Notes on the 1971
Summer Study Program
in
GEOPHYSICAL FLUID DYNAMICS
at
The WOODES HOLE OCEANOGRAPHIC INSTITUTION

Reference No. 71-63
Contents of the Volumes

Volume I  Course Lectures and Abstracts of Seminars
Volume II  Fellowship Lectures
Participants and Staff Members

Charney, Jule G. Massachusetts Institute of Technology
de Rivas, Eugenia E. Massachusetts Institute of Technology
Gierasch, Peter J. Florida State University, Tallahassee
Holton, James R. University of Washington
Howard, Louis N. Massachusetts Institute of Technology
Ingersoll, Andrew P. California Institute of Technology
Keller, Joseph B. Courant Institute of Mathematical Sciences
Kraichnan, Robert H. Dublin, New Hampshire
Leovy, Conway B. University of Washington
Malkus, Willem V.R. Massachusetts Institute of Technology
Moffatt, H. Keith University of Cambridge, England
Ooyama, Katsuyuki New York University
Prinn, Ronald G. Massachusetts Institute of Technology
Rasool, S. I. Institute of Space Studies, New York
Sakurai, Dr. Takeo N.C.A.R., Boulder, Colorado
Simmons, Adrian J. University of Cambridge, England
Smagorinsky, J. Princeton University
Stern, Melvin E. University of Rhode Island
Thompson, Rory Woods Hole Oceanographic Institution
Veronis, George Yale University
Welander, Pierre Massachusetts Institute of Technology
Williams, R. Terry Naval postgraduate School, Monterey, California
Young, Richard E. University of California at Los Angeles

Fellows

Baker, Louis Columbia University
Bennett, John R. University of Wisconsin
Cottrell, John W. Stanford University, California
Dubisch, Russell Cornell University
Roberts, Glyn O. Courant Institute of Mathematical Sciences
Saunders, Kim D. W.H.O.I. and Massachusetts Institute of Technology
Schneider, Edwin K. Harvard University
Simmons, Adrian M. University of Cambridge, England
Thiebaux, Martial L. University of Massachusetts
Editors' Preface

This volume contains the manuscripts of research lectures by the Fellows in the summer program. This year our computerized selection procedure produced more scientifically mature graduate students than is its wont. They arrived with excellent backgrounds in analysis and with many good proposals for research work. Hence the challenge to the staff was largely to trim these projects to a size which might prove tractable in a summer season.

We met this challenge with limited success. The reader will discover that only a few of the enclosed manuscripts represent a neatly-compassed problem. However, each author has reached a point in his work at which novel conclusions were drawn. It is our hope that, after returning to their universities, many of the Fellows will continue to explore the avenues opened by the summer discourse.

Because of time limitations it was not possible for the manuscripts to be edited and reworked. They may contain errors the responsibility for which must rest on the shoulders of the participant-author. It must be emphasized that this volume in no way represents a collection of reports of completed and polished work.

We who took part in this thirteenth summer of G.P.D. are grateful to the National Science Foundation for its continuing support of the program.

Mary C. Thayer
George Veronis
George Veronis, Director

Back row (left to right): Gierasch, Malkus, Kraichnan, Keller, Williams, Stern, Schneider.
Middle row: Thayer, Welander, Simmons, Henderson, Cottrell, Thompson, Ingersoll.
Front row: Bennett, Baker, Roberts, Saunders, Thiebaux, Dubisch.
# CONTENTS OF VOLUME II

Fellows' Lectures

<table>
<thead>
<tr>
<th>Page No.</th>
<th>Title</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Low Prandtl Number Convection</td>
<td>Louis Baker</td>
</tr>
<tr>
<td>6</td>
<td>On the Generation of Baroclinic Coastal Jets by Wind</td>
<td>John R. Bennett</td>
</tr>
<tr>
<td>15</td>
<td>Is the Great Red Spot of Jupiter a Taylor Column in Horizontal Shear Flow?</td>
<td>John W. Cottrell</td>
</tr>
<tr>
<td>32</td>
<td>Mars: &quot;Slope Winds&quot;</td>
<td>Russell Dubisch</td>
</tr>
<tr>
<td>44</td>
<td>Evaporation of Rainfall</td>
<td>Glyn O. Roberts</td>
</tr>
<tr>
<td>61</td>
<td>On the Formation of the North Atlantic Bottom Water</td>
<td>Kim D. Saunders</td>
</tr>
<tr>
<td>71</td>
<td>Thermodynamic Constraints on Penetration of Convection with Application to Venus</td>
<td>Edwin K. Schneider</td>
</tr>
<tr>
<td>90</td>
<td>The Role of Baroclinic Instability in the Momentum Balance of the Earth's Atmosphere</td>
<td>Adrian J. Simmons</td>
</tr>
<tr>
<td>124</td>
<td>Wiener-Mermite Expansion applied to Passive Scalar Dispersion in a Nonuniform Turbulent Flow</td>
<td>Martial L. Thiebaux</td>
</tr>
</tbody>
</table>
LOW \textit{PRANDTL} NUMBER CONVECTION

Louis Baker

\textbf{Introduction}

The problem considered here is the Bénard problem of a plane parallel layer heated from below, in the Boussinesq approximation. Non-dimensionalizing using the thermal conductivity $\kappa$, the equations are:

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{1}{\sigma} \frac{Du}{Dt} + \nabla p = \Theta R + \Delta u$$

$$\frac{D\Theta}{Dt} = \Delta \Theta + \beta w - \mathbf{u} \cdot \nabla \Theta$$

Here $R$ is the Rayleigh number $\propto \beta d^4 \kappa \nu$, $\beta \equiv -\partial T/\partial z$, where $\overline{T}$ is the mean temperature and $\Theta$ the fluctuating part of the temperature, and $\sigma \equiv \nu/\kappa$ is the Prandtl number. Defining $\Lambda = \sigma R$ we consider here the case $\sigma \ll \Lambda \ll 1$. This limit is relevant to the outer convective zones of stars, where the 'conductive' transfer is by radiation, the mean free path being smaller than a scale height and hence approximated by a Newtonian law of cooling; the Prandtl number is then very small and yet the Rayleigh number is large (for the sun, estimated at $10^{-9}$ and $10^{+11}$ respectively). The Prandtl number can be looked upon as a ratio of conductive to viscous time scales; this limit then implies that fluctuations lose their heat more rapidly than the time required by viscous damping to significantly affect the motion of a 'parcel' (which is acted upon by buoyant forces due to the temperature fluctuation). The ability of the conductive losses to smooth temperature fluctuations therefore will tend to inhibit convective motion, and for small enough $\sigma$ we might expect the dominant part of the heat flux to be carried by conduction, even for large $R$. 
Motivation

Dr. G. O. Roberts has a steady, two-dimensional cellular solution valid in this limit. It has an isothermal interior which also possesses constant vorticity, fast speeds (order $R^{2/3}$ in the above non-dimensionalization) and a Nusselt number (dimensionless heat transport sealed by the conductive transport) or order $R^{1/3}$. This is for cell sizes of order 1, with $\sigma \ll R^{1/6}$. As such it is quite counter to our intuitive expectations. Therefore we would expect it to be unstable and to decay. Mixing length arguments suggest for turbulent convection $N = 1 + c\Lambda^2$. For example (Kraichnan, 1962) if we balance kinetic energy and buoyant work at the middle of the layer, $w^2 \sim \Lambda \Theta$, and assume that conductivity dominates the convective heat transport, so that $\beta \sim 1$, then the heat equation gives $\beta w = w$, or $\Theta \sim w$, so that both $w$ and $\Theta$ vary as $\Lambda$ and $N$ is as above (since by definition $N \equiv 1 + w \Theta$). Kraichnan estimates $c$ as about $6 \times 10^{-4}$.

$R$ must be large enough for turbulence to occur, $R > 9000$. This is because we must invoke a cascade of energy into smaller scales until dissipated by viscosity to achieve a steady state. We can see this by writing the power integral

$$\Theta = \frac{1}{2} \frac{d}{dz} \int w^2 dz = \int w \Theta + w \Delta u \, dz$$

and observing that unless we have all scales in $u$, with the scaling shown and derivatives of order 1 then two terms on the right-hand side cannot balance. When this author first considered this problem, a mixing length model in which a parcel was followed in its motion was formed. Then the convective derivative becomes $w \frac{d}{dz}$ where $w$ is the vertical velocity.

Scaling in terms of a characteristic length for conductive diffusion, we
find the equations (D ≡ \( \frac{d}{dt} \) \( wD_\theta \) = -(\( \theta - \tau \)) and \( wDw = -\sigma sw \) \( \Lambda (\theta - \tau) \)

where \( \theta \) is the parcel temperature and \( \tau \) is the local temperature; \( s \) is the \([\text{squared}]\) ratio or thermal and viscous length scales, and so will be of order 1 for turbulence and hence for small \( \sigma \) this term can be dropped.

Assuming a conductive interior we find a \( 1 + cA^2 \) law with \( c \approx 0(10^{-2}) \).

Böhm-Vitense mixing length theory (which effectively replaces the operator \( D \) by \( \ell^{-1} \)) gives the same Nusselt number dependence.

The intent of this work was to look for solutions to the equations which were more in line with the fluxes predicted by these arguments.

The Problem

It will be more convenient to non-dimensionalize using \( \bar{\nu} \). Then the altered equations are:

\[
N = 1 + \sigma \bar{w}\bar{\theta} \\
\frac{du}{dt} + \bar{\nu} = \Delta u + \frac{\tau\bar{\theta}}{\sigma} \\
\sigma\frac{d\theta}{dt} = \Delta \theta + \beta w - \sigma u \cdot \bar{\theta}
\]

Note that velocities of order 1 in this non-dimensionalization are of order \( \sigma \) in the former; hence this scaling will imply smaller velocities if we expand \( u = u_0 + \sigma u_1 + \sigma^2 u_2 + \ldots \) with \( u_0 \approx 1 \) (we would need a term \( u_{-1}/\sigma \) to recover a solution such as Roberts'). If we expand \( \sigma \) similarly then there is no term to balance the \( \theta_0 \) term in \( \sigma \); it is so big that it must be zero. Then considering only \( u_0 \) and \( \theta_1 \) terms (the leading terms), dropping subscripts, and noting that \( \beta \) will be 1 to order \( \sigma^2 \) then

\( N = 1 + \sigma^2 \bar{w}\bar{\theta} \) and \( -w = \Delta \theta \), with the momentum equation being unchanged except for the disappearance of the \( \sigma \). We now have equations (see Spiegel, 1962) in only one parameter.
If we now look for laminar, two-dimensional, steady solutions we see that the altered form of the heat equation will probably preclude horizontal boundary layers. Using a stream function $\psi$ such that $w = \psi_x$ and $u = -\psi_y$ with $\partial_y = 0 = v$, we have $\Delta = -\psi_x$ and $J(\psi, \eta) + \Delta \eta + R \eta_x$ where $J$ is the Jacobian determinant and $\eta = -\Delta \psi$ is the $(y-)$ vorticity. Detailed calculation proves there are no solutions for cells of $O(1)$ size for (asymptotically large) $R$. In doing this calculation the power integral above is invoked as necessary for the single-valuedness of the streamlines of the cell (see Wesseling, 1969 or Roberts); free boundary conditions (assumed throughout this paper) are used, as well as heat equation balance in vertical boundary layers, or horizontal if there are no vertical, or else in interior; flux continuity from horizontal to vertical boundary layers, if both present, and balance between the larger of momentum or viscosity and buoyancy in the vertical boundary layer (plume), if any, or else the interior.

If we assume that cell size is $O(R^{-k})$ then we will be able to get an interior balance satisfying the power integral constraint; we must assume small amplitude, i.e., that $R$ is near $R_{\text{crit}}(k)$ where $k$ is the (horizontal) wavenumber of the cell. Then we will find $N-1$ is of order $\sigma^2 \varepsilon^2$ where $\varepsilon$ is of order $(R-R_{\text{crit}}(k))^{1/2}$, $R_{\text{crit}}$ being the critical wavenumber. The same length scale $R^{-1/4} = \delta$ is suggested as being significant by the turbulent solutions. Comparison of equations (49) and (51) of Ledoux, Schwartschild, and Spiegel (1961) on inserting a $k^2$ factor in the former integral to estimate $\overline{u \Delta u}$ will give a balance from contributions when the scaled wavenumber $q$ used is of order $\delta^{-1}$; Kraichnan's arguments above when the $\Delta$ operators are scaled as $\delta^{-2}$ and a viscous-buoyant balance is introduced to determine $\delta$.
we find that the above scaling for $\delta$ holds; this is independent of sealing the kinetic energy term (which arises from the $u \cdot \nabla u$) with or without $\delta$. The $N^{-1}$ values are altered, however. It would seem that this scale is significant in determining the viscous processes; fox the solar case it is only in the range of tens of meters and hence well below resolution. These rolls and other solutions in this range might profitably be examined by the perturbation technique of Malkus and Veronis (1958), but they are probably unstable and will 'run away' (Herring, 1970).

As pointed out by Dr. Roberts if we assume, guided by his solution, that the vorticity is constant on streamlines, i.e. $u \cdot \nabla \eta = 0$ or that $\mathcal{F}(\psi, \eta) = 0$, then we have a linear system subject to this constraint, in other words, an eigenvalue problem for $R$. Given this solution we could by the same perturbation techniques construct solutions with $u = \sigma^2(\varepsilon u_o + \varepsilon^r u_1 + \ldots)$ and similarly for $\theta$, where the eigenvalue is denoted $R_g$ and $\varepsilon \equiv \sqrt{(R - R_g)} \ll 1$. Then $N^{-1} = 0(\varepsilon^r) \ll 1$. This of course presupposes a solution for $R_g$ and stability is doubtful.

In summary, then we may have found an interesting length scale. But we must recognize that the turbulent solution must be of a very different nature than a laminar solution in terms of its (asymptotic) dependence on $R$. Galerkin schemes to be valid in the sense of giving physically relevant solutions must recognize this; expanding in such plume-like or roll-like solutions (with scale $\delta$) does not imply capturing the important physics. Clearly much more work must be done before any of the questions raised here can be esnsedered in any sense settled.
ON THE GENERATION OF BAROCLINIC COASTAL JETS BY WIND

John R. Bennett

I. Introduction

Near the shores of the Great Lakes, large variations in the depth of the thermocline, which last anywhere from a day to several weeks, are common. The detailed measurements of Birchfield and Davidson (1967) and Scott and Landsberg (1969) indicate that the longshore current during these periods is approximately geostrophic. The width of these currents is comparable to the internal radius of deformation and in many cases relative vorticity is comparable to f, the coriolis parameter.

The observed currents have been interpreted by Csanady (1967, 1968) as baroclinic "coastal jets" of the type discussed by Charney (1955). Csanady recognized that the coastal jet solution is a zero frequency Kelvin wave and that in a closed basin the solution must include traveling Kelvin waves,

References

Robert, G. O. 1970 Fast Viscous Convection (reprint)

Acknowledgements

This work was inspired by that of Dr. G. O. Roberts. He provided helpful suggestions and criticism at every step of the way. Prof. A. P. Ingersoll and Prof. W.V.R. Malkus provided helpful encouragement and advice, and useful criticism.

John R. Bennett
The purpose of this work is to study the effect of basin shape on the generation of coastal jets and to extend the theory to consider non-linear effects. Toward this goal, in Section II it is shown how the fact that the internal radius of deformation is small compared to the length scale of the basin can be exploited to derive an approximate equation for the depth of the thermocline near the shore. This equation is derived for an "equivalent one-layer model" for a basin of arbitrary shape and for an arbitrary distribution of wind stress. In Section III, the equation is used to study an initial value problem for a uniform wind stress on a circular basin.

II. The mathematical model

If, in a two-layer model, the depth of the upper layer is small compared to the total depth, one can neglect the pressure gradient in the upper layer in calculating the upper layer flow. Under this approximation, the pressure gradient in the upper layer is:

$$\frac{1}{\rho} \nabla p = -g \frac{\Delta \rho}{\rho} \nabla h$$

The depth of the thermocline is $h$ and $\frac{\Delta \rho}{\rho}$ is the relative density difference between the layers. The fluid in the upper layer is assumed to obey the shallow water equations and the wind stress is simulated by an acceleration term of magnitude $\frac{\tau_0}{\rho_0 H}$, where $\tau_0$ is a typical wind stress and $H$ is the equilibrium depth of the thermocline. The equations for the model are:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -g \frac{\Delta \rho}{\rho} \frac{\partial h}{\partial x} + F_x$$

(1)

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -g \frac{\Delta \rho}{\rho} \frac{\partial h}{\partial y} + F_y$$

(2)

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0$$

(3)
The boundary condition is that the normal component of the velocity must vanish at the wall. The initial condition is a state of rest where the thermocline is at its equilibrium depth:

$$t = 0, \quad u = v = 0, \quad h = H$$

In Csandady's (1968) linear problem for a circular basin the solution consists of inertio-gravity waves with frequency larger than $\mathbf{f}$ and steady or low-frequency Kelvin waves which are confined to the boundary region. We wish to isolate the low-frequency response and to take advantage of the fact that the solution has a boundary layer structure. Thus, we use as a time scale, $R/\sqrt{\rho \Delta p_H}$, the time scale of long Kelvin waves in a basin with a typical length scale $R$. The scale for velocity is given by a balance of the coriolis force with the forcing term in equations 1 - 2. The non-dimensional variables are

$$\begin{align*}
(u', v') &= (u, v) \frac{t}{|F|} \\
h' &= \frac{h}{H} \\
(x', y') &= \left(\frac{x, y}{R}\right) \\
t' &= \frac{t \sqrt{\rho \Delta p_H}}{R}
\end{align*}$$

In terms of these variables (dropping the primes) the equations are:

$$\begin{align*}
\lambda \frac{\partial u}{\partial t} + \epsilon (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) - \nu &= - \frac{\lambda}{\epsilon} \frac{\partial h}{\partial x} + F_x \\
\lambda \frac{\partial v}{\partial t} + \epsilon (u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) + u &= - \frac{\lambda}{\epsilon} \frac{\partial h}{\partial y} + F_y \\
\frac{\partial h}{\partial t} + \epsilon \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) &= 0
\end{align*}$$

The two non-dimensional parameters are...
\[ \varepsilon = \frac{F}{\mu R}, \] the Rossby number

and \( \lambda = \frac{\ell}{\mu R} \), the ratio of the internal radius of deformation to the length scale of the basin. Both will be considered small and \( \varepsilon/\lambda^2 \) will be considered of order unity. When \( \varepsilon/\lambda^2 \) is of order one the non-linear terms are important near the boundary. In the limit

\[ \varepsilon \to 0, \quad \lambda \to 0, \quad \varepsilon/\lambda^2 = o(1) \]

equations 4 - 6 are:

\[
\begin{align*}
-v &= -\varepsilon \frac{\partial h}{\partial x} + F_x \\
u &= -\lambda^2 \frac{\partial h}{\partial y} + F_y \\
\frac{\partial h}{\partial t} &= 0
\end{align*}
\]

Since \( h = 1 \) initially, the solution in the interior of the lake is

\[
\begin{align*}
-v &= F_x \\
u &= F_y \\
h &= 1
\end{align*}
\]

A boundary layer of width \( \lambda \) is needed to satisfy the condition that these be no normal flow. For this boundary region we introduce locally cartesian coordinates \( \lambda_n \) and \( S \). The coordinate along the coast increases in a counterclockwise direction and the normal coordinate \( n \) increases outwards and is zero at the boundary. The velocity component normal to the shore, \( V_n \), has the same scale as the interior flow and the longshore component is larger by a factor \( \lambda^{-1} \) so that the transport in the boundary layer is comparable to the interior transport. In short, the velocities and coordinates near the boundary are represented by

\[ \lambda_n, S; V_n, \lambda^{-1} V_S \]
where \( n, s, \nabla_n, \nabla_s \) are numbers of order one. In the same limit as that for the interior equations the boundary layer equations are:

\[
- \nabla_s = -\frac{\lambda^2}{\varepsilon} \frac{\partial h}{\partial \eta} \quad (7)
\]

\[
\frac{\partial \nabla_s}{\partial t} + \frac{\varepsilon}{\lambda^2} \left[ \nabla_n \frac{\partial \nabla_s}{\partial \eta} + \nabla_s \frac{\partial \nabla_n}{\partial \eta} \right] + \nabla_n = -\frac{\lambda^2}{\varepsilon} \frac{\partial h}{\partial \eta} + F_s \quad (8)
\]

\[
\frac{\partial h}{\partial t} + \frac{\varepsilon}{\lambda^2} \left[ \frac{\partial (\nabla_n h)}{\partial \eta} + \frac{\partial (\nabla_s h)}{\partial \eta} \right] = 0 \quad (9)
\]

In order for the solution to be consistent with the interior solution and for the boundary condition of no normal flow to be satisfied the following initial and boundary conditions must be satisfied:

\[
\nabla_n = 0 \text{ at } \eta = 0
\]

\[
\nabla_s \to 0, \ h \to 1, \ \nabla_n \to F_s \text{ as } \eta \to -\infty
\]

\[
\nabla_s = 0, \ h = 1 \text{ at } t = 0.
\]

It is consistent with the approximations used so far to assume that

\[
\frac{\partial F_s}{\partial \eta} = 0
\]

In other words, the forcing function varies appreciably only on a length scale comparable to the basin size, not over the boundary layer thickness.

The solution of this problem is

\[
h = 1 + Q(s,t)e^n \quad (10)
\]

\[
\nabla_s = \frac{\lambda^2}{\varepsilon} Q(s,t)e^n \quad (11)
\]

\[
(1 + Qe^n)\nabla_n = (1 - e^n)F_s + \frac{\lambda^2}{\varepsilon} (e^n - e^{2n}) \frac{\partial Q^2}{\partial s} \quad (12)
\]

where \( Q(s,t) \) satisfies

\[
\frac{\partial Q}{\partial t} + (1 + Q) \frac{\partial Q}{\partial s} = \frac{\varepsilon}{\lambda^2} F_s \quad (13)
\]
and
\[ Q = 0 \quad \text{when} \quad t = 0. \]

The form of the solution was suggested by the fact that the potential vorticity can be shown to be approximately uniform. The potential vorticity in the boundary region is
\[
\frac{\varepsilon - \frac{\partial v}{\partial n} + 1}{\lambda^2} = 1 + O(\lambda)
\]
This relation and equation (7) imply
\[
\frac{\partial^2 h}{\partial n^2} - h = -1
\]
which is satisfied by the form of the solution.

III. Solution for a circular lake

In this section, solutions are presented for the case of a circular lake driven by a uniform wind stress. If the wind comes from the direction \( \theta = -\pi \), the components of the forcing term are
\[
F_\theta = -\sin \theta \\
F_\gamma = \cos \theta.
\]
Since the non-linear effects become important for \( \varepsilon/\lambda^2 \) of order unity we set
\[
\varepsilon/\lambda^2 = 1.
\]
The solution of equation 13 was obtained by the method of characteristics. It is shown in Fig. 1 for \( t = 1.0 \) along with the solution to the linear problem in which the term \( Q \frac{\partial Q}{\partial \theta} \) was dropped. At \( t = 1.05 \) the depth of the thermocline at the shore, \( I + Q \), becomes negative in both theories, Therefore, in order to continue the solution, the wind stress was set equal to zero for \( t > 1.0 \).

At \( t = 1.5 \) the theory breaks down anyway since the characteristics
\[ \frac{\epsilon}{\lambda^2} = 1.0 \]
\[ t = 1.0 \]

Fig. 1

\[ \frac{\epsilon}{\lambda^2} = 1.0 \]
\[ t = 1.0 \]
Forcing stopped at \( t = 1.0 \)

Fig. 2
Fig. 3
cross. At this time the solution is shown in Figure 2. The approximations used here are no longer valid for this time, but perhaps the solutions of the full equations 1-3 may still be valid.

The solution for the motion at \( t = 1.0 \) is shown schematically in Figure 3. In the interior of the lake the current is to the right of the wind. Near the shore there is upwelling in the north-northwest portion and downwelling in the southeast with associated strong geostrophic currents. The radial component of the flow given by the second term on the right-hand side of equation 12 is also sketched in the boundary region. This flow would be zero in the linear theory and even in the non-linear theory the current is small compared to the interior flow until the time when theory fails.

The author wishes to thank Profs. Pierre Welander, Melvin Stern, Joseph Keller, Rory Thompson, George Veronis and R. Terry Williams for many helpful discussions of this work.

References

IS THE GREAT RED SPOT OF JUPITER
A TAYLOR COLUMN IN HORIZONTAL SHEAR FLOW?

