Notes on the 1973 Summer Study Program in GEOPHYSICAL FLUID DYNAMICS at THE WOODS HOLE OCEANOGRAPHIC INSTITUTION
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Editor's Preface

This volume contains the manuscripts of research lectures by the eleven fellows in the summer program. The lectures are the result of ten weeks of intensive effort in trying to formulate and carry out a successful research project. In some cases, the effort was successful. In other cases, the original problem is still unsolved, but a series of intermediate problems has been solved. In no case was the effort unsuccessful. It is hoped that the summer's experience will assist students and staff in future research projects. Most of us profit a great deal in our professional lives from the intense and stimulating atmosphere at Walsh Cottage.

These lectures have not been edited or reviewed in the manner appropriate for published papers. They should therefore be regarded as unpublished manuscripts, and readers who would like to quote or use the material should write directly to the authors.

Again we thank the National Science Foundation for financial support of this program.

Andrew P. Ingersoll

Malkus and Ingersoll - co-directors.
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1. Introduction

Internal gravity waves in an inviscid, stably stratified fluid incident upon a rigid boundary will experience a reflection at that boundary. Due to the anisotropic character of the wave motion, the amplitude and wavenumber of the reflected wave will be different from that of the incoming wave if the bounding wall is inclined at some angle to the vertical. (See Phillips, 1966). The angle of the wavenumber vector with respect to the vertical must be the same for both the incident and reflected waves, since this is determined solely by the frequency of the wave motion. If this angle is denoted by \( \theta \) and the angle of slope of the boundary given by \( \beta \), then the reflection condition at the boundary is given by

\[
 k \sin (\theta + \beta) = k' \sin (\theta - \beta) \tag{1.1}
\]

where \( k \) and \( k' \) are the incident and reflected wavenumber vectors, respectively, and \( k \) and \( k' \) their magnitudes.

The condition of zero normal velocity at the boundary leads to a relation between the incident and reflected wave amplitudes, \( A, A' \). This is

\[
 \frac{A}{k} = \frac{A'}{k'} \tag{1.2}
\]

From these two relations, it follows that, if the reflected wavenumber vector is parallel to the boundary, the reflected wave amplitude becomes infinite. In this case, the angle which the incident wave makes with respect to the vertical is called the critical angle for reflection, and the corresponding frequency is called the critical frequency.

In a viscous fluid, an oscillating boundary layer will be set up by the wave motion near the sloping wall. When the incident internal wave is near the critical angle, this boundary layer exhibits a rather striking instability. A train of evenly-spaced vortices is generated along the slope. The vortices grow over that half-cycle for which
fluid velocities along and normal to the plate, respectively
- pressure
- fluid density
- gravitational acceleration, directed in the negative y-coordinate direction.
- fluid kinematic viscosity
- diffusivity of the single diffusing component, e.g. heat or salt.
- the linear, stable density distribution away from the inclined plate.
- the Brunt-Väisälä frequency, assumed constant, associated with the interior density distribution.
- the density at $y = 0$

Subscripts have the usual meaning of partial differentiation with respect to the variable appearing in the subscript.

Since the plate is infinite, the flow should be similar at each station along the $\xi$-coordinate direction. This implies that $u_\xi = 0$ and that $u = 0$, so that Eqs. (2.1), (2.2) and (2.4) become

$$u_t = -p_\xi + \rho g \sin \theta + \nu \nabla^2 u_\eta$$  \hspace{1cm} (2.11)

$$\rho_\eta = -\rho g \cos \theta$$  \hspace{1cm} (2.12)

$$\rho_\xi + u_\xi = D \nabla^2 \rho$$  \hspace{1cm} (2.13)

It should be noted at this point that even when the plate is at rest in the fluid there does not exist a strict hydrostatic balance in the $\xi$-direction near the surface of the plate. Phillips (1970) and Wunsch (1970) have shown that the condition of zero normal flux at the sloping wall induces an upward velocity in a thin boundary layer near that wall. If the diffusivity $D$ is small, such as that for salt, this velocity is quite small. In the following analysis, we will neglect this small steady velocity, and so we assume that the fluid is in approximate hydrostatic balance in the rest state.

We therefore assume that the general solution can be written in the form,
where \( u', \rho', \varphi' \) represent the solutions to the time dependent part of the problem, and \( \rho_0(\xi, \eta) \) and \( \varphi_0(\xi, \eta) \) are the density and pressure distributions in the rest state. Substituting these equations into Eqs. (2.11) through (2.13), and the boundary conditions (2.5) through (2.10), we obtain equations governing the time dependent quantities. Dropping the primes, these are

\[
\begin{align*}
\rho &= \rho_0(\xi, \eta) + \rho'(\eta, t) \\
\varphi &= \varphi_0(\xi, \eta) + \varphi'(\eta, t) \\
u &= u'(\eta, t),
\end{align*}
\]

Equations (2.17) through (2.23) can be combined to give a single equation and boundary conditions for the velocity \( u \). In complex form, these are

\[
\begin{align*}
u_t &= \nu_0 \sin \theta + \nu \eta \eta \\
\eta &= -\rho_0 \cos \theta \\
\rho_t + u \rho = D \rho \eta \\
u(0,t) &= \delta \omega \cos \omega t \\
u(\eta, t) &= 0 \quad \eta \to \infty \\
\rho(\eta, t) &= 0 \quad \eta \to \infty
\end{align*}
\]

We now introduce the following nondimensionalization:

\[
\begin{align*}
(\xi, \eta) &= \delta (\xi', \eta') \\
t &= \frac{1}{\omega} t' \\
u &= \delta \omega u'.
\end{align*}
\]
We also define the Reynolds number \( \mathcal{R} \) as
\[
\mathcal{R} = \frac{\delta^2 \omega}{v},
\]
and the Prandtl number \( \alpha \) as
\[
\alpha = \frac{v}{D}.
\]
Equations (2.24) through (2.26) become, upon dropping the primes,
\[
\left\{ \left( \frac{\partial}{\partial t} - \frac{i}{\mathcal{R}} \alpha \right)(\frac{\partial}{\partial \eta} - \frac{i}{\mathcal{R}^2} \eta^2) + \frac{N^2}{\omega^2} \sin^2 \Theta \right\} u = 0 \tag{2.33}
\]
\[
u_0(0,t) = \frac{i}{\mathcal{R}} u_0(0,t) \tag{2.34}
\]
\[
u_0(\eta,t) \rightarrow 0 \text{ as } \eta \rightarrow \infty \tag{2.35}
\]
\[
u_0(\eta,t) \rightarrow 0 \text{ as } \eta \rightarrow \infty \tag{2.36}
\]

In the next section, we discuss the solutions to this system of equations.

3. Solution of the time dependent equations for the oscillating plate.

We assume a solution for \( u(\eta,t) \) of the form,
\[
u(\eta,t) = U(\eta)e^{it}, \tag{3.1}
\]
This yields the following ordinary differential system for \( U(\eta) \):
\[
U''(\eta) - i \mathcal{R} (1 + \alpha) U''(\eta) - \mathcal{R}^2 \eta \left( 1 - \frac{N^2 \sin^2 \Theta}{\omega^2} \right) U(\eta) = 0 \tag{3.2}
\]
\[
U(0) = 1 \tag{3.3}
\]
\[
U'(0) = - \frac{i}{\mathcal{R}} U''(0) \tag{3.4}
\]
\[
U(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \tag{3.5}
\]

The solution of (3.1) can be written in the form
\[
U(\eta) = C_1 e^{s_1 \eta} + C_2 e^{s_2 \eta} + C_3 e^{s_3 \eta} + C_4 e^{s_4 \eta}, \tag{3.6}
\]
where \( s_1, s_2, s_3, s_4 \) are roots of the characteristic equation
\[
s^4 - i \mathcal{R} (1 + \alpha) s^2 - \mathcal{R}^2 \eta \left( 1 - \frac{N^2 \sin^2 \Theta}{\omega^2} \right) = 0. \tag{3.7}
\]
These roots are given by
\[
1 = \pm \left( \frac{i s}{\mathcal{R}} \right) \left[ (1 + \alpha) \pm \sqrt{(1 + \alpha)^2 + 4 \alpha (K-1)} \right] \tag{3.8}
\]
where
\[
k = \frac{N^2 \sin^2 \Theta}{\omega^2}. \tag{3.9}
\]
For the case $\alpha$ finite and $K > 1$, the solution for $u(\eta,t)$, in complex form, is
\[ u(\eta,t) = e^{i(\eta \gamma + (1 - c) \eta \gamma)} e^{iK \eta} \eta \]
where
\[ \gamma = \left( \frac{R}{2} \right)^{1/2} \left[ i(1+\alpha) + \sqrt{(1+\alpha^2) + 4\alpha(K-1)} \right] \]
\[ \gamma \hat{\gamma} = \left( \frac{R}{2} \right)^{1/2} \left[ i(1+\alpha^2) + 4\alpha(K-1) - (1+\alpha^2) \right] \]
\[ \gamma = \frac{R \gamma + \gamma^3}{R(\gamma - \gamma) + (\gamma^2 - \gamma)} \]
and $\gamma$, $\gamma$, and $\gamma$ are all real.

For $\alpha$ infinite, that is, the nondiffusive case, the solution for $u(\eta,t)$ is $(K > 1)$,
\[ u(\eta,t) = e^{i(\eta \gamma + (1 - c) \eta \gamma)} e^{iK \eta} \eta \]
and the real part of this solution is
\[ \Re e \left[ u(\eta,t) \right] = e^{- \frac{\gamma}{2} \eta} \cos \left[ \left( \frac{R(K-1)}{2} \right) \eta + t \right]. \]
Also, for $\alpha$ infinite, the real part of the solution for $\rho(\eta,t)$ is
\[ \Re e \left[ \rho(\eta,t) \right] = e^{- \frac{\gamma}{2} \eta} \sin \left[ \left( \frac{R(K-1)}{2} \right) \eta + t \right] \]
In each case the solution fails at $K = 1$, that is, both the boundary condition at the plate and at infinity cannot be satisfied simultaneously.

