eq,(2.2)),

\[ \omega^r = O(\varepsilon^{3/4}) \quad \text{at } z = 0.1 \]

and from Eq. (3.1d)

\[ \omega_z = O(\varepsilon) \]  

(3.3)
an estimate for the interior vertical velocity is found
\[ \begin{align*}
\omega_r &= O(u, E^{1/4}) \\
\end{align*} \]

This estimate for \( \omega_r \) is used in Eq. (3.1b) for finding
\[ u = O(E, E^{1/4}) \]
whence Eq. (3.3) becomes
\[ \omega_r = O(E, E^{1/4}) \]

The basic dynamics at the lowest approximation are
\[ \begin{align*}
2v &= \varphi_r \\
T &= \varphi_z \\
\end{align*} \]
although these are not sufficient to determine the flow. Eliminating the pressure between (3.6a) and (b) yields
\[ 2v = T \]
- the thermal wind equation.

**Case 1** \( S = 0 \)  [Homogeneous Fluid]

If the fluid is homogeneous and the boundary conditions impose no fluctuations in temperature, there will be no fluctuations in temperature in the fluid
\[ T = 0 \]

Then Eq. (3.6c) gives a partial statement of the Taylor-Proudman theorem.

There is no vertical shear in the swirl
\[ \begin{align*}
v_z &= 0 \\
or & \quad v = v(r) \\
\end{align*} \]

Now the Ekman efflux into the interior has a magnitude \( E^{1/4} \), while the estimate (3.5) of the divergence of the vertical velocity is smaller than \( E^{1/4} \). Thus to the first approximation in
\[ \begin{align*}
\omega_r &= 0 \\
or & \quad \omega = \omega(r) \\
\end{align*} \]
Equations (3.7) and (3.8) express the strong restraint that the rotation imposes on the flow.

As the velocities $v$ and $w$ are independent of the vertical, they can be determined by evaluating them at $z = 0, 1$ where they are constrained by the Ekman boundary layer matching. Lecture #2, Eq. (2.2) yields

\[
\begin{align*}
\text{at } z = 1 & \quad w(r) = \frac{E^k}{2} \cdot \frac{1}{r} \frac{\partial}{\partial r} r(v_T - v(r)) \\
\text{at } z = 0 & \quad w(r) = \frac{E^k}{2} \cdot \frac{1}{r} \frac{\partial}{\partial r} rv(r)
\end{align*}
\]

Then eliminating $w$

\[
\frac{1}{r} \frac{\partial}{\partial r} r v(r) = \frac{1}{2} \cdot \frac{1}{r} \frac{\partial}{\partial r} rv_T
\]

i.e. $v(r) = \frac{v_T}{2} + \frac{C}{r}$

(3.9)

where $C$ is a constant to be determined.

Consider the radial mass flux in order to find $C$.

In the interior, (3.4) gives an estimate of $O(E_4^{1/2})$ for the radial flux, while in the Ekman layers it has a magnitude of $E_4^{1/2}$. Thus the radial mass fluxes in the Ekman layers must balance. These are found using (3.9) in Lecture #2, Eq. (2.4), at the top

\[
\int_0^\infty ru_T E^k dS = E^k \left[ \frac{v_T}{4} - \frac{C}{4} \right]
\]

at the bottom

\[
\int_0^\infty ru_b E^k dS = E^k \left[ - \frac{v_T}{4} - \frac{C}{4} \right]
\]
Hence $C$ must vanish,

Thus the interior swirl flow has been found

$$V = \frac{v_T(r)}{2}$$  (3.10)

and the interior vertical flow

$$u = O(E)$$  (3.12)

As there is no vertical shear, Eq. (3.7), then Eq. (3.1b) gives a radial flow

$$w = \frac{E^{1/4}}{r} \frac{1}{r} \frac{\partial}{\partial r} (r v_T)$$  (3.11)

The detailed structure of the flow in the Ekman layers can be found by using the solution (3.10) in Lecture #2, Eqs. (2.4) and (2.5). To complete the detailed description of the flow everywhere it would be necessary to discuss $E^{1/4}$ and $E^{1/3}$ layers on the side walls.

In the Ekman layers there are radial fluxes top and bottom

$$\pm \frac{E^{1/4}}{r} \frac{1}{r} v_T(r)$$  (3.13)

Unless $v_T$ vanishes on $r = r_i$ and $r_o$, there will be radial fluxes in the layers.
This cannot connect across the interior, and so there must be side layers to accommodate it. These are found possible. One interesting result is that when \( v_T(r) \) is specified (smooth) over the full cylinder then the flux up the layer on the inner cylinder is exactly that vertical flux in the interior of the excluded central cylindrical region. This is because the Ekman layers, due to the restraint of strong rotation, are a local phenomena: the radial fluxes in (3.13) depend only on the radius at which they are calculated. Thus at \( r = r_i \) the radial flux in the Ekman layer does not know whether it came from the interior of the completed cylinder or up the side wall layer of the annulus.

Case 2 \( S \neq 0 \) (Stratified Fluid)

The correct measure of the significant stratification can be estimated by comparing the vertical flow out of the Ekman boundary layer, assuming an \( O(1) \) discrepancy between the interior and imposed boundary vertical components of vorticity

\[
\omega = O(v^2)
\]

with the vertical advection of the stratification needed to balance the diffusion of temperature, (3.1e)

\[
\omega = O(\frac{E}{\sigma S})
\]

Thus it can be expected that the fluid will behave differently from when it is homogeneous when

\[
\sigma S \geq E^\frac{1}{2}
\]

(3.14)

It will be seen later (Lecture 5) that when the stratification exceeds this, it restricts the interior flow in such a way that the Ekman suction is not as strong as \( E^\frac{1}{2} \).

The linear problem is considered, a posteriori it is possible to find
the limitations on $\varepsilon$ for the nonlinear terms to be ignored.

$$\varepsilon \ll \delta$$

The linearized equations are

$$-2v = -p_r + E \left[ \nabla \cdot u - \frac{u}{r^2} \right]$$

$$2u = E \left[ \nabla \cdot u - \frac{v}{r^2} \right]$$

$$-T = -p_r + E \nabla^2 w$$

$$\frac{1}{r} (ru) r + w = 0$$

$$\sigma S = E \nabla^2 T$$

As before, assuming the sealing of the interior is correct, i.e. $v, p, T,$ $\frac{dr}{dz}$ are of order 1, it is possible to expand

$$v = v_1^{(a)} + \ldots$$

$$u = E u_1^{(a)} + \ldots$$

$$w = \frac{E}{\sigma S} w_1^{(a)} + \ldots$$

$$T = T_1^{(a)} + \ldots$$

$$p = p_1^{(a)} + \ldots$$

The governing equations for this first approximation to the interior then become

$$2 v_1^{(a)} = p_2^{(a)}$$

$$2 u_1^{(a)} = \nabla^2 v_1^{(a)} - \frac{v_1^{(a)}}{r^2}$$

$$T_1^{(a)} = p_2^{(a)}$$

$$\frac{1}{r} (ru_1^{(a)}) r + w_1^{(a)} = 0$$

$$w_1^{(a)} = \nabla^2 T_1^{(a)}$$

This is a linear system and can thus be reduced to an equation in a single variable, e.g. pressure

$$\nabla^2 \left[ \nabla^2 p_1^{(a)} + \frac{1}{\sigma S} \frac{\partial^2 p_2^{(a)}}{\partial z^2} \right] = 0$$

(3.15)
The previous equation in the elimination is familiar:

$$
\nabla^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r v^{(a)}_1 \right) + \frac{2}{\sigma^2} \frac{\partial T^{(a)}_z}{\partial z} \right] = 0
$$

which is the conduction equation for the potential vorticity. The constraint of symmetry with respect to the azimuthal direction has made the small diffusion effects dominant in determining the steady state interior flow even though $\mathcal{E}$ may exceed $E$.

The equation (3.14) is an "elliptic" partial differential equation, and so boundary conditions have to be imposed on both horizontal and vertical boundaries. The Ekman layers on the top and bottom have been studied already in Lecture 2. The matching conditions they impose on the interior are for the vertical velocity $w^{(a)}_1$:

$$
\begin{align*}
\omega^{(a)}_1 &= \frac{\sigma S}{E^2} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( v_T - v_I^{(a)} \right) \right\} \text{ at } z = 1 \quad (3.15a) \\
\omega^{(a)}_1 &= \frac{\sigma S}{E^2} \frac{1}{r} \frac{\partial}{\partial r} r v_I^{(a)} \quad \text{ at } z = 0 \quad (3.15b)
\end{align*}
$$

and for the interior temperature $T^{(n)}_z$ those conditions imposed on the temperature which in their general form are

$$
\begin{align*}
m' T^{(n)}_1 + \ell' \frac{\partial T^{(n)}_z}{\partial z} &= 0 \quad \text{ on } z = (1, 0)
\end{align*}
$$

The side boundaries are the subject of the next lecture, and the solutions for the interior flow will be presented in Lecture 5.

Notes submitted by
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Side Wall Boundary Layers

In investigating the regions close to the inner and outer cylinder walls we follow the same procedure as in the Ekman layer analysis. The Rossby number is provisionally taken to be small so that we can look at the linear problem. From this analysis we find the scales appropriate to the linear boundary layer and from these the restriction on the size of the Rossby number.

The field variables may be written as the sum of two parts. For example:

\[ u = u_I + u_B \]

where \( u_B \) is a boundary layer correction to the interior value chosen to help satisfy the boundary conditions on the wall and to vanish rapidly as we leave the layer. Because the equations are linear the equations governing these correction variables will be the same as for the total quantities, In the dissipative terms only the radial derivatives will be important and we are left with the following set of equations.

\begin{align}
-2v_B &= -\frac{\partial p_B}{\partial r} + E \left[ \frac{\partial u_B}{\partial r} \right] \\
2u_B &= \frac{\partial \omega_B}{\partial r} \\
-T_B &= -\frac{\partial p_B}{\partial r} + E \frac{\partial u_B}{\partial r} \\
\frac{\partial u_B}{\partial r} + \frac{\partial \omega_B}{\partial z} &= 0 \\
-\sigma S\omega_B &= E \frac{\partial T_B}{\partial r} 
\end{align}

By eliminating pressure between Eqs. (1) and (3) we get:
On differentiating by $z$ and using the continuity equation we have:

$$-2v_{Bz} + T_{Br} = E \frac{\partial}{\partial r} \left\{ \left[ \frac{\partial u_B}{\partial z} \right] - \frac{\partial u_B}{\partial r} \right\}$$

where the term within the square trace brackets can now be neglected in comparison with the next term. This equation shows the departure from a thermal wind relationship within the boundary layer.

By continuing to eliminate variables we finally arrive at the sidewall boundary layer equation:

$$E^2 \frac{\partial^2 v}{\partial z^2} + \sigma S \frac{\partial^2 v}{\partial r^2} + 4 \frac{\partial^2 v}{\partial z^2} = 0$$

which is reminiscent of the Ekman layer equation:

$$E^2 \frac{\partial^2 w}{\partial z^2} + 4 \frac{\partial^2 w}{\partial z^2} + \sigma S \nabla^2 w = 0$$

The relative importance of the individual terms in this equation depends on the thickness of the boundary layer. A balance between the first and second gives a thickness $\delta_1$ of

$$\delta_1 = \frac{E^2}{(\sigma S)\nu} = \frac{(\gamma \chi)\nu}{(\rho \alpha \Delta v \nu)\nu} \left( \frac{1}{R_s} \right)$$

and makes the last term of $O(E/((\sigma S)^{3/2})$ relative to the first. Therefore

$$\sigma S > E^{3/2}$$

for this type of layer to exist. Balancing the second and third terms produces a layer of thickness

$$\delta_2 = \sqrt{\sigma S}.$$  

The same condition $\sigma S > E^{3/2}$ is necessary to make the first term of smaller magnitude than the other two. In order to apply boundary layer techniques to this layer we need the additional condition that

$$\sigma S < 1.$$
Looking back at the interior equation we lose the $\nabla \cdot \mathbf{P}_2^w$ term. This vorticity diffusion is then restricted to the boundary layer region.

If $\sigma - \mathcal{S}$ is $0(1)$ there is only one boundary layer. When $\sigma - \mathcal{S} \leq \mathcal{E}^{\frac{1}{3}}$ the balance is between the first and third terms which yields an $\mathcal{E}^{1/3}$ layer. An $\mathcal{E}^{1/4}$ boundary layer also enters in this range of $\sigma - \mathcal{S}$ but the details will be ignored here. The different possible layers for the range of values of the parameter $\sigma - \mathcal{S}$ are represented by the chart in Fig. 1. We will restrict ourselves to

![Chart](image)

Fig. 1. Features of the Dynamics.

In the consideration of the upper and lower boundary conditions on the sidewall layers it is noteworthy that these layers are all thicker than the Ekman layers. As far as the Ekman layers are concerned the sidewall layers look like an interior and we may use the Ekman layer results to determine boundary conditions. Therefore, in terms of the boundary layer variables:
after integrating with respect to $r$.

We will first look in detail at the thin inner layer of thickness $\delta_1$ - the so-called buoyancy layer.

Buoyancy Layer

At $r = r_0$ let us define a stretched coordinate

$$\eta = (r - r_0) / \delta$$

(11)

and allow all B-subscripted variables to be explicit functions of $\eta$.

If we define the boundary layer temperature scale $\theta_B$ by

$$T_B = \theta_B \bar{T}_B(\eta)$$

(12)

where $\theta_B$ is still to be determined, and $\bar{T}_B(\eta)$ is 0(1), the scales of the other variables can be established from the equations of motion in the buoyancy layer to be:

$$w_B = \sqrt{\sigma S} \hat{w}_B$$
$$u_B = \theta_B \frac{E_k}{(\sigma S)^{1/4}} \hat{u}_B$$
$$v_B = \theta_B \frac{E_k}{(\sigma S)^{1/4}} \hat{v}_B$$
$$p_B = \theta_B \frac{E}{(\sigma S)^{1/4}} \hat{p}_B$$

We now have the following set of equations:

$$-2 \hat{\theta}_B = \hat{p}_{\eta\eta} + \sigma S \hat{u}_{\eta\eta\eta}$$

(13)

$$2 \hat{u}_B = \hat{v}_{\eta\eta}$$

(14)
As we are considering the case where the pressure term in the vertical momentum balance can be safely neglected to give a balance between buoyancy and diffusion.

Equations (15) and (17) now form an independent set from which we can determine \( \hat{T} \) and \( \hat{w} \). Elimination of \( \hat{T} \) yields an Ekman-like equation

\[
\hat{w}_{\eta\eta\eta} + \hat{w}_\eta = 0
\]

This is a local equation with no \( z \) derivatives and therefore the solution will depend only on local quantities and we do not really need the boundary conditions determined from the Ekman layers.

Once \( \hat{w} \) is determined, \( \hat{T} \) is obtained from (15), \( \hat{u} \) from (16), \( \hat{\phi} \) from (14), and finally \( \hat{\theta}_B \) from (13). The results are:

\[
\hat{w}_\eta = e^{-n/\sigma} \left\{ A(s) \cos n/\sigma + B(s) \sin n/\sigma \right\}
\]

\[
\hat{T}_\eta = e^{-n/\sigma} \left\{ -A \sin n/\sigma + B \cos n/\sigma \right\}
\]

\[
\hat{u}_\eta = \left\{ e^{-n/\sigma} \right\} \left[ \frac{dA}{dz} \left( \sin n/\sigma - \cos n/\sigma \right) - \frac{dB}{dz} \left( \cos n/\sigma + \sin n/\sigma \right) \right]
\]

Now if we are fixing the temperature on the sidewalls of the annulus, then \( \hat{\theta}_B \) must be \( O(1) \) in order that \( \hat{T}_B \) can correct the interior temperature field. Because \( \hat{w}_I \) is \( O(E/\sigma S) \) we have the ratio \( \hat{w}_I/\hat{w}_B \) being \( O(E/\sqrt{E_S}) \) which is much less than unity. This implies that \( \hat{w}_B \) must be zero by itself at the boundary. On the other hand if we specify the insulating condition:
\[
\frac{\partial T}{\partial r} = \frac{\partial T}{\partial r} - \frac{\partial T}{\partial \eta} (\sigma S) \frac{\partial T}{\partial \eta}
\]  
(22)

On the boundary we must have \( \theta_0 \) of \( O\left(E^2/\sigma S\right) \). The ratio \( \eta / \eta_B \) is now \( O\left(E^2/\sigma S\right) \) which is still much less than unity provided that \( \sigma S \gg E^2 \).

We again conclude that
\[
\eta_B = 0 \quad \text{at} \quad \eta = 0.
\]

(23)

This implies that
\[
\eta = 0.
\]

The unknown \( B(\eta) \) can now be determined from the other boundary conditions. Choosing the insulating temperature condition of Eq. (22) we find:
\[
B(\theta) = -\sqrt{2} \left( \frac{\partial T}{\partial \eta} \right)_{\eta=0}
\]
(24)

From (19) we calculate a mass transport of
\[
M_B = -\frac{E}{\sigma S} \left( \frac{\partial T}{\partial \eta} \right)_{\eta=0}
\]

and it seems that the buoyancy layer can compensate for the interior mass flux. From Eq. (21) evaluated at the boundary:
\[
\frac{\partial \theta}{\partial \eta} (\eta = 0) = \frac{1}{\sqrt{2}} \left( -\frac{d B}{d \theta} \right) \frac{E}{\sigma S}
\]

As the total radial velocity must vanish at \( r = r_o \) we find:
\[
u^{\infty}_r (r = r_o) = \frac{1}{\sigma S} \frac{d B}{d \theta}
\]
(25)

a condition on the interior field similar, in form, to the one derived for the Ekman layer. The last condition is that the total swirl velocity go to zero at the boundary. In the interior \( v_1 \) is \( O(1) \) while in the buoyancy layer \( v_B \) is \( O\left(E/\sigma S^{3/4} \right) \) which is much less than unity. Therefore \( v_1 \) must vanish by itself at the sidewall and from the thermal wind equation at \( r = r_o \):
\[
T_{1r} = 2 \frac{\partial v_1}{\partial \theta} = 0
\]

and therefore:
To this order there is no **buoyancy** layer!

From Eq. (25) we have the condition on the interior field that:

\[
0 = 2u_1^{(0)} = \nabla^2 v_1^{(0)} - \nu_1^{2}/r^2
\]

\[
= \frac{\partial}{\partial r} J_1^{(0)}
\]

(27)

where \( J_1^{(0)} = \frac{1}{r} \frac{d}{dr} \frac{d}{dr} F^{(0)}_1 \). In addition the swirl velocity vanishes so that

\[
v_1^{(0)} = \frac{\partial F^{(0)}_1}{\partial r} = 0.
\]

(28)

On the upper and lower surface we have the following results from the Ekman layer analysis,

On \( z = 0 \)

\[
\nabla^2 \frac{\partial F^{(0)}_1}{\partial z} = -\frac{\sigma S}{4E} \left[ \nabla^2 F^{(0)}_1 - J_1 \right],
\]

(29)

and on \( z = 1 \)

\[
\nabla^2 \frac{\partial F^{(0)}_1}{\partial z} = -\frac{\sigma S}{4E} \left[ \nabla^2 F^{(0)}_1 - J_1 \right],
\]

(30)

as well as the temperature conditions:

\[
m_0 \frac{T_1^{(0)}}{T_0} + \ell_1 \frac{\partial T_1^{(0)}}{\partial z} = 0 \quad \text{on} \quad z = (0)
\]

(31)

where \( m_0 \) and \( \ell_1 \) are specified functions of \( r \).

The previous considerations of the Ekman regions also allow us to calculate the interior swirl velocity at the top and bottom. For \( \sigma \Sigma >> E \), we have from (29) and (30) using the fact that \( 2v_1^{(0)} = \frac{\partial F^{(0)}_1}{\partial r} \),

On \( z = 0 \):

\[
\frac{1}{r} \frac{\partial v_1^{(0)}}{\partial r} = 0
\]

and on \( z = 1 \):

\[
\frac{1}{r} \frac{\partial v_1^{(0)}}{\partial r} \nu_1^{(0)} = \frac{\partial}{\partial r} J_1.
\]

These may be solved **easily** to yield the results:

\[
v_1^{(0)}(r, 0) = \frac{A_0}{r}
\]

(32)

and

\[
v_1^{(0)}(r, 1) = v_T t \frac{A_1}{r}
\]

(33)

where \( A_0 \) and \( A_1 \) are constants to be determined.
Equations (32) and (33) determine the relative vorticity between the interior and the top and bottom at the boundary. Therefore we can calculate the Ekman layer quantities. In particular the radial mass flux in each layer can be found. In the top layer we find a flux of $E^2 A_1/2$ inward and in the bottom $E^2 A_0/2$ inward. Because of the strong stratification constraint the Ekman layers are non-divergent and these fluxes must balance each other. This implies that:

$$A_0 = -A_1$$

However, this flux can only enter the sidewall which we have discovered cannot accept such a large flux. Therefore:

$$A_0 = A_1 = 0$$

and we have the surprising result that as well as no buoyancy layer there is also no Ekman layer!