John W. Cottrell

Introduction

Since Hide (1961) first suggested that the Great Red Spot of Jupiter is a natural manifestation of an inertial Taylor column, there has been renewed interest in this phenomenon. Previous investigations (Ingersoll, 1969, Hide and Ibbetson (1966)) have studied Taylor columns, both theoretically and in the laboratory, and have observed many properties characteristic of the Red Spot. All of these investigations assumed a uniform undisturbed flow past the obstacle. The Great Red Spot of Jupiter, however, is situated in a unique location, having the most highly retrograde flow and the strongest negative relative vorticity on the planet. Therefore, in this paper we consider the flow of a rotating fluid about an obstacle when the undisturbed flow is one of uniform horizontal shear. Throughout we imply a positive obstacle, mountain rather than depression, although the results are equally valid (with change of sign) to either case.

For simplicity of analysis, and also because the thermal structure of Jupiter is unknown, the effects of stratification are not included. We look at a homogeneous, incompressible fluid bounded above and below by rigid horizontal infinite planes. Although this models the Jovian atmosphere rather poorly with respect to vertical boundary conditions and thermal structure, it is likely that the property we wish to investigate, the effect of horizontal shear in the upstream flow, is insensitive to these deficiencies. We also consider only the steady state inviscid limit,
In the first part, flow on an f-plane is studied and it is found that two very different situations develop, depending on the sign of the relative vorticity. For positive vorticity and obstacles of small vertical extent, streamlines are constricted. If the height of the obstacle is large enough, a Taylor column (i.e., a region of closed streamlines) ultimately forms. In flows with negative shear, a Taylor column forms for all finite obstacles, a result with major implications for the Great Red Spot, which is embedded in a region of strongly negative vorticity.

In the second part, effects of spherical geometry are included by investigating the flow on a $\beta$-plane. As the governing equation is non-linear, solutions are obtained only in the asymptotic limit when the non-dimensional parameter $b$ is large (on Jupiter $b \approx 4$). In this case, valid solutions are found only when the critical streamline defining the Taylor column originates in zero or retrograde flow. Otherwise, for prograde flow, the assumed form of the asymptotic expansion proves inconsistent with the boundary condition at infinity and further analysis is required.

Finally, we conclude with a suggestion, based on the results of the analysis, explaining the formation of only one Taylor column in the atmosphere of Jupiter. It is argued that if the obstacles embedded in the atmosphere (whether material, thermal, or magnetohydrodynamic) are sufficiently small, then Taylor columns would only form in regions of zero relative velocity and highly negative shear, thus severely limiting their number and location.
**Formulation**

We consider a homogeneous, incompressible fluid in a state of uniform counterclockwise rotation about the vertical axis, and bounded vertically by rigid plane surfaces a distance $H$ apart. We assume an obstacle of height $h'$ small compared to $H$ and finite characteristic length $L$ is attached to the lower surface. If $(x', y', z')$ and $(u', v', w')$ are dimensional coordinates and velocities, and $U$ is a velocity characteristic of the variation of the steady flow from a state of uniform rotation, then the following non-dimensional quantities may be defined:

$$
(x, y) = \frac{(x', y')}{L} \quad z = \frac{z'}{H} \quad h = \frac{h'}{H} \\
(u, v) = \frac{(u', v')}{U} \quad w = \frac{w'}{L/H} 
$$

Together with the Rossby number

$$
\epsilon = \frac{U}{2 \Omega L}
$$

and the Taylor number

$$
R = \frac{\Omega H^2}{\nu}
$$

Ingersoll (1969) has formulated the problem of steady flow about an obstacle in a rotating fluid in the *inviscid* limit where:

$$
\epsilon \rightarrow 0, \quad \epsilon R^k \rightarrow \infty, \quad h/\epsilon = o(1) \quad (1)
$$

To lowest order in $\epsilon$, in the interior region of the flow outside of Ekman boundary layers, the flow is quasi-two-dimensional and quasi-geostrophic and satisfies

$$
\mathbf{q} \cdot \nabla (\zeta + h/\epsilon) = 0 \quad (2)
$$

where $\zeta$ is the vertical component of relative vorticity. This expresses the well-known fact that vorticity is created or destroyed by the stretching or compression of vortex lines as the fluid passes over the obstacle.
Equation (2) states that the quantity \((J + h/\varepsilon)\) is conserved along streamlines and may be \textit{reexpressed} in terms of the stream function as

\[
\nabla^2 \psi + h/\varepsilon = f(\psi)
\]

(3)

where \(u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}, \zeta = \nabla^2 \psi\) and the functional form of the right-hand side of (3) is \textit{determined} by the flow far upstream of the obstacle where \(h/\varepsilon\) is zero. As a result of the \textit{geostrophy} of the flow, to lowest order in \(\varepsilon\) the streamlines are lines of \textit{constant} pressure.

Equation (3) is the governing equation of the interior flow, solutions of which we wish to determine subject to boundary conditions on the undisturbed flow far from the obstacle.

In addition to the boundary condition at infinity, Ingersoll (1969) has shown that an additional condition is \textit{required} in the event closed streamlines occur in the solution. In the inviscid, steady state limit \((\nu \to 0, \psi \to \infty)\), the flow within a closed streamline must be stagnant in the steady state to \textit{preclude} accumulation of fluid by Ekman layer pumping. We therefore require zero velocity everywhere an the closed streamline and are led to the condition:

\[
\psi_x = \psi_y = 0 \quad \text{on} \quad \psi = \psi_c
\]

(4)

where \(\psi_c\) is the critical streamline defining the location of the Taylor column.

\textbf{Undisturbed flow of constant vorticity}

When the flow far upstream where the obstacle height is zero is one of uniform horizontal shear, equation (3) reduces to

\[
\nabla^2 \psi + h/\varepsilon = \zeta_{\infty}
\]

(5)

subject to the boundary condition
corresponding to mean flow $U_\infty$ and vorticity $\zeta_\infty$.

If we now specify an obstacle, cylindrical in shape and of constant height, such that

$$h_\varepsilon = \frac{h_0}{\varepsilon} \quad \text{for } r < 1$$
$$h_\varepsilon = 0 \quad \text{for } r > 1$$

the solution of (5) and (6) is straightforward and is given by

$$\psi = -\frac{h_0}{\varepsilon} y \ln r + \frac{\zeta_\infty y^2}{2} - U_\infty y \quad (r > 1)$$
$$\psi = -\frac{h_0}{4 \varepsilon} (r^2 - 1) + \frac{\zeta_\infty y^2}{2} - U_\infty y \quad (r < 1)$$

In the case when $\zeta_\infty$ is positive and the mean flow is zero, the streamline pattern is shown in Figs. 1-3. When the obstacle height is less

Fig.1 Streamlines (not drawn to scale) for flow about obstacle of height $h < 2 \varepsilon \zeta_\infty$ when $\zeta_\infty > 0$, $U_\infty = 0$. The asymptote angle $\Theta$ increases from $0$ to $\pi/2$ as height varies from zero to infinity.
Fig. 2 Streamlines for flow about obstacle of height \( h = 2 \epsilon F \) when \( F > 0, u_\infty = 0 \).

Fig. 3 Streamlines for flow about obstacle of height \( h = 2 \epsilon F \) when \( F > 0, u_\infty = 0 \). A region of closed streamlines forms over the obstacle.
than $2 \in \gamma_\infty$, fluid moving at low speed towards the center of the obstacle loses enough relative vorticity so that it is deflected across the center line and returns in the direction from which it came. Fluid traveling at higher speed and further removed from the obstacle center is deflected towards the axis but is able to ultimately continue in the same direction (Fig.1). As the height is increased, less fluid is able to pass the obstacle. Finally when $h = 2 \in \gamma_\infty$, all fluid which contacts the obstacle is returned (Fig.2). When the obstacle height surpasses $2 \in \gamma_\infty$, closed streamlines form with elliptical shape and major axes normal to the flow direction (Fig.3). However, in this case the added boundary condition expressed in equation (4) must be satisfied, and the solution (8) is no longer satisfactory. Because the added boundary condition (4) is non-

![Fig.4 Streamlines for flow about obstacle when $\gamma_\infty > 0$, $\nu_\infty = 0$. For any finite obstacle, all streamlines are closed.](image)
linear, in that it is applied on a free surface whose location is determined as part of the solution, complete solutions for this case have not been obtained.

Letting the mean flow again be zero and looking at solutions where $\zeta_{\infty}$ is negative, we find that all streamlines are closed for any finite value of $h$ (Fig. 4). Within the fluid region containing the obstacle the streamlines are elliptical and aligned with the direction of flow. However, once more (4) is not satisfied and the solution is not complete. In this case, not only must the non-linear boundary condition be satisfied, but since all streamlines are closed, some difficulty exists in determining the critical streamline.

Clearly, a solution of (5) which yields closed streamlines extending to infinity is not satisfactory in view of the boundary condition (4) which states that a steady inviscid solution must have stagnant flow inside closed streamlines. The resolution of this problem leads to a criteria for estimating the area of a steady state Taylor column. Returning to the partial solution (8), and neglecting for the moment the mean flow term, we note that the second term arises from the boundary condition at infinity and yields parallel streamlines. The closure of these streamlines is due to the first (logarithmic) term, created by the obstacle, and which acts as a negative source of vorticity. We now note that the existence of a Taylor column implies the necessity of an additional term in the solution which must satisfy

$$\nabla^2 \psi = -\zeta_{\infty} h / \varepsilon$$  \hspace{1cm} (inside Taylor column)

$$= 0$$  \hspace{1cm} (outside Taylor column)
with boundary conditions on the critical streamline defining the Taylor column given by

$$\nabla \psi = -\nabla \psi_o \text{ on } \psi = \psi_c$$

and boundary condition of zero velocity at infinity

$$\nabla \psi_i = 0 \quad \text{as } r \to \infty$$

It is evident that the sum of $\psi_o$ and $\psi_i$ satisfies (4), (5) and (6) and therefore constitutes the solution of the problem. Because the location of the critical streamline is unknown, the solution of (9) - (11) is difficult to obtain and so a complete closed form expression for the stream function is unlikely. However, realizing that the solution to (9) - (11) is that due to a source of vorticity distributed uniformly over the Taylor column, we can specify its behavior at Barge distances from the obstacle where it acts as a point vortex. We find

$$\psi_i \sim \frac{(T_0 + h \varepsilon)}{2 \pi} A_{Te} \log r + O(r^{-1}) \quad \text{as } r \to \infty$$

The criteria for determining the area of the Taylor column is now evident. At large distances, the complimentary solution (12) arising from the existence of the Taylor column must be great enough to cancel the logarithmic term in (8) due to the obstacle. When this is the case, open streamlines extending to infinity exist, and a credible solution is possible. Requiring that the sum of the first term in (8) and the complimentary solution (12) be greater than zero, we establish

$$A_{Te} \geq \frac{\pi}{(1 - \frac{h \varepsilon}{h_o})}$$
Remembering that \( \zeta_\infty < 0 \), the minimum area of the Taylor column increases from zero to the area of the obstacle as the height of the obstacle varies from zero to infinity, a result we would expect.

On physical grounds, we would expect the equality to hold in (13), since in the limiting cases where the obstacle height approaches zero or infinity, the inequality appears unrealistic. Further, it seems implausible to expect that fluid traveling from infinity towards a source of negative vorticity (the obstacle) would experience a trajectory consistent with a source of positive vorticity, as implied by the inequality in (13).

An important consequence of (13) is that in shear flows of negative vorticity, Taylor columns are to be expected for all finite obstacles. This is not so for uniform upstream flow (Ingersoll, 1969), or for positive shear flows [Figs. 1-31, where a minimum obstacle height (of order \( \varepsilon \)) is required for the formation of a Taylor column. Since the Great Red Spot of Jupiter lies in a region of high negative vorticity, only a very mild disturbance of the flow is necessary to form a Taylor column.

**Effect of mean flow**

When a mean component of the undisturbed flow is inclined \( (U_\infty \neq 0) \), its effect is to shift the streamline pattern in the \( y \) direction so that within the obstacle region \( (r < l) \), the hyperbola asymptote origin \( (\zeta_\infty > 0) \) or ellipse origin \( (\zeta_\infty < 0) \) is located at the latitude of zero relative velocity (Fig.5). Although complete solutions have not been obtained for cases when Taylor columns exist, it is reasonable to expect that their centers would also exhibit this behavior and lie near the latitude of zero velocity. The physical ground for this expectation is related to the
Fig. 5 Streamline pattern for flow about obstacle when $\tau_\infty < 0, u_\infty < 0$. All streamlines are closed with center at latitude of zero velocity.

reason that Taylor columns form with relative ease in flows of negative vorticity, namely, that fluid moving at low velocity is laterally deflected more readily than is the surrounding high speed flow. In other words, we would expect the Taylor column to form where the local Rossby number is smallest. This may explain why the red spot of Jupiter is situated at a latitude of nearly zero relative velocity (Fig. 7).

$\beta$-plane

When investigating Taylor column formation on geophysical scales, other effects, e.g., spherical geometry and stratification, are likely to be important. In his investigation of uniform flow about an obstacle,
Ingersoll (1969) found that in the limit when spherical geometry was very important (the obstacle size comparable to \( \epsilon \) times the radius of the planet), uniformly valid solutions over the entire region were only obtained for retrograde flow. In prograde flow, higher order corrections to the stream function were not confined to boundary layers, and the asymptotic expansion proved inconsistent with the applied boundary conditions. Similar difficulties are encountered in oceanographic studies when boundary conditions at the western shore must be matched to an eastward interior flow (Carrier and Robinson, 1962).

In order to verify that similar circumstances result when the upstream flow has uniform shear, an analysis was performed following Ingersoll's procedure on a \( \beta \)-plane. A brief outline is presented, from which similar conclusions are drawn, namely, that valid boundary layer solutions are only obtained when the critical streamline originates in a region of retrograde flow. Otherwise, the procedure fails, and for the present at least, the characteristics of Taylor column formation in prograde flow remain a mystery.

Looking at flow on a \( \beta \)-plane, we find that the governing equation (3) becomes

\[
\nabla^2 \psi + b \frac{\partial \psi}{\partial y} + \frac{h}{\epsilon} = f(\psi)
\]

where \( b = \frac{L}{(\alpha \epsilon)} \) and \( \alpha \) is the radius of the planet. We assume a flow with uniform horizontal shear far upstream so that

\[
u = -\frac{\partial \psi}{\partial y} = -\zeta_\infty y \quad \alpha \to \infty \quad r \to \infty
\]

where now the effect of any mean flow will be included by allowing the
obstacle to assume a variable location. The boundary condition (15) together with (14) implies that throughout the flow, the stream function satisfies

$$\frac{1}{b} \left( \nabla^2 \psi - \psi_\infty \right) + \sqrt{-\nabla^2 \psi} \psi_\infty = -y - h/\varepsilon b$$

(16)

Here the negative sign applies to streamlines originating \([\text{at } \infty]\) in retrograde flow and the positive sign to streamlines originating in prograde flow. In the limit when \(b\) is large, the stream function may be expanded in a series of the form

$$\psi = \psi_0 + b^{-\frac{\varepsilon}{2}} \psi_1(x^*, y^*) + \ldots$$

(17)

where \(x^* = b^{-\frac{\varepsilon}{2}} x\), \(y^* = b^{-\frac{\varepsilon}{2}} y\) and it is assumed that

$$\varepsilon << b^{-\frac{\varepsilon}{2}} << 1, \quad h/\varepsilon b = O(1)$$

(18)

Now the Lowest order term locates the critical streamline, on which the first order term is of proper magnitude to satisfy the non-linear boundary condition (Eq.4). To lowest order in \(b\) we find

$$\psi_0 = \frac{f_0}{2} \left( y + h/\varepsilon b \right)$$

(19)

Defining an obstacle with variable center \((x_0, y_0)\) and shape given by

$$\frac{h}{\varepsilon b} \frac{h_0}{\varepsilon b} \left[ 1 - (y - y_0)^2 \right] \quad r' < 1$$

$$= 0, \quad r' > 1$$

(20)

and \(r'\) is the distance from the center of the obstacle, a streamline pattern shown in Fig. 6 results. This same pattern applies no matter what the flow is at infinity, as it expresses a primary balance between the \(\beta\)-term and the obstacle height. The critical streamline is that which just intercepts the upper edge of the obstacle and is given by
Fig. 6. Streamline pattern for flow on $\beta$-plane in asymptotic limit for large $b$. Inertial boundary layers form on edge of both obstacle and Taylor column, if $\Psi_e$ originates in zero or retrograde flow. Here $h(r') = 1 - \frac{r^2}{r'^2}$.

To order $b^4$, we determine the equation for $\Psi_1$

$$\nabla^2 \Psi_1 + \left( \frac{2 \Psi_0}{\Gamma_0} \right) \frac{\nabla \Psi_1}{\Psi_0} = 0$$

with B.C.'s

$$\nabla \Psi_1 = -\nabla \Psi_0 \text{ on } \Psi_0 = \Psi_c \quad (23)$$

$$\nabla^2 \Psi_1 = 0 \quad \text{as } r^* \to \infty \quad (24)$$

When the critical streamline originates as a zero velocity streamline, the lowest order solution (19) identically satisfies the non-linear boundary condition (4). Otherwise, the next order solution, governed by (22), must be determined.

We see from (22) that when the critical streamline originates in retrograde flow, so that the negative sign applies, exponential B.L. solutions exist which satisfy (23, 24).
A critical streamline originating in prograde flow admits only wavelike solutions to (22), so that the boundary condition (24) cannot be satisfied. Certainly, a more thorough analysis of this situation should be undertaken. Until such an examination is complete, the possibility exists that no steady state solution is to be found admitting Taylor columns in prograde flow on a $\beta$-plane.

**Concluding remarks**

The formation of Taylor columns in rotating fluids has been investigated under conditions of uniform horizontal shear in the undisturbed flow for upstream. We have found that when the upstream vorticity is positive, the obstacle ($h < 2\varepsilon_{\sigma}$) constricts the streamlines (Fig.1). When the obstacle height is greater than $2\varepsilon_{\sigma}$, a Taylor column forms.

It is of interest to note that on Jupiter whereas no Taylor columns have been observed in regions of positive vorticity, these are multiple observations of constriction of various zones and belts. It is tempting to speculate that these are temporary manifestations of the influence of obstacles situated far beneath the visible surface.

In flow with negative relative vorticity, a Taylor column forms about any finite obstacle. While the determination of the exact shape of the Taylor column is unimportant, since it depends strongly on the assumed shape of the obstacle, criteria for determining the size and location of the Taylor column are of interest and have been established. The criteria for the location of the Taylor column, that its center be located at the latitude of zero relative velocity, was developed from heuristic arguments. On the other hand, its minimum size was determined rigorously and is expressed
in (13). This is the single most important result of this study, for it indicates that in regions of negative vorticity only very shallow obstacles, of order less than $\epsilon$, are necessary to form Taylor columns. This may have bearing on why only one Great Red Spot occurs on Jupiter, a matter discussed further below.

Finally it has been shown that when effects of spherical geometry are important, valid solutions are obtained only when the critical streamline originates in zero or retrograde flow. For prograde flow, no conclusions may yet be reached.

It is fitting, in view of its prominence as the universe's only readily observable natural manifestation of a Taylor column (if that it be), to conclude with a brief discussion of the Great Red Spot of Jupiter. For centuries its presence has mystified astronomers and only recently (Hide, 1961) has the Taylor column hypothesis been postulated. Although recent investigations (see Ingersoll, 1969) have shown that the Taylor column theory explains many observed characteristics of the Red Spot, e.g., the deflection of streamlines toward the equator (Fig.6), no satisfactory reason exists to explain the presence of only one Red Spot. Why not more?

If we look at the relative velocity of the cloud bands observed on Jupiter (Fig.7), we note that the Great Red Spot is situated in a unique location. It is embedded in a region of high negative vorticity and bounded on the northern edge by the most strongly retrograde flow on the planet. Furthermore, the critical streamline defining the edge of the Red Spot originates in flow upstream of nearly zero relative velocity. This is indicated by observations (Reese and Smith, 1968) of capture by the Red Spot of smaller
spots moving in both prograde and retrograde directions. While our theoretical analyses are still too incomplete to make any definite conclusions concerning the solitary nature of the Red Spot, the unique character of the surrounding flow leads to an overwhelming desire for speculation, a desire irreverently satiated in the concluding paragraph.

We have shown that Taylor columns form most easily in flows with negative vorticity and a zero velocity streamline. If we postulate that Jupiter has many obstacles located beneath the visible surface, but all very weak (of height less than the Rossby number) so that Taylor columns cannot easily exist, referring to Fig. 7 we see only three likely regions of Taylor column formation. Two of these, in the Northern hemisphere, only marginally satisfy the condition of zero velocity in negative shear. The third locates the Great Red Spot of Jupiter. If, in future investigations, we are able to determine the nature of Taylor column formation in prograde flow on a $\beta$-plane, more conclusive statements may be made.
Acknowledgments

The author thanks Dr. Andrew Ingersoll for suggesting the problem and for many valuable discussions on Taylor column phenomenon.

References


MARS: "SLOPE WINDS"

Russell Dubisch

I. General Introduction

A. Physio-chemical description

1. Mars has an orbit with semi-major axis \( a = 1.52 \) \( \text{AU} \), and an eccentricity of \( e = 0.093 \). This means that on the average it receives only about 43% of the solar flux as the earth, and receives — 1.4 times as much energy at perihelion as it does at aphelion. The inclination of the plane of rotation to the plane of the orbit is twenty-five degrees. As has been discussed earlier, the axis of rotation precesses at a rate of \( \sim 50,000 \) \( \text{yrs}^{-1} \).

2. The mass of Mars is about \( 1.077 \) of the earth, whereas its radius is \( \sim 0.53 \rho_{\oplus} \). Thus the density is about \( \frac{1.077}{0.53} \rho_{\oplus} \sim 3.96 \text{ gm/cm}^3 \), about the same density as the moon. The problem of its composition, as with the moon, is unresolved at present — both for similar reasons: materials, such as basalt, which have about this density at low pressures, undergo transition to denser forms at the high pressures found in a planetary interior.
The nature of these phase transitions is a difficult problem to tackle by either laboratory or theoretical methods.

The chemical nature of the immediate surface is a slightly more tractable problem for Mars. Its reddish color suggests iron oxides, and this is borne out by its optical reflection spectrum, which closely matches pulverized limonite ($\text{Fe}_\text{2}$)?; a recent attempt to attribute the spectrum to $\text{NO}_\text{2}$ absorption may generally be disregarded. Do not conclude, however, that the surface material is purely (or even predominantly) iron oxide - a mixture of quartz and limonite can give the same spectrum - with a small proportion of limonite. The same conclusion applies to the detection of $\text{H}_\text{2O}$ absorption features in the polar caps, this being consistent with the presence of only a small proportion of $\text{H}_\text{2O}$ ice in a matrix of $\text{CO}_\text{2}$ snow. It is easy to see why this might be so for any mixture of absorbing and non-absorbing materials.

Another unresolved question concerns the origin of the iron: is it indigenous to the planet, perhaps indicating that Mars has never undergone the differentiation that the earth has - or is it due to the high rate of meteoritic bombardment which Mars may have undergone, due to its proximity to the asteroid belt? These problems may be clarified this decade if we succeed in landing probes on Mars.

3. Atmosphere

a. Composition

The Martian atmosphere is composed mainly of $\text{CO}_\text{2}$, with a total abundance of $14 \pm 2 \text{ gm/cm}^2$. Water vapor is present with an abundance of $0.5$ to $2.5 \times 10^{-3} \text{ gm/cm}^2$. Nitrogen may be present ($\ll 0.8 \text{ gm/cm}^2$), as well as
argon ($< 3 \text{ gm/cm}^2$), but at present remain undetected. CO$_2$, O$_2$ and O$_3$ are produced by photochemical processes from CO$_2$. CO ($\sim 17 \times 10^{-2}$) has been detected, and O$_2$ and O$_3$ are inferred on the basis of chemical equilibrium. Ozone forms a layer similar to the terrestrial ozone layer, but optically thin in the U-V. Calculation of its abundance is difficult owing to the possibility of significant rates of reaction with materials on the ground.

b. Temperature

The temperature profile of the atmosphere has been determined by Rasool et al. (J.A.S. 27, 1970).