4. Formulation of the stability problem for the time-dependent basic state.

In terms of the dimensionless variables and parameters defined by Eqs. (2.28) through (2.32), Eqs. (2.1) through (2.4) may be written as
\[ u_t + uu_x + \nu u = -p_t + F^{-2} \sin \theta \rho + \frac{1}{R} \nu^2 \]
\[ v_t + uv_x + \nu v = -p \rho + F^{-2} \cos \theta \rho + \frac{1}{R} \nu^2 \]
\[ u_x + v_y = 0 \]
\[ \rho_t + uu_x + \nu u = \frac{1}{\alpha} \nu \]
where
\[ F^{-2} = \frac{\alpha}{\sigma \omega} \]
We now let \( U(\eta, t) \), \( R(\eta, t) \) represent the basic time-dependent solutions which were found in section 3. The flow is then assumed to be perturbed by velocity perturbations \( u'(\xi, \eta, t) \), \( v'(\xi, \eta, t) \) and density perturbation \( \rho'(\xi, \eta, t) \) so that

\[
\begin{align*}
    u &= U(\eta, t) + u'(\xi, \eta, t) \\
    v &= v'(\xi, \eta, t) \\
    \rho &= \rho(\xi, \eta) + R(\eta, t) + \rho'(\xi, \eta, t)
\end{align*}
\]

(4.6)

We may also define the perturbation stream function by

\[
\psi' = \psi'_p, \quad \nu' = -\nu'_p.
\]

(4.9)

The linearized system of equations governing \( \psi' \) and \( \rho' \), after dropping the primes, is

\[
\begin{align*}
    \left[ \psi'_{\xi\xi} + U\psi'_{\eta\eta} - U_{\eta}\psi'_{\eta} \right]_{\eta} + \left[ \psi'_{\xi\xi} + U\psi'_{\eta\eta} \right]_{\xi} &= \frac{1}{\pi \alpha} \sigma^2 \psi + F^{-3} \left[ \sin \Theta \rho + \cos \Theta \rho_\xi \right] \\
    \rho_{\eta} + U_{\xi} \rho_{\eta} - \psi'_x \left( \rho + R \right)_{\eta} &= \frac{1}{\pi \alpha} \sigma^2 \rho
\end{align*}
\]

(4.10)

(4.11)

The boundary conditions are,

\[
\begin{align*}
    \psi(\xi, 0, t) &= 0 \quad (4.12) \\
    \psi_{\eta}(\xi, 0, t) &= 0 \quad (4.13) \\
    \psi(\xi, \eta, t) &\to 0 \quad \eta \to \infty \\
    \psi'_{\eta}(\xi, \eta, t) &\to 0 \quad \eta \to \infty \quad (4.14) \\
    \rho(\xi, 0, t) &= 0 \quad (4.16) \\
    \rho(\xi, \eta, t) &\to 0 \quad \eta \to \infty \quad (4.17)
\end{align*}
\]

If we assume solutions to (4.10) and (4.11) of the form

\[
\begin{align*}
    \psi &= \varphi(\eta, t) e^{i k \xi} \quad (4.18) \\
    \rho &= \rho(\eta, t) e^{i k \xi} \quad (4.19)
\end{align*}
\]

we obtain

\[
\mathcal{R} \left[ \left( \frac{\partial}{\partial t} + i k U \right) (\varphi'' - k^2 \varphi) - i k U'' \varphi \right] = \varphi'' - 2 k \varphi'' + k^2 \varphi + \mathcal{R} F^{-3} \left[ \sin \Theta \rho + i k \cos \Theta \rho \right]
\]

(4.20)

\[
\frac{\rho''}{\rho} + i k \varphi \rho = \frac{1}{\pi \alpha} \left( \rho + R \right) \rho'' + i k \varphi \left( \rho + R \right)
\]

(4.21)
where the primes now denote partial differentiation with respect to $\eta$, and where we have made the approximation $\rho_0(\xi, \eta) \approx \rho(\xi, \eta)$.

Equations (4.20) and (4.21) have coefficients which are periodic in $t$. We assume, therefore, that the solutions are of the Floquet form, and write

$$\varphi(\eta, t) = e^{\mu t} \chi(\eta, t), \quad (4.22)$$
$$\bar{\rho}(\eta, t) = e^{\mu t} \bar{J}(\eta, t), \quad (4.23)$$

in which $\mu$ is a constant and both $\chi$ and $\bar{J}$ are periodic functions of $t$. Introducing (4.22) and (4.23) into (4.20) and (4.21), we obtain

$$= \chi'' - 2k^2 \chi + \bar{k}^2 \chi + \bar{\rho} F^{-2} \left[ \sin \theta J' + i \bar{k} \cos \theta J \right]$$

$$\frac{\partial^2 \bar{F}}{\partial t^2} + \mu \bar{J} + i \bar{k} \bar{U} \bar{J} + \frac{i}{\bar{k} \alpha} (k^2 \bar{J} - \bar{J}^0) = -\bar{\rho} \bar{F} \chi' + i k \bar{F} (\rho F \bar{\rho})$$ \quad (4.24)

Clearly, the stability problem as it is posed here is quite complex. We expect that the dominant mode of instability for the physical problem of interest will be a convective instability induced by downward displacement of the isopycnics during the downward phase of the plate motion. During this phase of the motion, the vertical density gradient in the boundary layer may become negative. In the static case, we would then have instability. However, since the basic density field is oscillatory, we expect that the frequency and amplitude of the oscillation would play a role in determining the stability in the dynamic case.

Several special cases of the full stability problem currently are being considered. In particular, the cases of uniform motion of the plane and of low frequency oscillations of the plane are of interest.

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Bibliography

7. Yang, C-C 1971 Viscous effect on waves reflected nearly parallel to the boundary, Phys. of Fluids, 14: 2562-2569.

GEOMETRICAL OPTICS AND WAVES IN CURRENTS
Richard D. Desautel

Introduction

The propagation of surface gravity waves in regions of non-uniform but steady currents and the subsequent modification of the wave structure have been studied by several authors, principally Longuet-Higgins and Stewart (1960, 1961), Whitham (1962). Within this subject, the particular problem of the interaction of surface and internal waves is of special concern and has been studied by Gargett and Hughes (1972) and Phillips (1973), among others. Surface wave/internal wave interaction is interesting as a possible source of internal wave amplification. Phillips (1969) has given a review of these topics.

Research on problems of surface waves in currents generally groups around two distinct approaches. The first treats short surface
waves propagating in a spatially slowly-varying current such as the current due to a long internal wave. With this approach, Longuet-Higgins and Stewart (1960, 1961) used a perturbation analysis of the flow dynamics in the case of irrotational flow. Phillips (1973) dealt with the kinematics and energy equation for short surface/long internal wave interaction and explicitly derived the conditions producing energy blockage and hence transfer. His treatment applies to an arbitrary current \( U(x) \), and indicates a singularity at the blockage point.

The second approach uses the theory of resonant wave interactions and applies specifically to surface/internal wave interaction. This approach will not be used or discussed here (see Phillips, 1969).

This paper develops a general geometrical optics formulation for the problem of waves in a current which is then applied to a simplified problem of surface waves in an internal wave current. The theory of geometrical optics and its ability to treat linear theory singularities (caustics) in wave problems has been developed and applied recently to water wave problems, principally by Keller (e.g. 1958, 1973). This analysis was initiated by Prof. Keller's suggestion that the blockage singularity indicated by Phillips (1973) could be treated by the theory of geometric optics.

The work reported here provides the groundwork for treating the singularity by developing the geometrical optics formulation and obtaining the first order surface wave solution away from the singularity. Further the regular solution is singular in a manner completely complementary to the work by Phillips.

**Formulation of the Problem**

We begin with the exact equations of motion for an inviscid, non-rotating, incompressible stratified fluid (asterisks denote dimensional variables):

\[
\begin{align*}
\text{Momentum} & \quad \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla \rho = -\rho \mathbf{f} \quad (1a) \\
\text{Incompressibility} & \quad \left( \frac{\partial}{\partial x} + \mathbf{u} \cdot \nabla \right) \rho = 0 \quad (1b) \\
\text{Continuity} & \quad \nabla \cdot \mathbf{u} = 0 \quad (1c)
\end{align*}
\]
together with the boundary conditions appropriate to a finite-depth fluid with a free surface:

\[
\begin{align*}
\mathbf{u}^* \cdot \mathbf{r}^* &= 0 \\
\mathbf{u}^* \cdot \mathbf{v}^* &= \left( \frac{\partial}{\partial t} + \mathbf{u}^* \cdot \nabla \right) \eta^* \\
p^* &= p^*_{\text{atm}} + \text{const} \\
\end{align*}
\]

(free-surface), (2c)

In these relations,

\[
\begin{align*}
\mathbf{u}^* &= \text{vector} \\
\mathbf{r}^* &= \text{unit vector}
\end{align*}
\]

and the \(x, y\) axes are horizontal, \(z\) is vertical and positive upwards and the free surface with the fluid at rest is at \(z = 0\).

Assume the flow consists of a basic flow (current) plus a small amplitude flow: \(\mathbf{u} = \mathbf{u}_0 + \mathbf{u}^*\), \(\mathbf{p} = p_0 + \mathbf{p}^*\). In this analysis, the basic flow is given (known) and satisfies equations corresponding to (1) and (2) when (2b, 2c) are evaluated at \(\mathbf{u}^* = \mathbf{r}^*\). Basic flow quantities are denoted \(\) and added flow quantities \(\). After substituting \(\mathbf{u}^*, \mathbf{p}^*, \mathbf{r}^*, \eta^*\) into (1) and (2), and linearizing about the basic state, we obtain the linearized equations for the small amplitude flow:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla \right) \mathbf{u}^* + \mathbf{u}^* \cdot \frac{\partial \mathbf{u}_0}{\partial z} + \left( \frac{\partial}{\partial z} + \mathbf{u}_0 \cdot \nabla \right) \mathbf{u}^* + \frac{\partial \mathbf{p}^*}{\partial z} - \frac{1}{g} \frac{\partial \eta^*}{\partial t} &= 0 \\
\left( \frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla \right) \mathbf{p}^* + (\mathbf{u}_0 \cdot \nabla) \mathbf{p}^* &= 0 \\
\mathbf{u}^* \cdot \mathbf{u}^* &= 0
\end{align*}
\]

and boundary conditions

\[
\begin{align*}
\mathbf{u}^* \cdot \mathbf{v}^* &= 0 \\
\mathbf{u}^* \cdot \frac{\partial \mathbf{u}^*}{\partial z} - \frac{1}{g} \frac{\partial \eta^*}{\partial t} &= - \mathbf{v}^* \\
\mathbf{p}^* &= \eta^* \\
\end{align*}
\]

To nondimensionalize (3) and (4), we will choose a characteristic length scale \(H\) and a velocity scale \(\sqrt{gH}\). Since spatial variations of the current and hence space/time variations of the surface wave structure will be slow compared to a surface wave cycle, \(x^*\) and \(y^*\) will be
scaled by \( \beta H \) and time, \( t^* \), by \( \beta \sqrt{\frac{H}{t^*}} \), where \( \beta \) is a large parameter. Thus, the variables are scaled as follows:

\[
\begin{align*}
\tilde{x}, \tilde{y} &= \frac{x}{\beta H}, \quad \tilde{z} = \frac{z}{\beta H}, \\
\tilde{u}_x &= \frac{u_x}{\beta \sqrt{\frac{H}{t^*}}}, \quad \tilde{t} = \frac{t^*}{\beta \sqrt{\frac{H}{t^*}}}, \\
\tilde{y}_t &= \frac{y_t}{\beta \sqrt{\frac{H}{t^*}}}, \quad \tilde{p} = \frac{p}{\beta \sqrt{\frac{H}{t^*}}},
\end{align*}
\]

where \( \beta \gg 1 \), \( \beta H = L \).