The problem for the interior is now fully posed, but for arbitrary $\sigma S$ is rather difficult to solve. We will look at the cases of $\sigma S$ small or large but still larger than the other small parameters.

For $\sigma S$ less than $O(1)$ the equation (3.15) (from Lecture #3, Eq.(3.15)) for the interior reduces to

$$\nabla^2 \frac{\partial^2 p^{(o)}}{\partial z^2} = 0$$

Integrating once with respect to $z$ we have:

$$\nabla^2 \frac{\partial p^{(o)}}{\partial z} = F(r) = w_z^{(o)}(r)$$

The interior vertical velocity $w_z^{(o)}$ is only a function of $r$ as a result of the constraint on $\sigma S$.

From the hydrostatic balance in the vertical:

$$\nabla^2 T_z^{(o)} = w_z^{(o)}(r)$$

The Ekman conditions at top and bottom are:
which when added together give:

\[
W^\omega (z=0) = \frac{\sigma S}{2E_k} \frac{1}{r} \frac{\partial}{\partial r} \left[ V^\omega (r, 0) \right]
\]

\[
W^\omega (z=1) = \frac{\sigma S}{2E_k} \frac{1}{r} \frac{\partial}{\partial r} \left[ V^\omega (r, 1) - V^\omega (r, 0) \right]
\]

(36)

Integration of the thermal wind equation over the fluid depth yields:

\[
V^\omega (r, 1) - V^\omega (r, 0) = \frac{1}{4E_k} \int_0^1 \frac{\partial T^\omega}{\partial r} \, dz.
\]

(37)

Substitution of (37) into (36) and the result into (35) produces the "interior-interior" equation.

\[
\nabla^2 T^\omega + \frac{\sigma S}{8E_k} \frac{1}{r} \frac{\partial}{\partial r} \left[ T^\omega \right] = \frac{\sigma S}{4E_k} \frac{1}{r} \frac{\partial}{\partial r} \left[ V_T \right]
\]

(38)

Notes submitted by
Nelson G. Hogg

AXIALLY SYMMETRIC CIRCULATION IN AN ANNULUS

Lecture #5 Joseph Pedlosky

In the interior our vorticity balance is of the form

\[
\nabla^2 \left[ \nabla^2 \Phi^\omega + \frac{4}{\sigma S} \frac{\partial}{\partial z} \Phi^\omega \right] = 0
\]

(1)

For \( \sigma S \ll 1 \) we saw in the previous lectures that there is a boundary layer of \( O(\sqrt{\sigma S}) \) and that the remaining "interior-interior" equation is

\[
\nabla^2 T^\omega + \frac{\sigma S}{8E_k} \nabla^2 \Phi^\omega \int_0^1 T^\omega \, dz = \frac{\sigma S}{4E_k} \frac{1}{r} \frac{\partial}{\partial r} \left[ V_T \right]
\]

(2)

We must now use the \( \sqrt{\sigma S} \) sidewall layer to set the swirl velocity to zero.

To solve the boundary layer equation we will define a stretched variable \( \eta = (r_0 - r)/\sqrt{\sigma S} \) and we will denote by bars the additive boundary
layer corrections. To first order

\[ \mathbf{v} = \mathbf{v}_{\infty} + \ldots + \mathbf{\bar{v}}(\eta) \]  

\( \mathbf{v}_{\infty} \) and \( \mathbf{\bar{v}} \) are of the same order since the existence of the boundary layer is to balance \( \mathbf{v}_{\infty} \) at the walls. The remaining variables will be scaled relative to \( \mathbf{v} \), thus

\[ p = p_{\infty} + \ldots + \sqrt{\sigma S} \bar{p}(\eta) \]
\[ u = \bar{u}_{\infty} + \ldots + \frac{E}{\sigma S} \bar{u} \]
\[ w = \frac{E}{\sigma S} \bar{w}_{\infty} + \ldots + \frac{E}{(\sigma S)^{3/2}} \bar{w} \]
\[ T = T_{\infty} + \ldots + \frac{E}{\sigma S} \bar{T} \]

The equations of motion in the \( \sqrt{\sigma S} \) layer are then

\[ -2 \mathbf{\bar{v}} = \bar{p}_\eta \]
\[ 2 \bar{u} = \bar{u}_\eta \eta \]
\[ -\bar{T} = -\bar{p}_\eta + \left(\frac{E \bar{w}}{\sigma S}\right)^3 \bar{w}_\eta \eta \]
\[ \bar{w} = \bar{T}_\eta \]
\[ \bar{u}_\eta = \bar{w}_\eta \]

We can neglect the last term in Eq. (4c) since \( \sigma S \gg \bar{E} \) and we have already studied the case in which the buoyancy was balanced by the viscous stresses. Again the boundary conditions at \( z = 0,1 \) are from the Ekman suction which yield the following relations:

\[ \text{at } z = 0 \quad \bar{w} = \bar{T}_\eta = -\frac{\sigma S}{E \bar{v}} \bar{v}_\eta = -2 \bar{v}_\eta \]
\[ \text{at } z = 1 \quad \bar{w} = \bar{T}_\eta = \frac{\sigma S}{E \bar{v}} \bar{v}_\eta = -2 \bar{v}_\eta \]
The last equality above is due to the thermal wind. Integrating the above two equations in $\eta$ yields the following condition on the swirl velocity:

$$\bar{v}_z = \pm \frac{\sigma S}{4 E V_s} \bar{v} \quad \text{at} \quad z = (0) \quad (1)$$

Elimination of $\bar{r}$, $\bar{p}$, and $\bar{w}$ from the equations of motion yields:

$$\bar{v}_z \eta + 4 \bar{v}_{z z} = 0 \quad (7)$$

Consider a solution of the form:

$$\bar{v} = \sum_k A_k e^{-\gamma_k \eta} \phi_k (\bar{z}) \quad (8)$$

Plugging this back into Eq. (7) yields:

$$\frac{d^2 \phi_k}{d \bar{z}^2} + \frac{\gamma_k^2}{4} \phi_k = 0 \quad (9)$$

The boundary conditions (6) reduce to:

$$\frac{d \phi_k}{d \bar{z}} = \pm \frac{\sigma S}{4 E V_s} \phi_k \quad \text{at} \quad \bar{z} = (0) \quad (10)$$

Note that if we multiply Eq. (9) by $\phi_k$ and integrate in $\bar{z}$, we get the following result:

$$-\int_0^1 (\frac{d \phi_k}{d \bar{z}})^2 d \bar{z} + \frac{\gamma_k^2}{4} \int_0^1 \phi_k^2 d \bar{z} - [\phi_k^2 (1) + \phi_k^2 (0)] = 0$$

Each of the terms is positive definite and this implies that $\gamma_k$ is real and that there are no oscillations, i.e., that all solutions have the proper boundary layer character.

The solution to Eq. (9) is:

$$\phi_k = \alpha_k \sin \frac{\gamma_k}{2} \bar{z} + \beta_k \cos \frac{\gamma_k}{2} \bar{z} \quad (11)$$

The boundary conditions (10) yield the following conditions:

$$\alpha_k \frac{\gamma_k}{2} = \frac{\sigma S}{4 E V_s} \beta_k \quad (12a)$$

$$\frac{\gamma_k}{2} \left[ \alpha_k \cos \frac{\gamma_k}{2} - \beta_k \sin \frac{\gamma_k}{2} \right] = \frac{\sigma S}{4 E V_s} \left[ \alpha_k \sin \frac{\gamma_k}{2} + \beta_k \cos \frac{\gamma_k}{2} \right] \quad (12b)$$
which lead to
\[
\left[ \alpha_k \cos \frac{\gamma_k}{2} \right] \frac{\alpha_S}{y} = \left[ \frac{\gamma_k}{4} - \left( \frac{\sigma_S}{y} \right)^2 \right] \sin \frac{\gamma_k}{2}
\] (12c)

In the limit of high stratification, \( \sigma_S \gg \epsilon \), \( \beta = 0 \) and \( \gamma_k = 2\pi n \), \( n = 0, 1, 2 \). This means that \( \mathbf{U} \) goes to zero at \( z = 0,1 \) and therefore the Ekman layers made no contribution, which is just what we expect in the high stratification limit. In the opposite limit \( \sigma_S \ll \epsilon \), we get two types of solutions. The first is a cosine series: \( \alpha_k = 0 \) and \( \gamma_k = 2\pi n \), \( n = 1, 2, \ldots \). The second is a solution for small \( \gamma_k \) which yields \( \gamma_k \sim \frac{1}{\sqrt{z}} \frac{\sqrt{\sigma_S}}{\epsilon \nu} \).

If we substitute this expression for \( \gamma_k \), we really have then an \( \epsilon^2 \) layer which is just the Stewartson layer.

We now have the swirl velocity in the \( \sqrt{\sigma_S} \) layer. From the thermal wind relationship we can calculate the temperature field:

\[
\frac{T}{T_0} = 2 \sum \frac{A_k}{\gamma_k} \phi_k(r) \exp(-\gamma_k \eta)
\] (13)

Let us now define a streamfunction \( \psi_i^{(\infty)} \) such that

\[
\omega_i^{(\infty)} = \frac{i}{\nu} \frac{\partial \psi_i^{(\infty)}}{\partial z}
\] (14)

Recalling that \( \omega_i^{(\infty)} \) is independent of \( z \) and reintroducing the scaling we have

\[
\psi_i = \frac{E}{\sigma_S} \psi_i^{(\infty)}(r)
\] (15)

In the boundary layer

\[
\omega = -\frac{1}{\nu} \frac{\partial \psi}{\partial \eta}
\] (16)

\[
\psi = \frac{E}{\sigma_S} \psi_i^{(\infty)}(\eta)
\] (17)

From the heat equations (4d) and (16) we have

\[
\overline{\psi} = -\frac{r_0}{\nu} \overline{T} \eta
\] (18)
The boundary condition on $\overline{v}$ at the sidewalls is that the total velocity go to zero, i.e.,

$$v(r, z) + \overline{v}(r, z) = 0$$

and similarly at $r = r_e$.

An additional boundary condition at the wall is that there be no radial flow, i.e., that the wall be a streamline. This implies that

$$\psi_z(r_e) + 2r_e \sum_k A_k \phi'_k (z) = 0 \tag{20}$$

Equations (19) and (20) combine to give the boundary condition on the interior swirl velocity, namely

$$2 \frac{\partial v^{(o)}_z}{\partial z} = \frac{1}{r_e} \psi^{(o)}_z (r_e) \tag{21}$$

Let us now convert this relationship into a condition on the interior temperature. The transport down in the boundary layer is given by

$$\frac{E}{\sigma S} \int_{\eta} \overline{w} \, d\eta = \overline{T}_T \frac{E}{\sigma S} = -2 \overline{v}_z (\frac{E}{\sigma S}) = 2 \frac{\partial v^{(o)}_z}{\partial z} \left( \frac{E}{\sigma S} \right) \tag{22}$$

Note that since $v_z^{(o)}$ is independent of $z$, the integral of $\int_{\eta} \overline{w} \, d\eta$ is independent of $z$ although $\overline{w}$ itself is not. The mass transport into the top Ekman layer is

$$M_1 = \frac{E}{\sigma S} \left[ v_T - v_T (r_0, 1) \right] \tag{23}$$

while the transport into the bottom Ekman layer is

$$M_2 = \frac{E}{\sigma S} v_T (r, 0) \tag{24}$$
Both the transports $M_1$ and $M_2$ must equal the transport in the $\sqrt{\sigma S}$ layer, namely $2E/\sigma S \times U^{(c)}_T$. This is due to the fact that the interior radial velocity is $O(E)$. Since the transport in the $\sqrt{\sigma S}$ layer is independent of $z$, we can add the two equalities to yield

$$
4U^{(e)}_T(r_0) = \frac{\sigma S}{2E} \left[ U_T - (U_T(r_0) - U_T^{(e)}(r_0, 0)) \right] 
$$

or

$$
2\frac{\partial T^{(e)}_I(r_0)}{\partial r} = \frac{\sigma S}{2E} \left[ U_T - \frac{1}{2} \int_0^r \frac{\partial T^{(e)}_I}{\partial r} \, dr \right] \quad (25)
$$

This is the boundary condition on the internal temperature. For the case of insulated walls it is actually unnecessary to find the boundary layer solution, we can easily find the transport in the $\sqrt{\sigma S}$ layer from the heat equation:

$$
\int d\bar{z} \, \overline{u_{BL}} = \int \frac{E}{\sigma S} T_{BL,rr} \, d\bar{z} = \frac{E}{\sigma S} T_{BL,rr} = -\frac{E}{\sigma S} T_{T}^{(e)}(r_0) \quad (26)
$$

where the subscript B.L. denotes the unscaled boundary layer variable.

We then equate this transport to the Ekman transports and proceed as above.

To solve the interior problem let us make a change of variable, let

$$
\Theta = T^{(e)}_I + \frac{\sigma S}{2E} \int_0^r T^{(e)}_I \, dz 
$$

Our differential equation (2) becomes

$$
\nabla^2 \Theta = -\frac{1}{r} \frac{\partial}{\partial r} r \nabla T 
$$

with

$$
\frac{\partial \Theta}{\partial r} = \frac{\sigma S}{2E} U_T \quad \text{on} \quad r = r_0 
$$

To find $T^{(e)}_I$ from $\Theta$ consider the following

$$
\int_0^r \Theta \, dz = (1 + \frac{\sigma S}{2E}) \int_0^r T^{(e)}_I \, dz
$$

and thus

$$
T^{(e)}_I = \Theta - \frac{\sigma S}{2E} \left( \frac{1}{1 + \frac{\sigma S}{2E}} \right) \int_0^r \Theta \, dz
$$
The solution to Eq. (28) is
\[ \theta = \frac{\sigma S}{4\pi \nu} \int_0^r \nabla T(r) \, dr + \Phi(r, z) \]
where \( \nabla^2 \Phi(r, z) = 0 \) and \( \frac{\partial \Phi}{\partial r} = 0 \) on \( r_0, r_1 \).

The boundary conditions on temperature are
\[ m_0 \, T_z^{(0)} + \rho_0 \, \frac{\partial T_z^{(0)}}{\partial r} = 0 \quad \text{on} \quad z = 0 \]  
These conditions determine \( \Phi \):
\[ \Phi = \sum \left[ \frac{J_0(\lambda_n r_0)}{J_1(\lambda_n r_0)} - \frac{Y_1(\lambda_n r_0)}{Y_0(\lambda_n r_0)} \right] \left[ a_n \sinh \frac{\lambda_n r}{r_0} (z - \frac{r_0}{2}) + b_n \cosh \frac{\lambda_n r}{r_0} (z - \frac{r_0}{2}) \right] \]

\( J_1, J_0, Y_1, Y_0 \) are Bessel functions, the \( \lambda_n \) are determined from the requirement that \( \frac{\partial \Phi}{\partial r} = 0 \) on \( r_0 \) and \( r_1 \) and the \( a_n \) and \( b_n \) are determined from Eq. (30). Let us call the first bracket in (31) \( \theta_n(r) \).

Now that we have \( T_z^{(0)} \) we can write \( v_z^{(0)} \) from the thermal wind relationship. Letting \( \lambda = \frac{\sigma S}{r_0} \), we have
\[ v_z^{(0)} = \frac{\nu(r)}{2} + \frac{\lambda}{r(1 + \lambda)^2} \nabla T(r) (z - \frac{r_0}{2}) \]
\[ + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\partial \Phi}{\partial r} \left[ a_n \left( \cosh \frac{\lambda_n r}{r_0} (z - \frac{r_0}{2}) - \cosh \frac{\lambda_n r}{2r_0} \right) \right] + b_n \left( \sinh \frac{\lambda_n r}{r_0} (z - \frac{r_0}{2}) - \frac{\lambda_n r (z - \frac{r_0}{2})}{r_0} \cosh \frac{\lambda_n r}{2r_0} \right) \]
In the ease of no vertical heat flux, due to the temperature produced by the mechanical driven flow, i.e., \( m_0 = m_1 = 0 \) all of the \( a_n \) and \( b_n \) are zero. As we increase the stratification, \( \lambda \), the swirl velocity goes from a constant value, \( \nabla T/r_0 \), coupled to Ekman suction, to a linear function of \( z \) in which the Ekman suction is choked off. In the more general case where \( a_n \) and \( b_n \) are not zero we get the same effect on the Ekman layer in the limit of large \( \lambda \), i.e., \( v(r_1) = \nabla T(r) \) and \( v(r_0) = 0 \).

The radial transport, in the upper Ekman layer is
and therefore $M_E$ is $O(E^2)$ for $S < O(E^4)$ but only $O(E/S)$ for $S > E^2$.

Therefore the mass transport in the sidewall layer is independent of the inner radius $r_f$ just as in the homogeneous case.

The circulation has the following structure:

Due to the discontinuity in the boundary condition there is a recirculating eddy in the upper corner region.

Finally let us look at the total dimensional temperature field $T^*$.

For the case of a rotating top with angular velocity $E\Omega$ we have

$$T^* = T_0 + \Delta T v \frac{z^2}{L} + \frac{E\Omega}{2gL} (r^2 - r_i^2)$$

when $T_z = 0$ at $z = 0, 1$ and the temperature contours have the following appearance.

Notes submitted by Joel E. Hirsh
LINEAR STABILITY OF THE SYMMETRIC FLOW (INVISCID THEORY) 1,

Lecture #6.  
Joseph Pedlosky

Some additional remarks concerning the symmetric flow.

1. In the case where \( \sigma^{-S} \gg 1 \), in the interior \( u = O(E) \) and \( w = O(E/\sigma^{-S}) \) and therefore \( w \ll u \). Physically, this means that the constraint of stratification is much stronger than the rotational constraint. To lowest order the continuity equation becomes

\[
\frac{1}{r} \frac{d}{dr} ru_I = 0,
\]

and the boundary conditions

\[ u_I = 0 \text{ on } v = r_0, r_1 \]

imply that \( u_I = 0 \) in the interior. Then the equation for the swirl velocity

\[
\nabla^2 v_I - \frac{v_I}{r^2} = 0.
\]

If the annulus is shallow this equation shows that the flow is essentially Couette and the plate velocity is transmitted to the interior of the fluid by viscous stresses.

The temperature (obtained from the thermal wind equation) will not satisfy the boundary conditions at the top and the bottom. Thus there is a thermal boundary layer of thickness \( O\left(\frac{1}{\sqrt{\sigma^{-S}}}\right) \) in which the thermal conditions are satisfied. In these boundary layers the thermal wind balance still applies and a small correction to the swirl velocity of \( O\left(\frac{1}{\sqrt{\sigma^{-S}}}\right) \) gives an \( O(1) \) change in \( T \).

2. In the situation where \( \sigma^{-S} = O(1) \) the existence of boundary layers depends critically on the side wall thermal conditions. In contrast to the case discussed above, when \( T = 0 \) on the side walls we have both buoyancy Payers and Ekman layers carrying a mass flux of \( O(E^3) \).
Linear stability of the symmetric flow.

**Inviscid theory.**

It has been shown that the potential vorticity of the motion is governed by the equation

\[
\varepsilon \frac{D}{Dt} \left( \zeta + \frac{2}{\sigma^2} \frac{\partial T}{\partial \theta} \right) + \mathcal{E} \nabla^2 \left( \zeta + \frac{2}{\sigma^2} \frac{\partial T}{\partial \theta} \right) = 0
\]

In the annulus the constraint of symmetry restricts the class of motions in such a way that even for \( \varepsilon > \mathcal{E} (\varepsilon \ll 1) \), the potential vorticity is (except in any boundary layers) governed by the conduction type equation

\[
\nabla^2 \left( \zeta + \frac{2}{\sigma^2} \frac{\partial T}{\partial \theta} \right) = 0.
\]

The possibility of non-symmetric motions removes this restriction and an asymmetric perturbation to the symmetric flow may produce an entirely different balance in the interior of the fluid.

In order to study the stability of the symmetric flow we write the symmetric state as

\[
\vec{q} = \vec{q}_s (r, \theta)
\]

\[
= (E\vec{u}_s (r, \theta), E\vec{v}_s (r, \theta), \frac{\partial}{\partial \theta} E\vec{w}_s (r, \theta)),
\]

\[
T = T_s (r, \theta).
\]

The perturbation velocities are scaled in the same way as the symmetric flow and it is assumed that Rossby and Ekman numbers of the perturbation are small, i.e., the perturbations are in hydrostatic and geostrophic balance to the lowest order. It is further assumed that the time scale of the perturbations is of the same order as the advective time scale of the flow. We write

\[
\vec{q}' = (u'(r, \theta, z, t), v'(r, \theta, z, t), \varepsilon w'(r, \theta, z, t)).
\]

(For \( \varepsilon \geq \mathcal{E} \), accelerations of the \( O(1) \) horizontal velocities give departures
from geostrophy sf $O(\varepsilon)$. This implies that the perturbation vertical velocity is $O(\varepsilon)$ in the interior.