The important features of the temperature profile are that (1) it is subadiabatic - as expected from a convective atmosphere in which heat transfer is dominated by radiation, and (2) the existence of temperatures below the saturation temperature for CO$_2$ (between $h = 3 \text{H}$ to $4 \text{H}$). Also of interest is the discrepancy between the equatorial temperature $\sim 250^\circ \text{K}$ and $h = 0$ and the surface temperature $T_s = 275^\circ \text{K}$ determined by infrared radiometer. This leads to a strong instability near the ground.

Pressure at the ground is $\sim 4.9$ to 7.6 mb, depending on the ground elevation. The scale height is $\sim 8-10 \text{ km}$.

c. Hazes

There are three kinds of haze that have been definitely observed on Mars - they may be related to each other. They are:

i) Thin haze observed at the limb. This was observed by the Mariner 7 flyby in several photographs - it appears as a light band at the daytime limb of the planet. The clear separation from the limb indicates that this is probably a condensation phenomenon. The nature of the particles
is not known,

ii) The dawn haze. This is a thick (τ~1??) haze seen both in ground-based photographs and Mariner photography, which persists for an hour(?) after dawn,

iii) Polar haze. This is a more-or-less permanent haze over the poles - during winter. It may be associated with precipitation. There is a w-shaped cloud in the region near and east of Nix Olympica, seen recurrently. There is some speculation that it may be water condensation (Leovy, et al. J.G.R. 76).

iv) In addition to these observed hazes, there is also the notorious blue haze, which has been thought to be a uniform layer of scattering particles just small enough to be optically thick only in the blue. The blue haze was postulated in order to explain the lack of observable features in blue light (from ground-based observatories). The haze explanation for this (unobservability of features in blue) suffers from some sizable difficulties, however,

(☆) The presence of slight limb brightening or a neutral disk in blue implies that atmospheric opacity would be due to scattering (significant absorption would lead to limb darkening), but this would lead to a very bright blue image (Mars would look blue!), just opposite to the observations,

(β) Blue clearings - occasional times when features are visible in blue. Such "clearings" may develop very rapidly - much too rapidly to be a meteorological phenomenon. The clearings show a tendency to occur near oppositions and may be partly a seeing effect.
Mariner photographs in blue light show the same craters and such as in red or green - no haze effect is observed,

It has recently been shown that in the blue we may expect a sharp drop in surface contrast, if that contrast is due to differentiation of dust particles by size. This could fully account for the "blue haze" effect, although not, perhaps, for the clearings.

d. Physical processes

Chief physical processes relevant to Martian climate are thermally driven wind; other kinds of winds; condensation and evaporation at the poles, and the processes involved in the long-term balance of water on the planet. Most of these have been discussed earlier.

B. Surface features and conditions

1. Large-scale features

a. Visual and photographic records

i) Maria. The maria are darkish regions visible on photographs of Mars. It is not clear to me why some dark regions are called maria and others not. Recent evidence indicates the maria are at a somewhat greater elevation than the rest of the surface. Elevations are determined by two methods: radar range - doppler technique, and CO₂ infrared absorption. The former is limited by overspreading, while the latter depends on the model for the atmosphere that is used. Unfortunately, the two methods are not in complete agreement.

ii) Canals. No photographs exist. Invented by Schiaparelli, nurtured by P. Lowell,

iii) Polar caps. The north cap is the larger, and never
disappears. The southern cap either disappears or nearly does so every summer. The depth of the frost layer is unknown, but probably ranges from a few centimeters at the periphery to as much as hundreds of meters at the thickest point, as judged by its obscuration of surface features.

2. Small-scale features
   a. Craters. The surface of Mars shows extensive cratering—presumably from meteoric impact, but possibly partly from volcanic activity. The craters shown are near the south pole, and show deposition of frost in the bottoms.

3. Features possibly associated with atmospheric processes.
   a. Polar caps. Earlier discussion has covered problems of water vapor transport, storage of CO₂, H₂O on polar caps, etc.
   b. Morains (?). Also possibly dunes, although scale and configuration is all wrong.
   c. Sink holes (?). We do not really know what this is, but appearance suggests sink hole, possibly caused by evaporation of underground water (permafrost).
   d. Connected with this speculation, cloud or haze-like features are occasionally seen in certain characteristic locations, suggesting evaporation from a thermally active region.
   e. Etch pits. So termed because of their resemblance to terrestrial features. Two explanations are likely, both involving a positive feedback mechanism. Darkness of polar c.p.'s suggests 1st explanation: differential sublimation:
The other possibility is

on earth, this occurs when grassy "cement" disappears - and is further undercut by wind. On Mars, ices might provide cement. These features may provide clues as to quantity of condensate in (or on) soil.

II. Winds and Dust

A. Observations of airborne dust. We are motivated to study the phenomena leading to the lifting and transport of dust on Mars by several observations.

1. Yellow clouds. The yellow clouds are localized transient features which may be seen during ground-based visual observation. Time scale: ~1 day (very roughly). These clouds, because of their color, are believed to be airborne dust. Transverse velocities > 100 km/hr have been reported (de Vancouleurs, 1967).

2. Dust storms. Planet-wide dust storms have been observed. Such storms occurred during the 1909 and 1956 oppositions. Such occurrences are of interest because (a) they appear to be nearly planet-wide - sometimes an entire hemisphere is obscured - which makes them rather unusual, in our
[limited] planetary experience, and (b) large quantities of dust may be transported; enough to alter the appearance of surface features.

a. Perihelion. A planet-wide event suggests a planet-wide cause. In this case the cause might be related to the position of the planet in its orbit, since we tend to do most of our observing at the perihelic oppositions, and dust storms have frequently been seen at such times. (In 1909, for example, when observations started, Mars was already displaying a lot of dust-storm activity.) No statistical study, to my knowledge, has been done regarding the time of occurrence of these storms - when such a study is done, great care will have to be taken to avoid observational bias, as seems to have afflicted the "blue haze" results.

B. Dust as an explanation for (variable) surface features.

There is good reason to believe that dust may play a big part in the appearance of the Martian surface. For example, take the crater (?) Hellas. By all indications, it is an old crater: it is quite large, hence statistically unlikely to occur in a given short interval of time. The walls show evidence of weathering, and are pockmarked with smaller craters; yet the interior of Hellas is almost totally featureless! Where are the craters? The most plausible explanation is that they are covered with dust, which obscures the little craters, Hellas is light-colored, and this is consistent with fine dust. The limited number of pictures available of other Barge light-colored regions also have a featureless appearance.

This idea that dust may determine which areas are light and which are dark on Mars can generate some interesting speculations. First of all,
we know that the southern hemisphere is darker on the whole: Hellas is the only large light area it has, outside of the pole, which doesn't count - it is not dust. Then again, the southern hemisphere has the hotter summers - it faces the sun during perihelic opposition. This generates the speculation, which I must jealously credit to Peter Gierasch, that the stronger summer winds blow most of the dust away from that area, depositing it in the northern hemisphere. In support of this hypothesis, during the 1909 opposition, dust clouds were observed at all times in the northern hemisphere, occasionally rendering the atmosphere so opaque that even Syrius Major was invisible (Syrius Major is the darkest extended region on Mars). Dust clouds were rarely seen in the southern hemisphere, however, and when seen, were in the general region of Hellas - just the place where we suspect dust to be anyway! So the speculation is, then, that during perihelion the winds are strong (due, perhaps, to greater equator-to-pole $\Delta T$'s) in the southern hemisphere, and that those strong winds are sufficient to keep the southern hemisphere denuded of dust. The dust is deposited in the northern hemisphere, where gentler winds allow it to settle out. The only exception is Hellas, where local topography may lead to less wind, and the polar regions, where the wind may not be as strong as near the equator.

So this is our motivation for finding out how wind transports dust, and what sort of winds are responsible. One possibility is that all the dust is sort of equally tucked away - the hard-to-lift grains are right out there exposed to the strongest winds, the easier ones in regions of moderate winds, and the easiest ones all tucked away in stagnant areas:
this would be a sort of "equilibrium" distribution. In that case, a slight rise in mean wind speed over an area of the planet would mean dust would be picked up all over:

On the other hand, it could be that when winds subside, they do so suddenly compared to the fall-out time, and all kinds of dust could be left in all kinds of places:

This would lead to a more gradual amount of airborne dust vs. average wind speed curve. On still another hand, it may not be that the actual wind mechanics which lifts the dust has a bimodal dependence on insolation. For example, if, as some people think, we need dust devils in order to lift dust from the surface, then the onset of dust devil formation may depend on reaching a certain critical rate of insolation.

1. Wave of darkening. This is a band of enhanced darkness that begins near the poles in the spring, and proceeds equatorward. (There are those who dispute the existence of this phenomenon.) Sagan and Pollack believe that this is consistent with the wind-blown dust hypothesis.

2. Polar collar - possibly nonexistent,
3. **Slope - darkness correlation,** Carl Sagan believes he has found a correlation between slope (determined by Haystack radar) and darkness. It appears somewhat marginal.

4. **Thermal wind:**

\[
\begin{align*}
-f_u + \left( \frac{\partial \phi}{\partial y} \right)_p &= 0 \\
-f_u + g \frac{\Delta p}{L} &= 0 \\
-f_u + g \frac{H (1 - T_{\text{pole}})}{L} &= 0 \\
\frac{1}{500} \left( 4 \times 10^{-5} \right) &= 4 \times 10^4 \text{cm/sec} \sim 40 \text{ m/sec}
\end{align*}
\]

\[
\phi = g z
\]

\[
\begin{align*}
L &\sim 5000 \text{ km} \\
H &\sim 10 \text{ km} \\
T_{\text{pole}} &\sim 150 \text{ K} \\
T_{\text{eq.}} &\sim 250 \text{ K} \\
g &\sim 4 \times 10^2 \text{ \text{cgs}} \\
f &\sim 10^{-5} \text{ s}^{-1}
\end{align*}
\]

Interestingly, this is just the same magnitude reported by de Vaucouleurs for some yellow clouds.

5. **Relief wind**

This is a slope-induced wind, coupled to the mean wind field, which may produce winds up to about 40 m/sec at some indefinite altitude. For details of this mechanism, see Gierasch and Sagan (CRSR report, Cornell University, 1970).

6. **Slope wind,** This is an effect which is due to the coupling of wind within the viscous boundary layer with the diurnal temperature variations,
through a small (large-scale) slope. Without going into great detail, we may express this coupling as the addition of a buoyancy term in the time-independent (non-dimentionalized) equations

\[ \rho \sin \phi + p_x + \nu = u_{ss} \]
\[ u = v_{ss} \]
\[ \frac{\partial p}{\partial s} = 0 \]

\( p \) a pressure perturbation.

What happens is that at night the near-surface layers of the atmosphere lose heat by radiation. This creates a down-slope buoyancy force

![Diagram of buoyancy force](image)

This mechanism was chosen for study because it may provide an easily-observable effect - if dust clouds originate at night, we may see them in the morning, and they will settle out during the day. Two problems are: we will not see the activity itself, and the dawn haze may obscure itself, and the dawn haze may obscure the phenomenon in the morning. On the other hand, surface changes taking place over night will almost certainly be observable.

When we analyze this mechanism, we also get a wind of the order of magnitude of the thermal wind. If we add all three wind fields we get (possibly) velocities up to 120 m/sec, certainly enough to lift dust, according
to the calculations of Sagan and Pollack (Nature 223: 791). For the earth, Holton (Tellus, 17: 199) (which contains detailed derivation of the equations) assumes coupling with an Ekman layer in radiative-convection equilibrium. For Mars, calculation of the relative importance of terms indicates that the convective heat flux may be ignored.

EVAPORATION OF RAINFALL

Glyn O. Roberts

1. Introduction

The macroscopic and microscopic processes of cloud physics are summarized. Several phenomena which may be important in the earth and planetary atmospheres are then described heuristically. Specific applications are made to large scale convection in the upper layers of Jupiter and in the lower layers of the Venusian atmosphere. Finally a simple mathematical model of the evaporation and condensation of rainfall is described in an appendix; this guiado might be useful in future for analytic treatment of some of the cloud phenomena described.

2. Cloud Physics

(a) Macroscopic processes. The illustration shows the saturation vapor pressure curve ZZ for water or some other moisture constituent of an atmosphere, in a log p - log T diagram. With a vapor partial pressure below the curve (at a given temperature) condensation will occur, while above it, any liquid present will tend to evaporate. Only on the curve are the processes in equilibrium.

Consider a parcel of atmospheric gas at the point A in the diagram.
Fig. 1

with its vapor partial pressure shown by A'. If this is cooled, the vapor will become saturated. Cooling in the atmosphere can be by radiation or through transfer to cooler surroundings, but the dominant process is by adiabatic decompression, as a parcel rises and is decompressed. In the figure the gas parcel is decompressed adiabatically to pressure $p_c$ and then compressed to the original pressure, presumably by falling again to the original atmosphere level.

From A to B in the diagram, no condensation occurs, and the ratio $P_{vapor}/P$ remains constant. Thus on a logarithmic plot the curve A'B' is parallel to AB. At B' saturation occurs, and further slow decompression results in condensation, with the partial pressure following the saturation vapor pressure curve along B'C'. The condensation releases latent heat, so that the temperature decreases more slowly as the pressure decreases. The
BC is called a moist adiabat; it is steeper than the dry adiabat AB.

The behavior on recompressing the gas parcel adiabatically from \( p_c \) to \( p_a \) depends on whether the condensed water remains as cloud droplets in the gas parcel. If it does, then these droplets will evaporate on compression and adiabatic cooling, and the parcel will go back along the path CBA in the diagram, with the vapor partial pressure following the curve C'B'A'. If, on the other hand, all the moisture is removed before compression begins, then the gas immediately becomes unsaturated on adiabatic heating, the parcel follows the dry adiabat CD in the diagram, and the vapor partial pressure follows the parallel curve C'D'. In practice, of course, only part of the condensed moisture is likely to be removed as rainfall, and the compression curves will be CE and C'E', as indicated.

(b) Microscopic processes. The physics of drop evaporation, condensation, aggregation, breaking, electrification, freezing and melting, and of snow and hail formation, are very complicated. Mason's review (1957) is probably the best introduction, to be supplemented by the more recent papers in the meteorological journals (Journal of the Atmospheric Sciences and Quarterly Journal of the Royal Meteorological Society). Only a summary is given here.

Condensation does not occur in clean gas unless the vapor pressure is roughly four times the saturation vapor pressure for a plane interface, shown in the figure. This is because the saturation vapor pressure increases without limit for vapor in equilibrium with drops of decreasing radius \( r \). However, the earth's atmosphere has a large number of "centers" of condensation, which ensure that the supersaturation is rarely more than a few
percent. Dust particles, of various origins, provide initial drop radii for condensation. Particles of soluble material, usually from the evaporation of salt spray over oceans, also provide initial radii, and in addition the solute concentration is high in the early stages of drop growth, which considerably lowers the saturation vapor pressure.

Speculations about the distribution of condensation centers in the atmospheres of other planets are difficult. On Mars, dust storms are likely, but center concentrations on Venus, Jupiter, Saturn, Uranus and Neptune are likely to be small because of the deep atmospheres.

Aggregation of droplets to form rain is probably most important within the tropics for the earth's atmosphere. At other latitudes the ice phase is important. Centers for ice formation are rare, and there are likely to be a lot of supercooled water droplets. However, ice has a lower saturation vapor pressure at given temperature, and once they are formed ice crystals will grow rapidly, taking water from neighboring droplets through the vapor phase. These ice crystals fall, usually melting to reach the surface as rain.

3. **Evaporation Phenomena**

(a) Thunderstorm temperature and humidity dip. At the ground just below the center of a thunderstorm there is always an area with anomalously low relative humidity (70% instead of 95%) and temperature, in just the region where the rain is strongest (Byers and Braham, 1949). Das and Subba Rao (1968) suggest that the reason is an unsaturated downdraft. At the cloud base the violently ascending air has reached and passed saturation. But suppose there is as little air there at say 90% saturation, entrained from
the sides or fallen through the middle. Evaporation of the heavy rain falling through such a parcel will cool it, and it will fall rapidly, and become warmer due to the pressure increase. The saturation vapor pressure will therefore rise much faster than the vapor partial pressure; the relative humidity will decrease, and the cold dry air will hit the bottom, and spread out, just below the thunderstorm.

(b) Heat transfer in a cloud of small horizontal extent. The condensation process occurs within a cloud, and releases latent heat wherever it occurs. The droplets formed are all falling under gravity, but the smallest ones are falling very slowly. Larger ones, formed by random aggregation and other processes, constantly form, fall, and evaporate below the cloud in dryer air, cooling this air. Thus any cloud cools the air below it and warms the air within it. It therefore tends to rise slowly, thus maintaining itself through adiabatic cooling and further condensation. The air below it falls slowly, becoming dryer in the process and capable of evaporating more rain. Thus the air around the cloud, on the same horizontal level, is sucked into the cloud base and below; the growth or decay of the cloud depending on the relative humidity of this air. These phenomena should be observable around nearly all isolated clouds,

(c) Horizontally uniform rain. In a cyclone the airflow is along pressure lines, anticlockwise in the northern hemisphere. In the bottom Ekman layers the flow converges on the low-pressure center of the cyclone. Thus there is a large region, up to a thousand miles in diameter, with a roughly uniform slow upward flow determined by the divergence of the Ekman flux. This upward motion results in decompression and cooling, saturation,
condensation, and rain. At first the rain will evaporate in lower layers,
but there will then be an accumulation of moisture, as more wet air rises
from the bottom, until with the evaporation of moisture all the air is
nearly saturated and slow precipitation reaches the ground, just sufficient
to balance the upward transport of moisture by the Ekman flux,

(d) Instability in a horizontally uniform evaporating rain layer.
A precise model is not presented here; instead the general features of an
instability are described. The line of argument can be illustrated from
parallel-plate convection. A cold spot implies downward motion, which con-
vects cold with it because of the temperature gradient, thus maintaining
or amplifying the cold spot,

Consider a horizontally uniform situation with rain falling, and
partly or completely evaporating in a layer. Now consider a cold spot,
with downward motion. This motion will have two effects, it will make the
air warm adiabatically and become dryer, and it will suck rain in from the
sides, above. Both effects increase the evaporation locally, and thus main-
tain or amplify the cold spot,

Cellular convection in layers below clouds can be observed by its
effect on the clouds themselves; cloud layers with a wave-like structure
or lower surface are frequently observed. Of course, other instabilities
might lead to this phenomenon, and it is necessary to relate observations
to calculations of, say, the preferred wave number, for a particular case,

(e) Structure within large cloud layers. It is generally known, from
accounts given by fighter pilots of wartime air battles over Europe, that
thick cloud layers have considerable structure, with clear air interspersed
with thick cloud. It is possible that much of this is due to entrainment of drier air at the cloud top; such air parcels are cooled by evaporation during their rapid fall through the atmosphere. This process is likely to occur especially in cumulus convection, where entrainment is nearly always a dominant feature at the billowy cloud top.

Here an alternative model will be described, with superposed steady convection cells leading to a steady cloud structure within the larger cloud. In Fig. 2 superposed cells are shown; we can think of symmetry boundaries on the side walls and a vertical array of perhaps five cells in this simplified model of a turbulent cloud.

The descending air at B is dry. There is condensation and thick cloud in the ascending regions A, with the rain evaporating at C and D. The transfer of moisture down from cell to cell by rain is balanced by turbulent upward transport of vapor and very small droplets, across the cell interfaces. This transport is possible because the moisture proportion in cells increases with depth and temperature, and because the des-
cending air from B towards C is dry, while the ascending air along AED is moist. The energy to maintain the motion is associated with a substantial heat flux in such a cloud system.

4. **Jupiter**

The present model for Jupiter is based on the following assumptions:

(a) A composition near the solar composition, and a moisture content leading to clouds of solid ammonia, ammonian hydrogen sulfide, ice, and a solution of ammonia in water, in descending order, roughly as described by Lewis (1969);

(b) The zones are cloudy regions where the atmosphere is generally rising, and the belts represent descent regions, with very little cloud; thus half of a zone and half of an adjacent belt can be thought of as a single convection cell, 5000 km wide;

(c) The cell can be thought of as extending up to the bottom (or to the top) of the stratosphere, and down to some level where solar radiation cannot penetrate and where turbulent motions transport the internal heat (from gravitational collapse) upwards; this level is assumed to be below Lewis' clouds.

Since the atmosphere is deep, rainfall cannot reach the bottom. Moisture and clouds are important in the heat balance in two ways, through latent heat effects and through their effect on radiative heat transfer in the visible and infrared. The convection cell is illustrated in the figure, with the main clouds in the ascending gas on the left. The descending air on the right is very dry.
The convection process will be studied using the energy and entropy equations and the conservation equation for each constituent. The dynamics of horizontal mass transfer in a planet rotating so rapidly will not be considered; baroclinic waves will transfer the mass.

The energy integral takes the form \( \sum \int Q \, dV = 0 \), where the sum is over the different types of heat supply \( Q \) to the dry gas per unit volume, including latent heat (which may not total to zero if rain falls out of the bottom of the cell and vapor diffuses in), solar heating, radiant heating, and cooling (mainly cooling), and interval heating at the bottom of the cell, but not viscous dissipation since this is equal to the loss of heat by transformation to mechanical energy.

The entropy integral takes the form

\[
\sum \int \frac{\theta}{T} \, dV + \int \frac{Q'}{T} \, dV < 0,
\]

where \( Q' \) is the viscous generation of heat. This equation expresses the second law of thermodynamics, that overall heat must be supplied at a high temperature and removed at a low temperature for a thermal engine to produce energy.
We now discuss the dominant level, and associated temperature, for the different types of heat supply and removal,

(a) The solar heating is likely to be weak above the cloud level because the atmosphere is transparent. The absorption in and above the clouds, on the left-hand side of the cell, is determined by the albedo, and is much less than the deep absorption, at the level A, on the right-hand side,

(b) The radiative cooling is mainly from above the cloud level, and does not vary much between zones and belts. At certain wavelengths in the infrared the opacity seems to be low, and thus radiation in these wavelength regions escapes from deeper down in the belts, in the neighborhood of A. In addition there is radiative heat transfer, from hot points to cold points, within the cell; this transfer makes a positive contribution to the entropy expression.

(c) The latent heating by condensation is high on the left-hand side; the latent cooling by evaporation is low down, at higher temperatures. So latent heating also makes a negative contribution to the entropy expression.

(d) Viscous dissipation goes with the turbulent intensity, which is likely to be much larger in the clouds than in the descending dry air. In addition there is turbulent diffusion of turbulence intensity across the lower boundary, but it is reasonable to include this as part of the internal heating, on the assumption that the kinetic energy dissipates quickly, near the lower boundary,

(e) The internal heating is at the lower cell boundary; the heat is carried there by turbulent diffusion from the planetary interior.

It remains to estimate and discuss the relative magnitudes of the
different Q terms. It is likely that the internal heat is three times the solar heat; so that the radiation from Jupiter is four times the solar input; this cannot be regarded as settled yet. If this is true, the entropy constraint is no real problem; on the other hand, if internal heat is negligible, it is unlikely that this constraint can be satisfied for a model of this kind, because the positive contribution from radiative transfer will be too large. The positive contribution to the entropy constraint from latent heating is roughly the product of the latent heat, the total moisture content of the cell, the cellular overturn frequency, and the difference between \( \frac{1}{T} \) at condensation and at evaporation (this difference will tend to increase with overturn frequency). So the overturn frequency may well be determined by the entropy constraint, if at this frequency the viscous dissipation is relatively negligible.

The other point of interest in this model for the Jovian clouds is the conservation of the moisture components. The gas coming over the top of the cell will carry a negligible droplet proportion, since flows are likely to be slow. So the descending gas in the belt is bone dry. But below the lower boundary the gas has moisture component content near the solar composition, and considerable heat is being transported across the boundary by turbulent diffusion. So moisture components diffuse upward across the lower boundary, at least between B and C. The total moisture content of the cell will therefore increase, for each component \( \text{H}_2\text{O}, \text{NH}_3, \text{H}_2\text{S} \), well above the solar composition of the deep atmosphere until either rain falls across CD or the cellular composition between C and D is roughly twice the solar composition (so that turbulent moisture diffusion across BC
and CD cancel). According to Lewis' model (1969), $H_2S$ only condenses, as $NH_4HS$, relatively high, and is unlikely to rain out until the concentration is enormous, it will evaporate. However, if the $H_2S$ proportion is say ten times the deep atmosphere proportion at the $NH_4HS$ evaporation layer, turbulent diffusion associated with the condensation and evaporation processes in the lower clouds may well produce an $H_2S$ proportion twice that in the deep atmosphere at the cell bottom CD.