With this scaling (3) and (4) become:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \hat{y}_t \cdot \nabla \right) \tilde{u}_x + \frac{\beta}{p_0} \left( \frac{\partial}{\partial x} + \hat{y}_t \cdot \nabla \right) \tilde{u}_x + (\hat{y}_t \cdot \nabla) \tilde{u}_x + \frac{\nabla \tilde{p}}{p_0} &= -\beta \frac{\nabla \tilde{h}}{p_0}, \\
\left( \frac{\partial}{\partial t} + \hat{y}_t \cdot \nabla \right) \tilde{p} + \left( \nabla \cdot \nabla \right) \tilde{p} &= 0
\end{align*}
\]

and, for future reference, the \( z \)-momentum equation for the basic flow is:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \hat{y}_t \cdot \nabla \right) \tilde{u}_x + \frac{\beta}{p_0} \frac{\partial \tilde{p}}{\partial x} &= -\beta \\
\beta \left( \nabla \cdot \nabla + \frac{\partial \tilde{y}_t}{\partial x} \right) \cdot \nabla \tilde{h} + (\nabla \cdot \nabla) \tilde{h} &= (\nabla \cdot \nabla) \tilde{h} + (\nabla \cdot \nabla) \tilde{h}
\end{align*}
\]

In these equations

\[
\begin{align*}
\tilde{\gamma} &= \frac{\partial}{\partial x} + \hat{y}_t \cdot \nabla, \\
\tilde{\gamma}_h &= \frac{\partial}{\partial x} + \hat{y}_t \cdot \nabla
\end{align*}
\]

We seek small amplitude wave solutions to (5) and (6) for a specified basic flow.

**Asymptotic Solution of Geometrical Optics**

To reduce complexity, we now consider two-dimensional flow problems in \( x \) the horizontal coordinate and \( z \) the vertical coordinate. Then

\[
\hat{y}_t = \hat{u}_x + \hat{u}_z \hat{h}
\]

Geometrical optics theory assumes that (5), (6) have an asymptotic solution for large \( \beta \), of the form
where \( s \), the phase function \( = s(x,t) \) and the amplitude functions are functions of \( (x,z,t) \) expanded in inverse powers of \( (i/\beta) \):

\[
U = u + \frac{\mathcal{U}}{\iota \beta} + \frac{(\mathcal{U})^2}{(\iota \beta)^2} + \cdots \\
W = w + \frac{\mathcal{W}}{\iota \beta} + \frac{(\mathcal{W})^2}{(\iota \beta)^2} + \cdots \\
P = p + \frac{\mathcal{P}}{\iota \beta} + \frac{(\mathcal{P})^2}{(\iota \beta)^2} + \cdots \\
R = \rho + \frac{\mathcal{R}}{\iota \beta} + \frac{(\mathcal{R})^2}{(\iota \beta)^2} + \cdots \\
\Phi = \gamma + \frac{\mathcal{\Phi}}{\iota \beta} + \frac{(\mathcal{\Phi})^2}{(\iota \beta)^2} + \cdots 
\]

The procedure consists of substituting (7), (8) and the basic current

\[
y = u_0 + \mathcal{K}(i w_0) \]

into (5) and (6), grouping the resulting terms in each equation into coefficients of each power of \( \beta \), and then setting each coefficient in each equation equal to zero. To each order in \( \beta \), four equations are determined in the 2-D case: \( x, z \) momentum equations, and incompressibility and continuity equations. There are also three boundary conditions determined. The lowest order equations, \( O(\beta) \), are:

\[
\begin{align*}
0(\beta) \\
\mu u + \frac{s x p}{\rho_0} + u_0 w + w_0 u_0 + \frac{p}{\rho_0} - w_0 u_0 = 0 \\
\mu w + w_0 w_0 + w_0 w - \frac{p}{\rho_0} - \frac{p}{\rho_0} (1-w_0 w_0) = 0 \\
\mu p + w_0 p_0 + p_0 w = 0 \\
s_x u + w_0 = 0
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
w &= 0 & \text{at } z = 0 \\
(u - w_0) n - m &= 0 & \text{at } z = \eta_0 \\
p + \rho_0 n &= 0 & \text{at } z = \eta_0
\end{align*}
\]
In these equations
\[ \mu = \frac{A}{U_0 S_x + S_t} \]
and
\[ D = \frac{1}{H} \]

In (9), \( u \) and \( p \) can be expressed in terms of \( W \) and \( p \), thus determining two coupled equations in \( W \) and \( p \) and their derivatives:

\[
\frac{w_{x}^{2} p_{o} w_{x}^{2} + a_{1} w_{e} + a_{2} w - w_{o} p_{x} + a_{3} p_{x} = 0}{w_{o}^{2} p_{o} w_{o}^{2} + a_{4} w_{e} + a_{5} w + a_{6} p_{o} - \frac{a}{p_{o}} p = 0} \tag{11a}
\]

and

\[
u = -\frac{w_{x}^{2}}{s_{x}}
\]

\[p = \frac{w_{o}^{2} p_{o}}{1 - w_{o}^{2} w_{o}^{2}} \left[ a_{x} + (w + w_{o} + w_{x} + w_{o} w_{x}) \right] \tag{11c}
\]

The coefficients \( a_{1} \) through \( a_{6} \) are somewhat lengthy functions of the basic flow variables, their derivatives, and \( s_{t}, s_{x}, \mu \). Expressions for \( a_{1} \) through \( a_{6} \) are given in the appendix.

The second order equations, 0(1), are:

\[
\begin{align*}
\mu w_{x} + \frac{w_{x}^{2}}{r_{o}} + w_{o} w_{e} + u_{e} w_{x} + \frac{\mu}{r_{o}} + u_{e} + \frac{w}{r_{o}} w_{x} u_{x} + \frac{p}{r_{o}} (w_{o} u_{o}) + \frac{p}{r_{o}} (u_{o} u_{o}) = 0 \tag{12a}
\end{align*}
\]

\[
\begin{align*}
\mu w_{x} + w_{o} w_{o} + w_{o} w_{x} - \frac{p}{r_{o}} (1 - w_{e} w_{e}) + w_{e} u_{x} w_{x} u_{x} + \frac{p}{r_{o}} (w_{o} u_{o}) = 0 \tag{12b}
\end{align*}
\]

\[
\begin{align*}
\mu p_{x} + w_{o} p_{o} + w_{o} p_{o} + p_{o} + u_{e} + a_{e} p_{o} = 0 \tag{12c}
\end{align*}
\]

and the boundary conditions are

\[
\begin{align*}
W_{x} &= 0 \tag{13a}
\end{align*}
\]

\[
\begin{align*}
\mu - w_{e} &= 0 \tag{13b}
\end{align*}
\]

\[
\begin{align*}
\eta_{0} = 0 \tag{13c}
\end{align*}
\]

As with the first order equations, \( u_{e} \) and \( p_{e} \) can be eliminated from (12), yielding two coupled equations for \( W_{e} \, p_{e} \):

\[
\begin{align*}
\mu w_{e}^{2} w_{e}^{2} + a_{1} w_{e} + a_{2} w_{e} - w_{o} p_{e} + a_{3} p_{x} = a_{1} \tag{14a}
\end{align*}
\]

\[
\begin{align*}
\mu w_{x}^{2} + a_{4} w_{e} + a_{5} w + a_{6} p_{o} - \frac{a}{p_{o}} p = a_{g} \tag{14b}
\end{align*}
\]

and we also have

\[
\begin{align*}
u_{e} &= -\frac{w_{x}^{2}}{s_{x}} \tag{14c}
\end{align*}
\]

\[
\begin{align*}
p_{e} = \frac{w_{o}^{2} p_{o}}{1 - w_{o}^{2} w_{o}^{2}} \left[ a_{x} + (w + w_{o} + w_{x} + w_{o} w_{x}) \right] \tag{14d}
\end{align*}
\]
In (14), $a_1$ through $a_6$ are the same coefficients as in (11) and $a_7$, $a_8$ are functions of the basic flow and first order variables, their derivatives, and $s_t$, $s_x$, $\mu$. Coefficients $a_1$ through $a_8$ are given explicitly in the appendix. Equations (11) with (10), and (14) with (13) are the first and second order solutions in the assumed solution form for the linearized equations of motion (5) and boundary conditions (6). The usual procedure at this point is to assume in (11) and (10) that $w$, $p$ solutions have the forms

$$w = A(x,t) \phi$$
$$p = A(x,t) \theta,$$

then solve (11) and (10) for $\phi$ and $\theta$. $A(x,t)$ is then determined by the second order equations (14) and (13) by first manipulating (14) and (13) to eliminate second order variables, then substituting $A\phi$ and $A\theta$ for $w$ and $p$. This yields a partial differential equation for $A$ which is then transformed into an ordinary differential equation for $A$ along a characteristic by the method of characteristics.

This procedure will not be pursued with these general equations because of the complexity of the coefficients in (11) and (14), and the fact that both sets are second order in $z$ derivatives.

Instead, we now consider two simple examples by which we can check our formulation through the first order solutions. Then we will investigate the problem of prime concern: surface waves propagating in an internal wave current.