Up to this point no restrictions have been placed on the size of the perturbation velocities relative to the symmetric flow velocities. However, the restriction $\varepsilon << 1$ implies that the azimuthal length scale of the perturbations is $O(1)$. The case $\varepsilon << E^k$ will not be considered here as all perturbations are found to decay in that case.

We write the total velocity and temperature fields as

$$\vec{q} = \vec{q}_s(r,\theta) + \vec{q}'(r,\theta, z, t),$$

$$T = T_s + T'$$

Denoting by $\zeta$ the vertical component of the total vorticity, we have that the equation of motion for the total field in the interior is

$$\frac{\partial}{\partial t} \left( \zeta + \frac{\partial}{\partial z} \frac{\partial T}{\partial z} \right) = 0$$

Here

$$\zeta = \frac{1}{r \frac{\partial}{\partial r}} r v_s + \frac{1}{r \frac{\partial}{\partial r}} r v' - \frac{1}{r} \frac{\partial u'}{\partial \theta},$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \left( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \right) + \left( - \frac{r}{\partial r} \right) \frac{\partial}{\partial \theta} + \left( \frac{\partial}{\partial \theta} \right) \frac{\partial}{\partial \theta} + \left( \frac{\partial}{\partial \theta} \right) \frac{\partial}{\partial \theta}.$$ 

Substituting for $\zeta$ and $T$ we have

$$\left[ \frac{\partial}{\partial t} + \frac{v}{r} \frac{\partial}{\partial \theta} \right] \left[ \zeta' + \frac{2}{5} \frac{\partial T}{\partial z} \right] + u' \frac{\partial}{\partial r} \left[ \zeta' + \frac{2}{5} \frac{\partial T}{\partial z} \right]$$

$$+ u' \frac{\partial}{\partial r} \left[ \zeta' + \frac{2}{5} \frac{\partial T}{\partial z} \right] = O(\varepsilon, \varepsilon, E/\sigma_s).$$

In the limit $\varepsilon << 1, E << 1$ the right-hand side is effectively zero. If the perturbation were symmetric the above equation would place no constraint on it and it would be necessary to go back to the conduction equation.
In the region of parameter space mentioned above

\[
\begin{align*}
  u' &= -\frac{1}{2\nu} \frac{\partial p'}{\partial \theta} + O(\epsilon), \\
  v' &= \frac{1}{2r} \frac{\partial p'}{\partial r} + O(\epsilon), \\
  T' &= + \frac{\partial p'}{\partial z} + O(\epsilon).
\end{align*}
\]

Substituting into (1) we have

\[
\begin{align*}
  \left[ \frac{\partial}{\partial t} + \frac{v}{r} \frac{\partial}{\partial \theta} \right] \left[ \frac{\partial^2 p'}{\partial \theta^2} + \frac{4}{5} \frac{\partial^2 p'}{\partial z^2} \right] - \frac{1}{r} \frac{\partial p'}{\partial \theta} \left[ \zeta' + \frac{2}{5} \frac{\partial T'}{\partial z} \right]_r = \\
  = \frac{1}{2} \left\{ \frac{1}{r} \frac{\partial p'}{\partial \theta} \left[ \frac{\partial^2 p'}{\partial \theta^2} + \frac{4}{5} \frac{\partial^2 p'}{\partial z^2} \right]_r - \frac{1}{r} \frac{\partial p'}{\partial r} \left[ \frac{\partial^2 p'}{\partial r^2} + \frac{4}{5} \frac{\partial^2 p'}{\partial z^2} \right]_\theta \right\}.
\end{align*}
\]

We now derive the boundary conditions on the flow.

**Vertical walls**

On \( r = r_o, r_i \)

\[ u = 0 \Rightarrow u' = 0 \Rightarrow \frac{\partial p'}{\partial \theta} = 0. \]

**Horizontal walls**

The thermal equation is

\[
\begin{align*}
  \omega' &= -\frac{1}{5} \left[ \frac{\partial^2 T'}{\partial t^2} + u' \frac{\partial T'}{\partial r} + u' \frac{\partial T'}{\partial \theta} + v' \frac{\partial T'}{\partial r} + v' \frac{\partial T'}{\partial \theta} \right].
\end{align*}
\]

It is necessary to have Ekman layers top and bottom to satisfy the no-slip condition on the perturbation. The vertical velocity pumped out by such an Ekman layer is

\[ \pm \frac{E \kappa}{2} \text{(vorticity of the perturbation)} \text{ on } z = (0). \]

This implies

\[
W' = \begin{cases} 
  \frac{E \kappa}{2 \epsilon} \zeta'(r, \theta, 0, t), \\
  -\frac{E \kappa}{2 \epsilon} \zeta'(r, \theta, 1, t).
\end{cases}
\]

This is a non-linear boundary condition, We see that the effect of viscous dissipation depends on the ratio \( E \kappa / \epsilon \).
An additional constraint on the flow

The azimuthal equation of motion may be integrated around the annulus
to give
\[ \epsilon \int_0^{2\pi} \frac{\partial v'}{\partial t} d\theta + \epsilon \int_0^{2\pi} u' \frac{\partial}{\partial r} \left( v_s + v' \right) d\theta + \int_0^{2\pi} 2u'd\theta = 0. \]

This holds for all \( r \) and in particular on the side walls. But \( u' = 0 \) on
the vertical walls (to lowest order) and it seems likely that
\[ \int_0^{2\pi} 2u'd\theta = 0 \]
to \( O(\epsilon) \). In other words, we postulate that the side walls boundary layers
cannot accept a mean radial flux of \( O(\epsilon) \). Then
\[ \frac{\partial}{\partial t} \phi v'd\theta = 0 \text{ on } r = r_0, r_1. \]

Notes submitted by
Paul F. Linden

QUASI-GEOSTROPHIC POTENTIAL VORTICITY EQUATION

Lecture #7                Joseph Pedlosky

The equation governing non-steady, non-symmetric perturbations when
\( \epsilon > \frac{\epsilon}{\epsilon} \) is the conservation of potential vorticity • hereafter referred to
as the quasi-geostrophic potential vorticity equation (Q.G.P.V.E.) because
\( u \) and \( v \) are evaluated geostrophically.

We can introduce a horizontal stream function \( \psi \) such that

\[ \begin{align*}
\psi' &= 2\psi \\
u' &= -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \\
v' &= \frac{\partial \psi}{\partial r} \\
T' &= 2 \frac{\partial \psi}{\partial r} \\
\end{align*} \]

with this definition, the Q.G.P.V.E. becomes
where \( \frac{D}{dt} = \left[ \frac{1}{2} \frac{d}{dt} + \frac{\partial}{\partial r} \right] \), the linearized total derivative and \( \frac{1}{r} \frac{\partial}{\partial r} r \mathbf{v}_s \),

the vorticity of the basic flow. The boundary conditions in terms of \( \psi \) are

\[
\frac{\partial \psi}{\partial \theta} = 0 \quad \text{on} \quad r = r_1, \ r = r_0
\]

\[
\frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial \theta} \right) = 0 \quad \text{on} \quad r = r_1, \ r = r_0
\]

and

\[
\frac{D}{dt} \left( \frac{\partial \psi}{\partial \theta} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial^2 \psi}{\partial \theta^2} \right) = \pm \frac{E \psi}{4 \epsilon} \nabla^2 \psi
\]

\[
\text{on} \quad z = (0)
\]

expanding \( \psi \) in a small parameter \( a << 1 \) where \( a \) can be regarded as the relative amplitude of the perturbation at \( t = 0 \) to the amplitude of the swirl flow. Then

\[
\psi = a \psi_1 + a^2 \psi_2 + a^3 \psi_3 + \ldots
\]

Inserting this expansion into the equation for \( \psi \) and considering those terms proportional to \( a \), we can investigate the stability of the symmetric flow; and the initial growth rate and initial structure of the perturbation if unstable.

For \( \psi_1 \):

\[
\frac{D}{dt} \left[ \nabla^2 \psi_1 + \frac{4}{3} \frac{\partial \psi_1}{\partial z} \right] - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \psi_1}{\partial r} \right) = 0
\]

where \( \Pi_z \) is the potential vorticity of the basic, symmetric state.

Boundary Conditions:

\[
\frac{\partial \psi_1}{\partial \theta} = 0 \quad \text{at} \quad z = (0)
\]

\[
\frac{\partial \psi_1}{\partial r} = 0 \quad \text{at} \quad r = r_o, r_i
\]

Since the coefficients are independent of \( \theta \) and \( t \) we can look for a solution of the form

\[
\psi_1 = R e^{i \phi \left( r, z \right)} e^{i (m \theta - \sigma t)}
\]

which amounts to specifying wavelike initial conditions. In reality this is not restricting the problem because when one considers a more general initial
condition, the non-wavelike portion always decays.

Using $v_i$ above, we obtain an equation for $\Phi$

\[
\left( \frac{\nu}{r} - \frac{a}{m} \right) \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial \theta^2} - \frac{m^2 \Phi}{r^2} = 0
\]

with boundary conditions

\[
\Phi = 0 \text{ on } r = r_1, r_0
\]

\[
\left( \frac{\nu}{r} - \frac{a}{m} \right) \frac{\partial \Phi}{\partial \theta} - \frac{1}{2} \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{i E_0}{m \xi} \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} - m^2 \Phi \right] a t \ z = (0)
\]

Considering $\sigma$ to have a real and an imaginary part

\[
\sigma = \sigma_r + i \sigma_i
\]

The condition for instability is $\sigma_i > 0$.

**Inviscid Linear Problem**

If $E_0 / \xi \ll 1$ we can set the right-hand side of the boundary conditions at $z = 0$ and $z = 1$ to zero. Before specifying $\nu_1$ and $\nu_2$ several general conclusions can be arrived at.

Suppose $\sigma_i \neq 0$

Since the complex conjugate of $\sigma$ will also be solution, $\sigma_i \neq 0$ insures an unstable mode.

Multiplying the equation for $\Phi$ by $\frac{r \Phi^*}{\left( \frac{\nu_2}{r} - \frac{a}{m} \right)}$,

where $\Phi^*$ is the complex conjugate of $\Phi$, and integrating

\[
-\int_{z=0}^{z=1} \int_{r_i}^{r_0} r d r d z \left[ \frac{\partial \Phi}{\partial \theta} \right] + \frac{m^2}{r^2} \Phi + \frac{\nu_2}{r} \frac{\partial \Phi}{\partial r} \left|_{r_1}^{r_0} \right. - \int_{r_i}^{r_0} \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} \right) r d r d z
\]

but $\frac{\partial \Phi}{\partial \theta} = \frac{\Phi}{\nu_2 - \frac{a}{m}} \frac{\partial \nu_2}{\partial \theta}$ at $z = 0, 1$

Substituting

\[
= \int_{r_i}^{r_0} \left[ r d r d z \right] \text{(real, positive definite)}
\]

\[
+ \frac{\nu_2}{r_i} \int_{r_i}^{r_0} \frac{\partial \nu_2}{\partial \theta} \left| \right|_{r_i}^{r_0} \left| \right|_{r_i}^{r_0} \left( \frac{1}{r} \frac{\partial \nu_2}{\partial r} \right) r d r d z = 0
\]
we can write
\[ \frac{1}{v_s - \frac{r \partial}{m}} \frac{1}{v_s - r \frac{\partial}{m}} (v_s - \frac{r \partial}{m}) \]

Then separating real and imaginary
\[ -\left( \int r dr d\theta \right) a \ \text{positive definite} \]

\[ + \frac{1}{m} \int \frac{r^2 | \frac{\partial}{\partial \theta} | dr}{| v_s - r \frac{\partial}{m} |} \left[ \int \left( | \frac{\partial}{\partial \theta} | r dr d\theta \right) \right] + 0 \]

Since \( \sigma_i \neq 0 \), we can conclude:
\[ \frac{1}{m} \int \frac{r^2 | \frac{\partial}{\partial \theta} | dr}{| v_s - r \frac{\partial}{m} |} \left[ \int \left( | \frac{\partial}{\partial \theta} | r dr d\theta \right) \right] = 0 \]

1. If \( T \) is fixed at \( z = 0 \) and \( z = 1 \), then \( \frac{\partial T}{\partial r} = 0 \) on \( z = 0, z = 1 \).

\[ \therefore \frac{\partial v_\theta}{\partial z} = 0 \] on \( z = 0 \) and \( z = 1 \) from the thermal wind equation. The second integral must vanish, which means that \( \frac{\partial v_\theta}{\partial r} \) must equal 0 somewhere in the fluid in order to have instability.

2. We can also consider the case where \( \frac{\partial v_\theta}{\partial r} = 0 \) everywhere. Then the first integral requires that the radial temperature gradient be of the same sign at \( z = 0 \) and \( z = 1 \).

It is possible to prove a second theorem which is a "semicircle theorem" for the eigenfrequencies.

Define \( \omega = \sigma / \beta_n \).

Then
\[ (v_s - \omega r) \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} (v_s - \omega r) \right) - \right. \frac{1}{r} \frac{\partial}{\partial r} \left( v_s - \omega r \right) - \frac{(v_s - \omega r)}{r^2} \right] = 0 \]

but we can write
\[ \frac{\partial^2 v_\theta}{\partial z^2} = \frac{\partial^2 v_\theta}{\partial z^2} (v_s - \omega r) \]

and
\[ \frac{\partial^2 v_\theta}{\partial z^2} = \frac{\partial^2 v_\theta}{\partial z^2} (v_s - \omega r) \]

Then for \( \sigma_i \neq 0 \) write: \( \Phi = W F(r, z) ; W = v_s (r, z) - \omega r \)

Insert into D.E.
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial F}{\partial r} \right) + \frac{3}{2} \frac{\partial}{\partial z} W^2 \frac{\partial F}{\partial z} \left( m^2 - 1 \right) \frac{\partial F}{\partial z} \frac{W^2}{r^2} = 0 \]
with boundary conditions

\[ F = 0 \text{ an } r = r_0, r_1 \]
\[ \frac{\partial F}{\partial z} = 0 \quad z = 0, 1 \]

Multiply \( D_z \), by \( rF \) and integrate in \( z \) and \( r \)

\[ \int_0^1 \int_0^r r dr dz \mathbf{W} \left\{ \left| \frac{3F}{3r} \right|^2 + \frac{4}{r} \left| \frac{\partial F}{\partial z} \right|^2 + \frac{m-1}{r^2} \left| F \right|^2 \right\} = 0 \]

where the \( \left| \frac{3F}{3r} \right|^2 \) and \( \left| \frac{\partial F}{\partial z} \right|^2 \) terms we obtained by integrating by parts and using the boundary condition at \( z = 0 \) and \( z = 1 \). The quantity inside \( \left\{ \right\} \) is positive definite, call it \( Q > 0 \). We can write

\[ w^2 = (v_z - r \omega_{real} - i r \omega_i)^2 = (v_z - r \omega_r)^2 - r^2 \omega_i^2 - 2 i r \omega_i (v_z - r \omega_r) \]

Then for the imaginary part

\[ w_i \int_0^r r^2 dr dz (v_z - r \omega_r) Q = 0 \]

or

\[ w_i \int_0^r r^2 dr dz \left( \frac{v_z}{r} - \omega_r \right) Q = 0 \]

which means that the real part of the angular velocity of the wave pattern is equal to the angular velocity of the symmetric flow field at some point in the fluid. This result can also be interpreted as showing that the time derivative of the wave field is of the same order as the advective time scale.

Turning our attention to the real part, we have

\[ \int_0^r r^2 dr dz \left[ v_z^2 - 2 r \omega_r v_z + r^2 \omega_r^2 - r^2 \omega_i^2 \right] Q = 0 \]

we just showed that \( v_z = r \omega_r \)

therefore we find

\[ \int_0^r r^2 dr dz \left( \frac{v_z}{r} \right) Q = \int_0^r r^2 \left( \omega_r^2 + \omega_i^2 \right) Q dr dz \]

\( v_z/r \) is always finite in the annulus, therefore we can write \( \Omega_1 < v_z/r < \Omega_0 \)

and
which leads to a condition

\[
\frac{\Omega_a - \Omega_b}{2} \geq \left\{ \omega_r - \frac{\Omega_a + \Omega_b}{2} \right\}^2 + \omega_i^2
\]

This condition can be represented graphically

This result shows that the energy available to the perturbation depends on the velocity gradients.

Also shows that the advective time scale of the basic state and the time derivative of the wave are of the same order.

We have found several necessary conditions for instability. We now proceed to solve the problem in order to find sufficient conditions. The problem we will solve is the one in which we have insulating boundary conditions at \( z = 0 \) and \( z = 1 \) and the mechanical driving is produced by a solidly rotating top

\[
\frac{\partial \mathbf{T}}{\partial z} = 0 \quad \text{on} \quad z = 0, \ z = 1
\]

with \( \sigma S < 1 \)

\[
v_T = \frac{v_T(r)}{2} + \frac{\sigma S / 8 \nu_k}{1 + \sigma S / 8 \nu_k} v_T(r) \left[ 2 - \nu_k \right]
\]

where \( v_T(r) = r \).

We define \( \delta \), a measure of the shear
Also when \( \nu_2(r) = r, \frac{\partial \nu_2}{\partial r} = 0 \)

For this specific case, the equation for conservation of potential vorticity becomes

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \tilde{\phi}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial \tilde{\phi}}{\partial \theta} \right) = 0
\]

with boundary conditions \( \tilde{\phi} = 0 \) \( r = r_0, \Gamma \)

and

\[
\left[ \frac{1}{r^2} + \delta (z-\frac{1}{2}) - \omega \right] \frac{\partial \tilde{\phi}}{\partial z} - \frac{\partial \tilde{\phi}}{\partial z} = 0 \text{ on } z = 0 \text{ and } \Gamma \text{ when } \varepsilon > > E^2.
\]

Choose \( \tilde{\phi} = \phi_{mn}(r) \Phi(z) \)

then

\[
\phi_{mn} = \frac{J_m(K_{mn} \gamma_{10})}{J_m(K_{mn})} - \frac{Y_m(K_{mn} \gamma_{10})}{Y_m(K_{mn})}
\]

with the condition

\[
0 = J_m(K_{mn} \alpha) Y_m(K_{mn}) - J_m(K_{mn}) Y_m(K_{mn} \alpha)
\]

where \( \alpha \equiv \gamma_{10}, J_n \) - Bessel function of first kind

\( Y_n \) - Neumann function.

\( D(z) \) must satisfy

\[
\frac{d^2 D}{dz^2} - K_{mn} \frac{d}{dz} D = 0
\]

then \( D(z) = A \sinh K_{mn} \frac{s^2}{2} (z - \frac{1}{2}) + B \cosh K_{mn} \frac{s^2}{2} (z - \frac{1}{2}) \)

with boundary conditions

\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi_{mn}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial \phi_{mn}}{\partial \theta} \right) \right] = \delta \left[ A \sinh \frac{K_{mn} s^2}{2} + B \cosh \frac{K_{mn} s^2}{2} \right] \text{ at } z = 1
\]

and

\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi_{mn}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial \phi_{mn}}{\partial \theta} \right) \right] = \delta \left[ -A \sinh \frac{K_{mn} s^2}{2} + B \cosh \frac{K_{mn} s^2}{2} \right] \text{ at } z = 0.
\]
In order to satisfy the boundary conditions

\[
\omega = \frac{1}{2} \pm \frac{\partial f}{\partial \nu} \left[ \frac{k_{mn} s^2}{4} - \tanh \frac{k_{mn} s^2}{4} \right] \left[ \frac{k_{mn} s^2}{4} - \coth \frac{k_{mn} s^2}{4} \right]^{1/2}
\]

Since \( \frac{k_{mn} s^2}{4} \geq \tanh \frac{k_{mn} s^2}{4} \) from \( x \geq \tanh x \)

the criteria for instability (\( \omega \) complex) is

\[
\frac{k_{mn} s^2}{4} < \coth \frac{k_{mn} s^2}{4}
\]

We note that stability depends on the parameter \( s \) not \( \alpha \), while the rate of growth does depend on \( \alpha \). Neutral behavior occurs when

\[
\frac{s^2}{4} = \coth \frac{k_{mn} s^2}{4}
\]

the intersection being \( \frac{k_{mn} s^2}{4} = 1.199 \).

The stability criteria can be illustrated by considering \( \gamma / c = 0.8 \). Then \( k_{mn} \approx 10 \) which results in an unstable mode when \( s < \frac{1}{2} \). Similarly, if \( \gamma / c = 0.4 \) instability occurs for \( s < 1.2 \).

One last comment can be made. \( \omega = \frac{1}{2} \) at the marginal point, which follows from the dispersion relationship. From the boundary conditions at \( z = 0 \) and \( z = 1 \) with \( \omega = \frac{1}{2} \) when \( \frac{k_{mn} s^2}{4} = \coth \frac{k_{mn} s^2}{4} \) we find \( A = 0 \).