These conservation arguments clearly make it possible for the cell bottom to be far below the lowest cloud level, since then neither rain nor turbulent diffusion can get the moisture components out of the cell again. The picture that emerges is of a mean cell composition with perhaps twice or four times the deep atmospheric $NH_3$ and $H_2O$, and perhaps ten or twenty times the $H_2S$ proportion. Clearly Lewis' calculations should be repeated with these processes in mind.

5. **Venus**

In spite of its greater proximity to earth and in spite of the Russian Verera probes, we seem to know less about Venus' atmosphere than about that of Jupiter. It is predominantly $CO_2$, 100 kg/cm$^2$, and with planetary radius and mean density like the earth this corresponds to a pressure of 100 atmospheres. The temperature gradient is adiabatic, below the stratosphere, within observational error, and the temperature is $750^0K$ at the solid surface. We observe a thin cloud layer at 10 mb pressure, $190^0K$, and we cannot see through the clouds at 300 mb, $270^0K$.

The existence of this very thick cloud layer makes it possible that solar radiation, in the visible spectrum, does not penetrate much below
pressures of a few atmospheres. For pure water, solar radiation could just penetrate through 100 kg/cm²; for the ocean, it penetrates 100 ft. (or 3 kg/cm²) at most. With thick clouds, the penetration through CO₂ might be even less.

With the assumption that solar radiation does not nearly reach the bottom, it is very difficult to understand how the observed adiabatic temperature gradient is maintained against radiative heat transfer. If clouds and dust were extremely thick all the way down, the internal heat might be sufficient, but this is unlikely. The other main possibility is penetrative convection: solar heating at height h_s and radiative cooling from a greater height h_r drive convective motions sufficiently violent to penetrate the lower regions which would otherwise have a zero or at least a very sub-adiabatic temperature gradient. Models suggested (de Rivas, 1971) have been of the Hadley cell type, with heating and cooling either at the subsolar and antisolar points or at the equator and the poles (depending on the size of the cell turnover time compared with the Venus day of about 117 days). The solar heating should be substantially deeper than the radiative cooling because of the entropy condition described in the previous section on Jovian convection.

In this section one comment is made concerning the possible effect of heavy rainfall on motions in the deep atmosphere. Clearly the moisture component forming the visible clouds (possibly water) must evaporate before reaching the surface. The depth to which heavy rain penetrates will depend on the size of the raindrops and the intensity of the rainstorm. Now on earth the most violent rain falls in the tropics where the upward convective
motions are strongest, and where the warmer air, at given pressure, can hold much more moisture. For the same reasons, heavy rain is likely near the subsolar point on Venus. Suppose solar heat is mainly absorbed at height \( h_s \); then the strongest convection will be at this level and above. But the violent rainfall produced will penetrate much deeper, if there is sufficient moisture on Venus.

Suppose the air in a cubical hole with side 60 km, near our equator, were kept well-stirred. Then the bottom temperature would be about \( 750^\circ K \), and the pressure would be about 100 atmospheres. Rain from the heaviest tropical storm might penetrate 10 km into the dry air in such a hole, before it evaporated. The air would be dry because the moisture proportion would be uniform and equal to that at the top, because of the stirring. The temperature at 10 km depth is about \( 100^\circ C \).

To come to the point at last, the deep atmosphere at the subsolar point on Venus may therefore be colder, and not warmer, than that at other points; the dominant thermal effect being the evaporation of the very heavy rain. In order to stop this cooled air from sinking to the bottom and staying there, other parts of the deep atmosphere would have to be heated by some part of the solar radiation reaching the bottom or by penetrative convection. In either case, the likely deep motion has two parts. The first is a subsolar cell, sinking at the position of the heaviest upper-atmosphere rainstorms and rising at perhaps 30° from the subsolar point. The second is the Hadley cell mentioned above, sinking at the antisolar point and rising at perhaps 40° from the subsolar point. Venus' rotation may be sufficiently fast to alter all this and make the deep motion axisymmetric,
sinking at the equator and poles, and rising at latitudes around $35^\circ$; the question is just one of cell overturn time.

A problem with moisture conservation, comparable to that with Jupiter, remains. But we can reasonably assume that the transfer of moisture back to the upper atmosphere is accomplished by turbulent transfer at the interface.

The diagram illustrates the suggested situation, for the case of deep-atmosphere-overturn time less than 117 days. Vigorous motions in the upper atmosphere are driven by the solar heating. Convective storms near the subsolar point, together with weak solar heating and turbulent cooling and possibly turbulent heating, produce the indicated convection motions in the deep atmosphere, which are sufficient to maintain the adiabatic temperature gradient against radiative transfer.

The diagram does not indicate the possibility that violent motions in the solar heating layer (the upper atmosphere) generate turbulence which penetrates through the deep atmosphere, at least in the subsolar hemisphere. This would involve a downward transfer of kinetic energy, and
a stirring, and a turbulent downward transfer of heat, i.e. small-scale penetrative convection. If this effect were important, weak solar heating would not be required to maintain the indicated deep atmosphere convection. It should be noted that the entropy result in the previous section does not exclude this possibility, since mechanical energy is being supplied, by the upper atmosphere convection, to maintain both the lower atmosphere turbulence and the lower atmosphere Hadley cells.

Appendix  A simple model for rain processes

In a first approximation, moisture influences the dry air only through its latent heat, its mass being negligible in the dynamics and its specific heat, as vapor or liquid, negligible in the thermodynamics.

We will make the further assumption that in a volume element large enough to contain many drops, but small compared with all cloud-length scales, all drops have the same mass \( m \), and that there are \( n \) drops per unit volume, \( n \) and \( m \) being functions of space and time. In a real cloud, of course, a whole spectrum of drop sizes will be present.

We will assume that drops always move with their terminal velocity

\[
V_t = -\frac{m}{3} \eta
\]

with respect to the vapor and dry gas, which themselves move at \( V \). \( V_t \) is a complicated function of \( m \), or of drop radius, given in the literature; it could be simplified for an analytic model. This assumption is excellent except for very large drops with important momentum, and for very large \( n \), when the drops influence each other.

Define \( \rho_d \), \( \rho_v \) and \( \rho_l \) as the dry air, vapor and liquid densities, and \( \xi_v = \rho_v / \rho_d \), \( \xi_l = \rho_l / \rho_d \) as the vapor and liquid proportions by mass. Define \( b = n / \rho_d \) as the number of drops per gram of dry gas. Then
\[ \xi = Nm, \quad \rho = nm. \]

Drops are conserved as they increase or decrease in size; clearly in assuming all neighboring drops have a single mass \( m \) we are ignoring aggregation and break-up of drops. Thus

\[ \frac{\partial n}{\partial t} + \nabla \cdot \left\{ (\nabla + \nu) n \right\} = 0. \]

Hence, using the equation of continuity

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\nu \rho d) = 0. \]

for the dry gas, we obtain

\[ \frac{dN}{dt} = \frac{\partial N}{\partial t} + (\nabla \cdot \nu) N = \frac{1}{\rho d} \frac{\partial}{\partial x} \left( Nv_T \rho d \right). \]

Consider the vapor \( \xi \) in a gram of dry gas. There is no transfer of vapor across the boundaries, so \( \xi \) changes only as a result of the evaporation of or condensation on the \( N \) drops within the gram of gas. Thus

\[ \frac{d\xi}{dt} = -N \frac{Dm}{Dt} = -N \left\{ \frac{\partial m}{\partial t} + (\nu + \nu_d) \cdot \nabla m \right\}. \]

Note that \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial t} \) are the derivatives following the gas and the liquid respectively.

Let \( Q \) be the external heating rate per unit mass, as a function of position and time, and let \( \phi_d \) be the entropy of the dry gas, per unit mass. The heating by condensation, per unit mass, is

\[ N L \frac{Dm}{Dt}, \]

where \( L \) is the latent heat. Thus all the thermodynamics is contained in the equation

\[ \frac{d\phi_d}{dt} = LN \frac{Dm}{Dt} + \frac{Q}{T}. \]

\( \frac{Dm}{Dt} \) is a complicated function of \( m \), the temperature, the thermal diffusivity and kinematic viscosity of the dry gas, and of the difference
between \( \rho_v \) and the saturation vapor density at that temperature (Watts, 1971). In an analytic application of these equations, a simplified model would be required.

Initial and boundary conditions, the Navier-Stokes equation and an equation of state for the dry gas, constitutive relations for the dry gas properties and the saturation vapor density, and an equation determining \( Q \), close the system of equations.

References


J. S. Lewis 1969 The Clouds of Jupiter and the \( \text{NH}_3\text{-H}_2\text{O} \) and \( \text{NH}_3\text{-H}_2\text{S} \) systems. Icarus 10: 365-378.


ON THE FORMATION OF THE NORTH ATLANTIC BOTTOM WATER

Kim D. Saunders

"Everything is drifting, the whole ocean moves ceaselessly . . . just as shifting and transitory as human theories."

Fridtjof Nansen

Abyssal circulation in the North Atlantic Ocean

Stommel's model (1) of the thermohaline circulation is generally accepted. In this model, water is cooled near the pole and sinks to form a dense narrow western boundary current moving south. This water spreads from the boundary current to the deep interior of the ocean. This must be
returned to the pole and the vorticity balance equation requires that there be an upward velocity from the bottom water to the upper layers (1,2). The water then returns to the northern sinking regions on the surface. The upward velocity is thought to maintain the main thermocline against downward diffusion of heat. The advantage of this model is that it is consistent with the equations of motion and the present observations of the ocean. One problem associated with this model is that it is very hard to test the vertical velocity requirement by observation of the ocean. This is due to the very small size of \( w \) in relation to the horizontal currents and the present state of the art in measuring currents.

The experiment and the mathematical model

In the light of the last statements of the previous paragraph, Prof. Stern wondered if there were other possible mechanisms for returning the deep water to the pole without requiring a net upward vertical velocity throughout the interior of the ocean. The motivation for this lies in the mode in which the bottom water is formed.

During the first three months of 1969, a detailed study was made of a sinking region in the Mediterranean Sea in the Gulf of Lyon (3,4,5). It appears that the cold bottom water is formed by a rapid convective process in a small region during a period of rapid cooling and high evaporation. There was some evidence that there was no net mass flux during the convective phase of the deep water formation (6).

In order to get a feeling for motions of the sort described above, we set up a crude experiment on the turntable in the basement of Walsh Cottage at the Woods Hole Oceanographic Institution. This consisted of a
Long narrow tank mounted on the turntable. The tank was filled about halfway with cool water and a partial barrier placed about 30 cm from one end of the tank and extending a few centimeters below the surface of the water. Warm, dyed water was introduced into the region between the barrier and the end of the tank and the whole apparatus spun up to a state of constant rotation. The system was allowed to settle a bit (although a state of solid body rotation was never achieved, probably due to thermal driving across the barrier). The barrier was then removed and the flow resulting from the density difference of the two fluids was observed.

The lighter fluid began to flow out from behind the position where the barrier had been. Because of the rotation, the fluid moved toward the leading edge of the tank and formed a (geostrophic?) jet near the wall. This jet moved away from the initial region, becoming unstable as it progressed, forming a number of cold core eddies whose dimensions were the size of the width of the tank. The jet continued past the axis of rotation, remaining on the same wall until it reached the other end of the tank. There it turned the corner reached the other wall, turned the corner and continued back toward the initial region without any obvious change in structure. After a short time, the jet was no longer discernable and the tank was filled with a series of cold core eddies which persisted a long time. Some of the lighter fluid remained in the formation region in an eddy.

This experiment suggests a way in which fluid may be moved in the abyssal circulation without requiring a net vertical velocity. Instead of balancing the $BY$ term against the $f\text{div}_H v$ term, it may be possible to balance the $\mathcal{G}V$ term against the $v' \cdot \text{grad} \zeta'$ term.
This experiment also raised the problem of determining how much fluid remained in the initial region compared to the amount escaping in the jet. In order to determine this we tried a simple mathematical model.

This model is a Rossby adjustment problem modified by the introduction of a lateral boundary. We hoped to determine the ratio of the flow out of the initial region to the flow in the initial region for the nonlinear case by means of some integral constraints.

The basic assumptions for this model were: 1) the flow obeys the shallow water equations, 2) the flow is inviscid, 3) the fluid is divided into two regions of constant potential vorticity, defined as \( \eta = \frac{5}{h} \). The question we wish to answer is: "Is the ratio of mass flux to the south to the mass flux to the north in region 1 determined by the conservation laws of the system and if so, what is it?"

The geometry of the system is shown in Fig.1. The fluid occupies the right half plane. The curve S is the streamline dividing the two regions of constant potential vorticity.

The time-dependent shallow-water equations are

\[
\begin{align*}
    u_t + uu_x + vu_y + gh_x - fv &= 0 \\
    v_t + uv_x + vv_y + gh_y + fu &= 0 \\
    h_t + (uh)_x + (vh)_y &= 0
\end{align*}
\]

These give rise to the conservation equations:

\[
\begin{align*}
    \frac{Dh}{Dt} &= 0 \\
    h_t + v \cdot vh &= 0 \\
    \frac{3}{\vartheta_t} (ghy^2 + hgh) + v \cdot \nabla (ghy^2 + gh^2) &= 0
\end{align*}
\]
Fig. 1  Geometry of the modified Rossby adjustment problem,

We will assume $\frac{\partial}{\partial t} = 0$ and let $E = \frac{1}{2} h \nu^2 + gh^2$. Then we have the following conservation laws:

- $\eta = \text{const}$ in each region,
- $\nabla \cdot \mathbf{v} = 0$. This implies $\int_A (\nabla \cdot \mathbf{v}) \, dx \, dy = \int_T \mathbf{v} \cdot \mathbf{n} \, ds = 0$
- $\mathbf{v} \cdot \mathbf{E} = 0$. This implies $\int_A (\nabla \cdot \mathbf{v} \mathbf{E}) \, dx \, dy = \int_T \mathbf{v} \cdot \mathbf{E} \cdot \mathbf{n} \, ds = 0$.

Other assumptions and boundary conditions on the flow are

- at $x = 0$, $u = 0$,
- at $y = \infty$, $x = \infty$, $h = H + \Delta H$, $v = 0$,
- at $y = -\infty$, $x = \infty$, $h = H$, $v = 0$,
- at $y = \pm \infty$, $\frac{\partial v}{\partial y} = 0$,
- at $x = \pm \infty$, $\frac{\partial v}{\partial x} = 0$.

$h$ and $v$ are continuous.
The constant-potential vorticity condition in each region, plus the invariance of the $x$- or $y$-derivatives at large distances from the origin determine the height profiles in Region 2 to within arbitrary factors. The Bernoulli integral along the dividing streamline, $S$, and the constant mass flux condition in Region 2 relate the heights on $AB$ and $BC$. The mass conservation equation applied to Region 1 then determines the flow across the bounding circuit $ABCD$ to a constant $x_0$, the distance between $S$ and the line $x = 0$ as $y = -\infty$. The height profiles in Region 1 on the segments of the bounding circuit are

<table>
<thead>
<tr>
<th>Segment</th>
<th>Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$h_B = H_D - K e^{\omega_1(x-x_0)}$</td>
</tr>
<tr>
<td>$BC$</td>
<td>$h_C = H_D - K e^{\omega_1(y)}$</td>
</tr>
<tr>
<td>$CD$</td>
<td>$h_D = H_D - K e^{\omega_1(x+x_0)}$</td>
</tr>
</tbody>
</table>

where

$$K = \frac{\omega_2}{\omega_1 + \omega_2}, \quad \omega_1 = f(gH_D)^{-\frac{1}{2}}, \quad \omega_2 = f(gH)^{-\frac{1}{2}}, \quad H_D = h + AH.$$

We thought the last integral constraint on $E$ would close the problem, but when the above relations were used, it was found to reduce to a zero identity. Thus, the problem as posed needs an additional constraint before it can be solved.

Prof. J. Keller suggested that the time dependent formulation might possess a unique solution. We considered working on this problem but decided to look at the situation in the real world before proceeding to see if the problem was relevant to the ocean.
The Norwegian and Greenland Seas

The Norwegian and Greenland Seas are ultimately the source for the North Atlantic Bottom Water. These seas occupy the region between Greenland and Norway and Iceland and Spitzbergen. The Norwegian Sea occupies the area to the south of Jan Mayen Island and the Greenland Sea occupies the area to the north of Jan Mayen. The North Atlantic Bottom Water is thought to form from water in the Norwegian Sea in late winter or early spring (although there is some speculation that it is formed in the shallow Barents Sea and cascades into the Norwegian Sea [7]).

Bathymetry

The bathymetry of the Norwegian and Greenland Seas is shown in Fig. 2. There are two main basins: the Greenland and Norwegian and two minor basins: the Iceland and Jan Mayen, separated from each other by ridges. The Norwegian Sea is separated from the North Atlantic Ocean by three straits: the Denmark Strait between Iceland and Greenland and the straits between Iceland and the Faroe Islands and between the Faroes and the Shetland Islands. The sill depth of the Denmark Strait is about 600 m and the sill depths of the other two are somewhat less. The Greenland Sea communicates with the Arctic Ocean over the Nansen Ridge between Greenland and Spitzbergen.

Hydrography

The surface currents are shown schematically in Fig. 3. This picture was obtained by Helland-Hansen and Nansen ($) in the early part of the century but the same pattern exists (9). Two gyres may be seen in the Norwegian and Greenland Seas. A current of Polar Water flows southwest
Fig. 2  Bathymetry of the Greenland-Norwegian Sea (12).

Fig. 3  Currents of the Norwegian-Greenland Sea (8).
along the coast of Greenland and a current of North Atlantic water flows in over the Faroes–Shetland channel,

The water in the East Greenland Current is mostly Polar Water which has temperatures in the range of \(-2^\circ - 0^\circ C\) and salinities in the range 30 - 34 \(\text{o/oo}\) (10). The water in the North Atlantic inflow is warm and saline with temperatures in the neighborhood of 8\(^\circ\)C and salinities in the neighborhood of 35.2 \(\text{o/oo}\) (11). There are two major water masses in the two gyres below 500 m. Greenland gyral water is found in the center of the Greenland Sea. This is surrounded by Norwegian gyral water which has the same salinity (34.92 - 34.93 \(\text{o/oo}\)) but is warmer. This is the same water found in the Norwegian Sea and which contributes to the formation of North Atlantic Bottom Water (12).

Worthington (11) has proposed that the influx of North Atlantic surface water is caused by the outflow of dense deep water over the sills of the southern straits bounding the Norwegian Sea. This dense water is mostly Norwegian gyral water which has been partly mixed with North Atlantic water. If Worthington's hypothesis is correct, it is very important to the mechanism controlling the outflow over the sills. The picture is further complicated by the observations that indicate the flow is a pulsing phenomena (13).

Conclusion and suggestions for future work

After reviewing some of the data relating to the Norwegian Sea, it is clear that the initial model was not realistic. It would seem that putting more effort into obtaining a solution for an initial value Rossby adjustment problem that reduces to ours in the long time limit is not justified. A more
realistic model should involve the flow over a sill in a rotating frame.

This has been studied somewhat by Welander (14).

Some questions raised by the data that seem worth investigating are:

1) What controls the outflow from the Norwegian Sea?
2) What are the details of flows over sills in rotating systems?
3) How and where is the deep water in the Norwegian Sea formed?
4) How is it distributed from the western boundary current to the interior of the deep ocean?
5) Does formation of this water at one time of the year affect the abyssal circulation and if so, how?
6) What are the causes and the effects of the pulsations of the sill overflow?
7) Does the time-dependent Rossby adjustment problem have a unique solution and if so, what is it?

References

Voorhis, A. D. Informal Lecture at M.I.T.
7) Worthington, L. V. Private Communication.

Acknowledgements

I would like to thank Prof. Stern for suggesting the problem and for his helpful comments. I would also like to thank Profs. Welander and Keller for helpful conversations.

THERMODYNAMIC CONSTRAINTS ON PENETRATION OF CONVECTION
WITH APPLICATION TO VENUS

Edwin K. Schneider

1. Introduction

It has been shown (Gierasch and Goody) that, given certain plausible physical assumptions and deductions from spectroscopic data, local turbulent convection cannot support reasonable clouds on Venus. One of the assumptions was that turbulent heat fluxes are always positive. This assumption, however, implies high local frictional dissipation of kinetic energy. Since little is known about the actual magnitude of frictional dissipation in an atmosphere, Gierasch (Gierasch) has derived thermodynamic integral constraints on the structure of the motion field valid for all magnitudes of frictional dissipation. These constraints are applied to a semi-grey
radiative-convective model in which horizontal variations are neglected, to find an upper limit to the depth of penetration of the upper convective zone. Arguments are then made to show that this model supports Gierasch and Goody's conclusion that the Venus clouds cannot be dust or deep condensation clouds, but indicates that it may be possible for local turbulent convection to support high-low vapor pressure condensation clouds made out of substances such as $\text{HgCl}_2$. Gierasch and Goody show that high vapor pressure condensates such as $\text{H}_2\text{O}$ seem to be ruled out by spectroscopic constraints.

This type of model does not specify the form of the motion field, as all motions which carry a significant heat flux compared to radiative fluxes are parameterized by assuming that the regions in which they occur have an adiabatic or observationally specified lapse rate. Therefore, momentum transfer is parameterized out of the model. But any geophysical flow must be energetically and thermodynamically consistent as well as dynamically consistent, and this model serves to indicate how the class of cloudy Venus atmospheres is limited by constraints of the first kind only.

II. The Mathematical Model

A radiative-convective atmospheric model makes the following assumptions:

1) In regions of the fluid in which a significant vertical heat flux arises from fluid motions, the lapse rate is specified.

2) (a) In regions of the fluid in which there is no significant vertical heat flux arising from fluid motions, the atmosphere is in radiative
equilibrium.

(b) There are no unstable temperature gradients.

3) The transition region between radiative and convective zones is sharp; i.e., small compared to a scale height.

4) Local thermodynamic equilibrium applies, and energy exchanges with the very high regions in which local thermodynamic equilibrium does not exist are negligible.

For a specific model of atmospheric composition, then, the problem is to locate the convective region consistent with the above assumptions. For Venus, the available observations indicate that the lapse rate is close to adiabatic throughout the bulk of the atmosphere. Therefore, adiabaticity is assumed in the convective region. Gierasch and Goody's calculations show that a dry adiabatic lapse rate for CO₂ is appropriate. On earth, an observed lapse rate that is more stable than the adiabatic is used in this type of calculation to determine a tropopause height. Assumption 3) seems reasonable for Venus based on the Venera temperature profiles and the fact that a tropopause is observed on the Earth. Obviously, 2b) describes only a time-averaged system.

The specific model for this calculation is one of the simplest possible radiative-convective models that may be applied to envisaged situations on Venus; however, care should be taken in not trying to draw strong conclusions from such a model. A semi-infinite homogeneous compressible CO₂ atmosphere is considered, in which particles, having constant mixing ratio with the gas, are seen as dominating the gas in solar and thermal absorption. Gierasch and Goody give optical depths of 3000 as plausible for dust vs.
500 far CO₂ for this model. Heat conduction is neglected, and a semi-grey parameterization is used to treat solar and thermal radiation with separate grey (i.e., frequency independent) absorption coefficients. That is, \( T_s = A T_T \) where \( T_s \) is the optical depth for solar radiation, \( T_T \) that for thermal radiation, and \( A \) a constant of proportionality \( \geq 1 \), depending on particle size, but not composition. The atmosphere is plane parallel, and radiative transfer in the horizontal is negligible in contrast to that in the vertical.

**A) Boundary Conditions for the Convective Zone; Gierasch's Entropy Integral**

The following derivation follows Gierasch. Consider the atmospheric momentum, mass, and energy equations:

\[
\begin{align*}
\rho \frac{d\mathbf{v}}{dt} + 2 \rho \nabla \times \mathbf{v} + \nabla p &= -f - \rho g \mathbf{z} \\
\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} &= 0 \\
\rho C_v \frac{dT}{dt} + \rho \nabla \cdot \mathbf{v} &= \rho Q + \rho Q_f
\end{align*}
\]

for an ideal gas of uniform composition, where \( Q_f \) is heating due to frictional dissipation, and \( Q \) is heating due to other processes, and \( f \) represents frictional forces. Write (3) in terms of the specific entropy, \( S \) by

\[
\rho C_v \frac{dT}{dt} = \rho C_v \frac{dS}{dt} + \rho \nabla \cdot \mathbf{v}.
\]

Take \( \mathbf{v} \cdot 1 \) and use 29 to find

\[
\frac{\partial}{\partial t} \rho \frac{v^2}{2} + \rho \mathbf{v} \cdot \nabla \frac{v^2}{2} + \mathbf{v} \cdot \nabla p = -\mathbf{v} \cdot \frac{f}{\rho} - \rho q w
\]

Integrate (5) over the whole volume of fluid in which there is motion, and denote this operation by an overbar, obtaining
and averaging over a long time, the operation \(\langle \rangle\),

\[
\langle -\nabla \cdot \mathbf{f} \rangle = \langle -\nabla \cdot \mathbf{f} \rangle
\]

Using 2) and 3), it is found that

\[
\langle \rho \nabla \cdot \mathbf{u} \rangle = \langle \rho Q \rangle + \langle \rho Q_f \rangle \\
= \langle -\nabla \cdot \nabla \mathbf{P} \rangle \\
= \langle \nabla \cdot \mathbf{f} \rangle
\]

and we may say that \(-\langle \nabla \cdot \mathbf{f} \rangle + \langle \rho Q_f \rangle = 0\) since this is just saying that the gain of internal energy by frictional heating plus the loss of kinetic energy by frictional dissipation is nothing. Then

\[
\langle \rho Q \rangle = 0 \tag{6}
\]

That is, the long term net heating of the atmosphere by radiation is zero.