Examples

1. Surface Waves in Zero Current

For the case of zero current and constant density,

$$u_0 = w_0 = \eta_0 = 0, \quad \rho = 1.$$

The first order equations (11) and (10) reduce to

$$s_t w - s_x \rho = 0 \quad (15a)$$
$$s_x \rho - s_t w = 0 \quad (15b)$$
$$u = -\frac{1}{s_z} w \quad (15c)$$
$$p = -\rho \eta + s_z wr \quad (15d)$$

and boundary conditions
The condition (16c) in view of (5d) becomes
\[ p = \eta \quad \text{at} \quad z = 0 \]  
(16d)

With
\[
\begin{align*}
\sigma_t & = -\omega \\
\sigma_x & = \kappa \\
\mu & = \eta \sigma_x + s_t = -\omega 
\end{align*}
\]
(17a) (17b)

we eliminate \( p \) from (15a,b) to obtain
\[
\begin{align*}
\omega_{zz} - k^2 \omega & = 0 \\
p & = -\frac{\omega}{k^2} \omega_z 
\end{align*}
\]
(18a) (18b)

In terms of the surface displacement amplitude, \( \eta \), the solutions to (18), (15c,d), and (16a,b) are precisely the amplitude function solutions of classical linear surface gravity waves:
\[
\begin{align*}
\omega & = \frac{\omega \sinh k(z + 0)}{\sinh k(0)} \\
p & = \frac{\omega}{k} \frac{\eta \cosh k(z + 0)}{\sinh k(0)} \\
u & = \omega \eta \frac{\cosh k(z + 0)}{\sinh k(0)} \\
p & = 0 
\end{align*}
\]
(19)

Finally, applying boundary condition (16d) we get the usual dispersion solution
\[
\omega^2 = k \tanh (kD). 
\]
(20)

2. Internal Waves in Zero Current

Suppose we have zero current, \( u_z = u_x = 0 \), and a density stratification in the basic state specified by a buoyancy frequency:
\[ \rho_0 = \rho_0 N^2 \]
(21)

where \( N = N(z) \), nondimensional buoyancy frequency. Substituting the zero current and \( \mu = s_t \) into (11), (10) we obtain
\[
\begin{align*}
(r_0 + s_t + px)\omega - s_t p & = 0 \\
\frac{s_t}{s_x} \omega - \frac{1}{\sigma_z} s_t p & = 0
\end{align*}
\]
(22a) (22b)
with boundary conditions

\[ w = 0 \quad \text{at } z = -D \]
\[ w = \frac{s_t}{s_z} \gamma \quad \text{at } z = 0 \]
\[ p = -p_0 \gamma \quad \text{at } z = 0 \]

As before, (23c) becomes

\[ p = n \quad \text{at } z = 0. \]

With the assertion that in the ocean \[ \frac{d}{dx} \left( \frac{1}{\rho} \right) \sim 0, \] \( p \) can be eliminated in (22a,b) to give, with \( s_t = -\omega, \ s_z = \rho \): \[
\frac{w z}{\omega^2} + \left( \frac{n^2 \omega^2}{\omega^2} \right) \frac{w}{z} = 0
\]

(24)

Also, boundary conditions (23b) and (23d) imply

\[ p = \frac{nu}{\omega} \quad \text{at } z = 0, \]

so that from (22b) the \( z = 0 \) condition becomes

\[ \omega^2 w_z - \frac{\partial}{\partial z} \left( \frac{nu}{\omega} w \right) = 0 \quad \text{at } z = 0 \]

(25)

and we still have

\[ \omega w = 0 \quad \text{at } z = -D. \]

Equations (24), (25a) and (23a) constitute the standard internal wave problem (e.g. Phillips, 1969).

Thus, in these two zero-current examples (one constant density, one stratified), we have recovered the well-known linear theory results. The general formulation presented here is valid for an inviscid, incompressible, nonrotating fluid. Irrotationality is not required of either the basic flow or the added flow.

Rather than pursue these examples further we now turn to the primary problem of interest.

Surface Waves in an Internal Wave Current

To make the problem tractable, we should choose the simplest internal wave current so as to reduce the complexity of (11), (10), and (14), (13) to a useful minimum. Fortunately, an appropriately simple internal wave current exists, even if it is somewhat idealized. Some discussion is needed concerning this current.
We take the case of the lowest mode internal wave and base our discussion on Phillips (1969). For this wave, the fluid is nearly homogeneous ($p_o = \text{constant}$) above the wave interface, and the motion away from the interface is irrotational. The solution in the region above the interface is available in terms of the velocity potential, and thus the $u_o, w_o$ velocity components.

We will not need the explicit form of $u_d$ and $w_c$ to carry out much of the analysis. However, the first and second order equations will be made tractable with certain approximations for $w_o, u_o$, and their derivatives. We can use these approximations if we consider the depth to the internal wave interface to be much greater than the depth of the flow field associated with the surface waves. In this case, from the solution for $p_o$

(Phillips, 1969, p.167)

$$u_o \sim \frac{U_o \cosh(k_i h)}{\sinh(k_i h)} e^{i(k_i x - w_i t)}$$
$$w_o \sim -i \frac{U_o \sinh(k_i h)}{\sinh(k_i h)} e^{i(k_i x - w_i t)}$$

where these are the forms resulting after nondimensionalizing $u_o^* = U_o$ and $w_o^* = w_o$, and $k_i, \omega_i$ are wavenumber, frequency of the internal wave. Here $h \equiv h_i/\beta$, depth to the wave interface. Then for $\left(\frac{k_i h_i}{\beta}\right) \ll 1$, we have the approximations from (26):

$$u_o \sim \frac{U_o}{\sinh(k_i h)} ; w_o \sim 0$$
$$u_{ox} \sim k_i u_o << u_o ; w_{ox} \sim k_i w_o << w_o \sim 0$$
$$u_{ox} \sim k_i u_o << w_o \sim 0 ; w_{ox} \sim k_i u_o << w_o \sim 0$$

and

Further, we assert $w_{ox}$ is negligible compared to $u$ because

$$\frac{u_{ox}}{w_{ox}} \sim \frac{u_{ox} - \omega}{k_i u_o} = \frac{k_i}{k_i} - \frac{k_i}{k_i} \frac{\omega}{u_o} = \frac{k_i}{k_i} (1 - \frac{\omega}{u_o}).$$

For the current we shall consider

$$\frac{\omega}{U_o} = 0(1)$$

and thus, since $k_i \gg k_i, \frac{\omega}{u_o} \gg 1$.

From (27) then, we make the approximations that only $u_o, u_{ox}$ and $w_{ox}$ are nonzero, and further

$$u \gg w_{ox}.$$
It will become apparent that the $U_0x$ terms in the first and second order equations are the key ones in the physics of the problem.

Under these approximations, the first order equations (11), (10) become

\[
\begin{align*}
\mu^2\omega - \mu \frac{p}{x} &= 0 \\
\frac{\partial^2 w}{\partial x^2} - s_x \omega\rho &= 0 \\
\omega &= -\frac{w}{s_x} \\
p &= \mu \omega - p_x
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
w &= 0 \quad \text{at} \quad z = D \\
\mu \eta - \omega &= 0 \quad \text{at} \quad z = 0
\end{align*}
\]

where $D = D/H$ as before, and as in the two previous examples $p_{0x} = -1$ from the basic flow momentum equation.

It will be noted that (28), (29) are formally identical to corresponding equations in (15), (16) with $\mu = U_0s_x + s_t$ in the former replacing $s_t$ in the earlier equations. The solutions of (28), (29) have the same formal similarity to (19) with $\mu$ replacing $s_t = -\omega$:

\[
\begin{align*}
w &= \eta(x,t) \left[ \frac{u \sinh k(z + b)}{\sinh(kb)} \right] = \eta(x,t) \phi \\
p &= \eta(x,t) \left[ \frac{u \cosh k(z + b)}{\sinh(kb)} \right] = \eta(x,t) \Theta \\
u &= \eta(x,t) \left[ -\frac{\mu \cosh k(z + b)}{\sinh(kb)} \right] = \eta(x,t) \left[ -\frac{k \theta}{\sinh(kb)} \right] \\
p &= 0
\end{align*}
\]

These solutions come from decoupling (28a,b) into

\[
\begin{align*}
\omega^2 - s_x^2 \omega &= 0 \\
p &= \frac{\partial^2 w}{\partial x^2}
\end{align*}
\]

setting $s_x = k$ and solving first (31a) for $w$. From (30b) and the boundary conditions (29c) we get the dispersion relation

\[
\omega^2 = \kappa \tanh(kb)
\]

which also bears formal resemblance to (20), with $\mu = s_x = -\omega$.

In (30) and (32) $\mu$, $k$, $\omega$ are in general functions of $x$.  


We now go to the second order equations (14) and (13) which will be used to determine $\eta(x,t)$ in the first order solution (30). The second order equations reduce to

\begin{align}
\eta_{ttx} - \eta_{xt} &= \mu \left( \eta_t + u_0 \eta_x \right) \\
(33a)
\eta_{ttx} - \eta_{xt} &= \mu \left( \eta_t + u_0 \eta_x \right)
\end{align}

with boundary conditions

\begin{align}
\eta_t = 0 \\
\eta_t - \eta_x &= - (\eta_t + u_0 \eta_x) \quad \text{at} \ z = 0 \\
\eta_t = \eta_x \\
(34a)
\eta_t - \eta_x &= - (\eta_t + u_0 \eta_x) \quad \text{at} \ z = 0
\end{align}

where we have again that $\rho = 1$. We eliminate $\rho$ between (33a, b) to obtain

\begin{align}
\eta_{ttx} - \eta_{xt} &= \alpha(x,t) \\
(35a)
\eta_{ttx} - \eta_{xt} &= \alpha(x,t)
\end{align}

and

\begin{align}
\rho_t - \rho_x &= \frac{\sigma_x^2}{\sigma_t^2} (\eta_x + u_0 \eta_x) - \frac{1}{\sigma_x^2} (\eta_t + u_0 \eta_x + u_0 \eta_x + u_0 \eta u)
\end{align}

(35b)

where

\begin{align}
\alpha(x,t) &= -\eta_x + \frac{\sigma_x^2}{\sigma_t^2} (\eta_x + u_0 \eta_x) - \frac{1}{\sigma_x^2} (\eta_t + u_0 \eta_x + u_0 \eta x + u_0 \eta u)
\end{align}

(36)

All the above second order equations are similar in form for $\omega, \rho, u, \eta$, as the corresponding first order equations in $\omega, \rho, u, \eta$. The difference in form between the two sets is the occurrence of basic current and first order variables in additional terms in the second order equations such as (33a, b) and (35a). These terms make the equations nonhomogeneous as opposed to the corresponding first order equations which are homogeneous.