Therefore \( \psi_1 = \Re \Phi_{mn} \coth \frac{k_{mn} s^2}{2} (z - 1) e^{i(m\theta - t/2)} \). With \( \psi_1 \) in this form, the non-linear terms in the quasi-geostrophic potential vorticity equation vanish and thus \( \psi_1 \) is a solution of the complete equation without regard to initial amplitude.

Notes submitted by
Frank M. Richter
LINEAR STABILITY OF THE SYMMETRIC FLOW (VISCOUS THEORY)

Lecture #8

Joseph Pedlosky

To repeat last lecture's results recall we assumed that $E'/\varepsilon \ll 1$, in which case the stability of the flow depended entirely on $S = \frac{g\alpha(\Delta T_v)}{\Omega^2 L}$.

Stability results if $S$ exceeds a certain value, depending on the aspect ratio of the tank. The condition for instability was

$$\frac{k_{mn} S^{1/4}}{4} \leq 1.2$$

or

$$S^{1/2} \leq \frac{4 \times 1.2}{k_{mn}}$$

And the lowest mode $k_{||}$ will give the least restrictive condition on $S$:

$$S_{crit}^{1/4} = \frac{1.2 \times 4}{k_{||}}$$

Now interpret this in terms of the Rossby radius of deformation

$$R = \sqrt{\frac{g\alpha(\Delta T_v)}{\Omega^2 L}}.$$

Then $S = \frac{R^2}{L^2}$, and $S_c << (number) \frac{R^2}{L^2}$ will give instability, i.e., the length scale $L$ must exceed the Rossby radius of deformation for instability. This agrees with yesterday's result that the flow may be stabilized by making the annulus narrower.

Note also that the stability did not depend on the magnitude of the shear, $S = \frac{\sigma S/\varepsilon E^2}{1 + \sigma S/\varepsilon E^2}$. The mean flow was $u_T = \frac{1}{2} + \delta (z - \frac{1}{2})$ and note last lecture's result that when $\delta \to 0$, $u_0 = \frac{1}{2}$, which just means that any perturbation on the interior flow is just carried around in the fluid interior.

Next we consider effects of viscosity, considering $\varepsilon, E^{1/2}$ to be of the same order. To see what this means,
\[
\frac{\varepsilon}{E} = \frac{U}{UL} \cdot \frac{L}{\sqrt{\nu}} \cdot \frac{U/L}{\sqrt{\nu}/L} = \text{Ekman dissipation time} \\
\text{or } = \frac{\text{spin up time}}{\text{circulation time}}
\]

This parameter is therefore a measure of the viscous dissipation rate compared to the energy input rate for a perturbation. Refer to the beginning of Lecture #7 for the derivation of the equations of motion in which we seek solutions for the stream function, \( \psi \equiv \frac{1}{2} \Phi' \), of the form

\[
\psi = R(r, z) e^{i(m \theta - \xi _0)}, \quad \omega = \frac{\xi}{\delta}
\]

\( m = \text{integer} \)

periodic in \( \theta \), the azimuthal angle

in the viscous, linear equation

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) - \frac{\partial^2 \Phi}{\partial z^2} = 0
\]

with boundary conditions

\[
\left[ \frac{1}{r} + \frac{1}{r} - \omega \right] \frac{\partial \Phi}{\partial z} - \lambda \Phi = \frac{i E \nu}{m \epsilon} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} - \frac{m^2}{r^2} \Phi \right) \right] (z = 1)
\]

\[
\left[ \frac{1}{r} - \frac{1}{r} - \omega \right] \frac{\partial \Phi}{\partial z} + \lambda \Phi = \frac{i E \nu}{m \epsilon} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} - \frac{m^2}{r^2} \Phi \right) \right] (z = 0)
\]

\[\Phi = 0 \text{ on } r_c, r_i\]

As before it is separable \( \Phi (r, z) = \Phi_{mn} (r) D (z) \) with

\[D (z) = A \sinh \frac{k_m s_k}{2} (z - \frac{1}{2}) + B \cosh \frac{k_m s_k}{2} (z - \frac{1}{2})\]

and some algebra yields the following eigenvalue equation

\[
\omega - \frac{1}{z} = \frac{i n}{2 z} \coth 2z \pm \frac{1}{2} \left\{ \frac{-r^2}{q^2 \sinh^2 q} + \frac{q^2}{q^2} (q - \cosh q) (q - \tanh q) \right\}^{1/2}
\]

where we have defined

\[
q \equiv \frac{k_m s_k}{4} \\
\tilde{r} \equiv \frac{E \nu}{E} \frac{s_k}{4m}
\]
Note here that when \( n = 0 \) (or \( \frac{E' k}{\varepsilon} \ll 1 \)) this result is the same as last time and viscosity only adds these \( n \neq 0 \) terms. Note also that the conditions for instability will depend this time on the shear \( \mathcal{S} \).

Consider what happens when \( \mathcal{S} = 0 \), then

\[
\omega - \frac{1}{2} = \begin{cases} 
- \frac{i n}{2 q} \tanh q, & (+) \text{ root} \\
- \frac{i n}{2 q} \coth q, & (-) \text{ root}
\end{cases}
\]

Thus \( \omega_i = 0 \) in both cases and recalling that \( \frac{U_{\infty}}{c} = \frac{1}{2} + \mathcal{S}(e^{-\frac{1}{2}}) \) this just says that any perturbation just rides with the fluid interior and decays slowly.

Next we would like to find the condition on \( \mathcal{S} \) which just makes \( \omega_i = 0 \) (i.e. that amount of shear which causes the flow to become unstable).

For marginal stability then

\[
\omega_i = 0
\]

\[
= - \frac{n}{2 q} \coth 2 q + \frac{1}{2} \sqrt{- \frac{n^2}{q^2} \sinh^2 2 q + \mathcal{S}^2 (q - \coth q)(q - \tanh q)}
\]

\[
= \frac{n^2}{2 q} \left( \coth \frac{2 q}{2 q} + \frac{1}{1} \right) + \frac{1}{2} \left( q - \coth q \right) (q - \tanh q) = 0
\]

or

\[
\mathcal{S} = - \frac{n^2}{(q - \coth q)(q - \tanh q)}
\]

or

\[
e \mathcal{S} = \frac{E' k S k_{\text{min}}}{H m} \left( \frac{1}{\sqrt{(\coth q - q)(q - \tanh q)}} \right)
\]

Plot...
and thus

\[
\frac{\varepsilon S}{E^{1/2}}
\]

where every scale on this curve is \(O(1)\). Thus the minimum value of shear \(S\) for instability is \(O(E^{1/2})\). The \(\mathcal{N}\) terms in the eigenvalue equation appeared due to the boundary conditions we had to satisfy. These boundary conditions give rise to a perturbation-caused Ekman boundary layer where the basic flow had none. As \(\mathcal{N} \to 0\) the unstable region fills all of the area \(q < 1.2\).

Now we will investigate the narrow gap case.

Re-scale the horizontal coordinates:
define
\[
\chi = \frac{r_t + r_i}{2} \theta - \left( \frac{1}{r_t - r_i} \right)
\]
\[
y = \frac{r - r_i}{r_t - r_i}, \quad 0 \leq y \leq 1
\]
to be a right-handed system

Then \( \Psi = R_e \Phi \left( y, z \right) e^{i(kx - \omega t)} \) where \( k = \frac{1}{2m} \), will replace the former \( R_e \Phi \left( r, z \right) e^{i(m \theta - \omega t)} \), and the equation of motion will become

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) = \frac{\partial^2 \Phi}{\partial y^2} + \Theta \left( \frac{r - r_i}{r_t - r_i} \right)
\]

Finally note that the notation will change here for \( u, v, w \):

We restate the problem (linear, viscous) with these new coordinates:

\[
(u_0(y, z) - C) \left[ \frac{\partial^2 \Phi}{\partial y^2} - k^2 \Phi + \frac{1}{\alpha} \frac{\partial^2 \Phi}{\partial z^2} \right] - \Phi \left[ \frac{\partial^2 u_0}{\partial y^2} + \frac{1}{\alpha} \frac{\partial^2 u_0}{\partial z^2} \right] = 0
\]

where
\[
u_0 = \frac{1}{\alpha} + \beta \left( z - \frac{1}{2} \right)
\]
\[G = \frac{g \alpha \left( \Gamma_v \right)}{4 \alpha^2 (\alpha - \Gamma_v)}\]
which takes into account the different horizontal and vertical scaling.

With boundary conditions

\[
(u_0(z = 0, 1) - C) \frac{\partial \Phi}{\partial z} = \frac{\nu}{\alpha} \left[ \frac{\partial^2 \Phi}{\partial y^2} - k^2 \Phi \right]; \quad z = 0
\]

where \( \beta = \frac{\gamma}{\alpha} \frac{1}{L} \frac{\nu (\alpha - \Gamma_v)}{\alpha^2 (\alpha - \Gamma_v)} \)

Now the results of this problem are essentially unchanged since the structure of the problem remains the same. Instead of Bessel function solutions,
we get \(\sin(n\pi y)\) eigenfunctions in the narrow gap.

We wish to investigate the mechanism for instability in the narrow gap problem. The equations of motion are

\[
\begin{align*}
e \left( \frac{\partial u}{\partial t} + u_s \frac{\partial u}{\partial x} \right) - v &= -p_x \\
e \left( \frac{\partial v}{\partial t} + u_s \frac{\partial v}{\partial y} \right) + u &= -p_y \quad (8.1) \\
T &= -pe \quad (8.2)
\end{align*}
\]

with \(E = \frac{\mu}{2\Omega t}, \sigma = 2\omega, \lambda = r_x - r_s, \Delta T = \frac{\mu l(2\omega)}{q_0 L^2}\), giving a slight scale change, let \(\omega = \epsilon \omega\), then

\[
\begin{align*}
\left( \frac{\partial ^2}{\partial t^2} + u_s \frac{\partial ^2}{\partial x^2} + u \frac{\partial ^2}{\partial y^2} \right) + \epsilon \omega = 0 \quad (8.4) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \epsilon \frac{\partial \omega}{\partial z} = 0 \quad (8.5)
\end{align*}
\]

Take \((0 \times u) + (2 \times v)\) and integrate over volume:

\[
\begin{align*}
e \frac{\partial}{\partial t} \left( \iiint \frac{u^2 + v^2}{2} \, dx \, dy \, dz \right) &= -\iiint (p_x u + p_y v) \, dx \, dy \, dz \\
\text{by parts:} &= \iiint p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, dx \, dy \, dz \\
\text{continuity:} &= -\epsilon \iiint p \frac{\partial \omega}{\partial z} \, dx \, dy \, dz \\
\text{(3) \Rightarrow} &= \epsilon \iiint (\omega^2) \, dx \, dy \, dz - \epsilon \iiint [p(x,y)\omega(x,y) - p(x,y)\omega(x,y)] \, dx \, dy \, dz
\end{align*}
\]

It should be remembered here that the \(\omega's\) refer to the interior problem and are not necessarily zero at \(z = 0,1\). To investigate the kinetic energy in the perturbation

\[
\langle K \rangle = \iiint \frac{u^2 + v^2}{2} \, dx \, dy \, dz
\]

\[
\begin{align*}
e \frac{\partial \langle K \rangle}{\partial t} &= \epsilon \iiint (\omega^2) \, dx \, dy \, dz + \frac{E}{2} \iiint dx \, dy \, dz \left[ p(i) \zeta(i) + p(o) \zeta(o) \right] \\
\frac{\partial \langle K \rangle}{\partial t} &= E \omega \iiint \left[ p(i) \zeta(i) + p(o) \zeta(o) \right] + \iiint (\omega^2) \, dx \, dy \, dz
\end{align*}
\]
Now, the geostrophic relation tells us that lower pressures will exist where $\zeta > 0$.

So Ekman suction is forcing fluid out of the Payer and from Bower to higher pressures:

![Diagram of flow](image)

(i.e. this does work, which damp the perturbing flow, and the first term on the right is negative.) The correlation of $\hat{\omega}$ and $T$ term must certainly be positive to get unstable flow.

Now investigate the $\langle \hat{\omega}T \rangle$ term. Multiply (4) $\times \frac{T}{G}$

$$\left[ \frac{1}{G} \langle \hat{\omega}T \rangle - \frac{\partial}{\partial t} \langle T \hat{\omega} \rangle \right]$$

giving

$$\frac{3}{\partial t} \left[ \langle K \rangle + \frac{\langle T^2 \rangle}{G} \right] = \iint \left[ \frac{V^2}{G} (-\frac{\partial T}{\partial y}) \right] + \text{(same term)}.$$  

(the $u_x \frac{2T}{\partial x}$ term integrates to zero since $u_x$ is independent of $x$ in the symmetric basic flow).

Using $-\frac{\partial T}{\partial y} = \frac{\partial u_x}{\partial z}$ (thermal wind), consider $\iint \langle vT \rangle (\frac{\partial u_x}{\partial z})$; we see that the correlation $\langle vT \rangle$ must be $> 0$ as well as $\langle \hat{\omega}T \rangle > 0$ in order to have energy in the perturbation increase. These correlations tell how the perturbation wave must move the fluid to gain energy: warmer fluid must be moved to a cooler region and vice versa.

Now let $\hat{\zeta} \equiv$ Lagrangian displacement in $z$ (not vorticity)

$\eta \equiv$ Lagrangian displacement in $y$

then

$$\left( \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} \right) (\hat{\zeta}) = (\hat{\omega}$$

$$\left[ \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} \right] (T + \eta \frac{\partial T}{\partial y} + \hat{\zeta}G) = 0 \text{ (heat equation)}$$
with solution $T = -\eta \frac{\partial T}{\partial y} - \frac{\hat{F}}{G}$.

Put this back into the integral

$$\iiint (\omega T) dx dy dz = \iiint dx dy dz \left[ \frac{d\hat{F}}{dt} \left( -\eta \frac{\partial T}{\partial y} - \frac{\hat{F}}{G} \right) \right]$$

Thus we need $\left( 1 + \frac{\eta}{G} \frac{\partial T}{\partial y} \right) < 0$ for instability.

But $\frac{\partial T}{\partial y} / G \approx \frac{\partial T}{\partial x} / G$ = slope of isotherms in the basic symmetric flow,

Putting this back in terms of dimensional (*) quantities,

$$\left( \frac{r}{r^*} \right) < \frac{\partial z^*}{\partial y^*} \bigg|_{T = \text{const.}}$$

In order for the perturbation wave to take energy from the basic flow, it must move in the above cross-hatched wedge: e.g. hotter fluid must rise, move in and into a cooler region to give energy up.

Recall that $T$ determines $\Delta\rho$ or the buoyancy. Thus as the hot and cold fluid elements get interchanged within the above wedge, the potential energy of the system on a whole is lowered. If $G$ is too large this can't be done - for example from the geometric constraint of too narrow an annulus.

Finally recall that we showed earlier that the Ekman boundary layer dissipation term must not be too great or motion even in this wedge will be damped.

Notes submitted by
Dennis Randolph Watts
NON-LINEAR STABILITY OF THE SYMMETRIC FLOW

Lecture #9  
Joseph Pedlosky


We want to find
(1) the final amplitude of a wave if it equilibrates
(2) how long it takes to reach final state
(3) the manner in which it reaches equilibration,

The basic flow is

\[ u_z(z) = \frac{1}{2} + \delta (z - z_0). \]

The full (non-linear) equation for the perturbation is

\[
\left( \frac{\partial}{\partial t} + u_z(z) \frac{\partial}{\partial x} \right) \left( \nabla^2 \psi + \frac{1}{\rho} \nabla \psi_{33} \right) + J(q, \nabla^2 \psi + \frac{1}{\rho} \nabla \psi_{33}) = 0
\]

with boundary conditions:

- At \( z = z_l \): \[ \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right] \nabla \psi + J(q, \nabla \psi_{33}) - \frac{\partial}{\partial z} \nabla^2 \psi = \pm \nu \nabla \psi \]

- At \( y = 0,1 \): \[ \frac{\partial \psi}{\partial y} = 0 \text{ and } \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \frac{\partial \psi}{\partial x} \, dx = 0 \]

where

\[ J(a, b) = a_y b_x - a_x b_y \]

\[ \nu = \frac{E x G}{\varepsilon} = \frac{1}{L} \left( \frac{\partial}{\partial x} \right)^\nu \frac{\partial}{\partial x} G \]

\[ G = \frac{\rho c (\Delta T) L}{4 \Omega^2 L^2} \]

\[ \ell = r_{0x} - r_{0y} \]

The integral \[ \int_{-\infty}^{\infty} \, dx \] may be taken over a wavelength.

We can't do the problem in general, but by allowing small non-linearity and slow growth from linear destabilization, the two can balance to give equilibration,

Inviscid Case: \( E^4 < \varepsilon \), i.e. \( A = 0 \).

For a given \( k \), linear theory yields a critical value of \( G \), i.e. \( G_c \).
for instability. Therefore let
\[ G = G_c + A \text{ where } \Delta < < 1. \]

The results of the linear problem were
\[
\mathcal{V} = R e \left\{ e^{i(ku-\sigma t)} \sin \pi \frac{m}{2} y \left[ A \cosh \frac{am}{2} + B \sinh \frac{am}{2} \right] \right\}
\]
where
\[
\sigma = \frac{k^2}{2} + \frac{k}{a_m} \left\{ \left( \frac{am}{2} \coth \frac{am}{2} - \sinh \frac{am}{2} \right) \right\}^\frac{1}{2}
\]
and
\[
a_m = \sqrt{\left( \frac{m^2 \pi^2 + k^2}{G} \right)}
\]

The critical value for \( a_m \) was
\[ a_m < 2.4 \text{ for instability} \]
so this gives \( G_c \) as a function of \( k \):
\[ G_c = \frac{2.4}{m^2 \pi^2 k^2} \]

This gives a "neutral" curve for fixed \( m \neq 0 \):

\[ G \]

stable

unstable

For a particular marginal wave number, say \( k \), the level of \( G \) is determined from \( G_c(k) \). For this level of \( G \) other waves may be unstable, but we restrict the initial conditions so that only this marginal wave is present.

The linear problem gives
\[
B = \frac{(1 + \frac{m}{2} - \frac{am}{2})(a_m \sinh \frac{am}{2} - \frac{am}{2} \coth \frac{am}{2}) A}{\sinh \frac{am}{2} - (1 + \frac{4}{2} - \frac{am}{2}) \left( \frac{am}{2} \coth \frac{am}{2} \right)}
\]
but for the marginal wave \( \left( \frac{am}{2} = \coth \frac{am}{2} \right) \), therefore, \( B = 0. \)
We can also see from the linear results that the growth rate for 
\[ G = G_c + \Delta \] 
will be the order of \( \Delta^{1/2} \). Thus there are two natural time scales in the problem:

1. a rapid scale from the real part of \( \sigma \)
2. a slow scale upon which the amplitude is growing,

The non-linear effects are weak and we expect them to "come in" on the longer time scale,

Assuming that we can separate the dependence on these time scales, we write

\[
\psi = \psi(x, y, z, t', T)
\]

where 
\[
t' = t \\
T = |\Delta|^{1/2} t
\]

Then the time derivatives must be transformed since

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + |\Delta|^{1/2} \frac{\partial}{\partial T}
\]

by the chain rule,

The function is expanded

\[
\psi = a_1 \psi_1 + a_2 \psi_2 + a^3 \psi_3 + \ldots
\]

where \( a \) is a scale for the magnitude of \( \psi \) and will be assumed small, \( a \) could be a measure of the initial amplitude of a wave-like disturbance.

If one does the strictly linear problem with this slow growth rate, one finds an amplitude equation in which

\[
\sigma \sim a \Delta
\]

where \( \Delta \) is the amount of destabilization. The non-linear contribution will result from the interaction of the wave with the change in the mean flow and will be proportional to \( a \times a^2 = a^3 \). Therefore, if \( a^3 \) balances \( a \Delta \), then \( a = 0(|\Delta|^{1/2}) \), and we can write
Substitution of this expansion into the equations leads to a sequence of problems according to the orders of $\Delta$:

The $O(|\Delta|^k)$ problem:

This is the linear, marginal problem; the solution is

$$\psi = \sum_{m} \{ A(\tau)e^{i(kx-ct)} \sin m\pi y \cosh am (z - \frac{1}{2}) \}$$

where $a_m = G_c (k^2 + m^2 \pi^2)$

$$c = \frac{k}{\beta} = \frac{1}{2}$$

The function $A(\tau)$ must be determined from the higher order problems.