Using 2), 3), and 4), an equation for the entropy is found:

\[
\frac{\partial}{\partial t} \rho S + \nabla \cdot \rho \nabla S = \rho \frac{Q}{T} + \rho f/T
\]

Then

\[
\langle \rho \frac{Q}{T} \rangle + \langle \rho \frac{Q_f}{T} \rangle = 0 \tag{7}
\]

relating the distribution of heat sources to the rate of generation of kinetic energy. Note that these are entirely global considerations, and have no validity in a local sense unless the motion fields are described.

The usual boundary conditions used to define the height of the radiation-to-convection transition zone is that temperature and radiative fluxes are continuous at the interface. It has been shown that temperature continuity is equivalent to assuming that the vertical convective heat flux is upwards only \((\geq 0)\), and kinetic energy is generated and dissipated locally.
The model atmosphere is one in which convection will occur only when solar radiation is deposited deep enough, and the atmosphere is opaque enough to thermal radiation, that the radiative equilibrium temperature profile is unstable somewhere. In the semi-infinite case there will be in general at least two "tropopauses". The desire is to find the maximum thermodynamically possible penetration of the lower of the top two tropopauses. To this end, the entropy integral 7) is used in the limit $Q_f \to 0$ to fix the location of the lower tropopause, given the location of the upper tropopause. Flux continuity at both interfaces, equivalent to $\langle \rho Q \rangle = 0$, has been retained. However, the two temperature continuity (high dissipation) boundary conditions have now been replaced by a single integral condition. That is, the problem is no longer well-defined. There may exist another thermodynamic constraint on the model, not yet found, or it may be true that the problem is not well-defined unless the motion field is known. At any rate, in ignorance of other thermodynamic constraints, we are free to place the top tropopause anywhere above the radiatively unstable region, as long as there is no unstable temperature discontinuity demanded by flux continuity. Kinetic energy is dissipated in the zero friction limit by work done against buoyancy by fluid overshooting into stable regions. The lowest possible top tropopause is picked, and one defined by the high dissipation boundary condition of temperature continuity with an upper radiative region, in order that all dissipative processes are occurring at the bottom of the convective zone, and maximum penetration is achieved. Convective flux is now allowed to be either upwards or downwards. Penetrative convection removes the
necessity for a "transition region" near cloud tops, which is necessary in the Gierasch and Goody high dissipation cloud models.

In this model, time dependence and horizontal temperature variation are neglected, because Gierasch and Goody include time dependence in their model and find its effects to be negligible and because horizontal temperature variations at the cloud tops are observed to be small. Incorporating time dependence with the above radiative-convective assumptions would be conceptually difficult anyway, because it takes a finite amount of time for convection to destroy a finite amplitude temperature instability, violating radiative-convective assumption 2b).

Judging from the Earth's tropopause, little can be said about variations of temperature below the cloud tops, given cloud top variations; however, the model is much simpler in the case of small horizontal temperature variations. In that instance, the entropy integral becomes much simpler:

\[ \iint \rho \frac{Q(x, y, z)}{T(z)} \, dV = \int \rho \frac{\bar{Q}(z)}{T(z)} \, d\bar{z} \]

where \( \bar{Q}(z) = \iint Q(x, y, z) \, dx \, dy \), and horizontally-averaged solar fluxes and heating rates may be considered.

B) Solution of the Problem in the Greenhouse Limit

1) The solar flux is represented by \( \mathcal{E} \),

\[
\text{solar flux} = F_S \left( \tau_T \right) = -F_S(0) e^{-\beta \tau_T}
\]

where \( \beta \ll 1 \). (See Gierasch and Goody, Appendix 4). \( A = 1 \) is chosen, as we are free to vary \( \beta \).

2) The top radiative equilibrium region:

\[ \rho Q = \rho \left( Q_T + Q_S \right) = -\frac{\partial}{\partial z} \left( F_T + F_S \right) = 0 \]
where the subscript $T$ refers to thermal quantities and the subscript $S$ to solar. The $Q's$ are heating rates and the $F's$ are vertical heat fluxes.

If there are no heat sources at $T_T = \infty$,

$$F_T + F_S = 0$$

$F_T, B_T$ must satisfy

$$\frac{\partial^2 F_T}{\partial T_T^2} - F_T = -2\pi \frac{\partial B}{\partial T_T}$$

(8)

where $B$ is the source function; $\gamma B = \sigma T^4$ if local thermodynamic equilibrium exists. The upper boundary condition is that

$$\frac{\partial F_T}{\partial T_T} - F_T = -2\pi B \text{ at } T_T = 0$$

in this report is $\tau_1$ where $\tau_1 = 1.66, \tau$ is the real optical depth.

In the limit $\beta << 1, \beta T_T << 1$, the solution for $B(T_T)$ is

$$B(T_T) = \frac{F_0}{2\tau} (1 + \tau T_T)$$

3) The convective zone:

Along an adiabat $\frac{T}{T_1} = \left(\frac{P}{P_1}\right)^{\gamma p_1}$ where $R$ is the gas constant. $C_P$ the specific heat at constant pressure of the gas, $P_1$ and $T_1$ reference pressure and temperature at a point on the adiabat. The reference point is chosen as the top tropopause, and the subscript 1 refers to that point.

Then

$$\pi B = \sigma T^4 = \sigma T_T^4 \left(\frac{T}{T_1}\right)^4 = \pi B_1 \left(\frac{P}{P_1}\right)^{\gamma p_1}$$

$$\frac{P}{P_1} = \left(\frac{T}{T_1}\right)^{\alpha \cdot \frac{4R}{C_P}}$$

where $\alpha = \frac{4R}{C_P} \text{ since optical depth is proportional to pressure.}$ Then $\pi B = \pi B_1 \left(\frac{T}{T_1}\right)^{\alpha}$ along an adiabat. $\alpha$ is a function of temperature, and hence of $T_T$; however, $\alpha$ is considered to be constant in these calculations, and the results are functions of $\alpha$.

43 The top tropopause:
The condition of flux continuity with the radiative zone implies

\[ F_R = F_S + F_T = 0 \] at \( \tau_1 \), \[ F_T = F_T^+ - F_T^- \] where \( F_T^\pm \) is the upward (+) or downward (-) component of the thermal flux at \( \tau_1 \), evaluated using

\[ F_T^\pm = \int \frac{B(t) e^{\mp (t - \tau_1)}}{\tau_1} \, dt \]

defined as positive definite, with the regions of integration including all \( B \)'s contributing to the appropriate component of the flux. Along with temperature continuity, the \textit{flux} condition then determines that

\[ \tau_1 = \frac{\alpha}{1 - \alpha} \]

As long as \( \tau_1 \gg 1 \), i.e. \( \alpha \) is close to and a little less than 1.

\[ \beta_i = \frac{F_{\phi}}{2 \pi \rho} \left( \frac{1}{1 - \alpha} \right) \]

5) The entropy condition; determining \( \tau_{\phi} \):

The subscript 2 refers to quantities evaluated at the lower radiative-convective interface. In the zero dissipation limit,

or

\[ \int_{\tau_1}^{\infty} \left( \frac{dF_T}{d\tau_2} + \frac{dF_S}{d\tau_2} \right) \left( \frac{1}{\tau_2 - \tau_1} \right) \, d\tau_2 = 0 \]

If \( \tau_1 \gg 1 \), \( F_T \approx 2 \pi \frac{\partial B}{\partial \tau_2} \). Using the relations derived above for \( \tau(\tau_1) \), \( B(\tau_1) \), this integral may be written as

\[ \int_{\tau_1}^{\infty} \left( \frac{dF_T}{d\tau_2} + \frac{dF_S}{d\tau_2} \right) \left( \frac{1}{\tau_2 - \tau_1} \right) d\tau_2 + \frac{1}{\tau_2 - \tau_{\phi}} \left( F_R(\tau_{\phi}) - F_R(\tau_{\phi} - 1) \right) = 0 \]

approximating the possibly large contribution to \( Q \) from a temperature discontinuity at \( \tau_{\phi} \) in the case where the temperature is approximately constant in the region where important contributions are occurring.

It is then found that
In the limit $T_z \gg T, e^{-\beta T_z} \ll 1$, we may write $T_z$ as a function of $\alpha$ and $\beta$:

$$T_z = \left[ \frac{1}{3}\left(1 - \frac{2}{3}\alpha + \frac{1}{4}\right) \left(1 - \frac{1}{4}\right) - \frac{1}{2} \alpha \left(1 - \frac{1}{4}\right) - \frac{1}{4} \alpha \left(1 - \frac{1}{4}\right) \right] \frac{4}{3\alpha - 4}$$

All assumptions used in evaluating the various aspects of this particular problem are evaluated a posteriori.

6) The lower radiative region:

$$\beta(T_z) = B(T_z + \varepsilon) + \frac{1 - \beta^2}{2\pi \beta} F_0 \left( e^{-\beta T_z} - e^{-\beta T} \right)$$

where $T_z + \varepsilon$ means below the interface. Use the condition $F_T^+ - F_T^- + F_S^0 = 0$ at $T_z$ to evaluate $B(T_z + \varepsilon) \equiv B_{\infty}$:

$$\Pi B_{\infty} = \pi B(T_z - \varepsilon)(1 - \frac{\alpha}{T_z}) + F_0 e^{-\beta T_z}$$

The temperature discontinuity is small for large $T_z$.

Similar calculations were performed to find the limits for penetrative convection in the high dissipation and Boussinesq low dissipation limits. The results are displayed on graphs.

C) Discussion of Behavior of Solutions

1) $\beta$: For $\alpha = .850$, $\beta \leq .01$, increasing $\beta$ has the effect of decreasing $T_z$, as is physically plausible, since the less solar penetration is, the less direct driving of the circulation by buoyancy forces there is. As $\beta$ becomes Barge, convection should cease altogether, as the radiative equilibrium state becomes stable everywhere. The approximations $\beta \ll 1, \beta T_z \ll 1$ do not really break down in this model until $\beta \sim .1$. The breakdown occurs because $T_z$ should be increasing with $\beta$, causing thermal
cooling to occur at a higher temperature. As $\beta \to 0$, the greenhouse limit of adiabaticity to $T_T \sim 3000$ is rapidly approached.

$\beta = 0.0234$ is assumed, because Gierasch and Goody calculate this value of $\beta$ as giving the correct bolometric albedo of $0.77$ for the model under discussion here.

2) $\tau$: As pointed out above, $\tau$ is not constant. To interpret these results, $\tau$ consistent with the mean temperature of the adiabatic region must be chosen. All of the results for $0.800 \leq \tau \leq 0.900$ are more or less consistent.

$\tau$ is somewhat too great as $\tau$ decreases below $0.800$, for much the same reason as the $\beta$ dependence fails. $\tau$ should actually be somewhat larger than $\frac{\tau}{1 - \tau}$, and thermal emission should occur at a somewhat higher temperature, making $\tau$ smaller as is easily seen from 9).

The actual calculations could easily be improved to be valid for all $\tau$ and $\beta$.

3) Qualitative aspects: The calculations performed indicate that for $\beta = 0.0234$, the adiabatic region can extend down only to about 200 optical depths as an upper limit, corresponding to a temperature somewhat lower than the $600^\circ K$ found in constant $\tau$ calculations. Thus the convective zone can reach down only to a distance of greater than a scale height above the ground. This is much deeper penetration than the high dissipation convective region can achieve. The high dissipation region can only extend down to $\beta T \sim 1$, whereas in the low dissipation case, only about $0.02$ of the solar flux is reaching the bottom of the convective zone. Frictional dissipation of any magnitude will tend to decrease the depth of penetration.
However, the integrand in the frictional integral is positive definite, so that

$$\int \rho \frac{\partial{\tilde{z}}}{\partial{t}} \, dz = -\int \rho \frac{\partial{\tilde{q}}}{\partial{t}} \, dz$$

From the solution for $\tau_z$ (equation 9), it is easily seen that for fixed $\tau_i \propto \beta$, and $\beta$, making the entropy integral negative implies that $\tau_z$ is decreasing from its value at zero dissipation. Then the zero dissipation limit really gives an upper bound on convective overshoot, which is hardly surprising.

It is also interesting to note that there seems to be an upper limit of $\propto$ very close to 1 before convection ceases in the low dissipation limit, defining convection to cease when $\tau_i = \tau_z$. In the high dissipation limit, $\propto k^{.96}$ defines a region of no convection, and $\propto \geq .98$ in the Boussinesq frictionless limit defines the ceasing of convection.

III. Deficiencies of the Model

The neglect of time dependence should have little effect on this type of model, due to the large thermal inertia of the Venus atmosphere and the fact that convective readjustment is parameterized as being instantaneous. Indeed, Gierasch and Goody calculate their cloud models with time dependence, and find that the time-dependent state is close to the time-averaged state. The validity of including any time dependence in a radiative convective model, in which the motion fields have been parameterized by a lapse rate is possibly questionable. A 4-day longitudinal stratospheric circulation would strengthen the feeling that time dependence cannot affect the results of the above model significantly, since the
4-day rotation would produce a sort of time-averaged stratospheric radiative equilibrium temperature profile.

The question of the effect horizontal variations in height and temperature have on the model is much more difficult to resolve, especially since there are significantly greater mathematical difficulties in working with the entropy integral. Horizontal variations, however, could easily be handled by machine, and the results compared to no horizontal variation results. An analytical solution of this problem would be of no more value than a numerical solution. It seems from physical grounds that in the case of \( \frac{\partial T}{\partial y} < 0 \), i.e. an equator-to-pole temperature decrease, \( T_2 \) rising near the equator and going deep near the pole, with an average \( T_2 \) of about the same height as calculated from a zero horizontal variation model satisfied the entropy integral. But that is in the realm of speculation. At least it must be pointed out that horizontal variations in the global model are potentially important, since the entropy integral is valid only over the whole motion field.

IV. Conclusions

It seems as though dust clouds on Venus are ruled out even more strongly by this upper bound calculation than by the Gierasch and Goody high dissipation calculation. The upper convective zone cannot penetrate to the surface, even in the limit of zero dissipation, at least if horizontal temperature changes and tropopause height variations can be neglected. A relatively small internal heat source might change this conclusion significantly if a zero-dissipation upper bound on the top of a lower convective region is found to be higher than the upper limit for the bottom of the upper
convective zone. Not much is known about pressure-absorption characteristics of CO$_2$ at pressures relevant to the deep Venus atmosphere. A significant increase in carbon dioxide thermal opacity at some level could increase penetration significantly.

However, on the optimistic side, this model indicates that condensation clouds could easily be supported by local convection, as seems intuitively plausible, removing the inconsistency of the Gierasch and Goody cloud models.

The following graphs illustrate the results I have obtained.

References


Acknowledgements

I would like to thank Dr. Andrew Ingersoll for helping me to find this problem and Dr. Peter Gierasch for solving it.
Table 1.  

"Values of $C_p$, $\Gamma$, $H$ and $C_p/4R$ for $\text{CO}_2$ (from the American Institute of Physics Handbook, 1963 ed.) The gas constant $R = 0.189 \text{ J gm}^{-1} (\text{K})^{-1}$"  

(from Gierasch and Goody).

<table>
<thead>
<tr>
<th>$T$ (°K)</th>
<th>$C_p$ [J gm$^{-1} (\text{K})^{-1}$]</th>
<th>$C_p/4R$</th>
<th>$\Gamma$ (°K km$^{-1}$)</th>
<th>$H$ (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0.745</td>
<td>0.98</td>
<td>11.4</td>
<td>4.4</td>
</tr>
<tr>
<td>250</td>
<td>0.801</td>
<td>1.06</td>
<td>10.6</td>
<td>5.6</td>
</tr>
<tr>
<td>300</td>
<td>0.852</td>
<td>1.13</td>
<td>10.0</td>
<td>6.7</td>
</tr>
<tr>
<td>400</td>
<td>0.941</td>
<td>1.25</td>
<td>9.0</td>
<td>8.9</td>
</tr>
<tr>
<td>500</td>
<td>1.015</td>
<td>1.34</td>
<td>8.4</td>
<td>11.1</td>
</tr>
<tr>
<td>600</td>
<td>1.077</td>
<td>1.43</td>
<td>7.9</td>
<td>13.3</td>
</tr>
<tr>
<td>700</td>
<td>1.126</td>
<td>1.49</td>
<td>7.5</td>
<td>15.6</td>
</tr>
</tbody>
</table>
Optical Depths of Top and Bottom Tropopauses as a Function of $\alpha$, $\beta = .0234$

Low Dissipation $\tau_z$

High Dissipation $\tau_z$

Boussinesq
\[ \frac{\Pi \beta^2}{F_0} \] as a Function of \( \zeta \), \( \beta = 0.0234 \)

- High Dissipation
- Low Dissipation
- Radiative Equilibrium
- Boussinesq
Temperature Profiles for $\alpha = 0.850$
THE ROLE OF BAROCLINIC INSTABILITY IN THE MOMENTUM BALANCE OF THE EARTH'S ATMOSPHERE

Adrian J. Simmons

1. Introduction

The classical baroclinic instability problem discussed by Eady (1949) has applications both to large-scale flows in the Earth's atmosphere, and to a class of laboratory flows. As such, it has been the subject of several subsequent investigations in which one or more of the simplifying assumptions contained in the original model have been relaxed. Thus, in the case of meteorological interest, Green (1960) has studied the inclusion of the $\mathbf{\beta}$-effect and compressibility, and illustrated the connection with the original study of Charney (1947). Phillips (1964) has commented on non-geostrophic effects, and non-linear effects have been investigated by Drazin (1970) for a marginally unstable mode. McIntyre (1970) has given a general perturbation scheme, and discussed its application when the basic flow is modified by the presence of horizontal shear. A similar body of theory has been developed for two-layer models, and, with further information from more complex numerical studies, baroclinic instability is now believed to describe the initial development of the travelling mid-latitude cyclones.

In most of the studies carried out to date, the boundaries of the flow have been taken to be rigid and horizontal. In this paper we investigate the extension to the case of a sloping upper boundary. The principal geophysical motivation behind the study of this idealised mathematical problem is a desire to understand the influence of the height variation of
the tropopause on the baroclinic instabilities which develop in the troposphere jet stream. In particular, we are interested in possible consequences regarding the angular momentum budget of the Earth's atmosphere. With this application in mind, we also study a perturbation of the mean flow posed by Eady.

A representative meridional cross section of the mean zonal wind in the lower regions of the Northern Hemisphere is presented on Fig. 1. A detailed picture, with references to source data, is given by Newell et al. (1969). In winter, surface winds are easterly south of $30^\circ$N, and westerly to the north. Consequently, the effect of surface friction is to remove eastward zonal momentum from the flow in mid-latitudes, and to return it to the atmosphere in equatorial regions. There is therefore a requirement for a net transfer of momentum across the $30^\circ$N latitude circle. In general, both the zonally-averaged meridional circulation and the eddy motions are active in transporting momentum, and the observations of

![Fig. 1. Representative cross section of mean zonal wind for Northern Hemisphere. Units: M/SEC. W denotes Westerly. Dashed lines show tropopause.](image)
Kidson et al. (1969) indicate that the former effect is of similar, or even greater, magnitude at individual locations. However, the vertically integrated transport due to the mean motion is small away from the tropics, and, in particular, it vanishes near 30°N. The eddies thus play an essential role in the angular momentum balance of the Earth's atmosphere. A five-year mean of the latitudinal variation of the eddy momentum transport for January is illustrated on Fig. 2. It is a preliminary result of a study by Oort and Rasmusson in collaboration with Starr, and further information may be seen in Smagorinsky (1969). Most of the transport takes place near the tropopause, at a location close to that of the jet stream maximum, which occurs almost at the latitude of zero surface wind. In summer, the jet stream is much weaker, and is displaced northwards; the same general considerations apply.

In the following section, we consider in general terms the possibility that the observed transport may be described by a linear baroclinic-instability theory, and mention some previous progress in this field. The height variation of the tropopause is discussed in Section 3, and this is followed by a simple preliminary investigation of the accuracy to which the Eady model represents the instabilities of a tropospheric flow in an atmosphere with a stratosphere of large, but finite, static stability. Sections 5 and 6 describe the progress which has been made with a study of the Eady model with a sloping upper boundary representative of the tropopause. Section 5 introduces a perturbation technique which illustrates the appearance in the limit of small slope of terms required for a realistic description of the momentum transport; it fails for height variations of the order of those
Fig. 2  Solid line: Observed 5-year mean of vertically integrated poleward eddy transport of angular momentum. Dashed line: Lowest order theoretical contribution, \( \sin \left( \frac{\pi \times \text{latitude}}{60} \right) \), with amplitude adjusted to coincide at maximum.

An alternative method for finite deviations of the upper boundary from the horizontal is described in Section 6, together with a partially-successful numerical example for which the 'eddy transports of momentum and heat are evaluated. Section 7 discusses the mathematics of the inclusion of the effects of compressibility and latitudinal variations of the coriolis parameter. In this study, the tropopause is not modelled by a rigid upper boundary;
the solution of the instability problem in this case is mentioned in Section 8. In Section 9, the perturbation formation developed for the tropopause problem is applied in a study of the effect of an asymmetry in the basic flow. We conclude with a summary and discussion of the results obtained to date, and the need for further action.

2. Baroclinic Instability and the Eddy Transport of Momentum

A convincing theoretical description of the influences which cause the baroclinic instabilities to transfer momentum northwards is lacking. In this section, we discuss in general terms the possibility that the principal features of the observed transport are described by a linear quasi-geostrophic instability on an f- or β-plane. Further effects of the geometry of the Earth may be important, but are not included.

Consider a flow bounded by vertical walls at $y = 0$ and $y = L$, with a normal-mode perturbation stream function:

$$\psi(x, y, z, t) = \phi(y, z) e^{ik(x-ct)}.$$  

The horizontal perturbation velocities are given by

$$u = -\frac{\partial \phi}{\partial y} e^{ik(x-ct)},$$  

$$v = ik \phi e^{ik(x-ct)}.$$  

We suppose that $\phi(y, z)$ may be represented by a Fourier expansion:

$$\phi(y, z) = \sum_{n=1}^{\infty} \phi_n(z) \sin \frac{n\pi y}{L}, \text{ and}$$  

$$u = -\frac{n\pi}{L} \left\{ \sum_{n=1}^{\infty} \phi_n(z) \cos \frac{n\pi y}{L} \right\} e^{ik(x-ct)}, \text{ and}$$  

$$v = ik \left\{ \sum_{n=1}^{\infty} \phi_n(z) \sin \frac{n\pi y}{L} \right\} e^{ik(x-ct)}.$$  

The northward eddy transport of zonal momentum, $\overline{u v}$, is the dominant
transfer within the quasi-geostrophic approximation, and is given by

\[ \frac{\mathbf{u} \cdot \mathbf{v}}{2} = \frac{1}{2} \mathcal{R} e \left\{ u v^* \right\} = \frac{\pi k e^{2 \xi m(n)}(\xi \sum_{r=1}^{n} m(\phi_r \phi^*_r) \left( \cos \frac{5 \pi}{L} \sin \frac{r \pi y}{L} - \cos \frac{5 \pi y}{L} \sin \frac{5 \pi x}{L} \right) \right\}. \]

Assuming a relatively rapid convergence of the Fourier series, the essential form of \( \frac{\mathbf{u} \cdot \mathbf{v}}{2} \) is given by the lower harmonics, and we obtain

\[ \frac{\mathbf{u} \cdot \mathbf{v}}{2} = \frac{\pi k e^{2 \xi m(n)}(\xi \sum_{r=1}^{n} m(\phi_r \phi^*_r) \sin \frac{5 \pi y}{L} - 4 \sum_{m=1}^{r} m(\phi_r \phi^*_r) \sin \frac{5 \pi y}{L} \cos \frac{5 \pi x}{L} + \ldots \right\}. \]

The form of the first component of \( \frac{\mathbf{u} \cdot \mathbf{v}}{2} \), with \( L \equiv 60^\circ \) of latitude, is illustrated on Fig. 2, for comparison with the observations for January. The difference between the two curves is much less than the year-to-year variation in the observed values, and, while this is in no way definite evidence, it suggests that a linear baroclinic instability is capable of reproducing the essential dynamical feature of the momentum balance of the Earth's atmosphere. Some further investigation is thus worthwhile.