The next step is the key one of determining $\eta(x,t)$ by suitable manipulation of (35a). The solution for $\eta(x,t)$ emerges from the integration of (35a) in $x$ from $-\infty$ to $0$. The steps will be outlined here in some detail.

We multiply (35a) through by $\omega$ and integrate in $z$:

\begin{align}
\int_{-\infty}^{0} \omega \eta_{ttx} \, dz - \int_{-\infty}^{0} \omega \eta_{xt} \, dz = \int_{-\infty}^{0} \omega \alpha \, dz.
\end{align}

(37)

The first integral is integrated by parts twice and (37) becomes

\begin{align}
\left[ \omega \eta_t \right]_{-\infty}^{0} - \int_{-\infty}^{0} \omega \eta_{tt} \, dz + \int_{-\infty}^{0} \omega \eta_x \, dz = \int_{-\infty}^{0} \omega \alpha \, dz
\end{align}

or by (29a), (31a) and (39a),
We also want to express (38) totally in terms of basic current and first order variables. To eliminate the second order quantities on the left-hand side of (38), we proceed as follows: The solution for \( \mathbf{u} \) (30a) indicates \((s_x = k)\)

\[
\begin{align*}
(w_{e_0})_e = \eta \mathbf{u}, \\
(w_{e_0})_e = \frac{\eta \mathbf{A}}{\tanh AD} \\
\text{so that}
\end{align*}
\]

From (33b), \[ w_{e_0} = \left[ \frac{\eta A}{A_k} - \frac{\mathbf{A}_k}{A_k} (p_x + u_k + u_0 - u_0) \right] \]

Also, from (32) \( \tanh (k0) = \frac{A_k}{A_k} \). Substituting \( w_{e_0}, w_i, \) and \( \tanh (k0) \) into (39), we find the \( p_i \) terms drop out and we are left with

\[
\begin{align*}
[w_{e_0} - w_i]_{e_0} = \eta \mathbf{u} \left[ \frac{\mathbf{A}_k}{A_k} (p_x + u_k + u_0) - u_0 \right]
\end{align*}
\]

From the first order solutions, (30), we note in general

\[
\begin{align*}
w &= \eta \phi, \\
\mathbf{p} &= \eta \mathbf{u}, \\
\mathbf{u} &= \mathbf{u} (-\frac{\mathbf{A}}{A_k}) \}
\end{align*}
\]

and at \( z = 0 \),

\[
\begin{align*}
\phi &= \mathbf{u}, \\
\theta &= \frac{\mathbf{A}}{A_k} \cdot
\end{align*}
\]

Substituting (41) and (42) into (40), we obtain after some rearrangement,

Looking at the right-hand side of (38) \((s_x = k)\),

\[
\begin{align*}
I = \int_0^\infty \mathbf{w} \, d\mathbf{z} = \int_0^\infty \mathbf{w} \, d\mathbf{z} + \int_0^\infty \mathbf{w} \, d\mathbf{z}
\end{align*}
\]

Substituting (41) into (44) yields, after some rearrangement,

Now, from (30a) \[ \int_0^\infty \phi \, d\mathbf{A} = \frac{\mathbf{u}}{A_k} \int_0^\infty \mathbf{A}_k \mathbf{A}_k \, d\mathbf{z} \]

Evaluating this we find \[ \int_0^\infty \phi \, d\mathbf{A} = \mathbf{u} \mathbf{B} \]

where \( B = \frac{1}{2} \left[ \cosh (\mathbf{k}0) - \frac{A_k}{\sinh^2 (\mathbf{k}0)} \right] \).

We note that in the deep water limit \((kD \text{ large})\),

\[
\begin{align*}
B = \frac{1}{2} \mathbf{u}
\end{align*}
\]

Thus, \[ \int_0^\infty \mathbf{w} \, d\mathbf{z} = \eta \mathbf{u} \mathbf{B} \left[ 2k \mathbf{n}_e - u_k \frac{A_k}{A_k} \right] \]

Putting the left-hand and right-hand sides of (38) together, we obtain
after some rearrangement the following partial differential equation

$$\frac{\partial^2\eta}{\partial t^2} + \left[ \frac{1}{2}\frac{\partial u_0}{\partial x} - \frac{1}{2}\frac{\partial^2 u_0}{\partial x^2} \right] \eta + \left[ \frac{1}{2}\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{2}\frac{\partial u_0}{\partial x} \right] \eta = 0 \quad (47)$$

By expressing $\eta(x,t)$ as a function of a single variable, (47) will become an ordinary differential equation for the amplitude $\eta$ whose solution will complete the first order solution.

Before doing this, however, we shall go to the deep water limit in (47). The deep water limit is justified on the same basis as the approximations of the basic current (27) were made. For the problem of short, small amplitude surface waves,

$$k^2 \sim \frac{k^*}{b} \sim 0.1 \quad (\lambda \sim 1 \text{ to } 10 \text{ meters}), \text{ and } k \sim \beta H^*.$$

Thus

$$H = \beta H^* \frac{dH}{dH} = \beta \lambda^* \lambda \gg 1 \text{ since } \beta, \lambda \gg 1.$$

The deep water limit by (46) and (32) reduces (47) to

$$(1 - 2u_0)\eta_x - 2u_0 \eta_t + u_{0x} \eta_x (\frac{k^*}{b} - 1) \eta = 0, \quad (48)$$

We now convert derivatives of $\eta$. By the method of characteristics (Courant and Hilbert), the dispersion relation (32) is reduced to the following system of characteristic equations:

$$\frac{d\xi}{d\lambda} = \lambda; \quad \frac{dx}{d\lambda} = \lambda \eta_q = \lambda \frac{d\eta}{d\lambda}; \quad \frac{dt}{d\lambda} = -\lambda \frac{d\omega}{d\lambda}; \quad \frac{d\eta}{d\lambda} = -\lambda(\omega - \lambda \eta_q). \quad (49)$$

Since the coefficients of $\eta_x, \eta_t, \eta$ in (48) are functions of $x$, we will use (49) simply to express $\eta$ in terms of $\eta_x$:

$$\eta_x = \frac{d\eta}{d\lambda} = \frac{d\eta}{dx} \frac{dx}{d\lambda} = \lambda \frac{d\eta}{dx} \eta_x.$$  

Hence, (48) becomes an ordinary differential equation for $\eta_x$:

$$\left[ (1 - 2u_0)u_0 - 2u_0 \eta_q \right] \eta_x = -u_{0x} (\frac{k^*}{b} - 1) \eta.$$  

We note in (50) that $u_{0x} = \frac{du_0}{dx} dx = du_0$. Separating variables and integrating in (50) gives:

$$\eta(x) \eta(x_0) = \exp \left\{ \int \frac{-u_0 (\frac{k^*}{b} - 1) du_0}{u_0 (\frac{k^*}{b} - 1)} \right\}. \quad (51)$$

Equation (51) gives the solution $\eta(x)$ in terms of an integral. It is obvious that the integrand will be singular when the denominator goes to zero. To investigate this possibility and tie it in with the blockage conditions shown by Phillips (1973), we define wave phase and group velocities and express the integrand in terms of these.
Appendix

The expressions for coefficients $a_1$ through $a_6$ in (11) are:

\[
\begin{align*}
    a_1 &= u_0 \left[ \frac{\rho}{\rho_0} (a_{12})^2 + \frac{\mu}{\rho_0} + A \right] \\
    a_2 &= \left( \frac{\rho}{\rho_0} \right)^2 (P_0 + P_0) \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \\
    a_3 &= \frac{\rho}{\rho_0} \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \\
    a_4 &= \frac{\rho}{\rho_0} \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \\
    a_5 &= \frac{\rho}{\rho_0} \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \\
    a_6 &= \frac{\rho}{\rho_0} \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right]
\end{align*}
\]

The expressions for $a_7$ and $a_8$ in (14) are:

\[
\begin{align*}
    a_7 &= -\left( a_2 + a_3 \right) \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \\
    a_8 &= -\left( a_2 + a_3 \right) \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right]
\end{align*}
\]

and

\[
\begin{align*}
    a_8 &= \frac{\rho}{\rho_0} \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right] \\
    a_7 &= \frac{\rho}{\rho_0} \left[ \frac{\mu}{\rho_0} + \frac{\rho}{\rho_0} \right]
\end{align*}
\]

In these equations,

\[
\begin{align*}
    A &= \frac{\rho}{\rho_0} + \frac{\rho}{\rho_0} \\
    M &= \frac{\rho}{\rho_0} + \frac{\rho}{\rho_0}
\end{align*}
\]

References

1. Introduction

In certain geophysical and astrophysical cases convection occurs in the presence of a phase change in the convecting fluid. A change of phase provides additional parameters to the classical convection problem, since both the latent heat and the density difference between phases are available as sources of instability. The effects of phase change on the marginal stability problem have been described by Busse and Schubert (1970), Schubert and Turcotte (1971), and Peltier (1973). Their results have shown that the phase change may cause the onset of convection to occur at a different Rayleigh number than in an ordinary fluid, depending upon the physical arrangement of the system and the properties of the fluid.

The following is a description of an experiment performed at the 1973 Geophysical Fluid Dynamics summer program to investigate the effect of a phase change on simple Rayleigh-Bénard convection. A brief review of the relevant parameters and the marginal stability problem will be given to serve as an introduction to the experiment.

2. Marginal stability and relevant parameters

A horizontal layer of fluid of thickness $d$ will be considered, in which the temperature at the bottom ($z = z_1$) is $T_1$ and at the top ($z = z_2$) is $T_2$. The fluid consists of two phases which coexist at $z = 0$ with certain values of pressure $P$ and temperature $T$. The slope of the phase change boundary is given by the Clausius-Clapeyron relation

$$(dP/dT)_c = \frac{q}{\rho_1 - \rho_2} > 0,$$

where $q$ is the latent heat of transformation, $\rho_1$ and $\rho_2$ are the densities of the two phases, and $\Delta \rho$ is the difference of the densities. It will be assumed that the denser phase corresponds to a higher pressure at a given temperature, that $\Delta \rho/\rho \ll 1$, and that $q > 0$ so that $\gamma = q \rho_1 - \rho_2 / \Delta \rho > 0$.