The $O(|\Delta|)$ problem:

The equation for $\psi_L$ is

$$\left[ \frac{\partial^2}{\partial z^2} + \left( \frac{i}{2} + \delta(3 - \frac{1}{2}) \right) \frac{\partial^2}{\partial x^2} \right] \psi_L = - \frac{\partial}{\partial x} \left( G_c \nabla^2 \psi + \frac{\partial^2}{\partial y^2} \psi \right) - J(\psi, G_c \nabla^2 \psi + \frac{\partial^2}{\partial y^2} \psi)$$

Both terms on the right-hand side are zero since the marginal wave, i.e., $\psi_L$, has zero potential vorticity. Hence, a particular solution for $\psi_L$ has the same form as the linear solution:

$$\psi_L = B_0 \left( \sum_{m} \{ A(\tau)e^{i(kx-ct)} \sin m\pi y [A_m \cosh am (z - \frac{1}{2}) + B_m \sinh am (z - \frac{1}{2})] \} \right)$$

The boundary conditions for $\psi_L$ are

$$\left[ \frac{\partial^2}{\partial z^2} + \left( \frac{i}{2} + \delta(3 - \frac{1}{2}) \right) \frac{\partial^2}{\partial x^2} \right] \psi_L = - \frac{\partial}{\partial x} \frac{\partial \psi_L}{\partial z} \quad \text{at} \quad z = 0, 1$$

Using the above solution, the boundary conditions become

at $z = 1$: \[ i \frac{\partial}{\partial z} \left[ A_m \sinh \left( \frac{am}{2} \right) + B_m \cosh \left( \frac{am}{2} \right) \right] - i \delta \left[ A_m \cosh \left( \frac{am}{2} \right) + B_m \sinh \left( \frac{am}{2} \right) \right] = - \left( a_m \sinh \left( \frac{am}{2} \right) \right) \frac{\partial A_m}{\partial \tau} \]

at $z = 0$: \[ i \frac{\partial}{\partial z} \left[ -A_m \sinh \left( \frac{am}{2} \right) + B_m \cosh \left( \frac{am}{2} \right) \right] - i \delta \left[ A_m \cosh \left( \frac{am}{2} \right) - B_m \sinh \left( \frac{am}{2} \right) \right] = \left( a_m \cosh \left( \frac{am}{2} \right) \right) \frac{\partial A_m}{\partial \tau} \]
For the marginal wave, $\frac{d}{dt} = \coth \frac{d}{dt}$, and these two equations become redundant. In fact, $%$ drops out altogether, and then we can solve for $B_2$ in terms of $\frac{d}{dt}$. $A_2$ is arbitrary but inclusion of it would merely reproduce the basic linear solution. If we assume that the amplitude of the basic wave is given entirely by $A(T)$, we may set $A_2 = 0$. This is simply a normalization condition. Thus

$$B_2(T) = \frac{i a_m \sinh a_m}{a_m} \frac{dA}{dT}$$

There is another solution for $\Psi_1$ that must be included at this order: any function of $y$, $z$ and $T$ only, say $\phi_2(y, z, T)$. This is a slowly-varying mean flow that will be determined in the next order problem.

The total solution to this order is

$$\Psi = \Re \left\{ |A|^{\alpha} \sin \pi y \ e^{i k(x \pm 0)} [A(T) \cosh a_m(z \mp y) + \frac{|A|^2}{R} i \frac{d}{dT} a_m \sinh \frac{2a_m}{z} \sinh a_m(z \mp y)] \right\} + |A| \phi_2(y, z; T) + O(|A| %)$$

We note that the harmonic contribution that has arisen at this order gives a phase shift to the wave that is proportional to the rate of change of amplitude. In the linear problem, this phase shift is constant since the growth rate is constant. (That is, for the linear wave, $\frac{1}{A} \frac{dA}{dT} = \kappa$, a constant.) But for the non-linear problem, $\frac{1}{A} \frac{dA}{dT}$ is not constant and so the phase of the wave will vary slowly with time.

We still have not determined $A(T)$ so we must press on to the next order.

The $O(|A|^2)$ problem.

Now the perturbation in $G$, i.e. $\Delta$, enters also the non-linear interaction between the wave and the mean flow, $\phi_2$. 
The equations for \( \psi \), after the non-linear products have been calculated, are

\[
\begin{align*}
\left\{ \frac{\partial}{\partial t} + \left[ \frac{1}{2} + A(3-\frac{1}{2}) \right] \right\} \left( \frac{\partial^2 \phi}{\partial x^2} + G_c \frac{\partial \psi}{\partial x} \right) &= -\frac{\partial}{\partial t} \left( G_c \frac{\partial \psi}{\partial x} \right) + \frac{\partial^2 \psi}{\partial x^2} + \frac{a_m}{\alpha} \int \left( \frac{\partial \phi}{\partial t} \right) \left( \frac{\partial \phi}{\partial x} \right) e^{-i \omega_m t} \alpha_m^2 \cos \alpha_m y + \text{complex conjugate} \\
+ \frac{\partial^2 \phi}{\partial y^2} \left( \frac{\partial \phi}{\partial y} \right) &+ (\frac{1}{2}) \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial y} \right) e^{i \omega_m t} \alpha_m^2 \cos \alpha_m y + \text{complex conjugate}
\end{align*}
\]  

\[ \text{at } z = 1: \left( \frac{\partial}{\partial t} + \left[ \frac{1}{2} + A(3-\frac{1}{2}) \right] \right) \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial t} \frac{\partial \phi}{\partial y} - a_m \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \frac{a_m}{\alpha} \int \left( \frac{\partial \phi}{\partial t} \right) \left( \frac{\partial \phi}{\partial y} \right) e^{-i \omega_m t} \alpha_m^2 \cos \alpha_m y + \text{complex conjugate} \]  

plus a similar equation at \( z = 0 \).

Eliminating resonance in Eq. (A) we get an equation for \( \phi \) :

\[
\frac{\partial}{\partial t} \left[ \frac{\partial^2 \phi}{\partial y^2} + G_c \frac{\partial \phi}{\partial y} \right] = 0
\]

Similarly, we have boundary conditions on \( \phi \) from Eqs. (B):

\[
\frac{\partial}{\partial t} \frac{\partial \phi}{\partial y} = -a_m \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \psi}{\partial y} \left( |A|^2 - |A_o|^2 \right) \text{ at } y = 0, 1
\]

Or, integrating with respect to \( T \) and using \( \phi = 0 \) at \( T = 0 \) as an initial condition on \( \phi \) :

\[
\phi = -a_m \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \psi}{\partial y} \left( |A|^2 - |A_o|^2 \right) \text{ at } y = 0, 1
\]

where \( A_o = A(0) \).

On the side walls there is another condition on \( \phi \) :

\[
\frac{\partial}{\partial t} \frac{\partial \phi}{\partial y} = 0 \text{ at } y = 0, 1
\]

These equations give a solution for

\[
\psi(y, z) = \left[ \frac{\partial}{\partial y} \psi \right] \frac{\partial \psi}{\partial y} \left[ \frac{\partial \psi}{\partial y} \right] + \chi_2(y, z)
\]

where \( \chi_2 \) is a solution of

\[
\frac{\partial^2 \psi}{\partial y^2} + G_c \frac{\partial \psi}{\partial y} = 0
\]

needed to satisfy the boundary conditions,

Substituting for \( \psi \) and \( \phi \) in Eqs. (B) and removing secular terms leads to the amplitude equation:
\[
\frac{d^2 A}{d T^2} + \frac{A}{|A|} \sigma \sigma^* A + k^2 N A \left[|A|^2 - |A_0|^2\right] = 0
\]

The coefficient of the second term is just the linear growth rate squared. Note that \( \Delta \) is negative for linear instability. The constant factor \( N \) in the non-linear term has been found to be always positive.

If we let \( A(T) = R(T) e^{i \phi(T)} \), where \( R = \|A\| \), substitute for \( A \), and separate real and imaginary parts, two equations for \( R \) and \( \phi \) emerge:

\[
\frac{d^2 R}{d T^2} = \text{constant} = L, \text{ say}
\]

\[
\frac{d R}{d T} - \frac{A}{|A|} \sigma \sigma^* R - k^2 N R \left[ R^2 - R_0^2 \right] + \frac{1}{R^2}
\]

We have a second order equation for \( A \) so we need initial conditions on \( A \) and \( \frac{dA}{dT} \). Let us consider the case where the wave conforms to the linear unstable solution at \( T = 0 \). Then \( \frac{1}{A} \frac{dA}{dT} \) would be real to \( T = 0 \). But

\[
\frac{1}{A} \frac{dA}{dT} = \frac{1}{R} \frac{dR}{dT} + i \frac{d\phi}{dT}
\]

Therefore we have \( \frac{d\phi}{dT}(0) = 0 \). This gives us \( L = 0 \) so the equation for \( R \) is

\[
\frac{d^2 R}{d T^2} = -\frac{A}{|A|} \sigma \sigma^* R - k^2 N R \left[ R^2 - R_0^2 \right]
\]

A first integral can be found:

\[
\frac{1}{2} \left( \frac{dR}{dT} \right)^2 + V(R) = E, \text{ a constant}
\]

where

\[
V(R) = -\frac{k^2}{2} \left[ \frac{\sigma^2_0}{k^2} + N^2 R_0^2 \right] R^2 + \frac{k^2 N}{4} R^4
\]

\[
E = \frac{1}{2} \left[ \frac{dR}{dT}(0) \right]^2 - \frac{\sigma^2_0}{2} R_0^2 - \frac{k^2 N}{4} R_0^4
\]

after taking \( \Delta = -|\Delta| \).

This gives us the following qualitative picture of the behavior of \( R \):
\[ R \text{ will oscillate with time between the maximum and minimum amplitudes which are determined by the initial "energy" } E. \text{ Note that the amplitude can become less than the initial amplitude. Note also that the shape of the curve, and thus the minimum point, depends on the initial amplitude } R_0. \]

As a function of \( T \) can be calculated in terms of elliptic integrals. Qualitatively, it looks like

The period can also be calculated. The limiting amplitudes are given by

\[ R_{\text{max}} = \frac{1}{N} \frac{\sigma_{\text{lo}}^2}{k^2} + R_0 \pm \frac{\sigma_{\text{lo}}^2}{NK^2} \left[ 1 + \frac{2Nk^2}{\sigma_{\text{lo}}} R_e^2 \right] \]

The energy transfer process is as follows: As the wave grows, \( \frac{1}{A} \frac{dA}{dT} \) does not remain constant. Initially \( \frac{1}{A} \frac{dA}{dT} = \sigma_{\text{up}}^2 \) and the wave extracts energy from the mean field as in linear theory. As the wave grows and its amplitude departs from its initial value, \( A_0 \), the zonal flow is altered.

When \( R = R_e \) the altered zonal flow is stable according to linear theory but
$\frac{1}{A} \frac{dA}{dt} \neq 0$. Hence the wave still possesses a phase tilt with height (Pines of constant $p$ sloping upstream with height) which allows continued extraction of energy from the zonal flow. The wave continues to grow, its phase shift decreasing as it approaches its maximum amplitude $R_m$. At this point $\frac{1}{A} \frac{dA}{dt} = 0$, the wave no longer can gain energy from the mean flow (which is now quite stable). The wave amplitude then decreases with time, the phase shift is reversed and the wave returns its energy to the mean flow until the wave achieves its minimum amplitude (which is less than $A(0)$) at which point the wave proceeds to grow again starting another cycle of the oscillation.

Notes submitted by Michael A. Weissman

NON-LINEAR INSTABILITY OF SYMMETRIC FLOW, concluded:
THE EFFECTS OF DISSIPATION

Lecture #10
Joseph Pedlosky

In all the previous analysis, the effects of viscosity have been treated as negligible. We now ask what happens if a little dissipation is present, and in answer we shall quote a few results without derivation. These results do not apply directly to the continuously stratified Payer, but are from a two-layer model in which dissipation occurs at the rigid top and bottom. In this model, the interface tilts in the same sense as do the isotherms of the continuously stratified model, and the two models are indeed essentially similar in all imperrant respects except tractability.
For the two-layer model, let

\[ \Delta \rho \] be the density difference,

\[ L \] be the layer height, and

\[ \ell \] be the gap width.

The important parameter is then

\[ F = \frac{\Delta \rho L^2}{g \ell^3} \]

and the form of the amplitude equation is the same for both models. The models differ slightly in the values of the coefficients \( \sigma^2 \) and \( N \).

We shall first discuss the ease of large dissipation (\( \varepsilon = o(E^2) \)). We have

\[
\frac{dA}{dt} - \sigma(r)A + N(r)A|A|^2 = 0, \quad (1)
\]

where \( r = \frac{E^3}{\varepsilon} \)

and \( T = |A|t \).

As in the inviscid case, the first two terms of this equation are the linear results, and the third term is the non-linear effect, which here leads to equilibration. In the inviscid case, two modes grew slowly. Here, however, the order of the equation is reduced (phase shift is proportional to amplitude rather than rate of change of amplitude), and only one mode grows.

The graph depicts initial growth via linear theory leveling off to a steady finite-amplitude solution \( |A_\infty|^2 = \sigma(r\sqrt{N(r)} \) which is independent of initial conditions. The behavior is thus very different from that of the inviscid case. There the wave "remembers" its initial amplitude; here its initial phase.
We now consider an intermediate case. A critical value is \( z = 0(A)^{1/2} \), for which viscosity and growth have comparable effects on phase. With \( T = |\Delta|^{1/2}t \), the equations are

\[
\frac{d^2A}{dt^2} + \frac{3T}{|\Delta|} \frac{dA}{dT} - \frac{3T}{|\Delta|} A - \frac{\kappa^2(U_i - U_p)}{2(m^2 + \pi^2 + k^2)} A \int \sin^2 m \pi y \frac{d^2k}{dy^2} dy = 0
\]

(2)

\[
\frac{d}{dT} \left[ \frac{3T}{|\Delta|} \frac{dA}{dT} \right] = \frac{2r}{|\Delta|} \left( \frac{\kappa^2 m^2 \pi^2}{2(U_i - U_p)} \left( \frac{d}{dT} |A|^2 + \frac{2r}{|\Delta|} |A|^2 \right) \sin 2m \pi y \right)
\]

(3)

This may be integrated for \( r = 0 \). The problem reduces to the inviscid one.

If \( \frac{r}{|\Delta|} \) is large, we obtain the small-\( r \) limit of Eq. (1). In general, one can calculate the equilibrium value \( A_\infty \) by solving (2) neglecting \( T \) derivatives, using \( \Phi \) from (3) viz.,

\[
|A_\infty|^2 = \frac{A}{|\Delta|} (U_i - U_p)^2 / m^2 \pi^2 (\kappa^2 + m^2 \pi^2)
\]

To gain insight into (2), we relax the boundary conditions on the part of the flow that is independent of \( x \). Then, assuming a natural \( (\sim \sin 2m \pi y) \) periodicity in \( y \), we solve for \( \Phi \) in terms of \( A \) and obtain the following equation for \( A \):

\[
\frac{d^2A}{dt^2} + \frac{r}{|\Delta|} \frac{dA}{dT} - \frac{3r}{|\Delta|} A + \Phi \left[ A - A(0) e^{-\gamma t} + \frac{2r}{|\Delta|} k \left( \frac{\kappa^2 m^2 \pi^2}{2(U_i - U_p)} |A|^2 e^{-k^2 t} \right) \right] = 0,
\]

(4)

where

\[
P = \frac{m^2 \pi^2 k^2}{2(4m^2 \pi^2 + k^2)}
\]

and

\[
\gamma = \frac{r}{|\Delta|} \frac{4m^2 \pi^2}{\kappa^2 m^2 \pi^2 + k^2}
\]

and

\[
k_m = m^2 \pi^2 + k^2
\]

From (4), we see that the wave gradually forgets its initial conditions as time goes on and the integral causes the memory of distant past history to fade. The early growth is oscillatory, and the final steady state in the small-\( r \) limit of the previously calculated \( |A_\infty|^2 = \sigma(r)/N(r) \).

We note that the final state \( A \) is independent of the initial conditions.
and may, therefore, actually be smaller than the initial amplitude.
Finally, we reluctantly remark that numerical solutions of (4) should be informative."

To summarize briefly all the lectures, we recall that we first examined the importance of potential vorticity, its conservation and its diffusion. Next we studied the importance of boundary layers in determining the interior flow of symmetric states. Lastly, we determined that such states may be unstable (for sufficiently small Ekman number) to wave states which can grow via baroclinic instability and equilibrate via non-linearity.

All of these results have been found in a very simple thought experiment. In real geophysics, the same mechanisms are present, but in vastly more complicated forms. If we actually perform the experiment, we shall expect to find confirmation of these results. Being experienced in this field, we shall expect surprises too.

*Calculations done since have shown that when \( k^2 > 3 \frac{m^2}{\pi^2} \) the possible steady state is in fact unstable and then a limit cycle rather than a fixed amplitude is obtained for large T when \( \sqrt{\Delta} \) is small.

Notes submitted by
Richard C. J. Somerville
ABSTRACTS OF SEMINARS
INSTABILITIES OF CONVECTION ROLLS IN A HIGH PRANDTL NUMBER FLUID

Friedrich Busse

It has been shown by Schlüter, Lortz and Busse (1) that rolls represent the only stable form of convection in a fluid layer heated from below provided that the fluid satisfies the Boussinesq approximation and that the Rayleigh number exceeds the critical value by a sufficiently small amount. The range of wave number for which rolls are dynamically possible solutions of the basic equations increases strongly with $R$. The range is bounded by the curve $R_0$ in Figure 1. Only for a relatively small subrange of wave numbers $\alpha$, however, rolls are stable. The region of stability is roughly centered around the critical wave number $\alpha_c$ and is bounded by the curves B and C which correspond to different instability mechanisms: Rolls with a too small wave number become unstable to the "zig-zig instability", By bending the original rolls this instability leads to a pattern of shorter wave length. Rolls with a sufficiently high wave number become unstable to the "cross roll instability" which induces rolls perpendicular to the original ones. For Rayleigh numbers above 8000 the latter instability is also responsible for the boundary on the left side of the stability region in Fig.1.

The characteristic wave number of the marginal cross roll instability at the stability boundary is shown by the dashed line in Fig. 1. The fact that the line crosses the stability boundary at $R \approx 15000$ indicates that the cross roll instability can not induce rolls as stable stationary solution for $R \geq 15000$. Instead a three-dimensional form of convection, called bimodal convection, is induced. Figure 2 shows the streamlines for the horizontal component of the velocity of bimodal convection near the boundary,
Fig. 1

Stability region.

Fig. 2

Streamlines for the horizontal component of the velocity of bimodal convection near the boundary.
Experimental evidence suggests that bimodal convection represents the only stable form of stationary convection for $R \geq 22600$ when rolls are no longer stable. An experimental investigation of the instabilities of convection rolls has been carried out recently by Busse and Whitehead (2) in extension of the earlier work by Chen and Whitehead (3). The observations show reasonable agreement with the theoretical predictions displayed in Fig.1 which are taken from (4).

(1) Schlüter, A., D. Lortz and F. H. Busse 1965 J.Fluid Mech., 23: 129,

ON THE EARTH'S CORE-MANTLE INTERFACE

Raymond Hide

The tightness of the coupling between the core and mantle of the Earth that is implied by the occurrence of irregular fluctuations of up to about $5 \times 10^{-3}$s in the length of the day on time scales of a few years and upwards raises important questions concerning the nature of the horizontal stresses at the core-mantle interface. Electromagnetic coupling does not suffice unless very rapid magnetic fluctuations in the core - on time scales well below the "skin effect" cut-off due to the mantle - are invoked. An alternative suggestion is that the core-mantle interface is bumpy, typical undulations being a kilometre or so in height, and recent laboratory experiments on "Spin-up" in irregular containers remove one theoretical objection to this suggestion. Planetary-scale undulations of the core-mantle interface would contribute significantly to the Earth's regional gravita-
tional field, and they would also distort the magnetic field. The recent discovery of evidence of a previously unsuspected correlation between global features of the Earth's gravitational and magnetic field lends weight to the suggestion that undulations are present on the core-mantle interface. The study of theoretical models of interactions between these undulations and magnetohydrodynamic motions in the core will be a useful development in geophysical fluid dynamics.


THE PERIODS OF FINITE-DISK GENERATED GRAVITY WAVES

Louis N. Howard

Gravity waves in a cylindrical tank of water excited by vertical oscillation of a centrally placed disk exhibit resonant frequencies different from those of the cylindrical tank with a completely free surface, The lowest axisymmetric resonance has been found experimentally by Kaiser and Murty (Phys.Fl. L2: 1144(1969)) to be nearly the same as the computed natural frequency of an annulus whose inner cylinder lies just below the exciting disk.

In this lecture it is shown that resonances observed in this manner should be accurately given by natural frequencies of free surface modes in a cylinder with a partial rigid cover coinciding in position with the forcing disk. Using the general variational characterization of free surface waves, the lowest axisymmetric frequency of the "capped" cylinder is estimated, and it is found to be bounded from below by that of the annulus. A slightly more complicated but basically similar estimate from above shows that in cir-
eumstances like those of the experiments, the eapped cylinder frequency also cannot be more than a few percent above the annulus frequency, even when the "cap" is as large as 3/4 of the tank radius and the annulus frequency differs markedly from that of the free cylinder. This gives a theoretical rationalization of Kaiser and Murty's observation.