We first note that the basoeline instability problem is separable in the absence of latitudinal variations in the mean flow or boundaries; solutions are then of the form

\[ \phi_n(z) \sin \frac{n \pi y}{L}, \]

and there is therefore no horizontal transport of zonal momentum in the linear quasi-geostrophic approximation. This is a feature of the prototype problems of Charney and Eady, and the extensions studied by Green.

We must therefore investigate meridional variations in the basic state. The most apparent effect is that of the horizontal shear of the mean flow. For a symmetrical jet with a maximum at \( y = \frac{L}{2} \), normal-mode
solutions have either an even or an odd symmetry about its axis. The former case gives a momentum transport with odd symmetry, which is consistent with the findings of Stone (1969) and McIntyre (1970), who showed how the eddies act to reinforce an existing jet. The principal harmonic description of the symmetric jet in this case is the $\sin \pi y / L$ mode, which appears to be present in the observations, but not dominant. A normal mode with odd symmetry has a much shorter length scale than the lowest even mode, and is unlikely to possess a fast growth rate; in any case the associated momentum transport vanishes at $y = L / 2$, and thus does not correspond to the observed state.

The removal of the symmetry of the horizontal jet leads to a more general solution. Lipps (1966) has shown by means of an initial value problem that a weak asymmetry such as is a feature of the observed wind field can cause a tendency for a momentum transport similar to that shown on Fig. 2. However, his initial state comprised only the $\sin \pi y / L$ harmonic, and the introduction of a relatively weak $\sin 3\pi y / L$ term leads to additional terms which may significantly modify the resulting horizontal Reynolds stress. It is thus not clear that a normal-mode baroclinic instability can develop such that the momentum transport in the presence of a small asymmetry differs considerably from that found for a symmetrical jet alone. There have been several published accounts of studies of the latter case, but, as far as the author is aware, none has investigated the more general problem. An extension of existing techniques to this case is a priority in the understanding of the eddy momentum transport. We return to this aspect later.

The effect discussed in the preceding paragraph is not the only one which may be of importance. An alternative latitudinal variation which can
introduce the harmonics required for a complete description of the momentum transfer is the poleward decrease in height of the tropopause. In winter, the principal height change occurs near the jet maximum, and may lead to a significant modification of the developing instability. Before commencing a detailed mathematical study of this problem, we discuss the principal features of the tropopause.

3. The Height Variation of the Tropopause

A typical vertical temperature profile in the Earth's atmosphere is such that there is a relatively sharp change in its gradient at a height which varies from 8 - 10 km in polar regions to 16 - 18 km at the equator. This level is termed the "tropopause", and its position is illustrated on Fig. 1. It divides the lower atmosphere, with a lapse rate between 6 and $7^\circ C/km$, from the relatively stable stratosphere, whose lapse rate is weak, with a sign varying both with the season and with latitude. The principal discontinuity occurs at about $30^\circ N$, less marked and transient breaks occur elsewhere (Murgatroyd (1969)).

The observed form of the tropopause is the result of both radiative and dynamical effects. In this respect, radiative/convective models of the atmosphere which neglect the influence of the large-scale dynamics on the temperature field are of particular interest. Such a calculation is that of Manabe and Strickler (1965). These show a single unbroken tropopause rising from about 10 km at the summer pole to 13 km at the equator. This height is maintained into the mid-latitude winter hemisphere, where the tropopause becomes ill-defined owing to a serious underestimate of the temperature in the polar winter stratosphere. The latter deficiency is probably a conse-
quence of the neglect of the heat transported northwards by the long-
wave disturbances to the polar-night jet, Hemispheric numerical models
have shown the high equatorial tropopause to be a result of adiabatic
cooling in the rising branch of the meridional cell (Manabe and Hunt
(1968)).

Such models also reveal that the level of the tropopause inter-
mediate between pole and equator is controlled by the heat transported
polewards by the eddies which form on the mid-latitude jet stream.
This produces a cooling, and thus a higher tropopause, on the equatorial
side of the axis of the jet, with a lowering towards the pole. The pro-
blem of the influence of the height variation of the tropopause is thus
in essence non-linear. However, for the purposes of investigating the
consequences relating to the eddy momentum transport, we regard its shape
as a fixed consequence of external influences and previous eddy activity,
and study the growth of an infinitesimal normal-mode disturbance on a
zonally symmetric current.

4. **A Preliminary Problem**

The formulation and solution of a detailed realistic model of the
height variation of the tropopause poses many problems. Accordingly, in
this paper, we restrict our attention to a particularly simple model ap-
propriate to the limit of infinite static stability in the stratosphere.
Evidence that the eddy momentum transport is not significantly dependent
on the nature of the wind system in the upper atmosphere is to be found
in the 9-level general circulation model described by Smagorinsky et al.,
(1965). Despite its inadequate resolution of the stratospheric flow, the
calculation was successful in simulating the momentum balance. In addition to this result, as a preliminary to a study of the sloping tropopause problem, we investigate the sensitivity of the Eady problem to an alteration of the upper boundary condition.

The basic state comprises a zonal flow

\[ u = \lambda \dot{x}, \quad 0 < z < H, \]

in a Boussinesq atmosphere with constant static stability \( N^2 \). We work on an f-plane, with a rigid horizontal boundary at \( z = 0 \), and vertical boundaries at \( y = 0 \) and \( y = L \). We replace the rigid lid at \( z = H \) by an additional flow of infinite vertical extent, with static stability \( N_s^2 \), and shear \( \frac{h_f}{N_s} \). The model tropopause is taken to be horizontal; for the basic state to be consistent, it must in fact slope by

\[ \frac{H}{L} \frac{N^2}{N_s^2} < \text{Rossby number}, \]

but this effect is negligible within the quasi-geostrophic approximation.

The lowest mode then takes the form

\[
\psi(x, y, z, t) = \phi(z) \sin \frac{H y}{L} e^{i \kappa x - \omega t},
\]

\[
-\frac{f^2}{N_s^2} \phi_{zz} - \left( \kappa^2 + \frac{f^2}{N_s^2} \right) \phi = 0, \quad 0 < z < H
\]

\[
-\frac{f^2}{N^2} \phi_{zz} - \left( \kappa^2 + \frac{f^2}{N^2} \right) \phi = 0,
\]

where

\[
\kappa = \frac{H}{L} \frac{N^2}{N_s^2} \text{Rossby number}
\]

and

\[
\text{for } z > H.
\]

The boundary conditions are

(i) vertical velocity, \( w_z = 0 \) on \( z = 0 \).

(ii) \( \phi, w_r \) continuous at \( z = H \), and

(iii) \( \phi \to 0 \) as \( z \to \infty \).

In this section, we scale
\[ z \quad \text{with} \quad \alpha^{-1} \cdot \frac{f}{\sqrt{\frac{k}{\alpha}}} - \]
\[ C \quad \text{with} \quad \Lambda \alpha^{-1}, \quad \text{and introduce} \]
\[ \tau = \left( \frac{N}{N_0} \right) \text{and} \quad s = \frac{\Lambda s}{\Lambda}. \]

Then
\[ \phi = \phi_0 (\sinh z - c \cosh z), \quad 0 < z < \alpha H, \]
\[ = \phi_0 (\sinh \alpha H - c \cosh \alpha H) e^{-\frac{1}{2} (z - \alpha H)} \]
for \( z > \alpha H \), and the matching condition on \( \omega \) gives
\[ c = a \pm \sqrt{\frac{1}{4} a^2 - b}. \]

\( a \) and \( b \) are given by
\[ a = \frac{(2p+r) + r(2p+sr) \coth 2p}{1 + r \coth 2p} \]
and
\[ b = \frac{2pr + r^2 - 1 + 2p \coth 2p}{1 + r \coth 2p} \]
where \( p = \frac{1}{2} \alpha H. \)

The limit \( r \to 0 \) corresponds to a stratosphere of infinite static stability, which acts like a rigid lid, and we recover Eady's result:
\[ c = p \pm \sqrt{1 + p^2 - 2p \coth 2p} \]
\[ = p \pm \sqrt{(p - \tanh p)(p - \coth p)}. \]

There is instability if \( \Im(c) > 0 \) for \( 0 < p < p_0 \), where
\[ p_0 = \coth p_0 \approx 1.2. \]

The non-dimensional growth rate, \( \Im(c) \), and phase speed, \( \Re(c) \), are illustrated on Fig. 3, which also depicts the unstable solution for the cases
\[ r = \frac{1}{\sqrt{3}}, \quad s = 0, \quad \text{and} \]
\[ r = \frac{1}{\sqrt{3}}, \quad s = -1. \]
Fig. 3 Growth rate and phase speed as a function of wave number for models discussed in Section 4.

- - - - - - Eady model, \( r = 0 \).
- - - - - \( r = \frac{1}{\sqrt{3}} \), \( s = 0 \).
++ + + + \( r = \frac{1}{\sqrt{3}} \), \( s = -1 \).

Some differences in the three cases are apparent; the reduced growth rates for the more general models are a consequence of a redistribution in the
stratosphere of a portion of the energy released in the troposphere by the instability mechanism. The inclusion of a simplified stratosphere flow is seen, however, to cause little significant modification of the solution in the neighborhood of the maximum growth rate. We thus proceed with an investigation of the sloping tropopause problem in the limit of a highly stable stratosphere.

5. The Eady Model with a Sloping Upper Boundary

(i) The formal problem

We investigate the solution of the baroclinic instability problem for the zonal flow $U = U_0$, with boundaries in the meridional plane as shown on Fig. 4. We consider in detail the case of a Boussinesq atmosphere on an f-plane. These assumptions may be relaxed without formal difficulty, but the introduction of compressibility and the $\beta$-effect will lead to a

Fig. 4. Meridional boundaries for the perturbed Eady problem.
greater complexity in the functional form of the \( z \)-dependence, with confluent hypergeometric functions replacing the hyperbolic forms. The problem is then immediately in the realm of automatic computation. Some mention of it is made later.

Case must be taken with the condition to be applied at the upper boundary. Consider the general case, when the surface

\[ z = H \left( 1 - \gamma h \left( \frac{y}{L} \right) \right) \]

is an interface between two differing mean states. The appropriate matching conditions are continuity of pressure and normal velocity. The former is equivalent to continuity of the stream function, and thus of the northward velocity component. The latter condition then implies continuity of the vertical velocities, and, in the limit of infinite static stability in the stratosphere, we obtain the condition \( \omega = 0 \) as that appropriate to the tropopause problem. In the quasi-geostrophic approximation, this is quite different from the case of a rigid upper boundary, which is discussed later.

The mathematical problem for the stream function

\[ \psi = \phi(y, z) e^{ik(x-ct)} \]

is

\[ (\Lambda^2 - c)^2 \left[ \frac{1}{N^2} \phi_{yy} - \Lambda^2 \phi \right] = 0, \]

with

\[ \phi = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad y = L, \quad \text{and} \]

\[ (\Lambda^2 - c) \phi_z - \Lambda \phi = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad z = H \left( 1 - \gamma h \left( \frac{y}{L} \right) \right). \]

We scale

\[ \bar{\psi} \quad \text{with} \quad \alpha^{-1} = \left( \frac{Nk}{J} \right)^{-1}, \]

\[ y \quad \text{with} \quad L, \]

\[ c \quad \text{with} \quad \Lambda \alpha^{-1}, \quad \text{and introduce the parameter} \quad \varepsilon = \frac{\Pi^2}{k^2 L}. \]
problem then reduces to

$$\phi_{zz} + \frac{\epsilon}{n^2} \phi_{yy} - \phi = 0,$$

with $c \phi_x + \phi = 0$ on $z = 0$, and

$$[\alpha H \{1 - \chi h(y)\} - c] \phi_z - \phi = 0 \text{ on } z = \alpha H \{1 - \chi h(y)\}.$$

(ii) A perturbation approach

We are interested in the effect on the baroclinic instabilities of a tropopause constrained externally to be at a high level towards the equator, and at a low level in polar regions. As there is little tropopause slope, and tropospheric wind shear, north of $60^\circ$N, we consider the idealised problem with $L = 60^\circ$ of latitude. The model is most closely related to the situation in winter, when the maximum shear occurs at $30^\circ$N in conjunction with the principal height change of the tropopause.

The extreme values for the height of the upper boundary are taken as 18 km at $y = 0$ and 8 km at $y = 1$.

The magnitude of $h(y)$ is chosen such that

$$\max_{0 < y < 1} \{|h(y)|\} = 1.\text{ Then } \chi \leq \frac{5}{13} \approx 0.38, \text{ and we investigate solutions of the form}

$$c = c_0 + \chi c_1 + \chi^2 c_2 + \ldots, \quad \phi = \phi_0 + \chi \phi_1 + \chi^2 \phi_2 + \ldots.$$

The remaining arbitrariness in the definition of $h(y)$ is removed by requiring

$$\int h(y) \sin^2 \pi y dy = 0$$

this ensures that the correction to the lowest order eigenvalue occurs only to $O(y^2)$. Our technique only utilizes Fourier coefficients of $h(y)$ and may in principle handle discontinuities in the upper boundary. However,
the limit of an infinitely stable stratosphere is inappropriate near
the observed break, and, as we do not expect to obtain a detailed agree-
ment with observation, it suffices to consider a continuous boundary
decreasing from equator to pole.

We note that each \( \phi_r \) satisfies

\[
\phi_{rzz} + \frac{\varepsilon}{n^2} \phi_{ry} - \phi_r = 0
\]

with \( \phi_r = 0 \) on \( y = 0, L \). Differences only arise in the boundary condi-
tions at the upper and lower boundaries,

(iii) The zeroth order solution

The lowest order problem is that of Eady, with the growing solution

\[
\phi_0(y, z) = Z_1(z) \sin \pi y
\]

where, in the scaling of this section,

\[
Z_1(z) = \sinh \sqrt{1+E} z - C \sqrt{1+E} \cosh \sqrt{1+E} z,
\]

\[
C \sqrt{1+E} = \frac{\sqrt{p+1}(1+\tanh p)(\cosh p - p)}{2 \pi i H \sqrt{1+E}} \quad \text{and}
\]

\[
p = \frac{1}{2} \ll H \sqrt{1+E} < p_0.
\]

(iv) The \( O(\varepsilon) \) Solution

The boundary conditions for \( \phi_1 \) are

\[
C_0 \phi_{1z} + 4 \phi_1 = -C_1 \phi_{0z} \quad \text{at} \quad z = 0,
\]

\[
(\ll H - C_0) \phi_1 - \phi_1 = C_1 \phi_{0z} + 2 \varepsilon_0 (1+E) \ll H (\ll H - C_0) h(y) \sin \pi y \quad \text{at} \quad z = \ll H.
\]

We expand \( h(y) \sin y \zeta = \sum \phi_r \cos r \zeta \), the \( n = 1 \) term vanishing by the definition of \( h(y) \), and obtain the solution

\[
C_1 = 0
\]

\[
\phi_1 = a \int \tilde{Z}_r(z) \sin r y, \quad \text{where}
\]
Provided $b_2 \neq 0$, which will be the case for representative choices of $h(y)$, we note the appearance of the $\sin 2\pi y$ harmonic to first order in $y$. As the values of $y$ to be used in the application are not particularly small, we postpone a numerical discussion until further terms have been calculated.

(v) The $O(\gamma^2)$ solution

$$\phi_2 \quad \text{satisfies}$$

$$c_0 \phi_{22} + \phi_2 = -c_2 \phi_0$$

at $z=0$ and

$$(\omega H - c_0) \phi_{22} - \phi_2 = c_2 \phi_0 + \omega H (\omega H - c_0) h(y) \phi_1 \quad \text{at } z = \omega H$$

The solution is of the form

$$\phi_2 = f(z) \sin \pi y + \frac{\infty}{z} kr \sin \pi r y.$$ 

$\tau \in \mathbb{R}$ has an undetermined part proportional to $Z_1(z)$, without loss of generality we take this to be such that

$$f(z) = -c_2 \sqrt{1 + \varepsilon} \cosh \sqrt{1 + \varepsilon} z.$$
This has been chosen to satisfy the lower boundary condition; \( C_2 \) is then determined by equating to zero the coefficient of \( \sin \pi y \) in the expansion of the condition at \( \beta = \infty H \).

We form the Fourier series

\[
\{h(y)\}^2 \sin \pi y = \sum_{r} r \sin \pi r y
\]

\[
h(y) \sin \pi r y = \sum_{s} g_{rs} \sin \pi s y, \quad g_{15} = b_5,
\]

\[
g_5 = 2 \cdot 9m(c_0)(1+\varepsilon) = 4 p^2 \left\{1 - c_0 \sqrt{1+\varepsilon} \coth 2p\right\} x
\]

\[
x \left\{-\varepsilon + (2p - c_0 \sqrt{1+\varepsilon}) \sum_{s} \frac{b_5 g_{rs} (1+\varepsilon) Z_1 (\gamma H)}{k_s a \sinh h \sqrt{1 + s^2}} \right\}
\]

and

\[
h_r = \frac{4 p^2 (1+\varepsilon)}{k_r \sinh h \sqrt{1 + t^2} \varepsilon H} \left\{1 - c_0 \sqrt{1+\varepsilon} \coth 2p \right\} x
\]

\[
x \left\{-\varepsilon + (2p - c_0 \sqrt{1+\varepsilon}) \sum_{s} \frac{b_5 g_{rs} (1+\varepsilon) Z_1 (\gamma H)}{k_s a \sinh h \sqrt{1 + s^2}} \right\}
\]

(vi) An Example

The above formulae take a particularly simple form when

\[
h(y) = \cos \pi y
\]

which bears some resemblance to the actual form of the tropopause. We then have

\[
b_2 = -1/2, \quad b_r = 0 \quad r \neq 2,
\]

\[
\xi_1 = \xi_3 = 1/4, \quad \xi_r = 0 \quad r \neq 1, 3, \text{ and}
\]

\[
g_{r2} = -1/2 \quad r = 1, 3
\]

\[
= 0 \quad \text{otherwise},
\]

The solution then takes the form
For this case, we investigate a numerical example.

We take

\[
\begin{align*}
H &= 13 \text{ km} \\
N^2 &= 1.5 \times 10^{-4} \text{ sec}^{-2} \\
L &= 6.700 \text{ km} \\
f &= 2\pi \sin 30^\circ = 7.3 \times 10^{-4} \text{ sec}^{-1}
\end{align*}
\]

The lowest order growth rate is a maximum when \( p \sim 0.9 \). This corresponds to zonal wavenumber 4, which is lower than the dominant observed wave.

Both the inclusion of the \( \beta \)-effect (Green (1960)), and horizontal curvature in the mean flow (Stone (1969)), cause some shift towards higher wave numbers; in addition, this value of \( p \) is not necessarily that appropriate to the fastest growing wave in a finite-\( \gamma \) calculation. We proceed with this value for the moment. We find

\[
\begin{align*}
C_0\sqrt{1+\varepsilon} &= 0.9 + 0.3i \\
\varepsilon &= 0.48 \\
d_2 &= -0.093 - 0.56i \\
h_3 &= 0.038 - 0.051i \\
\text{and} \quad C_2 &= -0.85 - 0.63i
\end{align*}
\]

To this point, the convergence of the perturbation solution appears reasonably good for the range of \( \gamma \) of practical application. However, the situation is seriously worsened by proceeding to the next order in \( \gamma \). The algebra now becomes somewhat lengthy, and is not presented here. For the example \( h(y) = -\cos \pi y \), the solution takes the form

\[
\phi_3 = (d_2 Z_2(x) - c_2 d_2 \cosh \sqrt{1+4\varepsilon} z) \sin 2\pi y + 4 Z_2(x) \sin 4\pi y.
\]
The numerical example gives
\[ \ell_2 = -5.93 + 3.67i, \]
and thus
\[ |\delta \ell_2| \sim |d_2| \] when \( \gamma = .3. \)

In the case of practical interest, this solution method thus fails. It may be seen to be in part due to the division at various stages of the expansion procedure by the parameter \( k_2 \), which takes the value \(-.55\) in the numerical example. Quite apart from the difficulties which will arise owing to the rapid variation with height of the components with a fine horizontal structure, we can therefore only expect the method to be successful when
\[ \gamma \ll \frac{.35}{b_2 \approx H} \sim .3. \]

The situation does not appear to be much improved by a different choice of wave number, and the difficulty will arise for any \( h(y) \) such that the \( \sin 2\pi y \) mode is excited. We must therefore seek some alternative technique for evaluating the solution when numerical values are appropriate to the real atmosphere. The use of the perturbation approach is limited in the present context to an explicit demonstration of the appearance of both even and odd harmonics in the limit of small height variations of the upper boundary. We note that its application for larger values of \( \gamma \) may be improved in a more detailed model including the \( \beta \) -effect and compressibility, but this is unlikely to be significant. Variations of the basic method may also help, but these will probably not yield sufficient advantage to justify the additional work involved. A successful application of the general perturbation technique developed in this section is given later in Section 9.
6. **An Alternative Method of Solution**

Several other solution methods are open to us; we consider one of them in this section. The perturbation approach fails initially in its determination of the coefficient of the \( \sin 2\pi y \) Fourier component. The higher harmonics appear to have a much reduced amplitude, suggesting that we may obtain a better approximation to the solution by retaining the exact form of the upper boundary condition as regards terms in \( y \), but truncating all Fourier expansions after a finite number of terms.

The exact solution of the governing equations is

\[
\phi = \sum_{i} \int \left( \sinh \sqrt{1+\epsilon} z - c \sqrt{1+\epsilon} \cosh \sqrt{1+\epsilon} z \right) \sin \pi ry,
\]

where we take \( d_1 = 1 \) for convenience. The eigenvalue \( c \), and the \( d_r (r \geq 2) \) are determined by the equation

\[
\sum_{i} \int \{ g_r(y) \cosh g_r(y) + \sinh g_r(y) \left[ c^2 (1+r^2 \epsilon) - c g_r(y) - 1 \right] \} \sin \pi ry = 0,
\]

where

\[
g_r(y) = \sqrt{1+r^2 \epsilon} \alpha H (1-\gamma h(y)).
\]

We then form the Fourier series of

\[
(\sin \pi ry) x \begin{cases} g_r(y) \cosh g_r(y) \\ g_r(y) \sinh g_r(y) \\ \sinh g_r(y) \end{cases}
\]

and truncate all expansions after \( N \) terms, which reduces the problem to a set of \( N \) equations of the form

\[
\sum_{i} \int \left( \eta_i + \chi_i c + \delta_i c^2 \right) = 0 \quad i = 1, 2, \ldots, N.
\]

The solubility requirement for the \( d_1 \) gives a polynomial equation of order \( 2N \) for \( c \), and for each root we may obtain the set \( \{d_r\} \). In the limit \( \gamma \to 0 \), these correspond to the \( 2N \) eigensolutions for the Eady problem.
with horizontal structure \( \sin \pi y, r = 1, 2, \ldots, N \).

The approach described above is most suited to a machine calculation, but we have determined the most unstable solution for the case considered in detail in the preceding section:

\[ h(y) = -\cos \pi y, \quad p = .9, \text{ for } N = 2 \text{ and } N = 3, \]

The 2-mode expansion involves the solution of a quartic equation, and we find

\[ C = .62 + .22i \quad \text{and} \quad d_2 = -.15 - .10i. \]

Using this as the basis for an iterative determination of the result when \( N = 3 \),

\[ C = .58 + .23i, \quad d_2 = -.27 - .18i, \quad \text{and} \quad d_3 = .01 + .02i. \]

The corresponding solution to the Eady problem is \( C = .74 + .251 \).