In order to facilitate description of the physical processes it will be convenient to define some nondimensional parameters. The ordinary thermal Rayleigh number is $R_T = \alpha d$, where $\beta = \frac{T_2 - T_1}{d}$, $\alpha$ is the
coefficient of thermal expansion, $\kappa$ and $\gamma$ are the molecular diffusivities of heat and momentum, and $g$ is the acceleration of gravity. A similar parameter in which the temperature difference of the boundaries is replaced by the characteristic temperature of the phase change is

$$R_q = \frac{\alpha g d^2 q / \kappa}{\gamma K}.$$ 

The ratio between the fractional change of density due to phase change to that due to temperature change is

$$S = \frac{\Delta \rho / \rho}{\alpha d (q^2 - \beta)}.$$ 

so that the parameter which corresponds to $R_T$ and $R_q$ is

$$SR_T = R_{\Delta \rho}.$$ 

The quantity $R_T$ gives a comparison between the production of energy by the imposed boundary conditions ($T_1 > T_2$) and its dissipation by diffusion. When $R_T$ is increased to some critical value $R_{T_C}$, the effects of diffusion are overcome and the point of marginal stability is reached at which any further increase in $R_T$ would cause convective motion. As long as $T_1 > T_2$ an increase in $R_T$ is always destabilizing, that is, an increase in $R_T$ always brings the fluid closer to the point of marginal stability when $R_T < R_{T_C}$.

Similarly, increasing $R_{\Delta \rho}$ is also always a destabilizing influence. When the heavy fluid is on the top it is obviously so, since the basic state is gravitationally unstable. When the heavy fluid is on the bottom the phase boundary $\epsilon = \eta$, $\eta < 1$, moves down due to a positive temperature fluctuation, since the light phase corresponds to a higher temperature at a given pressure. Because vertical velocity $\omega$ and temperature $T$ are positively correlated, $\omega > 0$ and $\eta < 0$ occur at the same position. So because of $\Delta \rho$ gravity tends to aid the upward vertical motion by opposing the distortion of the interface.

In contrast to the above two parameters the effect of increasing $R_q$ depends upon whether the heavy fluid is on the top or bottom. Latent heat is released when the fluid goes from the light phase to the heavy one. So with the heavy fluid on top, latent heat is being released where $\omega > 0$ and $T > 0$, thus $q$ is a destabilizing effect. When the light fluid is on top the opposite is true, and $q$ represents a stabilizing effect.
The marginal stability analysis proceeds in a similar manner to that for a fluid without phase change. The effect of the phase change can be isolated by writing the specific heat and expansion coefficient in the form

\[ c_p = c_{p0} + q_0 \frac{\partial f}{\partial T} \delta(f) \]

\[ -\frac{1}{\rho_0} \left( \frac{\partial p}{\partial T} \right)_p = \alpha + \frac{\partial p}{\partial T} \frac{\partial f}{\partial f} \delta(f) \]

where \( f(p, T) = 0 \) defines the phase change boundary and \( \gamma = \frac{\partial f}{\partial T} \frac{\partial f}{\partial p} \).

When \( d, d/k, \) and \( q/c_p \) are used as scales for length, time, and temperature the stationary perturbation equations of motion and energy can be reduced to

\[ \ddot{\omega} = \frac{\partial^2}{\partial z^2} \omega + \left( S \delta(z) + I \right) R_0 \frac{\partial^2}{\partial z^2} \theta \]

\[ c = R_0 \frac{\partial^2}{\partial z^2} \theta + (R_0 \alpha_{12} \delta(z) + R_T) \omega \]

where \( \alpha_{12} = \pm 1 \) for \( \frac{q_0}{\alpha} - \gamma \leq 0 \) i.e. \( \alpha_{12} = 1 \) when heavy phase is on the top and \( \alpha_{12} = -1 \) when heavy phase is on the bottom. It can be seen that in each layer these equations reduce to those for an ordinary fluid. The term \( S \delta(z) \) represents the effect of \( \Delta \rho \) at the boundary, and \( R_0 \alpha_{12} \delta(z) \) represents the latent heat effect.

The solution is obtained by solving the equations in each layer and matching the solutions at the interface. Matching conditions (in addition to the obvious kinematic ones) are obtained by integrating the equations w.r.t. \( z \), the first relating \( \Delta \rho \eta \) to the perturbation pressure and the second relating \( w \) to \( \frac{\partial \theta}{\partial z} \).

In the case of free-free boundary conditions

\[ w = \frac{\partial \eta}{\partial z} = \theta = 0 \text{ at } z = z_1, z_2 \]

the above problem has been solved for the case in which the heavy phase is on the bottom and \( z_1 = z_2 \) (Schubert and Turcotte). The results of these calculations are summarized in the following figure, showing \( \frac{R_T}{R_q} \) vs. \( R_q \) at different values of \( S \). We see that \( R_q \) is a stabilizing effect and \( S \) is a destabilizing one. In the limit \( S \rightarrow \infty \) the critical state is reached at \( \frac{R_T}{R_q} \), which is only slightly different from \( \frac{R_T}{R_q} = 655 \), when \( S = R_q = 0 \).

When the heavy fluid is on top the significant effects due to phase with respect to
change can be obtained by considering \( R_T / R_T \rightarrow 0, R_T^2 \rightarrow 0, S \rightarrow \infty \) (Busse and Schubert). The minimum value of \( R_T S \) occurs for the case \( R_T = R_T^2 \) which is shown below. The horizontal wave number is \( a \).
This material is crystalline at room temperature, melts to a liquid-crystal at $119^\circ\text{C}$ and changes phase to liquid-liquid at $135^\circ\text{C}$.

While the liquid-crystal properties of this fluid are not of primary interest in the present case, it is necessary to discuss them briefly to determine their effects on the results. A liquid crystal lies somewhere between solids which can sustain static shear stresses and an isotropic liquid which can sustain no static shear stresses at all. Liquid crystal materials are composed of anisotropic molecules and it is the ordering of their axes which gives them their "crystal" properties. Nematic liquid crystals can sustain only a static shear stress acting on the direction of the molecules.

The liquid crystal phase will therefore be distinguished by anisotropy because of the molecular orientation. This anisotropy will not be expected to alter the mechanical properties significantly, but may be important as a nonlinear effect. For example, convection may be present in rolls instead of hexagons if alignment of the molecules is more important than nonlinearity of viscosity.

The convection apparatus was made from a 6.25 cm dia. by 5 cm block of aluminum in which a 5 cm dia. by 1.25 cm hole was milled. Nichrome wire taped to the opposite side of the block served as the heat source. The convection layer was of variable height, 5 cm dia., was bounded on bottom and sides by the heated aluminum and on the top by glass 6.3 mm thick. Thermocouples were placed in the block and on the top of the glass to measure the temperature of the block and difference across the layer.
The amount of temperature drop in the glass was determined at one gap height by sandwiching two thermocouples between three identical glass plates. The ratio of conductivities between glass and fluid was found to be \( \frac{k_g}{k} = 4.3 \). From this relation the temperature drop in the fluid was determined for different gap heights. Maximum temperature drops of 16°C could be maintained across the whole system when the top glass surface was exposed to room air, with some amount of regulation by enclosing the experiment within a small volume.

4. Experimental results

Convection was observed for three values of \( \mathbf{d} \). The conditions in each case are summarized in the following table.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \mathbf{d} ) (mm)</th>
<th>( T_1 - T_2 ) (°C)</th>
<th>( R_q )</th>
<th>( R_T )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.8</td>
<td>11.0</td>
<td>1750</td>
<td>17000</td>
<td>.43</td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
<td>8.5</td>
<td>314</td>
<td>2400</td>
<td>.58</td>
</tr>
<tr>
<td>3</td>
<td>0.9</td>
<td>1.7</td>
<td>58</td>
<td>86</td>
<td>2.6</td>
</tr>
</tbody>
</table>

In each case the temperature \( T_1 \) was raised to the melting point and then varied slowly to determine the effect on the observed convection when phase change was present. The essential observed results were as follows:

a. Case 1. Strong convection rolls are present in the heavy phase. If left long enough the rolls form a regular pattern, nearly straight with wavelength \( L \approx 2d \). No change is observed when \( R_T \) is decreased by a factor of two. As \( T_1 \) is increased so that the second phase is approached, the rolls turn abruptly into "hexagons" with updraft in the center. The light phase is initially present as a bubble in the center, with the amount of light phase increasing as \( T_1 \) is increased. The "hexagons" observed have between five and nine sides initially but if left long enough nearly all become six-sided figures.

As \( T_1 \) is increased more light phase is present in the center of the cells. The cells become more circular until the heavy phase is confined to a doughnut shape around each cell. When all light phase is reached strong convective motions are present, but the horizontal patterns cannot be discerned because the fluid is clear.
b. Case 2. Strong convection rolls are present in the heavy phase at \( R_T \sim 2400 \). The rolls become regular and nearly straight with \( L \sim 2d \). As \( R_T \) is reduced to \( R_T \sim 1300 \) the rolls become weaker. At \( R_T \sim 1300 \) the regular pattern disappears, leaving a convective motion, but no orderly rolls.

As \( T_1 \) is increased so that the second phase is approached the rolls turn into hexagons. The process is more gradual than in Case 1, with a zig-zag roll pattern developing first, then hexagonal cells forming. As \( T_1 \) is further increased the hexagonal cells become more rounded and gradually longer until large rolls of \( L \sim 4d \) are formed. Convective motion can be seen when all light phase is present, but no horizontal pattern can be discerned.

c. Case 3. No single phase convection rolls are present if \( T_1 \) is increased very slowly from the melting point. As \( T_1 \) is increased two-phase rolls are formed with \( L \sim 2d \). These rolls form a very irregular pattern and persist as \( T_1 \) is lowered even down to the freezing point. No motion is observed when the layer is all light fluid.

5. Discussion of results

In both case 1 and case 2 for one-phase convection \( R_T > R_T^C \). The rolls formed indicate that at least the nonlinearities of viscosity are not important (since rolls are formed instead of hexagons), and perhaps that the anisotropy of the liquid-crystal phase is important.

The development of hexagons as the phase change boundary is approached is due to the second phase coming in as an asymmetry in the vertical direction. The result is similar to the effect caused by temperature nonlinearities in ordinary convection. It has been demonstrated that convection will occur in two-dimensional rolls unless there is some asymmetry in the vertical direction (Busse, 1967). With vertical asymmetry the favored horizontal pattern is hexagons with flow at the center going either up or down, depending on the sign of the temperature nonlinearities. The effect can be explained physically by considering either \( \frac{d^2 p}{d T^2} \neq 0 \) or \( \frac{d \mu}{d T} \neq 0 \). In liquids generally \( \frac{d^2 p}{d T^2} \leq 0 \) and \( \frac{d \mu}{d T} \leq 0 \), while the opposite is true in gases.
It can be seen that the liquid finds it more favorable to have a jet-like flow at the bottom where more instability is available \( \left( \frac{dp}{dz} \right) \) is greater) and \( \mu \) is less. The opposite is true for gases.