SOAP FILMS

Louis N. Howard

Primarily a popular talk centered around demonstrations of large soap films (O(1 m²)), this lecture dealt with some less obvious but everyday manifestations of surface tension phenomena, a few mathematical curiosities regarding minimal surfaces and surfaces of constant mean curvature, and the role of soap in the actual production of films. With respect to the latter, emphasis was placed on the necessity of variations in surface tension with macroscopic decay times, both for the existence and the stability of films. A simple experimental demonstration of this variation in a vertical soap film together with a discussion of the physical mechanism by which it can be produced with the aid of soap, were presented,

INERTIAL TAYLOR COLUMNS

Andrew P. Ingersoll

A homogeneous fluid is bounded above and below by horizontal plane surfaces in rapid rotation about a vertical axis. An obstacle is attached to one of the surfaces, and at large distances from the obstacle the flow is uniform and horizontal. Steady solutions are obtained as power series ex-
pansions in the Rossby number, uniformly valid as the Ekman number approaches zero.

If the height of the obstacle is greater than the Rossby number times the depth, there is a region of closed streamlines in the vicinity of the obstacle. The effect of viscosity, however, is to prohibit flow within such regions in the steady state. Thus a stagnant region (Taylor column) forms over the obstacle. Outside this region the fluid obeys inviscid equations of motion, and there is a net circulation in a direction opposite the rotation.

The shape of the Taylor column is not known a priori. This shape is determined from the free surface boundary condition on the critical streamline: free shear layers do not exist when the slope of the obstacle is much less than one. Velocity is therefore continuous in the inviscid limit. Thus velocity must vanish on the closed portion of the critical streamline. This condition applied to the inviscid exterior flow uniquely determines the shape of the Taylor column and the circulation round it in the steady state.


VENUS: THE GREENHOUSE THAT RAN AWAY

Andrew P. Ingersoll

Venus, Mars and the earth have nearly the same average density, which suggests that they have nearly the same average composition. Nevertheless, the atmospheres of the three planets do not resemble each other in any obvious way. A reconciliation is possible if one assumes that large amounts of H₂O and CO₂ have been exhaled from the interiors of all three planets, and that
differences in their atmospheres have evolved because of differences in the amount of solar heating.

On Mars, the temperatures are so low that both $\text{H}_2\text{O}$ and $\text{CO}_2$ exist mainly in solid form. The partial pressure of these gases in the Martian atmosphere is controlled by vapor-solid equilibrium at the Martian surface. On the earth the temperatures are such that $\text{H}_2\text{O}$ exists mainly in liquid form, and the partial pressure of $\text{H}_2\text{O}$ in the atmosphere is controlled mainly by vapor-liquid equilibrium at the surface. $\text{CO}_2$ in the earth's atmosphere is controlled by equilibrium with dissolved $\text{CO}_2$ in the oceans, which in turn is controlled by equilibrium with metallic carbonates in sedimentary rocks.

On Venus, the situation is less clear. Both $\text{H}_2\text{O}$ and $\text{CO}_2$ can only exist in the vapor phase, but there appears to be very little $\text{H}_2\text{O}$ on Venus at present. One explanation is that $\text{H}_2\text{O}$ has been destroyed by ultraviolet light from the sun at a rate which is several orders of magnitude greater on Venus than it is on the earth or Mars. Such a circumstance could arise if $\text{H}_2\text{O}$ were once the major constituent of the Venus atmosphere. The quantum efficiency for destruction of $\text{H}_2\text{O}$ on Venus would then have been close to one.

On the earth the quantum efficiency is about $10^{-5}$, since $\text{H}_2\text{O}$ is only a minor constituent of the earth's atmosphere. If large amounts of $\text{H}_2\text{O}$ have been dissociated on Venus, the hydrogen must have escaped into space, and the oxygen must have gone into oxidation of surface rocks.

The remainder of the argument is to show why $\text{H}_2\text{O}$ should have been a major constituent of the early Venus atmosphere, although it is a minor constituent of the earth's atmosphere. Water vapor is the major source of
infrared opacity in the earth's atmosphere, and therefore the depth of the radiating layers, where the infrared optical depth is less than one, is determined by the amount of water vapor present. However, if the amount of water vapor is determined by the vapor pressure of the liquid, then the temperature of the radiating layers cannot rise above a certain point. This means that equilibrium with the sun is impossible if the sunlight absorbed exceeds a critical value, as long as the water vapor pressure is controlled by the liquid. Calculations indicate that Venus and the earth are on opposite sides of this equilibrium, and therefore H₂O could never have existed in liquid form on Venus. Thus H₂O may have been the major constituent of the early Venus atmosphere, and this may account for the absence of H₂O on Venus today.


**TURBULENT BURSTS**

Mårten T. Landahl

Shear flow turbulence is discussed starting from the mathematical model

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 \nu - \nabla \cdot \nu \nabla^2 \nabla + C(\nu) = g + C(\nu)
\]

Here \( u, v, w \) are the fluctuating velocity components of a parallel shear flow of velocity \( U(y) \) and

\[
q = \frac{\partial^2}{\partial x_i \partial x_j} \left[ \frac{\partial}{\partial x_j} (u_j u_i - u_i u_j) \right] - \frac{\partial}{\partial x_i} \left[ \frac{\partial^2}{\partial x_i \partial x_j} (u_j u_i - u_i u_j) \right]
\]

\( (u_1 = U, \ u_2 = v, \ u_3 = w) \)

may be considered a source term due to the fluctuating Reynolds stresses (see Landahl, *J.F.M.* 29: 441). Except for the term \( C(\nu) \), the left-hand
side is the Orr-Sommerfeld operator. The model may be described as "linear ringing" (the modified Orr-Sommerfeld response) caused by "non-linear banging" (the fluctuating Reynolds stress terms). The operator $C(\nu)$, is added to include in some fashion the scattering effect of the background turbulence on the linear response (predominantly damped Orr-Sommerfeld waves, see J.F.M. 29: 441). The most important terms in $q$ (i.e., the ones containing the highest derivatives in y) are

$$q \simeq \frac{\partial^2}{\partial y^2} \left[ \frac{\partial}{\partial x} (u \nu) + \frac{\partial}{\partial z} (\nu w) \right]$$

These can be expected to be large wherever there is a locally high growth rate of fluctuations. Such is known to occur in turbulent spots. It is therefore of great importance to understand the conditions leading to bursts.

The observations by Klebanoff et al. (J.F.M. (1962), 12: 1) of bursts in a laminar boundary layer undergoing transition are, so far, the best ones to use as a basis for discussion as the experimental situation was very well controlled. These experiments showed that the bursts occurred very suddenly once a critical value of the amplitude of the primary wave is exceeded. The growth rate of the fluctuation amplitude is far too high to be explained by secondary instability. A different mechanism must therefore be at work. The explanation proposed is based on a discussion of the relationship between the wave propagation velocities of the primary and the secondary waves. When the primary wave amplitude increases, the group velocity of the secondary wave will (probably) decrease. A criticality condition is reached when the group velocity of the secondary wave becomes equal to the phase velocity of the primary one somewhere along the primary wave. At that condition, secondary disturbances travelling into the primary wave...
wave are trapped and must eventually cause complete breakdown of the primary wave to remove the critical flow region. An analogy with choking in a Laval nozzle, in which deceleration from supersonic to subsonic flow is attempted, can be drawn. A corollary of the proposed explanation is that criticality should depend exclusively on the local instantaneous velocity profile, which is in excellent accord with the observations. Furthermore, the criticality condition is calculable on basis of linear stability theory for a parallel flow.

Some consequences of the proposed model were discussed. The Reynolds stresses in the flow will be produced almost exclusively by the bursts whereas the linear random waves will contribute very little. Hence a reasonable assumption to determine approximately the operator is to assume that the fluctuations in a burst are simply convected in a "frozen" manner by the wave. This assumption leads to

\[ C(\nu) = \frac{\partial^3}{\partial x^2 \partial y} (\delta \nu^2) + \frac{\partial^3}{\partial x \partial y^2} \left[ \delta \frac{d(\nu \nu)}{dy} \right] \]

where \( \delta \) is the streamline displacement by the wave which is connected to \( \nu \) by

\[ \nu = \frac{\partial \delta}{\partial t} + \nu \frac{\partial \delta}{\partial x} \]

and the bar denotes average.

TURBULENT SHEAR FLOW STRUCTURE

Mårten T. Landahl

Some model problems believed relevant for understanding the mechanics of turbulent shear flow are proposed and discussed. Recent experimental investigations, primarily by Kline's group at Stanford, indicate that the strong turbulence production that takes place in the wall layer is associated with
local inflexional instability of strongly concentrated shear layers produced by local interaction of longitudinal vortices and the fluctuations due to the large eddies. As inflexional instability is predominantly an inviscid phenomenon, it is proposed that the non-linear behavior of a strongly unstable inflexional region can be modelled by a distribution of concentrated potential vortices interacting with each other in the presence of a rigid wall. The simple case of a concentrated two-dimensional shear layer was set up on a computer, and the development of the vortex layer following an initial small deformation in form of a bump studied. The numerical results show that a "hole" in the shear layer develops very rapidly and that it starts to roll up on each side of the hole very much in the fashion of a roller curtain. Such behavior might be consistent with the recent observations by Corino and Brodkey (J.F.M. 37: 1) of turbulent breakdown showing the formation of vertical jets on the edge of the low-speed flow region resulting in violent turbulent mixing with the higher velocity fluid above.

The "passive" region of the turbulent boundary layer is believed to react in a linear fashion to the perturbations excited by the local breakdowns in the active layer. Dominating statistically will be wave propagation modes (see Landahl, J.F.M. 29: 441). Of importance is to determine the effects of background turbulence on the wave propagation. The formulation of this problem is considered and some possible approximations discussed,
ODE TO TURBULENCE

by Märten T. Landahl
(sung to the Marine Hymn)

FROM THE LAMINAR SOLUTION
INTO FINITE AMPLITUDE
WE SHALL STRIVE FOR UNDERSTANDING
WITH MATHEMATICS NEAT OR CRUDE

BUT IF THE FLUID BECOMES TURBULENT
WITH EDDIES AND RANDOM BURSTS
WE SHALL RALLY 'ROUND THE PHYSICS
AND LET COMPUTORS DO THEIR WORST,

"WHITHER SST - BOOM OR BUST?"

Märten T. Landahl

The main economic-engineering argument for the supersonic transport is explained and aerodynamic consequences for the design demonstrated. The basic fluid dynamic reasons for the sonic boom are reviewed, and the parameters influencing its strength discussed. The physics of the phenomenon makes the chances of a substantial reduction of boom strength limited. The results of sociological investigations show quite clearly that the annoyance effect, in particularly the startle (illustrated), is so large that it is highly unlikely that any democratic government would ever allow regular supersonic flights over inhabited territory. Thus, present proposed designs are intended to operate supersonically over water, only. Some of the implications of this limitation are discussed.
CYTHERIAN ENERGETICS: HOW DOES VENUS KEEP HER COOL?

Willem V. R. Malkus

Recent work on the atmosphere of Venus is reviewed. The observed four-day circulation of the visible clouds led Schubert and Whitehead to suggest that, given a sufficiently small effective Prandtl number, the zonal flow could be a Halley circulation. In 1686 Halley proposed that a heat source moving over a layer of fluid would induce an average horizontal flow, Stern and Davey have constructed quasi-linear theories supporting this proposal. Here, the work of Davey is extended to the non-linear regime by the construction of an Oseen-type solution. Quite fortuitously, it is found that the maximum zonal velocity is insensitive to the atmospheric mixing processes, depending on the one-sixth power of the Prandtl number. The computed velocity for Venus is in keeping with the observations. Other aspects of the theory relate potential observables, such as the solar-antisolar temperature contrast and the depth of the high velocity region, to the unknown magnitudes of the turbulent transport of heat and momentum. The inference from available data is that an effective thermal diffusion coefficient for Venus is orders of magnitude larger than previously suggested. Among the unresolved problems which emerge from this study is the thermal-tidal resonant interaction. Such interaction could produce long term torques on the planet, perhaps explaining her retrograde rotation.
THE STRUCTURE AND STABILITY OF VORTEX RINGS

Tony Maxworthy

A series of observations on experimentally produced vortex rings was described. Velocity and vorticity fields were observed using dye and hydrogen bubble techniques. It was concluded that stable rings are unique entities which grow in such a way that their vorticity is always contained within the fluid mass moving with the ring. As the vorticity diffuses out of the ring into the outer irrotational fluid it causes this fluid to be entrained into the interior of the bubble. It is possible that an exponentially small amount of vorticity escapes reentrainment and is shed into a very small wake behind the moving fluid mass. It has been found that the velocity of these stable rings varies as \( t^{-1} \); where \( t \) is the time measured from a virtual origin of the motion at downstream infinity. A simple entrainment model containing all of these features of the observed flow predicts such a behaviour.

Sufficiently energetic rings become unstable and significant vorticity is then shed into a wake. Under some circumstances a new more stable vortex emerges from this shedding process and continues with less vorticity than before. Eventually, all motion should cease as the ring circulation decays to zero under the diffusing action of viscosity.

LES TOURBILLONS DES FUMEURS

Tony Maxworthy

A lecture honouring Prof. Henri Bouasse who came close to correctly describing the flow field of non-buoyant vortex rings. This presentation described work extending the previous lecture on "smoke rings". An entrainment model was presented (Fig.1) which correctly predicts the behaviour of
vortex rings. As vorticity diffuses from the main ring structure it is picked up by irrotational fluid flowing around the ring; some is entrained into the ring the rest is shed as a wake. The amount of fluid being entrained is:

\[ \frac{d a^2}{d t} \sim U a^2 \left( \frac{\nu}{a} \right)^2 \]

We can show that at high Reynolds number the impulse is closely constant even though a small amount of momentum is shed into the wake; whence \( I \sim U a^2 = \text{const.} \) Substitution gives

\[ a^2 \sim (I \nu)^{1/2} t \]

and

\[ U \sim (I \nu)^{1/4} t^{-1} \]

The decrease in ring circulation can be accounted for by the loss of vorticity to the wake.

The model can be extended to buoyant rings, for which:
and the circulation is constant, i.e., as much vorticity is being shed into a wake as is being produced by buoyancy forces. $F$ is the total buoyancy force acting on the ring.

Alternative explanations for this behaviour have been given by J. S. Turner using similarity arguments. More refined experiments are needed in order to distinguish between the two models.

The lecture concluded with demonstrations of the major features of vortex rings. In particular it was shown that two rings of closely equal initial velocity do not pass through each other, as commonly believed, but the slower entrains the faster and they combine.

**TOPOGRAPHIC EFFECTS ON THE WIND-DRIVEN OCEAN CIRCULATION**

Elliott E. Schulman

We consider the linear wind-driven flow of a homogeneous ocean in which bottom friction is the dominant mechanism of dissipating the wind-induced vorticity. The bottom topography varies only along latitude lines and the effects of an oceanic ridge, plateau and continental rise are studied. Solutions are obtained both for the inviscid case and those in which the bottom frictional parameter are small.

We find that a mid-oceanic ridge (such as the Mid-Atlantic Ridge) tends to reduce the Sverdrup transport in the basin and direct the westerly drift to the south. The diversion is the strongest where the bottom frictional parameter is the smallest. In general, closed streamline patterns
can be found over variable topography without diffusing vorticity to the ocean floor. As the ocean rises onto a plateau (such as the Blake Plateau) the circulation is blocked from riding onto the plateau. A western continental rise (such as the rise from the Blake Plateau to the eastern coast of North America) strongly reduces the magnitude of the westerly intensified northward flow at the coast; the maximum transport of the northward current appears at the foot of the continental rise.

HEAT TRANSFER IN STEADY TWO-DIMENSIONAL BÉNARD CONVECTION

Richard C. J. Somerville

Consider the following dimensionless Boussinesq system (Chandrasekhar, 1961):

\[ \nabla \cdot \mathbf{V} = 0 \]  
\[ \nabla \cdot \mathbf{V} + \nabla p - \sigma \mathbf{R} \mathbf{T} \cdot \mathbf{k} - \sigma \nabla^2 \mathbf{V} = 0 \]  
\[ \nabla \cdot \mathbf{T} - \nabla^2 T = 0 \]  

Here \( \sigma \) is the Prandtl number, \( \mathbf{R} \) is the Rayleigh number, \( \mathbf{k} \) is the vertical unit vector, and \( \mathbf{V}, p, T \) are scaled velocity, pressure, and temperature variables respectively. This system may be solved numerically in two space dimensions \((x, z)\) subject to the boundary condition

\[ \mathbf{V} = 0, \ T = (0) \ \text{on} \ z = (\mathcal{H}) \]  

and the periodicity condition

\[ (\mathbf{V}, T)(x, z) = (\mathbf{V}, T)(x + \lambda, z) \]  

where \( \lambda \) is a parameter.

To each solution of this system there corresponds a value of the Nusselt number \( N \):
Physically, \( N \) is a heat transport, which is invariant with depth \( z \), as may be seen by integrating (3), subject to (1) and (5). In experiments (e.g., Krishnamurti, 1970), \( N \) and \( \lambda \) have been measured in approximately steady roll convection in the parameter ranges

\[
\sigma^* = 0.7 \text{ (air) and } \sigma^* \geq 6.8 \text{ (water, oils) } \quad (7)
\]

\[
R_c = 1708 < R \leq 22,600, \quad (8)
\]

The upper bound in (8) is from Busse's (1967) theory. Observed values of \( \lambda \) depend on \( R, \sigma^* \), the history of the flow, and several experimental details; but seem to lie between 2 (at \( R = R_c \)) and values slightly in excess of \( \lambda_{\text{max}}(R) \), Busse's maximum permitted wavelength for infinite \( \sigma^* \):

\[2 < \lambda \leq \lambda_{\text{max}}(R) \quad (9)\]

The purpose of this note is to give an accurate explicit formula for \( N(R, \sigma^*, \lambda) \), based on numerical solutions of the above system, so as to facilitate comparisons with experimental and theoretical determinations of \( N \).

Let \( \delta(\sigma) = \begin{cases} 1 & \text{for } \sigma^* = 0.7 \\ 0 & \text{for } \sigma^* \geq 6.8 \end{cases} \)

Then the Nusselt number \( N(R, \sigma^*, \lambda) \), as determined by numerical solutions of (1) - (6) in the parameter ranges (7) - (9), is given by

\[
N(R, \sigma^*, \lambda) = \left( \frac{R_c}{R_c^*} \right)^{0.385 + 0.9 R_c^*} \left[ \delta(\sigma) + 2.3 [2 - \lambda] \left[ 1 + \frac{\delta(\sigma)}{3} \right] \right] \quad (10)
\]

The formula (10) may be misleading outside the parameter ranges (7) - (9). It is meaningless, for example, that the exponent \( \rightarrow 0.385 \) as \( R \rightarrow \infty \). Inside the ranges (7) - (9), however, the formula and the numerical models
used to solve (1) - (5) are sufficiently accurate so that the error in the
formula (10) is at most about 2%.

References:

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PREFERRED MODES IN CONVECTION

Richard C. J. Somerville

This study is concerned with steady, two-dimensional (roll) convec-
tion in a Boussinesq fluid layer bounded by rigid horizontal plates main-
tained at different constant temperatures. This regime of Bénard convection
is experimentally observed to occur for Rayleigh number $R$ and Prandtl num-
ber in the ranges $1708 \leq R \leq 20,000$ and $\sigma \geq 1$, in accordance with a
theory of Busse (1967) for $\sigma \rightarrow \infty$. This theory also predicts that rolls
are stable only if their wavelength $\lambda$ lies inside a certain closed region
(interior to the neutral curve of marginal stability theory) in the $R, \lambda$
plane. Many experiments (e.g., Krishnamurti, 1970) at finite $\sigma$, while dif-
fering in detail, confirm this prediction, but also show that:
(a) the wavelength exceeds its critical value \((\lambda > \lambda_c)\), where \(\lambda_c = 2.016 = \lambda\) at \(R = R_c = 1708\); 
(b) the excess increases with Rayleigh number \((\frac{\partial \lambda}{\partial R} > 0)\); 
(c) this increase becomes more marked as \(\sigma\) is decreased \((\frac{\partial^2 \lambda}{\partial R^2} < 0)\).

In the absence of adequate theoretical explanations for these properties, the author and F. B. Lipps (ESSA) have sought numerical solutions of the Boussinesq equations. When initial-value problems are integrated in a domain of horizontal dimension \(L >> \lambda_c\), we find:

1. Integrations in two space dimensions do not display these properties;
2. Preliminary results in three space dimensions display at least property (a).