The expressions for \( C \) indicate a good convergence. This is less obvious in the expressions for \( d_2 \); in general, however, the inclusion of the \((N + 1)\)th mode will probably be most effective in modifying the \( N \)th mode, with progressively weaker effects for the lower harmonics. If so, the expression for \( d_2 \) from a three-mode expansion may be reasonably accurate, the \( d_3 \) term is in more doubt. Further computations are required to verify this. We note that the phase of \( d_2 \), which largely determines the vertical structure of the lowest order eddy momentum transport, is unaltered by the inclusion of the third mode.

We now discuss the northward eddy transports of zonal momentum and heat as given by the results of the three-mode expansion. With regard to the former, an uncertainty arises quite apart from doubts regarding the accuracy of the solution method. The lowest order momentum transport is of the form
where \( \phi_0 \) is the amplitude of the \( \sin \gamma \) mode at the time under consideration. Values of \( f(z) \) are illustrated below, together with an estimate of the maximum error arising from the slide-rule calculation. As the form of \( f(z) \) is determined for large \( Z \) by the difference between two quantities of similar magnitude, the uncertainty is considerable:

<table>
<thead>
<tr>
<th>( z )</th>
<th>( f(z) )</th>
<th>Error estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>- .15</td>
<td>.01</td>
</tr>
<tr>
<td>.2 &lt; H</td>
<td>- .07</td>
<td>.04</td>
</tr>
<tr>
<td>.4 &lt; H</td>
<td>- .03</td>
<td>.02</td>
</tr>
<tr>
<td>.6 &lt; H</td>
<td>- .02</td>
<td>.04</td>
</tr>
<tr>
<td>.8 &lt; H</td>
<td>.02</td>
<td>.08</td>
</tr>
<tr>
<td>&lt; H</td>
<td>.11</td>
<td>.18</td>
</tr>
<tr>
<td>1.2 &lt; H</td>
<td>.40</td>
<td>.40</td>
</tr>
<tr>
<td>1.4 &lt; H</td>
<td>.90</td>
<td>.90</td>
</tr>
</tbody>
</table>

We are clearly not in a position to draw any detailed conclusions, however, we note that the principal transport occurs in the upper troposphere, as in the observed state. Its maximum magnitude is close to that estimated by Lipps (1966). In his initial-value estimate of the contribution from the asymmetry in the horizontal structure of the upper-level flow, a comparison with observation indicates that this value gives a reasonable estimate for the momentum transport by a single zonal wavenumber component. The negative values of \( f(z) \) for small \( z \) are not observed. The sign of the \( \sin^3 \gamma \cos \gamma \) component of \( \overline{UU^*} \) is opposite to the contribution from the symmetric part of the horizontal jet.

Our calculations have failed to give definite evidence of the importance of height variations of the tropopause in influencing the momentum
transport by baroclinic instabilities, but are such as to encourage a
more detailed and accurate study. This view is strengthened by consider-
ation of the northward eddy transport of heat, given in non-dimensional
form by \( \frac{L}{2} \mathcal{R}_L \left\{ -i \phi^* \phi \right\} e^{2k_m \cos \theta} \) which is significantly modified by the
introduction of the \( \sin 2\pi y \) component of \( \phi(z,y) \). The lowest order con-
tribution is proportional to

\[
(1 + \varepsilon) g_m(c) \sin^2 \pi y, = 0.28 \sqrt{1 + \varepsilon} \sin^2 \pi y,
\]

which is independent of height, as found in the Eady problem. The inclusions
of the \( \beta \)-effect is such as to give a low-level maximum to this trans-
port. Observations do indicate such a maximum, but with a secondary max-
imum near the tropopause. These are reproduced by the next order contribu-
tion in the tropopause problem:

\[
\sqrt{1 + \varepsilon} \sin \pi y \cos \pi y \ell(z),
\]

where

\[
\sqrt{1 + \varepsilon} \ell(z) = 9 m \left\{ c^* Z^* \frac{d}{dz} (Z_0) + Z^* \frac{d}{dz} (Z_0) \right\}.
\]

Values of \( \ell(z) \) are:

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \ell(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.36</td>
</tr>
<tr>
<td>0.2cH</td>
<td>-0.25</td>
</tr>
<tr>
<td>0.4cH</td>
<td>-0.16</td>
</tr>
<tr>
<td>0.6cH</td>
<td>-0.23</td>
</tr>
<tr>
<td>0.8cH</td>
<td>-0.27</td>
</tr>
<tr>
<td>cH</td>
<td>-0.31</td>
</tr>
</tbody>
</table>

These numbers are less sensitive to inaccuracies in the calculation than
are the values for the momentum transport. The horizontal structure is
such that the principal heat transport will occur north of 30\(^\circ\)N, with rela-
tively little in low latitudes, as found in the observations (Lorenz (1969)).

The form of the eddy transport of heat is thus seen to be significantly
improved by the inclusion of a form of tropopause height variation.

7. The Inclusion of Compressibility and the $\beta$-effect

The generalization of the model discussed in the previous sections is formally straightforward, with the solution now being given in terms of confluent hypergeometric functions. Their properties are given by Abramowitz and Stegun (1964), whose notation is initially followed here. We give the general form of the solution.

The model we now consider is for a compressible atmosphere with constant density scale height, $H_1$, on a $\beta$-plane. We introduce the parameters

$$\mu_r = \frac{1}{2H_1 + \mu \epsilon},$$
$$\chi_r = \frac{\mu \epsilon}{2(1 + \mu \epsilon)} \left( 1 + \frac{\beta N^2 H_1}{\bar{r}^2} \right),$$
$$\lambda_r = \sqrt{1 + \mu \epsilon} \left( \sqrt{1 + \mu \epsilon} - \mu \right),$$

and $\eta_r = 2 \sqrt{1 + \mu \epsilon} / \mu$. For convenience, we modify the general notation for the confluent hypergeometric functions to

$$U_{rS}(z) = U(1 - \chi_r, \eta_r, \eta_r(z - c)),$$
$$M_{rS}(z) = M(1 - \chi_r, \eta_r, \eta_r(z - c)).$$

The basic solution is then

$$\Phi(y, z) = \sum_{r = 2}^{\infty} d_r Z_r(z - c) \sin m y,$$

where

$$Z_r(z - c) = \eta_r(z - c) e^{-\lambda_r z^2} \left\{ U_{r2} + A_r M_{r2} \right\},$$

and the condition $w = 0$ on $z = 0$ gives

$$A_r = \frac{\chi_r U_r(0) - (1 - \lambda_r c) U_{r2}(0)}{(1 - \lambda_r c) M_{r2}(0) - M_{r1}(0)}.$$

The $d_r$, $r \geq 2$, and $C$, are determined by the upper boundary condition.

The principal difference in the form of the solution in this case is
the appearance of the eigenvalue, c, in the two independent hypergeometric functions. Thus, in a numerical scheme for the truncated Fourier expansion described previously, the functions have to be evaluated at each iteration in the determination of the components of the Fourier series. This is likely to be a lengthy computation unless the convergence of the scheme turns out to be rapid.

Most of the objections to a small-$\gamma$ expansion that were mentioned in Section 5 apply also here. The analysis presented there has been extended to the more general case, but yields little additional information apart from illustrating the essential similarity in the horizontal structure for the two cases.

8. **A Rigid Upper Boundary**

The model introduced in Section 5 imposes the condition $w = 0$ on the upper boundary. In view of the predominantly horizontal nature of quasi-geostrophic motions, a quite different solution results if we consider the Eady problem with a rigid sloping upper boundary. The problem is then removed from atmospheric relevance, although it may be related to laboratory studies, and there is some oceanographic interest if the slope occurs at the lower boundary. For example, Orlanski (1969) has investigated the influence of bottom topography on the baroclinic instability of a two-layer model, and discussed its application to the Gulf Stream. In this section, we present general results for the perturbed Eady problem, with no specific application in mind.

We now consider the slope of the boundary to be small, formally

$$\text{Slope} = O(\text{Rossby Number}) \times H/L.$$
This ensures that the vertical and horizontal velocities are of equal importance at the sloping boundary. Moreover, the boundary condition may be applied at \( Z = \infty H \) to the accuracy with which the quasi-geostrophic equations apply. A larger slope imposes the stabilizing condition \( v = 0 \). We discuss briefly the results for two cases.

(i) A constant slope

Take \( h(y) = (2 \frac{Y}{L} - 1) \).

In terms of the scaling introduced in Section 5, the problem is

\[
\begin{align*}
\frac{\partial \phi}{\partial z} + \frac{c}{H^2} \frac{\partial \phi_y}{\partial y} - \phi &= 0, \quad \text{with} \\
\phi &= 0 \quad \text{on} \quad y = 0, 1, \\
w &= 0 \quad \text{on} \quad z = 0, \quad \text{and} \\
w + \frac{Y}{L} v &= 0 \quad \text{on} \quad z = \infty H, \quad \text{as long as} \quad w \quad \text{and} \quad v \quad \text{are scaled with the same velocity.}
\end{align*}
\]

Introducing \( \delta = 2N^2 H \), which is \( \lesssim 0(1) \) by assumption, the fastest-growing solution to this separable problem is

\[
\phi = (\sinh \sqrt{1+\delta} z - c \sqrt{1+\delta} \cosh \sqrt{1+\delta} z) \sin \pi y,
\]

where

\[
c \sqrt{1+\delta} = p - \frac{\delta}{2} \coth 2p \pm \sqrt{(p - \frac{\delta}{2} \coth 2p)^2 + (1+\delta) - 2p \coth 2p},
\]

and \( p = \frac{1}{2} \infty H \sqrt{1+\delta} \).

For \( \delta = 0 \), we recover Eady's result. For non-zero \( \delta \), there is both a long- and a short-wave cutoff, as found by Bretherton (1966) for a two-layer model. The maximum growth rate is reduced such that

\[
\max \{ \gamma \} = (1+\delta) - \frac{\delta}{2} \sqrt{1+\delta} e^{-(2+\delta)} + O(\delta^2 e^{-(2+\delta)}),
\]

and occurs for \( p = 1 + \frac{\delta}{2} - (6 + 7\delta) e^{-4+2\delta} + O(\delta^2 e^{-(6+4\delta)}) \).

The range of unstable wavenumbers is of dimensionless width \( H \sqrt{1+\delta} e^{-(2+\delta)} \), and decreases as \( \delta \) increases.
(ii) A varying slope

The boundary condition at $Z = \infty H$ is, in the general case,

$$\frac{\partial H - c}{\partial \eta} - \phi = \delta h' \phi, \quad \text{As with earlier work, the solution is of the form} \quad \phi = \sum d_r \{a \sin h_1 + r \cos \eta \} \sin \eta \sin \gamma.$$

A set of infinitely coupled equations for the $d_r$ and $c$. An expansion procedure in powers of $S$ may be developed in a straightforward manner, and certain $h(y)$ may again be found such that the solution takes a particularly simple form. We find various contributions to the eddy momentum transport in terms of the coefficients of the Fourier series of $h'(y) \sin \pi \gamma$.

9. A Horizontal Variation in the Mean Flow

To this point, we have been concerned with the solution of the baroclinic instability problem with a sloping upper boundary. In this section, we adapt the perturbation technique described in Section 5 to investigate the modified Eady problem in which the upper boundary is horizontal, and the mean flow is perturbed by a velocity distribution which simulates the westerly surface winds north of $30^\circ N$. Since it is surface friction acting on this low-level wind system that balances the eddy momentum transport in the real atmosphere, a demonstration that the inclusion of this effect directly results in a realistic transport would be of great logical satisfaction. Time has not permitted a detailed solution of this problem, but we are in a position to compare preliminary results with those obtained as the main theme of this study.

We consider the zonal flow which, in terms of dimensional variables, is given by
At the ground \((z = 0)\), winds are easterly in \(0 < y < \frac{1}{2}L\), and westerly for \(\frac{1}{2}L < y < L\). The flow, whilst neglecting the principal horizontal curvature of the observed state, contains an asymmetry of the order of that observed for values of \(\delta\) between \(1/2\) and \(1/4\). This differs from the asymmetry studied by Lipps, who was interested in deviations from symmetry only in the high-level jet stream.

Some of the mathematical simplicity of the Eady problem is retained. Since the mean flow potential vorticity gradient involves the horizontal curvature, and not the horizontal shear, it again vanishes. The differential equation and boundary conditions on \(y = 0\) and \(y = L\) are as in Section 5, and, on the horizontal boundaries, we have, in dimensionless form,

\[
(c - \delta(y - \frac{1}{2}) - \alpha H)\phi_x + (1 - \delta(y - \frac{1}{2}))\phi = 0 \text{ on } z = 0, \\
(\alpha H - c)\phi_x - (1 - \delta(y - \frac{1}{2}))\phi = 0 \text{ on } z = \alpha H,
\]

This problem does not generalize in any simple manner to include the \(\beta\)-effect and compressibility.

We seek a solution in the form of an expansion in powers of \(\delta\). Experience suggests that this will be feasible, for the case \(p = .9\) discussed earlier, if \(\delta \ll \frac{16}{9\pi^2} \approx 1.6\), which is well satisfied.

\(\frac{16}{9\pi^2}\) is the coefficient of \(\sin 2\pi y\) in the Fourier expansion of \((y - \frac{1}{2})\sin 2\pi y\), and appears in a similar fashion to the parameter \(b_2\) in Section 5.

We pose the expansion \(c = c_0 + \delta^2 \hat{c}_2 + O(\delta^4)\), and

\[
\phi = \phi_0 + \delta \phi_1 + \delta^2 \phi_2 + O(\delta^3),
\]

the zeroth
order problem for \( \phi_0, C_0 \) is as previously discussed. The boundary conditions which determine \( \phi \) are

\[
C_0 \phi_{1z} + \phi = [(y-\frac{1}{2})\alpha H \phi_{1z} + (y-\frac{1}{2}) \phi_0] \text{ at } z = 0,
\]

and

\[
(\alpha H - C_0) \phi_{1z} - \phi_1 = -(y-\frac{1}{2}) \phi_0 \text{ at } z = \alpha H.
\]

Now \((y-\frac{1}{2}) \sin \pi y = -\frac{1b}{\pi^2} \sum \frac{1}{(2n+1)^2} \sin 2\pi y\),

so \(\phi_1 = \sum \left(d_r \cosh \sqrt{1+\varepsilon} z + 2r \sinh \sqrt{1+\varepsilon} z \right) \sin 2\pi ry\),

where

\[
\frac{c_2 \kappa_r \sinh \sqrt{1+\varepsilon} z \alpha H}{\pi^2 (\pi^2 - 1)} \sum \frac{1}{(2n+1)^2} \sin 2\pi y,
\]

and \(\kappa_r\) is as defined previously. \(\frac{\kappa_r}{(\pi^2 - 1)}\) decreases rapidly with increasing \(r\), and the ratio of the value when \(r = 2\) to that when \(r = 1\) is \(\frac{9x^2}{225} \approx 0.08\). This is illustrated in the case \(p = 0.9\), for which

\[
\begin{align*}
\ell_1 &= -0.094 + 0.46i, \\
d_1 &= 0.15 - 0.49i, \\
e &= -0.0096 - 0.0022i, \\
d_2 &= 0.0053 + 0.016i.
\end{align*}
\]

The solution for \(\phi_2\) is determined by the boundary conditions.

\[
C_0 \phi_{2z} + \phi_2 = [(y-\frac{1}{2})\alpha H \phi_{2z} + (y-\frac{1}{2}) \phi_0] - C_2 \phi_{2z} \text{ at } z = 0,
\]

and

\[
(\alpha H - C_0) \phi_{2z} - \phi_2 = -(y-\frac{1}{2}) \phi_0 + C_2 \phi_{2z} \text{ at } z = \alpha H.
\]

and has a solution of the form

\[
\phi_2 = f_1 \cosh \sqrt{1+\varepsilon} z \sin \pi y + \\
+ \sum \frac{1}{\pi} \sin \pi (2r+1)y \left[g_r \sinh \sqrt{1+\varepsilon} (2r+1)z + h_r \cosh \sqrt{1+\varepsilon} (2r+1)z\right].
\]

\(f_1\) and \(C_2\) are determined by the coefficients of \(\sin \pi y\) in the Fourier
expansions of the boundary conditions at $Z = 0$ and $Z = \infty$, and $g$ and $h$ from the coefficients of $\sin \pi (2r + 1)y$.

In our numerical example, we find

$$c_2 \approx -0.01 + 0.05i, \quad g_3 \approx \sinh \sqrt{1 + 9\delta} \approx H, \quad h_3 \approx \sqrt{1 + 9\delta} \approx H$$

are of the same order of magnitude, indicating a rapid convergence. The difficulty encountered with the $O(\delta)$ term in Section 5 is absent here, so we proceed with an evaluation of the principal terms in the eddy momentum transport.

$$\frac{U}{V} \approx \frac{\pi \delta}{L} \left( g \sin^2 \frac{\pi y}{2} \{ 4.6 S, S_2 - 4.9 S_1 - 4.9 C_1 C_2 + 4.4 C_2 S_1 \} \right),$$

where

$$S_r = \sinh \sqrt{1 + \gamma^2 \delta} \approx \frac{S_r}{\gamma^2} \cos \sqrt{1 + \gamma^2 \delta} \approx \frac{S_r}{\gamma^2}.$$

We note first a positive poleward transport at low levels, in comparison with the negative values found earlier. The numerical uncertainty for larger $Z$ discussed in Section 6 again applies, with the transport in the upper troposphere being given in terms of the difference between large quantities. Similar considerations apply to the heat transport. A more accurate calculation involving higher harmonics and a larger number of powers of $\delta$ is thus required. This calculation is suited to computer methods. For the present, we note that the numerical coefficients in the $\sin 2\pi y$ term are comparable when $\delta \approx \frac{1}{2}$ with those found for the tropopause problem. Moreover, a combination of the eddy momentum transports due to both processes shows a masked similarity with the observed form. The effect considered in this section thus also merits further study,
10. **Summary and Conclusions**

The results of the investigations described herein contain aspects of both success and failure. We have seen from a general argument that the principal features of the observed eddy transport of zonal momentum may possibly be described by a linear theory of baroclinic instability on an \( f \) - or \( \beta \) -plane. In this respect, the work of Lipps is of interest, and worth pursuing in further detail.

Two alternative modifications of Eady's model which may also influence the transfer of momentum by baroclinic instabilities have been studied. Our main attention has been directed towards the consequences of the height variation of the tropopause. The limit of infinite static stability in the stratosphere was found to differ little from more realistic cases when the tropopause is approximately horizontal, and we proceeded on the basis of this approximation. A perturbation technique illustrated the appearance of the modes required for a description of the observed state in the limit of small tropopause slope, but proved inadequate for numerical values appropriate to the Earth's atmosphere. The truncated Fourier expansion appears more suited in this case, but requires further study in order to obtain detailed results. An example was of insufficient accuracy to draw positive conclusions, but indicates that the momentum transport resulting from the influence of the sloping tropopause may account for a significant part of the net observed transfer. An improved description of the northward eddy heat transport also results. A relaxation of the \( f \) -plane and Boussinesq assumptions is possible, but an assessment of its practicality must await additional numerical work.
The perturbation technique was modified successfully in a highly simplified model of the asymmetry of the low-level wind system. It is estimated that this effect is at least of similar importance to that of tropopause height variations. Since it is this phenomenon which is directly linked with the momentum transfer, an explicit demonstration of its direct influence on the detailed nature of the baroclinic instabilities is of fundamental interest.

In order to formulate tractable mathematical models, several essential features have been neglected. Probably the most significant of these is an absence of a horizontal jet structure in the mean flow. However, the calculations suggest that a full numerical solution of the non-separable baroclinic instability problem for a realistic basic state including the phenomena which we have discussed will reproduce the essential horizontal transfer process involved in the angular momentum budget of the Earth's atmosphere.

Acknowledgements

Dr. M. E. McIntyre originally pointed out the momentum transfer associated with a sloping boundary. Subsequent encouraging comments, and suggestions for future improvements and extensions, from various staff members are much appreciated, and financial support from the National Science Foundation is gratefully acknowledged.

References

Murgatroyd, R. J. 1969. "The Structure and Dynamics of the Stratosphere".
WIENER-HERMITE EXPANSION APPLIED TO
PASSIVE SCALAR DISPERSION IN A NONUNIFORM TURBULENT FLOW

Martial L. Thiebaux

ABSTRACT

The dispersion of a passive scalar advected by a turbulent incompressible flow characterized by inhomogeneous and nonstationary statistics is investigated by expanding the velocity and scalar concentration fields in Wiener-Hermite functions. The time evolution equation for the ensemble mean concentration is determined in terms of Eulerian one-point, two-time and one-point, three-time correlation functions defined in the coordinate frame moving with the mean velocity. The example of a plane-parallel mean flow is studied in detail and eddy diffusivity tensors are extracted from the formalism.

1. Introduction

The dispersion of a scalar contaminant passively advected by a turbulent flow is a problem that has attracted considerable attention in fluid mechanics (Taylor, 1921; Batehelor and Townsend, 1956; Corrsin, 1964). Much of the interest in this problem has centered on its fundamental role in unravelling the relationship between Eulerian and Lagrangian statistics, for which the assumption of stationary, homogeneous, isotropic velocity field statistics makes a difficult enough problem, as it is usually formulated (for example, Lumley, 1964; Patterson and Corrsin, 1966). However, the advective dispersion of a passive scalar in a turbulent flow possessing non-uniform (inhomogeneous and nonstationary) statistics is a problem having interesting geophysical applications. The desirability of relating a statis-
tical description of the transport of atmospheric contaminants or tracers to realistically nonuniform statistics of turbulent atmospheric flow motivates this paper.

Mean seasonal and spatial horizontal wind data could provide a lowest order basis for understanding atmospheric transport of passive scalars. Turbulent mixing accounts for additional and often highly significant transport which must have a statistical basis. Assigning an eddy diffusivity to account for the mixing may be an appealing approximation but it is difficult to relate such a diffusivity to velocity field statistics. We shall investigate this problem and be especially interested in determining how an eddy diffusivity (or eddy diffusivity tensor) may depend on the interaction between the characteristics of a nonuniform mean flow and nonuniform statistics of the random part of the flow. We have in mind an ensemble of atmospheres, each member of which, say, is a year record of global winds. A mean wind function of space and time as well as the statistics of the random part of the flow are then definable over a period of one year in this example.

The Eulerian statistics of the velocity field \( u_i(x,t) \) together with its ensemble mean \( \bar{u}_i(x,t) \) are assumed known. The concentration \( \Gamma(x,t) \) of a passive scalar satisfies the advection equation

\[
\frac{\partial \Gamma}{\partial t} + u_i \frac{\partial \Gamma}{\partial x_i} = 0
\]  

appropriate to an incompressible flow with no molecular dispersion. We limit this investigation to determining the ensemble mean concentration \( \Gamma(x,t) \) given the flow statistics and an initially specified concentration \( \Gamma(x,0) = \Gamma(x,0) \), the same for all members of the ensemble,
2. Review of Wiener-Hermite functions

The Wiener-Hermite expansion provides a useful representation for random functions governed by or occurring as coefficients in a differential equation [Meecham and Siegel, 1964; Imamura, Meecham and Siegel, 1965; Meecham and D.-t. Jeng, 1968]. The function is expanded in a series of statistically orthogonal functionals which are constructed from the ideal random function. The first term in the series is the nonrandom part or ensemble mean of the function. The next term represents the normally distributed part of the function. Higher terms in the expansion depart more and more from Gaussianity.