In the convection experiment the light phase, which is distinguished by smaller \( \rho \) and \( \mu \), is first present at the hot lower boundary, since the equivalent temperature change due to the depth

\[
\rho g d \gamma - T_c \ll L T_1 - T_2 \sim 10 C.
\]

The effect of density and viscosity variation, therefore, looks like:

which is favorable to liquid-like hexagons.

As more light phase is present the above explanation no longer applies, and once \( R_T > R_T \) an explanation for the observed finite amplitude convection patterns (hexagons in case 1 and rolls in case 2) is not apparent.

For the observations of case 3 \( R_T \ll R_T \) and single phase convection is not possible. The applicable two-phase case is the latter one discussed in section 2, with light fluid on the bottom and
$R_q \to 0$, $R_T \to 0$. For such conditions it was found that the marginal stability analysis gives $(R_bS)_c = 118$ and $a_c = 2.4$ for free-free boundary conditions. It is seen that $a_c$ is nearly equal to the value for ordinary convection ($a_c = 2.22$). Peltier (1973) finds no significant change in horizontal wavelength due to phase change in fixed-fixed boundary conditions. This suggests that the horizontal wavelength will phase change and fixed-fixed boundaries will be nearly equal to that for no phase change and fixed-fixed boundaries. Anticipating the different value of $(R_bS)_c$ with fixed boundaries, the marginal state should be given by

$$(R_qS)_c \sim 300a_c \sim 3.1, \ L \sim 2d.$$  

The observations of case 3 were at $R_qS \sim 200$ and showed two-phase rolls of $L \sim 2d$, which is consistent with this marginal stability argument.

The strong hysteresis observed indicates the presence of a finite amplitude instability in two-phase convection. Once established, it persists at $T_1$ well below that for initiation.

6. Summary and suggestions for further work

An experiment has been performed to investigate the effects of a phase change on thermal convection. The results of the experiment show that two-phase convection is possible at Rayleigh numbers well below the value for ordinary convection. The value of the critical parameter and observed wavelength agree well with the linear theory for the case in which the light fluid lies on the bottom of the layer. At Rayleigh numbers above the critical value the phase change sets in as a vertical asymmetry and induces hexagonal convection cells. The effect is similar to that caused by liquids with nonlinear properties.

In the present experiment, as in most laboratory work, the effects of pressure are small compared to thermal effects. This results in the light phase occurring near the lower boundary under the heavy phase. In geological flows the opposite is true, and the heavy phase is below. By increasing the gap size by a factor of four and decreasing the temperature drop, the effect can be reversed with pressure effects
becoming ten times greater than thermal ones. This would allow experimental investigation of the geologically interesting case in which the stabilizing effect of latent heat at the phase boundary is offset by the density difference.

An interesting theoretical problem which remains to be solved is the finite amplitude effect due to the phase boundary. The additional difficulty of finite amplitude disturbances on the boundary distinguish this case from that of ordinary convection.

References

A SUBHARMONIC SURFACE GRAVITY WAVE PHENOMENON

Michael H. Frese

Introduction

One of the attractions of fluid mechanics is that even in experiments for which the theory is most well-developed some truly startling effects may be seen. This is also one of the discipline's less subtle tortures. One such occurrence provides the motivation for the analysis to be presented below.

The experiment involved is a demonstration of forced small amplitude standing surface gravity waves in a rectangular wave tank (see Fig.1).

Fig.1 Schematic of Apparatus
The tank has a wave maker at one end which is a wedge 3/4 inch by 3 inches in cross section, which is driven vertically with an amplitude of 1 inch at a frequency $\omega$. Linear theory predicts that the resulting wave will be an oscillation also at frequency $\omega$ with some wave form which will be determined by the value of $\omega$ and its relation to the natural frequencies of the linear modes. When $\omega$ is near one of these natural frequencies the wave form should be approximately that of the associated linear mode. But in reality, in a range of frequency near the second natural frequency, the wave apparently has some component of frequency $\frac{1}{2} \omega$. Particles in suspension near the middle of the tank clearly exhibit even a small component at this frequency by surging to the left during one cycle of the wedge and then returning during the next! At some frequencies within this band the wave seems to have no component at frequency $\omega$ at all, and the wave form is not the second linear mode, but much more like the first linear mode. In this circumstance the phenomenon is very striking as the wave maker pushes first into the crest and then into the trough of the wave.

The work which follows is an attempt to wring from surface gravity wave theory some behavior similar to that described above. Many of the assumptions which would otherwise seem rather ad hoc have at their root the observation.

The mathematical technique is a two-time asymptotic analysis, in which an asymptotic solution is sought having the form of an expansion in terms of the linear problem eigenfunctions, the coefficients of which are slowly time-varying, amplitude-dependent functions.

**The Interaction Equations**

The model to be explored is shown in Fig. 2. For boundary conditions at the free surface take

![Fig. 2 Model](image-url)
\[ \frac{\partial^2 u}{\partial t^2} + (\phi_x^2 + \phi_y^2) = p \quad \text{at} \quad y = \eta \]
\[ \phi_y = \frac{D_n}{D_t} \quad \text{at} \quad y = \eta \]

where \( p(x,t) \) is some imposed pressure distribution intended to model the forcing.

Since \( \phi \) is a harmonic function it is analytic in \( y \). Therefore (1) will be considered as equivalent to the condition on the infinite series gotten by expanding \( \phi \) in powers of \( y \) and evaluating at \( y = \eta \). Manipulation of this and the similar series condition gotten from (2) would yield a single condition on \( \phi \) alone. However, since the analysis must include some effects of nonlinearity, but no more than necessary, only the linear and quadratic parts of this infinite series are desired. This allows an alternative approach. Taking the convective derivative of (1), remembering to apply the chain rule for the evaluations at \( y = \eta \), and using (2) to replace \( \frac{D_n}{D_t} \) by \( \phi_y \), eliminates \( \eta \) from the linear part of the condition, and \( \eta^2 \) appears only in the cubic terms. Therefore, using \( \eta = \phi_y - \phi_y \), explicit dependence on \( \eta \) can be removed from the quadratic terms as well:

\[ \phi_{tt} + (\phi_x^2 + \phi_y^2) + q \phi_y = p_x + p_y + p_x + \text{(cubic terms)} \quad \text{at} \quad y = \eta. \]

Now expanding in \( \eta \) is easy, since only the first and third terms produce corrections, and these multiplications by \( \eta \) may be removed, since they are in quadratic terms, by using \( \eta = \frac{1}{2} (p_x + p_y) + \text{(quadratic terms)} \) at \( y = 0 \). So

\[ \phi_x + q \phi_y = p_x - (\phi_x^2 + \phi_y^2) + \frac{1}{2} (p_x + p_y)(p_x + q \phi_y) + p_y \quad \text{at} \quad y = 0. \]

If \( p \) is on the order of the quadratic terms then

\[ \phi_x + q \phi_y = p_x - (\phi_x^2 + \phi_y^2) + \frac{1}{2} (p_x + p_y)(p_x + q \phi_y) \quad \text{at} \quad y = 0. \]

Taking \( \phi(x,y,t) = \sum \limits_{n=1}^{\infty} A_n(t) \frac{\cosh n(y + H)}{\cosh nH} \cos nx \) satisfies \( \nabla^2 \phi = 0 \) and all boundary conditions except the nonlinear one.

For \( p(x,t) = \sum \limits_{n=1}^{\infty} P_n(t) \cos nx \) and \( \omega_n = qn \tan h \pi n H \), that condition is

\[ \sum \limits_{n=1}^{\infty} \left[ A_n^2 + \frac{q}{\omega_n} A_n P_n \right] \cos nx - \sum \limits_{n=1}^{\infty} \sum \limits_{m=1}^{\infty} \left[ A_n^2 \frac{m^2}{q^2} + m^2 A_m \right] \cos nx \cos mx + \sum \limits_{n=1}^{\infty} \sum \limits_{m=1}^{\infty} \frac{1}{\omega_n^2} (A_n A_m)(nm \sin nx \sin mx + \frac{\omega_n \omega_m}{q^2} \cos nx \cos mx) = 0. \]
If the fact that $A^n_m = -\omega^2 A^n_m$ (quadratic terms) is used, and the double summations rewritten to explicitly indicate the Fourier decomposition of the products such as $\cos n\xi \cos m\xi$, then the coefficient of $\cosh n\xi$ in this condition is

$$A^n_m + \omega^2 A^n_m - \frac{L}{2} \sum \left( A^n_m A^m_n \right) \left[ \frac{1}{2} \left( \cos \omega t + \omega \sin \omega t \right) - A^n_m A^m_n \left( \frac{\omega^2}{2} - \omega t \right) \right] + \frac{1}{2} \sum \left( A^n_m A^m_n \right) \left[ \frac{1}{2} \left( \cos \omega t - \omega \sin \omega t \right) - A^n_m A^m_n \left( \frac{\omega^2}{2} - \omega t \right) \right] = 0$$

The observed phenomena seem to indicate that there may be solutions with $A_{m,n}$ for $k \neq l$, much smaller than at least one of $A_1$ and $A_2$. Applying this, the two most significant conditions are approximately

$$A^n_m + \omega^2 A^n_m - \frac{L}{2} A^n_m A^m_n + m A^n_m A^m_n = 0 \\ A^n_m + \omega^2 A^n_m - \frac{L}{2} A^n_m A^m_n + 2k A^n_m A^m_n = 0$$

where

$$L(H) = \frac{3}{2} + \frac{1}{k} \tanh H (4 \tanh 2H + \tanh H)$$

$$m(H) = \frac{3}{2} + \frac{1}{k} \tanh H (4 \tanh 2H + \tanh H)$$

$$K(H) = -\frac{2}{3} (1 - \tanh^2 H)$$

Now let $H$ be small, so that $2 \omega - \omega_2 \equiv E$ is small.