These results not only indicate that the wavelength selection mechanism is intrinsically three-dimensional, but also suggest that it exists within the standard theoretical framework, and that it is not necessary to invoke effects of sidewalls, imperfectly conducting boundaries, or other experimental "contaminants". We have also studied the parametric dependence of two-dimensional solutions on \(\lambda\) (forcing any desired stable value of \(\lambda\) by fixing initial conditions and/or \(L\)) and find:

3. No obvious extremum principle leads to properties (a, b, c). Maximum heat transport, for example, violates (a);
4. Nusselt number \(N\) depends sufficiently strongly on \(\lambda\) to explain much of the discrepancy between experimental determinations of \(N(R)\) and many previous numerical determinations which assumed \(\lambda = \lambda_c\), thus ignoring (a) and (b);
(5) Result (4) and property (c) explain why early numerical studies also failed to find the well-known experimental dependence of N on $\sigma^*$. References:


Acknowledgement:

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ESTIMATE OF CONVECTIVE TRANSFER

Edward A. Spiegel

The following results are offered for the Nusselt number in Boussinesq convection. They have been obtained in collaboration with D. O. Gough and J. Toomre.

Let the function $f^i_{a'i}(x, y)$ be a convective plan form for a horizontal wave number $a_i$; thus

$$\nabla^2 f^i_{a'i} = \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right) f^i_{a'i} = -a^2_i f^i_{a'i}. \quad (1)$$

For a given $a_i$, the $f^i_{a'i}$ span the appropriate subspace of allowable plan forms. Let $f^i_i$ be the most general linear combination of the $f^i_{a'i}$ for fixed $a_i$.

Note that for $a_i \neq a_j$, $\overline{f^i_i f^j_j} = 0$ where the bar denotes horizontal average. Normalize so that

$$f^i_i f^j_j = \delta_{ij}. \quad (2)$$

Let us write
where \( \boldsymbol{w} \) is vertical velocity component and \( \Theta = T - \overline{T} \). Then the Boussinesq equations become

\[
\frac{1}{\sigma} \partial_k \frac{\partial W_k}{\partial t} + \frac{1}{\sigma} \sum \frac{C_{km} \partial W_m}{\partial z} \left[ a_{mk} \partial W_k + a_{km} \frac{\partial W_m}{\partial z} \right] = - \beta a_k^2 \Theta + \partial_k W_k, \\
\frac{\partial \Theta}{\partial t} + \sum \frac{C_{km} \partial W_m}{\partial z} \left[ a_{km} \partial W_k + 2 a_k^2 a_m^2 W_m \frac{\partial \Theta}{\partial z} \right] = \beta W_k + \partial_k \Theta_k, \\
\frac{\partial \Theta}{\partial t} + \sum W_m \Theta_m = \frac{\partial \Theta}{\partial t}
\]

where

\[
D_k = \frac{\partial}{\partial x_k} - a_k^2, \quad a_{km} = a_k^2 (a_m^2 + a_m^2 - a_k^2), \quad C_{ij} = \frac{1}{2} f_i f_j f_k
\]

and, as usual,

\[
\sigma = \frac{1}{K}, \quad R = \frac{2 \varepsilon \Delta d^3}{K}. 
\]

We have proceeded by retaining only a few terms in the expansion (3) and solving (49 - 6) numerically. In the case where only one term is retained, a fairly simple set of equations, similar to one reported previously, is found. These one-mode equations have been solved numerically and for a variety of initial conditions the solutions tend to a steady state. Accordingly, the steady equations have been examined in detail. These are

\[
(D^2 - a^2) W = Ra^2 \Theta + \frac{C}{\sigma} [2 (DW) (D^2 - a^2) W + W (D^2 - a^2) DW], \\
(D^2 - a^2) \Theta = - \beta W + C [2 WD \Theta + \Theta DW], \\
\beta + W \Theta = N_i,
\]
where
\[ D = \frac{d}{dx}, \beta = -D \bar{T}, C = C^\text{int}, \alpha = \alpha_1, W = W_1, \Theta = \Theta_2 \] (12)
and \( N_1 \) is the one-mode Nusselt number, or non-dimensional total heat transfer. These equations have been obtained by Roberts using a method of Prigogine. If simple approximations for \( \int_1^\alpha \) are used C can be explicitly found. For rolls and rectangles, \( C = 0 \); for hexagons \( C = 6^{-\frac{1}{2}} \).

For \( R \to \infty \) and \( \alpha, \sigma \), all \( O(1) \) we find from matched asymptotic expansions that

\[ N_1 \sim \left[ \frac{\varphi}{3} \left( \frac{2}{\pi} \right) \frac{\sigma}{\sigma + 4 C_0^2} \right] \frac{R a^2}{(\ln R a^2 - 9 \ln \ln R a^2)}^{\frac{1}{2}}. \] (13)

Further study reveals that this approximation holds for large \( \alpha \) so long as \( \alpha \) does exceed \( qR^{1/4} \) where the number \( q \sim \frac{1}{2} \). Thus the most heat transport obtained from one mode varies like \( R^{-3} (\ln R)^{-2} \), which is reasonably representative of some experiments. The choice of \( \alpha = O(R^{1/4}) \), though it leads to a fair \( R \)-dependence of \( N \), does not seem to be indicated in other respects by experiments, though the evidence is not convincing. However, at low \( R \), \( \alpha \) seems, if anything, to decrease with increasing \( R \). With \( \alpha = O(1) \) we have \( N_1 \sim R^{-2} (\ln R)^{-2} \) which does not accord with experiments at large \( R \). It seems reasonable to expect to remedy this deficiency with further modes. Let us do this in an approximate way,

We take \( \alpha = O(1) \) (for example = \( \alpha_6 \)).

For one mode at large \( R \) we have

\[ \alpha \sim (R \ln R)^{\frac{1}{8}} \] (14)

where \( \alpha \) is given in (13). Introduce successively more modes taking at each step the mode that will most modify \( N \). The modes that do this will be those which can break down the thermal boundary layer. Such modes will be excited both by the fluctuating and mean field interactions of (4) - (6).
If the viscous boundary layer is thinner than the thermal boundary layer, the fluctuating terms will probably excite small scale modes that can do this effectively when the Reynolds number of the first mode is high enough. If the thermal boundary layer is thinner, the mean-field term, $\beta W$, will be the crucial one. The latter case occurs at large enough $\sigma$ and let us consider only that. We may then proceed with the help of Malkusian arguments. The second mode sees $\alpha T$ like:

Thus it derives its input from layers which in total are $N^{-1}$ the total layer thickness. It therefore sees an effective Rayleigh number

$$R_{\text{eff}} = \frac{R}{N_i^3}.$$  
We use (14) and find a heat transport for 2 modes like

$$N_2 = N_i \times \left[ \alpha \left( R_{\text{eff}} \ln R_{\text{eff}} \right)^{1/2} \right] = \alpha \left( \frac{R}{N_i^3} \ln \frac{R}{N_i^3} \right)^{1/2} \left( 1 + \frac{3}{5} \right).$$  

Likewise a third mode sees $R_{\text{eff}} = \frac{R}{N_2^3}$ so for 3 modes

$$N_3 = \alpha \left( \frac{R}{N_2^3} \ln \frac{R}{N_2^3} \right)^{1/2} \left( 1 + \frac{3}{5} + \frac{3}{7} \right).$$  

Evidently, for $n$ modes,

$$\left( \alpha \frac{R}{N_i^3} \ln \frac{R}{N_i^3} \right)^{1/2} \approx \alpha \frac{R}{N_i^3} \ln \frac{R}{N_i^3}.$$

Of course, excitation of an infinite number of modes by mean fields will not be possible. We can only have $n$ modes where for the $(n+1)^{th}$ mode

$$R_{\text{eff}} = \frac{R}{N_n^3} \leq R_c \approx 1708.$$

This leads us to

$$\left( \alpha \frac{R}{N_i^3} \ln \frac{R}{N_i^3} \right)^{1/2} \approx \alpha \frac{R}{N_i^3} \ln \frac{R}{N_i^3}.$$
and hence to

\[ N \approx \left( \frac{R}{R_0} \right)^{\frac{4}{3}} \approx \left( \frac{R}{1708} \right)^{\frac{4}{3}}. \]

a result which enjoys a certain popularity. It is worth noting that the \( a_i's \) seem reasonable as well. The \( a \) of the first mode, call it 4, is arbitrarily taken \( o(1) \), \((a_c \text{ say})\). For the second mode the unit of length was seen to be \( O(N^{-1}) \). Hence for the second mode, \( a = N_i a_i = O(R^2 (2n R)^{i/4}) \) etc. For the \( n^{th} \) mode \( a_n = N_n a_n = O(R^{2i}) \) for large \( R \). These results should not yet be compared with experiment since (13) holds only for \( R \) much larger than is usually attainable in experiment. For this purpose we must use our numerical solutions of (9) - (11). These details will be presented elsewhere in extenso if Douglas and Jüri can ever be distracted from their more pressing responsibilities.

**THERMODYNAMICS AND COSMOLOGY**

Edward A. Spiegel

In cosmology one considers in the simplest models a perfect fluid in homogeneous, isotropic expansion. The state of the system is then characterized by a single scalar function of the cosmic time, \( t \). Thus, if at \( t = t_0 \) two infinitesimal elements of the cosmic fluid are a distance \( r_0 \) apart, at a later time their separation is

\[ r = \frac{R(t)}{R(t_0)} r_0 \]

where \( R(t) \) is the scale factor of the universe. The evolution of \( R \) is a dynamical problem of cosmology and evidently this depends on your theory of gravity. Even the interpretation of \( R \) depends on the theory. Here I have given a Newtonian interpretation but \( R \) appears also in relativistic cosmology.
with a different interpretation. For the purposes of this discussion we shall just assume that R is a given function. Evidently with the interpretation given here, the cosmic density varies like

$$\rho(t) = \frac{R^2(t)}{R(t)} \rho(t_0).$$

A question of some interest is, What can be said of the thermodynamics of this cosmic fluid? Of course, this can be discussed as realistically as you wish but with no guarantee of success. Here I want to mention some problems that arise in one of the most idealized models, namely, that of a gas of structureless particles which collide elastically. The problems that arise even in this case have been discussed by only a few authors and Engelbert Schüking and I have been puzzling over them for a while (Comments on Astrophysics and Space Physics, 11: 121 (1970)). Some people do not agree that these problems are to be worried about, but here they are,

For a (special) relativistic gas which is not expanding, one can use the usual equilibrium distribution

$$f = A e^{-\beta E}$$

where $$\beta = \frac{(kT)^{-1}}{}$$ and E is the energy of a particle. The relativity enters in the formula

$$E = \sqrt{m^2 + p^2}$$

where the units are chosen so that c = 1 and where m is the rest mass of a particle and p is its momentum. In that case the normalization becomes

$$A = \frac{N}{4\pi m^2 \sqrt{4\pi k_B T}} \text{K}_0(m\sqrt{k_B T})$$

where N is the particle density and $$K_0$$ is a special Hankel function. In the limit $$p^2 < m^2$$ we find

$$f = \frac{N}{(2\pi m k_B T)^{1/2}} e^{-\beta p^2/m}.$$
which is the classical expression. In the opposite limit we have

\[ f = \frac{n_e N}{(2\pi k T)^3} e^{-\beta |p|}. \]

This would be the correct distribution for photons were it not for E-B statistics.

Now we turn to the expanding case with \( R(t) \) specified. In a Minkowski space expanding like \( R(t) \) we let \( \mathbf{p} \) be the momentum relative to the expanding coordinates. If you want to visualize the situation kinematically picture two sets of observers. One set remains fixed with respect to each other; the members of the other set move apart from each other with their mutual distances increasing like \( R(t) \). Let \( \mathbf{p} \) be the momentum of a free particle moving relative to the expanding set of observers. It can be shown that for such a particle \( |\mathbf{p}| R \) is conserved.

In cosmology this result is used as follows: For the classical gas equilibrium is maintained if \( \beta E \propto |p|/T \) is invariant. Since \( |p| \propto R^{-1} \) we have \( T \propto R^{-2} \) which describes the adiabatic cooling. \( N / T^2 \) then is also constant since \( N \propto R^{-2} \). Likewise for an ultrarelativistic gas to have \( \beta |p| \) invariant we need \( T \propto R^{-1} \) and this ensures \( N / T^3 \) is invariant. Thus in cosmology a classical gas is said to cool with \( T \propto R^{-2} \) while a photon gas with \( T \propto R^{-1} \). This is a primitive description, but it can be made precise with the use of the relativistic Boltzmann equation.

What of the relativistic gas? There seems to be no choice of \( T(R) \) which makes \( \beta E \) invariant. Hence and there are more rigorous arguments it has been concluded that an expanding relativistic gas has no equilibrium in the quasistatic sense described here. It follows that the expanding relativistic gas is dissipative. The assumption of a perfect fluid for the cosmological problem would then be unjustified, though it is probably a good
approximation most of the time, Macroscopically the dissipation must result from bulk viscosity since the fluid is homogeneous and shear viscosity and conduction do not occur. You can think of an origin of this bulk viscosity like that for a classical fluid. The usual reason for bulk viscosity is that different degrees of freedom of the particles adjust to changes on different time scales. If the classical and ultrarelativistic particles can be visualized in this way the bulk viscosity seems more plausible.

Still we may wonder whether the bulk viscosity is really an artifact of the Boltzmann equation for the relativistic case. Suppose we widen the Boltzmann notions a bit, There, the distribution depends only on the momenta. But there is another property of the particles that differs from particle to particle. This is the proper time.

As an example consider a relativistic gas in a box. At \( t = 0 \) give each particle a clock reading \( 0 \). Clearly at any later time \( t \) the clock readings will not all be the same since the particles move differently. Indeed, you can ask what the distribution of clock readings is at any time, \( t \). For an ergodic gas, as \( t \to \infty \) the dispersion of clock readings tends to zero since all particles have the same average history in some sense.

Since the ages or proper times differ for different particles we may ask what happens if we include this parameter as a statistical variable. This complicates things a bit but one simple fact emerges,

Consider the model

\[
R(t) = R_0 (1 + \alpha t)
\]

where \( R_0 \) and \( \alpha \) are constants. Let \( \tau \) be the age of a particle reckoned from \( 0 \) at \( t = 0 \). Then for a free particle in this space
\[ I = \frac{\mathcal{P}}{\mathcal{R}_0} E - m \alpha \mathcal{T} \]

is conserved. Note also that if two particles collide and if the collision lasts no time then the sum of their \(I\)'s is conserved since the sum of the \(E\)'s is conserved and the \( \mathcal{T} \)'s do not change.

Thus

\[ f = A e^{-\mathcal{A} I} \]

is the solution of the appropriate "Boltzmann equation", though the details are messy. Note that in the non-expanding case \( I = E \) and the usual equilibrium is recovered. Thus there does seem to be a sort of equilibrium for a relativistic expanding gas, but we are not sure we understand it at all.

One curious point is that in a vague sense this kind of thing is not without precedent. In viscoelasticity it is known that an alternate way to discuss dissipative effects is through history terms. History terms also play a role in two component fluids such as dusty gases. Can there be a significance to these vague similarities?

GENERALIZATIONS OF THE ROTATING FLAME EFFECT
WITH APPLICATION TO TORNADO GENESIS
Melvin E. Stern

Heat sources and sinks propagate in an azimuthal direction with frequency \((\omega)\) relative to a basic fluid state of vertical thickness \((h)\), static stability \((s)\) and rotation rate \((f/2)\). If \((f^2 - \omega^2)(\omega^2 - gs)^{-1} > 0\) the forced azimuthal perturbations pump kinetic energy and angular momentum \((\omega\)-directed) radially outwards. Thus the Reynolds stress, computed from linear theory, leads to a mean relative spin of the air in the vicinity of the heat sources and this is in the opposite sense to the relative motion of the sources. In this generalization of the "rotating flame effect" viscosity
is neglected. It is suggested that such a horizontal vortex can be amplified as the air "passes" through a vertically convecting cumulonimbus cloud. The kinetic energy and heat generated by the mother cloud is pumped into the far-field by the azimuthal perturbations of the vortex, and angular momentum is thereby separated. It is suggested that the angular momentum accumulating within the radius of the mother cloud is dissipated by the smaller scale tornado funnel.

**PHASE BEHAVIOR OF LIGHT GAS MIXTURES AT HIGH PRESSURES**

William B. Streett

If solid surfaces exist beneath the visible clouds of the major planets, they may be expected to exist at depths and pressures at which the component gas mixtures solidify under their own weight. An understanding of the solid-fluid phase behavior in mixtures of light gases at high pressures is therefore essential to the solution of the problem of deep atmosphere structures in these bodies. For Jupiter and Saturn, the mixture of primary interest is hydrogen-helium. Although experimental results for this system are limited to low pressures, several possible extrapolations of the phase diagram are suggested by the results of high pressure experiments on other helium binary systems. The suggested phase diagrams have been used to develop a structural model for a hydrogen-helium atmosphere. In this model, gravitational separation of coexisting phases results in a layered structure, and it is shown that masses of hydrogen-rich solid can exist in dynamic and thermodynamic equilibrium with a fluid layer of equal density but higher helium content. This picture may be relevant to the floating raft concept of Jupiter's Great Red Spot.
MOTION OF JUPITER'S RED SPOT
(OCEANOGRAPHY OF JUPITER)
George Veronis

A thermodynamic justification can be made for the hypothesis that a layer of hydrogen-rich solid can form at a level of a few thousand km below the visible surface of Jupiter and that the solid is gravitationally stable (in the dynamic sense) at the level of formation. If a topographic or other such feature of the solid surface of Jupiter causes an accumulation of this material into a massive object and if the motion of this stable Cartesian diver is analyzed, the observed motion of the Red Spot can be rationalized in terms of the model. Vertical oscillations of the Cartesian diver are accompanied by longitudinal changes of position because of the associated changes in the angular momentum of the object. Speculation about the fluid motions caused by the motion of the diver leads to predicted changes in the size and location of the Red Spot which agree with observed changes.

THE MICRO-OCEANOGRAPHY OF BERMUDA
Carl Wunsch

An attempt to measure internal waves in the main thermocline at Bermuda (Wunsch and Dahlen, 1970) leads us to some peculiar physical processes that appear to go on around this mid-ocean island. The island has been explored in 3 cruises (ATLANTIS II, Cruise 47 in October 1968, GOSNOLD Cruise 144 in July 1969, and GOSNOLR Cruise 147 in July 1970), using an STD, current meters on buoys, and parachute drogues.

The internal wave measurements indicated no detectable coherence between temperature sensors 1 km apart for periods shorter than about 8-10
hours, a disconcerting feature when one calculates theoretical wavelengths exceeding 1 km for internal waves of periods longer than 30 min. The data show no inertial peak, no diurnal peak, a strong semi-diurnal tide with random phase and large amplitude (about 20 m displacement of isotherms in a small frequency band around 12.42 hours) and an unexplained strong peak at about 18-20 hours period.

The hydrographic surveys of the island revealed that the temperature and salinity microstructure is a strong inverse function of distance from the island, apparently accounting for the low internal wave coherence. It was also found that the large scale current field around Bermuda is very strong, with high shear and very time-dependent. In the first two cruises it was noted that the mixing region (defined as the area of maximum microstructure) was strongest on the northern side of Bermuda. On the second cruise, the current field (basically from the west) was also much more intense there. The buoy data was also quite perplexing as two current meters placed a mile apart at the same depth show notably different mean displacements. Thus, the whole island area appears to be exceedingly complex and variable.

Some progress toward understanding the problem of an island can be obtained by two hypotheses. We can postulate an internal wave field propagating into the island and breaking (Cacchione, 1970) so as to generate a microstructure. The corresponding change in the mass field generates a geostrophic current penetrating into the interior, a distance given by the baroclinic Rossby radius of deformation. A mechanism like this was used by Wunsch, 1970, to explain the large scale ocean mixing. The Reynolds stress effects of the incoming internal waves can then also generate apparent mean
Eulerian velocities leading to Barge differences between current meters not very Ear apart, (Wunsch, 1971). The intensification of the microstructure and currents on the north is then accounted for by postulating a stronger internal wave field from the south, a rather unattractive ad hoc hypothesis. However, estimates of the energy flux into the island based on the temperature measurements give reasonable agreement with the change in potential energy in the current near the island, and the estimated time a given water parcel is in the vicinity of the island,

A perhaps more attractive, if less analytically tractable problem, is to hypothesize that the current shear flow past the island becomes unstable. The density gradients are then such that the mixing that occurs through the instability tends to intensify the (geostrophic) current when the island is to the right, and weakens it when the island is to the left. Thus, a current from the west would be strongest on the north side, and weakest on the south side, with the reverse being true for a flow from the east. The hypothesis is consistent with the flow features in July, 1970, and 1969.

The 15-20 hour spectral peak is difficult to explain. An analysis of an infinite cone in a rotating stratified ocean showed no non-singular trapped modes for periods between the inertial and Brunt, with many modes above and below this range. The inertial period at Bermuda is 22.3 hours. Analysis of the tide gauge records at Bermuda shows no sign of the inertial resonance predicted by Longuet-Higgins, and the drogue data show no discernible circulation around the island in the presence of the mean flow.