The ideal random function \( a(z) \) is the normally distributed random function with zero ensemble mean and covariance

\[
\langle a(z) a(z') \rangle = \delta(z-z'). \tag{2}
\]

We shall need the generalization of \( a(z) \) which is a vector function of position and time. This, the first order Wiener-Hermite function, has zero mean

\[
\langle H^{(1)}_1 (z, t) \rangle = 0
\]

and covariance

\[
\langle H^{(1)}_1 (z_1, t_1) H^{(1)}_j (z_2, t_2) \rangle = a_{i j} \delta(z_1-z_2) a(t_1-t_2). \tag{3}
\]

It is helpful to adopt the compact notation

\[
H(1) = H^{(1)}_1 (z_1, t_1) \tag{4}
\]

where the single argument on the left-hand side is the subscript carried by the arguments \( z_1, z_2, \) and \( t_1, t_2 \). Thus we have \( \langle H(1) \rangle = 0 \) and we write (3) as

\[
\langle H(1) H(2) \rangle = \delta(12). \tag{5}
\]

The second order Wiener-Hermite function is defined by

\[
H^{(2)}_{1122} (z_1, t_1; z_2, t_2) = H(12) = H(1)H(2) - \delta(12). \tag{6}
\]
In addition to the zeroth order function, \( H^{(0)} = 1 \), we list the other functions that we shall need:

\[
H^{(3)} = H(123) = H(1)H(2)H(3) - H(1)\delta(23) - H(2)\delta(13) - H(3)\delta(12),
\]
\[
H^{(4)} = H(1234) = H(1)H(2)H(3)H(4) - H(1)\delta(234) - H(2)\delta(134) - H(3)\delta(124) - H(4)\delta(123) +
\]
\[
- \delta(12)\delta(34) - \delta(13)\delta(24) - \delta(14)\delta(23) - \delta(23)\delta(14),
\]

By construction, the Wiener-Hermite functions are symmetric in their arguments, and satisfy statistical orthogonality,

\[
\left< H^{(m)} H^{(n)} \right> = 0 \quad \text{for} \quad m \neq n.
\]

The expansion of a random scalar function of space and time, \( f(x) = f(x; t) \), is given by

\[
f(x) = \sum_{n=0}^{\infty} f^{(n)}(x),
\]

with

\[
f^{(n)}(x) = F(x; 12...n)H(12...n).
\]

In (11), we suppress the summation notation

\[
\sum_{i_1=1}^{3} \int dx_1 \int_{i_2=1}^{3} dx_2 \ldots \sum_{i_n=1}^{3} \int dx_n
\]

that should precede the right-hand side to indicate the appropriate sums over repeated symbols. We shall refer to the symbols \( 12...n \) as the Wiener-Hermite arguments, while \( x \) in (10) and (11) will be called the real space-time argument. The kernels \( F(x; 12...n) \) are nonrandom, continuous functions symmetric in the Wiener-Hermite arguments.

3. Advection equations

The Wiener-Hermite expansion has been applied to the turbulent diffusion problem in a low-order determination of the diffusion coefficient in
a homogeneous, stationary flow (Saffman, 1969). This result, when compared to the classical result for large time diffusion (Taylor, 1921), showed that the integrated, Lagrangian correlation function is effectively given by the integrated Eulerian one-point, two-time correlation function defined in the frame moving with the mean velocity. We shall extend this result to higher orders and shall incorporate nonuniform statistics.

The expansion (10) applied to the velocity and scalar fields gives, respectively,

\[
\begin{align*}
    u_i &= \overline{u}_i + \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \cdots, \tag{12} \\
    \Gamma &= \overline{\Gamma} + \epsilon \Gamma^{(1)} + \epsilon^2 \Gamma^{(2)} + \cdots. \tag{13}
\end{align*}
\]

A perturbation ordering through the parameter \( \epsilon \) is built into these expansions. This ordering forms a basic premise on which our study is to be conducted. It is only fair to point out that another ordering choice for the velocity,

\[
    u_i = \overline{u}_i + u_i^{(1)} + \epsilon u_i^{(2)} + \epsilon^2 u_i^{(3)} + \cdots, \tag{14}
\]

may be justifiable since the lowest order random part (purely Gaussian) is not necessarily a small perturbation superimposed on a mean flow. However the procedure followed in Sec. 4 cannot work consistently for choice (14).

Inserting expansions (12) and (13) into the advection equation (1), using the notation

\[
    D_t \equiv \left( \frac{\partial}{\partial t} + \overline{u}_i \frac{\partial}{\partial r_i} \right),
\]

and retaining up to \( \epsilon^4 \) terms, we get
We now introduce the velocity and scalar kernels,

\[ u_i(n) = K_i(12...n)H(12...n) \]

and

\[ \Gamma(n) = G(12...n)H(12...n), \]  

in which the real space-time argument \( x \) is suppressed, and only the Wiener-Hermite arguments are exhibited.

When representations (17) are substituted in (16), products of Wiener-Hermite functions occur which, from the definitions given in Sect. 2, can be decomposed uniquely into sums linear in the Wiener-Hermite functions according to

\[
\begin{align*}
H(1)H(2) &= H(12) + \delta(12), \\
H(1)H(23) &= H(123) + H(2)\delta(13) + H(3)\delta(12), \\
H(1)H(234) &= H(1234) + H(34)\delta(12) + H(24)\delta(13) + \\
&\quad + H(23)\delta(14), \\
H(12)H(34) &= H(1234) + H(24)\delta(13) + H(23)\delta(14) + \\
&\quad + H(14)\delta(23) + H(13)\delta(24) + \delta(13)\delta(24) + \\
&\quad + \delta(14)\delta(23).
\end{align*}
\]

Following Meecham and Jeng (1968), the results of the above substitutions are then separated into the following statistically orthogonal parts:

\[
D_t\Gamma = -\varepsilon^2K_i(1)G_i(1) - 2\varepsilon^4K_i(12)G_i(12),
\]
The subscript \(i\) on \(\overrightarrow{r}\) and the \(G_i's\) always denotes a gradient component. The subscript \(S\) denotes symmetrization over the Wiener-Hermite arguments. The initial conditions on the \(G_i's\), according to the assumption stated at the end of Sect. 1, is that at \(t = 0\),
\[
G(1) = G(12) = G(123) = G(1234) = 0. \tag{27}
\]

4. Elimination of concentration kernels

Using the presumed smallness of \(\varepsilon\) to guide the way, it is now possible to eliminate the \(G_i's\) from Eqs. (22)-(26), and isolate a single differential equation governing the time evolution of \(\overrightarrow{r}\). Should we wish to include only the \(\varepsilon^2\) term in (22), we could drop \(G(12)\) from consideration and compute \(G(1)\) to lowest order from Eq. (23). This procedure yields, with the real space-time arguments displayed
\[
D_t G(\mathbf{r}, t; 1) = - K_i(\mathbf{r}, t; 1) \overrightarrow{r}_i (\mathbf{r}, t) \tag{28}
\]
which is of the form
\[
(\delta_t + \mathbf{u}_i \mathbf{v}_i)g(\mathbf{r}, t) = f(\mathbf{r}, t) \tag{29}
\]
with initial condition \(g(\mathbf{r}, 0) = 0\). The solution of (29) is
\[
g(\mathbf{r}, t) = \int_0^t dt' f(\mathbf{r}', t') = \int_0^t dt' f(x') \tag{30}
\]
where \( \mathbf{x}' \) is the position at time \( t' \) of the fictitious particle that follows the given mean motion of the velocity field and passes through the point \( \mathbf{x} \) at time \( t \). It is convenient in what follows to simplify the notation in (30) by introducing an advective integral operator \( S \) such that

\[
g = S(t, t') f',
\]

(31)

where \( f' = f(\mathbf{x}', t') \). Thus we write the solution of Eq. (28) as

\[
G(l) = -S(t, t') K_1'(l) \mathbf{F}'_1 .
\]

(32)

We now establish two rules regarding the operator \( S \). The expression

\[
S(t, t') f' S(t', t'') g''
\]

(33)

is a function of \( \mathbf{x}', t' \) and hence the operand of the left-hand \( S \) is evaluated at points \( \mathbf{x}', t' \) lying on the mean trajectory labeled by \( \mathbf{x}, t \). In particular, the part \( S(t', t'') g'' \) is a function of \( \mathbf{x}' \) and \( t' \), which are equally good labels for the same trajectory. Hence \( g'' \) is evaluated at points \( \mathbf{x}'', t'' \) also lying on the same trajectory. The point is that in a chain of the form \( S f S g S h \ldots \), only one trajectory is involved. Once the trajectory is defined, say by the endpoint labels \( \mathbf{x}, t \), then any other point on the trajectory is uniquely defined simply by specifying its time. While the even shorter form

\[
S f S g
\]

(34)

is unambiguous and will be used wherever possible, we shall later want to commute the \( S \)'s to the left in which case we must write (34) as

\[
S(t, t') S(t', t'') f' g'' .
\]

(35)

The second rule concerns commuting \( S \) through a gradient operator. We note that
where

\[ \frac{\partial}{\partial t_1} \delta_{i j}(t', x'') \frac{\partial f(x'')}{\partial x_j'} = \int_0^{t'} dt'' \delta_{i j}(t', x'') \frac{\partial f(x'')}{\partial x_j'} = S(t', t'') \delta_{i j}(t', x'') \frac{\partial f(x'')}{\partial x_j'} \]

which must be considered a function of the independent variables \( t' \) and \( x'' \).

The derivative on the right-hand side of (37) is carried out for fixed \( t' \) and \( t'' \). The tensor \( \delta_{i j}(t', x'') \) is completely determined by the mean velocity field.

Thus the notation defined by (31) together with the rule (36) make it possible to commute \( S \)'s all the way to the left in chain expressions like

\[ S f \nabla S g \nabla S h \ldots \]

Since we wish to incorporate at least third order correlation functions in this work, it turns out to be necessary to include the \( \epsilon^3 \) term in Eq. (22). To be consistent we should therefore include the \( \epsilon^2 \) correction in \( G(1) \). But we then need \( G(12) \) to lowest order only in Eqs. (22) and (23). Hence we shall compute \( G(12) \) from Eq. (24) to lowest order by inserting into (24) the lowest order \( G(1) \) given by Eq. (28). Integrating Eq. (24) with this substitution, we find

\[ G(12) = -S K_i(12) T_j + \frac{1}{2} S K_i(1) v_i S K_m(2) T_j + \frac{1}{2} S K_i(2) v_i S K_m(1) T_j \]  

(38)

Inserting (32) and (38) into (23) and integrating we get \( G(1) \) with its \( \epsilon^2 \) correction,
Finally we insert (38) and (39) into (22) to get

\[ \mathcal{D}_t \Gamma = A_1 + A_3 + A_4 \]  

(40)

where

\[ A_2 = e^{2K_1(1)\nu_j S_k(1)\Gamma_j} + 2e^{4K_1(12)\nu_j S_k(12)\Gamma_j}, \]  

(41)

\[ A_3 = -2e^{4K_1(1)\nu_j S_k(2)\nu_j S_k(12)\Gamma_m(1)\Gamma_m(2)} + \]  

(42)

and

\[ A_4 = e^{4K_1(1)\nu_j S_k(2)\nu_j S_k(12)\Gamma_m(1)\Gamma_m(2)} + e^{4K_1(1)\nu_j S_k(2)\nu_j S_k(12)\Gamma_m(1)\Gamma_m(2)} \]  

(43)

It is now possible to commute the S's to the left in each of

Eqs. (41), (42), and (43). After some manipulation, we find

\[ A_2 = S(t,t')d_{ij}(t,x') \left[ e^{2K_1(1)\nu_j K_m(1)} + 2e^{4K_1(12)\nu_j K_m(12)} \right] \]  

(44)

and

\[ A_3 = -2e^{4K_1(1)\nu_j S_k(2)\nu_j S_k(12)\Gamma_m(1)\Gamma_m(2)} + \]  

(45)

As elsewhere in this paper, the gradient operators \( \nabla \) operate on everything to their right in Eqs. (44) and (45). The notation

\[ \frac{\partial K_{ij}}{\partial r_{ip}} = K_{ij,p} \]  

(46)
is used in these equations. The corresponding result for A₄ will not be displayed at this point. In the above derivations we have used the fact that products of d's collapse according to

\[ d_{ip}(t,x')d_{pj}(t',x'') = d_{ij}(t,x'') \]  (47)

We can now relate the kernel combinations appearing in Eqs. (44) and (45) to the ordinary statistics of the velocity field. Let the random part of the velocity field be denoted by

\[ \tilde{u}_{\bot} = u_{\bot} - \bar{u}_{\bot} \]  (48)

The means of \( \tilde{u}_i(x)\tilde{u}_m(x') \) and \( \tilde{u}_j(x')\tilde{u}_m(x'') \), the second and third order Eulerian correlation functions, respectively, are evaluated by taking the appropriate products of (17) and using relations (18)-(21). We find

\[ \langle \tilde{u}_i\tilde{u}_m' \rangle = \varepsilon^2 K_i(1)K_m'(1) + 2\varepsilon^4 K_i(12)K_m'(12) \]  (49)

and

\[ \langle \tilde{u}_i\tilde{u}_j\tilde{u}_m'' \rangle = 2\varepsilon^4 \left[ K_i(1)K_j''(12) + K_i(12)K_j'(12)K_m''(1) + K_i(12)K_j'(12)K_m''(1) \right] \]  (50)

From Eq. (49), we also obtain

\[ \langle \tilde{u}_i\tilde{u}_n'' \rangle = \langle \tilde{u}_j\tilde{u}_n''' \rangle + \langle \tilde{u}_i\tilde{u}_n''' \rangle \]  (51)

which is evidently the Gaussian part of a fourth-order correlation function. The above results (49), (50), and (51) are correct up to terms in \( \varepsilon^4 \). Inserting these results into Eqs. (44) and (45) and reverting to a more explicit notation, we obtain

\[ A_2(x) = \int_0^t dt'd_{ij}(t,x') \langle \tilde{u}_i(x)\tilde{u}_m(x') \rangle \frac{d}{m} \langle \tilde{u}_m(x') \rangle, \]  (52)
The steps leading from (41) and (42) to (52) and (53) can also be carried out for \( A_4 \) starting from (43). We omit the lengthy details and give the final result only:

\[
A_4(x) = A_4'(x) + A_4''(x)
\]

where

\[
A_4'(x) = \int_0^t dt' \int_0^{t'} dt'' \left\{ \begin{array}{l}
\frac{d_i p(t, x')}{\partial t} \frac{d_j q(t', x'')}{\partial t''} < \tilde{u}_i(x) \tilde{u}_j(x') \frac{\partial}{\partial x} \tilde{u}_m(x'') > \\
+ \frac{d_i p(t, x'')}{\partial t} \frac{d_j q(t', x''')}{\partial t'''} < \tilde{u}_i(x) \tilde{u}_j(x') \frac{\partial}{\partial x} \tilde{u}_m(x'') > \\
+ \frac{d_i p(t, x''')}{\partial t} \frac{d_j q(t', x''')}{\partial t'''} < \tilde{u}_i(x) \tilde{u}_j(x') \frac{\partial}{\partial x} \tilde{u}_m(x'') > \\
\end{array} \right\}
\]

while \( A_4''(x) \) is the same as \( A_4'(x) \) except for the interchanges \( x \leftrightarrow x' \) and \( i \leftrightarrow j \) in the velocity correlation functions.

While the statistics involved in these expressions appear to be the 2-point, 2-time and 3-point, 3-time Eulerian correlations, they are evaluated under the integrals at points along the mean trajectory passing through \( x \). Hence, in the frame moving with the mean trajectory, the statistics are actually 1-point, 2-time and 1-point, 3-time Eulerian correlation functions.
We conclude this section by pointing out that there is another systematic way to proceed, given the ordering hypothesized in (12) and (13). We could initiate the procedure by ignoring the right-hand side of (22) and simply take

$$D_t \overline{r}(z,t) = 0$$

Eq. (56) has the solution

$$\overline{r}(z,t) = \overline{r}(\overline{u}z(z,t;0),0),$$

where \( \overline{r}(\overline{u}z(z,t;0),0) \), completely determined by \( \overline{u}_i(\overline{r},t) \), is the location at time 0 of the particle that follows the mean motion and passes through the point \( \overline{z} \) at time \( t \). This solution can then be inserted into (23) to compute \( G(1) \) explicitly to lowest order, and the entire procedure leading to Eqs. (52)–(55) can be repeated in very nearly the same way. This would lead to the same expressions except that \( \overline{r} \) would be replaced by the known function \( \overline{r}^0 \).

One more advective integral performed on \( A_2 + A_3 + A_4 \) would then give an explicit solution for \( \overline{r}(\overline{z},t) \). On the other hand, the same explicit solution is obtained by integrating Eq. (40),

$$\overline{r}(x) = \overline{r}^0(x) + S(t,t') \left[ A_2(x') + A_3(x') + A_4(x') \right],$$

and then carrying out the first two iterations of (58). However, the differential equation (40), rather than its first iterated solution, is a more suitable basis for extracting eddy coefficients, as will be seen in Sect. 5.

5. Example: Plane-parallel mean flow

A simple application of the results of Sect. 4, exhibiting an interaction between the mean flow and the velocity field statistics, is that of a plane-parallel mean flow defined by
Points on the same trajectory, \((\mathbf{r}', t')\) and \((\mathbf{r}, t)\) say, are then related by
\[
\mathbf{r}_1' = \mathbf{r}_1 + \int_t^{t'} \mathbf{i}(\mathbf{r}_2, t'') dt'',
\]
\[
\mathbf{r}_2' = \mathbf{r}_2, \quad \mathbf{r}_3' = \mathbf{r}_3.
\]

The tensor \(d_{ij}(t, x')\) has the components
\[
d_{11} = d_{22} = d_{33} = 1,

d_{21} = c(t, t'; \mathbf{r}_2) = \int_t^{t'} \frac{\partial u(\mathbf{r}_2, t'')}{\partial \mathbf{r}_2},
\]
\[
d_{ij} = 0 \quad \text{otherwise},
\]
which we write all together for this example as
\[
d_{ij}(t, x') = d_{ij}(t, t'; \mathbf{r}_2).
\]

We now assert some relations very likely satisfied by velocity correlation functions in this type of flow. It is natural to suppose that the origin of the frame in which the correlations are defined is moving with the mean flow. While it is unrealistic to assume isotropic statistics in a shear flow, it is quite reasonable to assume invariance under inversion through the origin, at least in a local sense in the neighborhood of the origin. Suppressing the time dependence, we then have for infinitesimal \(\mathbf{z}'\) and \(\mathbf{z}''\),
\[
\langle \mathbf{u}_i(0) \mathbf{u}_j(\mathbf{z}') \rangle = \langle \mathbf{u}_i(0) \mathbf{u}_j(-\mathbf{z}') \rangle,
\]
\[
\langle \mathbf{u}_i(0) \mathbf{u}_j(\mathbf{z}') \mathbf{u}_m(\mathbf{z}'') \rangle = -\langle \mathbf{u}_i(0) \mathbf{u}_j(-\mathbf{z}') \mathbf{u}_m(-\mathbf{z}'') \rangle
\]

Thus, odd derivatives of the second order function and even derivatives of the third order function should vanish at the origin.

We limit the study of this example to deriving expressions for \(A_2\)
and $A_3$. Inserting (62) into (52) and invoking inversion invariance, we find

$$A_2 = \int_0^t dt' \left[ \langle \tilde{u}_i(x) \tilde{u}_j(x') \rangle \frac{\partial^2}{\partial x_i' \partial x_j'} + c(t, t'; r_2) \langle \tilde{u}_i(x) \tilde{u}_j(x') \rangle \frac{\partial^2}{\partial x_i' \partial x_j'} \right] \tilde{f}(x') \, . \tag{64}$$

If the mixing has proceeded for a long enough duration, say long in comparison to a correlation time scale, so that $\tilde{f}(x')$ within the integral is essentially constant over the recent history during which the correlations are significantly different from zero, the argument $x'$ can be updated to current time. This updating procedure then gives effective eddy diffusion coefficients. Assuming homogeneity in the $r_1$ and $r_3$ directions, we write

$$\langle \tilde{u}_i(x) \tilde{u}_j(x') \rangle = R_{ij}(t, t'; r_2) \tag{65}$$

for the nonstationary Eulerian correlation function evaluated at the moving origin. The updated form of (64) is then

$$A_2(r, t) = D_{ij}^{(2)}(r_2, t) \frac{\partial^2}{\partial x_i' \partial x_j'} \tilde{f}(r, t) \tag{66}$$

in which the effective eddy diffusivity tensor for $A_2$ is

$$D_{ij}^{(2)}(r_2, t) = \int_0^t dt' \left[ R_{ij}(t, t'; r_2) + c(t, t'; r_2) R_{ij}(t, t'; r_2) \right] \tilde{f}(r, t) \tag{67}$$

In a similar fashion, but omitting the details, we find the updated form

$$A_3(x, t) = \left[ D_{i}^{(3)}(r_2, t) \frac{\partial}{\partial x_i} + D_{ij}^{(3)}(r_2, t) \frac{\partial^2}{\partial x_i' \partial x_j'} \right] \tilde{f}(r, t) \, , \tag{68}$$

where

$$D_{i}^{(3)}(r_2, t) = -\int_0^t dt' \int_0^{t'} dt'' \frac{\partial c(t', t''; r_2)}{\partial x_2} \langle \tilde{u}_2(x) \tilde{u}_2(x') \tilde{u}_i(x''') \tilde{u}_i(x''') \rangle \tag{69}$$

and
Various specializations of the general plane-parallel flow are easily derived from the above results. A uniform shear mean flow with homogeneous, stationary statistics at the moving origin is particularly interesting from the standpoint of simplicity. Let $\omega$ = the mean rate of shear; i.e. assume $u(r, t) = \omega r$. The quantity $D_i^{(3)}$ is then zero, and we find from the updated forms (67) and (70),

$$D_{ij}^{(3)}(r, t) = -\int_0^t \int_0^{t'} \left\{ \begin{array}{l} d_{pq}(t, t'; r) \delta_{mi}(t, t'; r) \\
\times <\hat{u}_p(x)\hat{u}_m(x')\hat{u}_j(x'')> \\
+ \left[ d_{pi}(t, t''; r) d_{mq}(t, t''; r) + d_{pj}(t, t''; r) d_{mi}(t, t''; r) \right] \times <\hat{u}_p(x)\hat{u}_m(x')\hat{u}_j(x'')> \end{array} \right\} - (70)$$

The constant diffusivity tensor in (71) is given by

$$D_{ij}^{(3)} = D_{ij}^{(2)} + D_{ij}^{(3)} - (72)$$

where

$$D_{ij}^{(2)} = \int_0^\infty dt \left[ R_{ij}(t) - \omega t R_{ij}(t) \delta_{ij} \right] - (73)$$

$$D_{ij}^{(3)} = -\int_0^t dt_1 \int_0^{t_1} dt_2 \left[ R_{mji, m}(t_1, t_2) - \omega t_2 R_{mji, m}(t_1, t_2) \delta_{ij} + 1 \right.$$

$$- \omega t_1 R_{ij, i}(t_1, t_2, t_2) + \omega t_2 R_{ij, j}(t_1, t_2) - \omega t_2 R_{ij, 2}(t_1, t_2)$$

$$+ R_{imj, m}(t_1, t_2, t_2) - \omega t_2 R_{imj, m}(t_1, t_2) \delta_{ij} -$$

$$- \omega(t_1+t_2) R_{mji, m}(t_1+t_2, t_2) \delta_{ij} + \omega^2 t_2 (t_1+t_2) R_{22j, i}(t_1+t_2, t_2) \delta_{ij} +$$

$$+ R_{mij, m}(t_1+t_2, t_2) - \omega t_2 R_{mij, m}(t_1+t_2, t_2) \delta_{ij} -$$

$$- \omega(t_1+t_2) R_{2ij, i}(t_1+t_2, t_2) + \omega^2 t_2 (t_1+t_2) R_{22j, i}(t_1+t_2, t_2) \delta_{ij} \right]. - (74)$$
The above results involve combinations of moments of the stationary Eulerian correlations

\[ R_{ij}(t) = \langle \tilde{u}_i(t)\tilde{u}_j(0) \rangle \quad (75) \]

and

\[ R_{ijm,p}(t_1,t_2) = \langle \tilde{u}_i(t_1)\tilde{u}_j(t_2) \frac{\partial}{\partial x} \tilde{u}_m(0) \rangle \quad (76) \]

6. Concluding Remarks

The general structure of the right-hand side of Eq. (40) has been revealed as an expansion involving the Eulerian one-point, n-time correlation functions. The fact that we obtain such explicit forms as (52), (53) and (55) seems to depend rather crucially on the ad hoc ordering hypothesis exhibited in Eqs. (12) and (13). Before results such as the plane parallel flow equations of Sect. 5 can be taken very seriously (results which follow rigorously from the ordering hypothesis and the assumed statistical description of the velocity field), the ordering hypothesis should be examined critically. One way to tackle this question is based on the speculation that Eqs. (52) and (53) may be correct to all orders in \( \varepsilon \), and therefore correct no matter what the statistical distribution of the velocity field may be. This is suggested because the combination of kernels that arise in (44) and (45) are precisely the combinations that occur in (49) and (50), the ordinary velocity correlation functions, when carried out to the same order in \( \varepsilon \). However, the structure of the right-hand side of Eq. (55) is probably not persistent when higher orders of \( \varepsilon \) are included. While mean products of four velocities are present in (55), they are purely Gaussian. The absence of fourth cumulants or contributions from a non-Gaussian part of the velocity distribution is contrary to what one would expect in a more nearly complete analysis.
Acknowledgements. Much of this work was carried out while the author was a participant in the 1971 Woods Hole Oceanographic Institution summer program in Geophysical Fluid Dynamics which is supported by the National Science Foundation.

References