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MICRO-SYMPOSIUM ON TURBULENCE
MODEL CONSTRUCTION

Robert H. Kraichnan

A number of statistical approximations proposed for turbulence in recent years can be characterised by generalized \textit{Langevin-type} equations for a model velocity field. These model equations have in common that the \textit{actual} nonlinear terms of the Navier-Stokes equation are replaced by a dynamical damping term (generalized eddy viscosity), and a random forcing term, acting on each Fourier mode (or other appropriate mode) of the system. The damping and forcing terms are \textit{determined wholly} by ensemble averages of the turbulent velocity field and thus are insensitive to the fluctuations in any one realization. As a result, the model amplitude equation, in contrast to the Navier-Stokes equation, is effectively linear in stochastic quantities. Thereby it leads to \textit{closed} statistical equations which determine first and second order moments.

A model representation of the direct-interaction approximation is obtained by a particular choice of random forcing and damping terms, such that the latter describes a damping with memory. In the case of \textit{isotropic} turbulence, this model is most simply stated as follows. The \textit{Fourier-space form} of the Navier-Stokes equation is

$$
\frac{\partial}{\partial t} + \nu k^2 \mathbf{u}_1(k,t) = -\frac{1}{2} \sum_{p+q=k} P_{ijm}(k) \mathbf{u}_j(p,t) \mathbf{u}_m(q,t),
$$

where

$$
P_{ijm}(k) = k_i P_{ij}(k) + k_j P_{im}(k), \quad P_{ij}(k) = \delta_{ij} - k_i k_j / k^2.
$$

The direct-interaction \textit{model equation} is

$$
\left( \frac{\partial}{\partial t} + \nu k^2 \right) \mathbf{u}_1(k,t) + \int_0^t \eta(k,t,s) \mathbf{u}_1(k,s) \, ds = -i P_{ijm}(k) \sum_{p+q=k} \xi_j(p,t) \xi'_m(q,t),
$$

\text{(2)}
where the damping-with-memory function is

$$\eta(k,t,s) = k\int_\Delta b_{kpq} G(p,t,s) U(q,t,s) pqdpdq$$ \hspace{1cm} (3)

In these equations, $\xi$ and $\xi'$ are random solenoidal vector fields statistically independent of each other and of the initial velocity field $u(k, t=0)$, but constrained by the requirement

$$\langle \xi_1(k,t) \xi_1^*(k,t') \rangle = \langle \xi_1^l(k,t) \xi_1^* \rangle = \langle u_1(k,t) u_1^*(k,t') \rangle$$ \hspace{1cm} (4)

$G$ and $U$ are respectively the response and covariance scalars of $u_1(k,t)$, $\int_\Delta$ denotes integration over all $p,q$ which form a triangle with $k$, and

$$b_{kpq} = (p/k)(xy + z^2)$$, where $x,y,z$ are the cosines of the internal angles opposite $k,p,q$ respectively.

Given the requirement (4) on the random forcing term (the right-hand side of (2)), the form of $(k,t,s)$ is uniquely determined to be (3) (apart from a slight ambiguity in the value of $b_{kpq}$) by the requirement that the ensemble-averaged energy input from the random forcing balance the drain from the damping when summed over all $k$, thus preserving the conservation property of the Navier-Stokes equation. Equations (2)-(4) easily yield the closed direct-interaction equations for $G$ and $U$.

Simpler model amplitude equations, in which the damping term has no memory, are obtained by making the random forcing term have the same general form as in (2) but, in contrast to (4), requiring it to be a white noise in time and, again, fixing the damping term by requiring energy conservation. A particular model of this kind yields Edwards' turbulence theory and extends the latter to non-steady-state turbulence. In contrast to the direct-interaction model, these white-noise models do not give a qualitatively
faithful representation of the full space-time covariance of the $u$ field. However, they embody more flexibility and can be constructed to yield the Kolmogorov inertial range spectrum, which the direct-interaction model does not.

The generalized Langevin-type models promise to be useful in three ways. First, they can serve directly as approximations for low-order statistical properties of turbulence. Here the fact that a model amplitude equation underlies the eventual statistical equations insures important consistency properties (under suitable precautions). Second, they can serve as the zeroth approximation in a systematic expansion scheme for the exact statistical dynamics, for example by introducing a parameter whose value measures the proportions of a mixture of the true right-hand side (1) and the model right-hand side (2). Here the fact that a model amplitude equation exists makes the analyticity properties of an expansion in the parameter accessible. Third, the models can be used to give representations of sub-grid motions in turbulence simulations, with the advantage, over simple eddy-viscosity devices, that energy conservation is maintained.

Reference

STOCHASTIC MODELS

Cecil E. Leith

An important class of turbulence approximations is based on the construction of stochastic models. We examine such models for a quadratically non-linear system of $N$ variables evolving according to

$$
\dot{u}_\alpha(t) = \sum_{\beta, \gamma} A_{\alpha \beta \gamma} u_\beta(t) u_\gamma(t)
$$

with constraints on the non-linear interaction coefficients $A_{\alpha \beta \gamma}$ to provide conservation laws. The single-time statistical properties of ensembles of solutions of Eq. (1) is completely determined by a Liouville equation for the ensemble probability distribution $P(u, t)$ defined in the phase space of the system, namely,

$$
\frac{\partial}{\partial t} P(u, t) = -\sum_{\alpha \beta \gamma} A_{\alpha \beta \gamma} u_\beta u_\gamma \frac{\partial P(u, t)}{\partial u_\alpha}
$$

In practice for large $N$ we cannot solve this linear differential equation and must seek approximations.

A stochastic model for the non-linear system (1) is a system evolving according to the random linear equation

$$
\dot{u}_\alpha(t) = -\int_0^t \gamma_\alpha(t, s) u_\alpha(s) ds + f_\alpha(t)
$$

where $\gamma_\alpha(t, s)$ is a non-random eddy visco-elasticity and $f_\alpha(t)$ a random Gaussian eddy forcing term with moments $\langle f_\alpha(t) \rangle = 0$ and $\langle f_\alpha(t) f_\alpha(s) \rangle = \delta(t-s)$ specified. The functions $\gamma_\alpha(t, s), F_\alpha(t, s)$ are chosen to simulate as well as possible the non-linear coupling between modes.

The direct-interaction approximation uses the choice

$$
F_\alpha(t, s) = 2 \sum_{\beta, \gamma} A_{\alpha \beta \gamma} U_\beta(t, s) U_\gamma(t, s)
$$

$$
\gamma_\alpha(t, s) = -2 \sum_{\beta, \gamma} A_{\alpha \beta \gamma} \left[ A_{\alpha \gamma \beta} q_\beta(t, s) U_\alpha(t, s) + A_{\alpha \beta \gamma} q_\gamma(t, s) U_\beta(t, s) \right]
$$
where $g_{\alpha}(t,s)$ is the Green's function for Eq. (3) and $U_\alpha(t,s) = \langle u_\alpha(t) u_\alpha(s) \rangle$.

The eddy-damped Markovian approximation is based on the even cruder stochastic model in which the random forcing is white and the eddy viscoelasticity becomes an eddy viscosity. The model equation becomes

$$\dot{u}_\alpha(t) = -\frac{1}{2} \chi_{\alpha}(t) u_\alpha(t) + f_{\alpha}(t)$$

with

$$F_{\alpha}(t) = 4 \sum_{\beta \gamma} D_{\beta \gamma}(t) A_{\alpha \beta \gamma} U_\beta(t) U_\gamma(t)$$

$$\chi_{\alpha}(t) = -4 \sum_{\beta \gamma} D_{\beta \gamma}(t) A_{\alpha \beta \gamma} [A_{\alpha \beta \gamma} U_\beta(t) + A_{\alpha \beta \gamma} U_\gamma(t)]$$

where $D_{\beta \gamma}(t)$ is an eddy-damping time for triple moments. Here $F_{\alpha}(t)$ measures the strength of the white forcing, $\langle f_{\alpha}(t) f_{\alpha}(s) \rangle = F_{\alpha}(t) \delta(t-s)$.

The Markovian stochastic model Eq. (6) induces the random Liouville equation

$$\frac{\partial}{\partial t} \rho(y,t) = \sum \delta \left[ \left[ -\chi_{\alpha}(t) u_\alpha + f_{\alpha}(t) \right] \rho(y,\alpha,t) \right] \frac{\partial \rho(y,\alpha,t)}{\partial u_\alpha}$$

but since $f_{\alpha}(t)$ is white the average $\langle \rho(y,t) \rangle$ over the $f_{\alpha}(t)$ ensemble satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t} \langle \rho(y,t) \rangle = \sum \frac{\partial}{\partial u_\alpha} \left[ \left( \chi_{\alpha}(t) u_\alpha + \frac{1}{2} F_{\alpha}(t) \frac{\partial}{\partial u_\alpha} \right) \langle \rho(y,t) \rangle \right] .$$

(10)
The numerical simulation of turbulence usually involves simulation of complicated flows in simple geometries with simple boundary conditions. In this case, it is found that Galerkin approximation procedures have some important advantages over finite-difference approximations. Two cases of importance are examined: flows in rectangular geometries using Fourier expansions, and flows in spherical geometries using surface harmonic expansions. Transform tricks are described that make the number of numerical operations involved in evaluation of the Galerkin approximations competitive with those involved in finite-difference approximations (cf. Orszag, Phys. Fluids Suppl., 12: 250 (1969) for the rectangular case; Orszag, J. Atmos. Sci., Sept. 1970 for the spherical case).

The principal advantage of the Galerkin approximations over finite-difference approximations is the absence of spatial-difference phase-speed errors and aliasing errors. It is shown that to attain the same accuracy with (high-order accurate) finite-difference methods as with the Galerkin method requires that about $3^n$ times as many degrees of freedom (where $n$ is the number of space dimensions) be retained in the finite-difference simulations as in the Galerkin simulation. Roughly, the argument is that finite difference simulations of a wave with wavenumber $k$ on a uniform grid spaced by $\Delta x$ requires $k\Delta x \sim 1$ in order that phase speed errors not be large. In other words, there should be at least 6 grid points per wavelength, while a Fourier representation requires only two degrees of freedom (the real and imaginary parts of the complex wave) to represent the same wave without
error, Several less trivial examples were cited to substantiate the claim of increased accuracy of the Galerkin approximations. Remarkably, the Galerkin approximations even do a better job of describing the evolution of localized flow structures than do finite-difference approximations, despite the fact that each of the Galerkin basis functions is non-local in physical space.

In three space dimensions, the Galerkin methods require about 10-30 times fewer degrees of freedom to describe accurately the evolution of a given complicated flow.

Finally, the limitations of numerical simulation are discussed. An 'honest' simulation requires that all scales through the dissipation scale be described accurately. It is shown that this requires that the number of arithmetic operations involved in numerical simulation of a flow with Reynolds number $R_\lambda (\lambda = Taylor$ microscale) scales as $R_\lambda^6 \log R_\lambda$, implying that numerical simulation of high Reynolds number flows is not likely to be feasible on digital computers in the near future. It is suggested that 'honest' numerical simulations are useful for (a) testing theories of turbulence at low and moderate Reynolds numbers, (b) investigating properties of moderate Reynolds number flows which are then applied by assuming (hoping) for Reynolds number independence in extrapolating to high $R_\lambda$, and (c) visualizing features of turbulence as in a laboratory experiment, with the advantage that the numerical experiments may be set up for more idealized flow situations than laboratory fluids permit and the results may be analyzed more fully because numerical simulations provide complete detailed flow fields,
UPPER BOUNDS

Willem V. R. Malkus  =  Louis N. Howard

Various studies seeking upper bounds on heat transport by turbulent (or non-turbulent) convection are reviewed, with primary emphasis on the recent thesis of S.-K. Chan (Massachusetts Institute of Technology, January, 1970). The paper treats the case of infinite Prandtl number, and uses the full momentum equation as well as the continuity equation and the entropy flux integral as constraints in the bounding variational problem. Results are obtained by boundary layer methods analogous to those used by Busse in the case of the power integral constraint. With a single horizontal wave number the asymptotic bound is proportional to $R^{3/10} \left( \log R \right)^{1/5}$, and the overall bound is $0.152 R^{1/3}$. The optimal fields also bear a striking resemblance to observed profiles of mean temperature and r.m.s. temperature fluctuations,
CONVECTION AT ZERO PRANDTL NUMBER

Jackson Herring

At low Prandtl number, $\sigma$, a hot blob of fluid loses its temperature excess rapidly to the ambient fluid by conduction. Consequently, convection is not an efficient mode of heat transport even though the temperature stratification is so unstable as to cause vigorous motion. In the limit $\sigma \to 0$, it seems likely that rapid conductive losses will entirely prevent motions from distorting the temperature profile. The gross energetics of statistically steady flow is then a balance of energy input at low wave number and viscous dissipation at large wave number. The non-linear advection term breaks up the large scale modes, and conservatively transfers their energy to small scales where viscous dissipation takes effect. Equations appropriate to this regime have been derived by Ledoux, Schwarzschild, and Spiegel (1961), who expand the velocity-temperature field in a power series of $\sigma$. Suitably non-dimensionalized, they are,

$$\frac{1}{\sigma t} \nabla^2 \overline{u} = -\nabla p - (\overline{u} \cdot \nabla) \overline{u} - \hat{k} R \frac{1}{\sigma^2} \omega,$$

and

$$\nabla \cdot \overline{u} = 0,$$

$$\omega = \hat{k} \cdot \overline{u}.$$

Here, $\sigma$ is the Prandtl number, $R$ the Rayleigh number, $\hat{k}$ the vertical unit vector, and $\left( \frac{1}{\sigma^2} \right)$ is the inverse operator to $\nabla$ for free boundaries.

We have begun a numerical investigation of this regime of thermal convection, and report here on some preliminary results for the free boundary initial value problem, both for the amplitude problem and the statistical moment problem. The particular statistical approximations considered
are the direct interaction approximation [Kraichnan, 1964], the self-consistent field approximation (Herring, 1966), and one of the Markovian type approximations considered earlier this week by Dr. Kraichnan. In the present calculations $R$ ranged from $R_c \approx 657$ to $R = 10^4$. In the numerical simulations up to 5 vertical wave numbers and 24 horizontal wave number vectors were used to represent the velocity field. The horizontal wave vectors were chosen by the rule, $\alpha = 2^{-k_2}$, with any $|\alpha|$ greater than a cut-off value $\alpha_m$ discarded. In most calculations $\alpha_m \leq 3\alpha_o$, although for two dimensional flows $\alpha_m = 10\alpha_o$ was used. Initial values for the Fourier components of $\vec{V}$, $\vec{V}_m$, $\vec{V}_n$, were picked either by a Gaussian random number generator, or assigned values which represented the regular plan forms, squares and rolls.

In the statistical theories, only two vertical wave numbers and three horizontal wave number bands were used to represent the velocity covariance. Horizontal homogeneity and isotropy were assumed. These results are to be compared to the randomly excited amplitude experiments in the limit of large numbers of waves.

Results for the randomly excited amplitude experiment suggest a runaway behavior for the system. Thus the total kinetic energy, after an initial decrease, eventually increases without apparent bound. We were able to follow the energy amplification by about three orders of magnitude with good accuracy. A similar runaway behavior was also found for rolls. The square plan-form, on the other hand, stabilized at a finite value of energy; however, these static solutions are not stable with respect to the runaway type if they are perturbed by a three dimensional disturbance.
The D.I. approximation also has a \textit{runaway} behavior. Both the total kinetic energy and the Green's function which had a positive growth rate eventually grew indefinitely. \textbf{For} the two other statistical theories considered, the S.C.F. and the Markovianized model the behavior was different; the energy tended to stabilize at a finite value, although the linearly unstable Green's function remained finite at large times.

It then appears that the D.I. approximation faithfully reproduces the qualitative behavior of this free boundary truncated system, whereas the other methods do not. We have observed a similar behavior \textit{for} isotropic turbulence in which a region of (small) wave numbers destabilized by negative viscosities.

With regard to the \textit{runaway} character of these free boundaries solutions, it should be observed that Eq. (1) does not prohibit such behavior. This may be seen by examining its energy equation:

\begin{equation}
\dot{E} = \frac{d}{dt} \langle \mathbf{\nabla}^2 \mathbf{v} \rangle = -\langle (\mathbf{\nabla} \cdot \mathbf{v})^2 \rangle + R \langle \mathbf{v} \cdot \mathbf{\nabla} \mathbf{v} \rangle
\end{equation}

Here, the angular brackets indicate a total volume integration. If \( \mathbf{v} \) becomes large there is no obvious way in which one can show \( \dot{E} \leq 0 \). This contrasts with the finite \( \mathbf{\nabla} \) convection equations in which the last term in (2) is replaced by \( R \langle \mathbf{w} \mathbf{\theta} \rangle \), where \( \mathbf{\theta} \) is the temperature field. One may show from the equation of motion for \( \mathbf{\theta} \) that it is bounded by its initial distribution. Hence as \( |\mathbf{v}| \) becomes large, \( \langle \mathbf{w} \mathbf{\theta} \rangle \sim \sqrt{\langle \mathbf{w}^2 \rangle} \), and \( \dot{E} \leq 0 \) at least in a large region of \( \mathbf{v} \)-phase space.

The detailed dynamical reason for the runaway character may be due to the fact that for free boundaries the "linear harmonics" non-linear terms \( i.e., \) terms that couple \( \mathbf{v}_{m} \) to \( \mathbf{v}_{m+1} \) make no contribution...
to the equation of morfon for $\mathbf{V}_m z$. This has the consequence that an exponentially growing roll is an exact solution to Eq. (1):

$$\mathbf{V}(t) = A e^{\lambda t} \left\{ \frac{2}{a} \cos a x \cos \pi z + R \sin a x \sin \pi z \right\} \nabla^2 \left( \rho e^{\lambda t} \left( -\frac{\pi^2}{2a^2} \sin a x + \frac{1}{2} \sin^2 \pi z \right) \right) = 0, \quad \frac{\partial \rho}{\partial m} = 0,$$

(3)

The fact that the roll solutions runaway may be understood in terms of the dynamical stability of (3). These roll type solutions are not, however, stable with respect to the class of three dimensional motions, but these three dimensional solutions also runaway.

Finally we note that qualitatively different results may be obtained for rigid boundaries. For this case, the exponentially growing roll is not a solution to Eq. (1) and the linear harmonies manifest themselves in non-linear terms in the $\mathbf{V}_m z$-equation. Thus it may be possible that (1) is sensible for rigid boundaries but not for free boundaries.

These numerical results and conclusions based on them must be considered preliminary until calculations using a much larger set of wave number points confirm or negate them. We plan to improve this aspect of the calculation soon.

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FINITE AMPLITUDE EXPANSIONS

Bernard J. Matkowsky

We treat a number of stability problems governed by non-linear equations by a new method proposed by the author (1, 2, 3). The method, which is formal, yields the complete dynamical description of the system, and is believed to be the first to treat arbitrary initial perturbations. As an example we consider the initial-boundary value problem

\[ -u_t + u_{xx} = \lambda f(u) \quad 0 < x < \pi, \quad t > 0 \quad (1) \]
\[ u(0,t) = u(\pi,t) = 0 \quad (2) \]
\[ u(x,0) = h(x) \quad (3) \]

which is a mathematical model for the temperature distribution in a bar with a non-linear heat source (sink) of magnitude \(-\lambda f(u)\). We study the stability of the equilibrium solution \(u = 0\), and obtain a complete description of the dynamical behavior of the system uniformly for \(x\) in \((0,1)\) and \(t \geq 0\), valid in a neighborhood of \(u = 0\) and \(\lambda\) near a critical value \(\lambda_c\). Here \(\lambda_c\) is the value of \(\lambda\), which when exceeded, yields the onset of instability. It is in addition the first bifurcation point of the associated non-linear eigenvalue problem

\[ v_{xx} = \lambda f(v) \quad 0 < x < \pi \quad (4) \]
\[ v(0) = v(\pi) = 0 \quad (5) \]

whose solutions are stationary (time independent) solutions of (1) - (2). As a consequence of obtaining the complete dynamical description, we are able to trace the evolution in time of arbitrary initial perturbations until they either decay or grow into (new) stationary solutions of (1) - (2).
The stationary states for this problem were described asymptotically by Millman and Keller (4). We therefore see how these stationary states are generated from initial perturbations. Another problem touched upon is the Bénard problem. The asymptotic analysis for the stationary problem was treated by Malkus and Veronis (5). We discuss the time evolution to these states as well. In addition the stability or instability of the new equilibrium solutions is determined, without the necessity of performing an additional perturbation analysis.

As a final example, we discuss the nsn-linear stability problem for the buckling of a compressed elastic column (9). This example illustrates the importance of treating arbitrary initial data, since the resulting motion of the column depends very strongly on the character of the initial data. The essential feature of the theory is a scaling of the time variable. Then, using a two-time method, we systematically derive the asymptotic expansion of the solution in power of a small parameter $\varepsilon$, which is alternatively a measure of a norm of $u$ or the nearness of $\lambda$ to $\lambda_c$. The method seems to enjoy certain computational advantages over existing methods (6, 7, 8) and these are compared and contrasted.

References