Then let $P_n(t) = e^{2\pi n} \cos(\omega_2 + \alpha \xi) t$ where $\alpha_2$ is a real constant, and in particular let the origin of $t$ be such that $\alpha_2$ is negative. As stated above, $A_1$ and $A_2$ are $O(\xi)$ while $A_k = O(\xi^2)$ for $k > 2$. In addition, take

$$A_1(\xi, t) = e(a_1(\xi, t) e^{i\omega t} + a'_1(\xi, t) e^{-i\omega t}) + O(\xi^2)$$

$$A_2(\xi, t) = e(a_2(\xi, t) e^{i\omega t} + a'_2(\xi, t) e^{-i\omega t}) + O(\xi^2)$$

These have the alternate forms

$$A_1(\xi, t) = e(a_1(\xi, t) e^{i\omega t} + a'_1(\xi, t) e^{-i\omega t}) + O(\xi^2)$$

$$A_2(\xi, t) = e(a_2(\xi, t) e^{i\omega t} + a'_2(\xi, t) e^{-i\omega t}) + O(\xi^2)$$

Using the first form of $A_1$ and the second form of $A_2$ in (3) and using the second form of $A_1$ and the first form of $A_2$ in (4) satisfies (3) and (4) to order $\xi^2$. But the remainders have secular terms (terms proportional to $e^{i\omega t}$ in (3) and proportional to $e^{i\omega t}$ in (4) at order $\xi^2$.

If the amplitudes are chosen to suppress these terms we have
This nonautonomous system can be transformed to an autonomous one by taking

\[ a_i(T) = \frac{2}{k(2L-m)} e^{-i\frac{\omega T}{2L-m}} \quad a_2(T) = \frac{2}{2L-m} e^{i\omega t} \]

so that

\[ \alpha' + i \left( \frac{\alpha}{2} \right) \alpha + \alpha^* \beta = 0 \]

\[ \beta' + i \alpha \beta + \beta^* = R \]  

(5)

For \( H \) small \( k \) and \( 2L-M \) are negative so that \( R \) is positive.

The Critical Points and their Stability

The interpretation of \( \alpha \) and \( \beta \) rests on the following:

\[ A_1(t) = \frac{2}{k(2L-m)^{1/2}} \left( \alpha(\omega t) e^{-i\left( \frac{\omega}{2} + \xi \right) t} + \beta(\omega t) e^{-i\left( \frac{\omega}{2} - \xi \right) t} \right) \]

\[ A_2(t) = \frac{2i}{2L-m} \left( \beta(\omega t) e^{-i\left( \frac{\omega}{2} + \xi \right) t} + \alpha(\omega t) e^{i\left( \frac{\omega}{2} - \xi \right) t} \right) \]

They are, therefore, slowly varying amplitudes of fast time oscillations with frequencies, respectively, half the forcing frequency and the same as the forcing. So steady solutions of the amplitude equations (5) represent oscillatory waves.

These equations have the following critical points:

I. for all \( R, \alpha \neq 0 \)

\[ \alpha_1 = 0 \quad \beta_1 = -\frac{R}{2 \omega} \]

II. for \( R > -\frac{1}{2} \alpha \) (\( 1-\alpha \))

\[ \alpha_\Pi = \pm \sqrt{R + \frac{1}{2} \alpha (1-\alpha)} \quad \beta_\Pi = \frac{1}{2} i (1-\alpha) \]

III. for \( R < \frac{1}{2} \alpha \) (\( 1-\alpha \))

\[ \alpha_\Pi = \pm i \sqrt{-R + \frac{1}{2} \alpha (1-\alpha)} \quad \beta_\Pi = -\frac{1}{2} i (1-\alpha) \]

For a fixed value of \( R \), the critical points \((\alpha_1, \beta_1)\) bifurcate from the critical point \((\alpha_\Pi, \beta_\Pi)\) at \( \alpha = -\frac{1}{2} \sqrt{1+8R} \), traverse an ellipse, and rejoin \((\alpha_1, \beta_1)\) on its other branch in a bifurcation point at \( \alpha = \frac{1}{2} \sqrt{1+8R} \). If \( R \) is less than \( \frac{1}{8} \), the critical points \((\alpha_\Pi, \beta_\Pi)\) bifurcate from \((\alpha_1, \beta_1)\) at \( \alpha = \frac{1}{2} \sqrt{1-8R} \) and rejoin it at \( \alpha = \frac{1}{2} \sqrt{1-8R} \), traversing an ellipse between these values. Figure 3 shows the modulus of \( \alpha \) (above the axis) and the modulus of \( \beta \) (below the axis) for the various critical points, as a function of \( \alpha \) for a typical value of \( R > \frac{1}{8} \), while
Fig. 3 Amplitude of critical points for $R > \frac{1}{8}$.

Fig. 4 Amplitude of critical points for $R < \frac{1}{8}$.
Fig. 4 shows the same for $R < \frac{1}{8}$. For $R = 0$ the two ellipses coincide, and as $R$ increases to and exceeds $\frac{1}{8}$ they draw apart, the inner one shrinking to a point and disappearing. As $R$ increases in Fig. 3, the ellipse expands about $\alpha = \frac{1}{4}$ and the $\alpha$ curve increases proportionally with $R$. This change keeps the intersection of the $\alpha$ and $\beta_{II}$ curves at the same value of $\alpha$ as the intersection of the $\alpha_{III}$ curve and the $\alpha_{II}$ curve (the latter is the $\alpha$-axis).

By the remarks above, at $\alpha = 1$ $(\alpha_{II}, \beta_{II})$ represents a purely subharmonic oscillation, though at all values of $\alpha$ the wave associated with it has some subharmonic component.

Were the solutions of Eqs. (5) known for general initial conditions, and for all values of $\alpha$ and $R$, much information about the behavior of this wave-tank model would be apparent. Some of this information can be obtained by examining the stability of the critical points of (5) and how it changes with $\alpha$ and $R$. But stability or instability of one of these means only that the wave associated with that critical point is stable or unstable with respect to perturbations of the specific kind allowed by the amplitude space. And these include only the first and second free oscillations of the model.

Let $\alpha = p + iq$ and $\beta = r + is$ where $p, q, r,$ and $s$ are real and separate (5) into real and imaginary parts to get

\[
\begin{pmatrix}
 p \\
 q \\
 r \\
 s
\end{pmatrix} = \begin{pmatrix}
 -\frac{(\lambda - \alpha)}{2} q - pr - qs \\
 \frac{\lambda - \alpha}{2} p - ps + qr \\
 \lambda - p^2 + q^2 + R \\
 -\alpha r - 2pq
\end{pmatrix} = F_{\alpha, R}(x) \cdot x,
\]

If $y$ is a critical point of this equation (i.e., $F_{\alpha, R}(y) = 0$) then $y$ is unstable for those values of $\alpha$ and $R$ for which $\det(dF_{\alpha, R, y})$ has at least one root with positive real part, where $dF_{\alpha, R, y}$ is the linearization of $F_{\alpha, R}$ about $y$.

For the critical point $y_{II}$ which is derived from $(\alpha_{I}, \beta_{I})$ the spectrum of the linearization is
so this critical point is unstable for \( R > \frac{1}{2} \alpha \left( 1 - \alpha \right) \). The critical point \( \lambda_{II} \) has associated with it the spectrum \( \lambda_{II} (\alpha, R) \) where
\[
\lambda_{II} (\alpha, R) = \frac{1}{2} \left[ \pm (\alpha - 4R') \pm \left[ (\alpha - 4R')^2 + 16R^2 \right]^{1/2} \right]
\]
in which \( M = -R + \frac{1}{2} \alpha \left( 1 - \alpha \right) \), and therefore it is always unstable since \( \lambda_{II} \) always has one positive value. The criterion for stability of the II critical point is that
\[
\lambda_{II} = \frac{1}{2} \left[ -(\alpha - 4R') \pm \left[ (\alpha - 4R') - 8N^2 (2N^2 - \alpha \left( 1 - \alpha \right)) \right]^{1/2} \right]
\]
be real and negative, where \( N = R + \frac{1}{2} \alpha \left( 1 - \alpha \right) > 0 \). Hence \( \lambda_{II} \) is stable only when it exists and the following condition is met:
\[
(\alpha < 0, \alpha > 0) \text{ and } (R < \frac{\alpha (3x - 2)^2}{B(2x - 1)})
\]

Figure 5 shows the regions of stability in the \( R, \alpha \) plane for the I and II critical points. Note the region where neither is stable.

**Fig. 5** Stability of the critical points.

**Conclusions**

The I critical point will be referred to as the fundamental solution and the II critical point, as the subharmonic solution. This terminology is justified since the wave associated with I has the same least period as the forcing while the wave associated with II has least period twice that of the forcing.
Suppose that the wave maker is oscillating at some frequency and amplitude for which \( \lambda_i > \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8R} \) and that the fundamental wave is present in the tank. Then the amplitude is held fixed and the frequency abruptly reduced some small amount to \( \lambda_2 \). The change that occurs in the amplitudes of the fundamental and subharmonic is given by the solutions to Eqs. (5) with \( \lambda = \lambda_2 \) and initial conditions \( a = 0 \), \( \beta = \frac{-1R}{\lambda_1} \). If the fundamental critical point is stable at \( \lambda = \lambda_2 \) and the initial conditions are close to this critical point, i.e., if \( \lambda_2 \) is sufficiently close to \( \lambda_1 \), so that this is true, then presumably these solutions will go, as \( T \) increases, to this critical point. Then the frequency is reduced again, to some value \( \lambda_j < \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8R} \). But for this value of \( \lambda \), the fundamental solution is no longer stable. The nearest stable solution if \( |\lambda_j(1 + \sqrt{1 + 8R})| \) is small, is the subharmonic critical point. So the amplitudes may go to this solution. This can be seen in the experiment. In this way then, by using very small frequency changes the critical points can be explored experimentally.

But the theory so far has given no indication of what happens when the system is at the subharmonic critical point, and the frequency is changed a small amount to a value where the subharmonic loses stability. Of course, if \( R \) is sufficiently small, Fig. 5 indicates that the fundamental may have regained its stability. Perhaps the system would return to this behavior. But for \( R > \frac{1}{8} \), the fundamental is still unstable when the subharmonic becomes so. For such \( R \), the question remains open.

The bifurcation of the subharmonic from the fundamental solution occurs precisely when the fundamental solution loses stability, and this loss of stability occurs by two pure imaginary eigenvalues of the linearized problem coalescing at zero and then emerging real, one positive and one negative.

It is quite possible that the loss of stability of the subharmonic solution, which occurs by two real negative values of \( \lambda_2 \) coalescing at some negative real value and then emerging as a complex conjugate pair, results in the bifurcation of a periodic solution from the subharmonic critical point. This periodic solution, representing a wave with period-
ically slowly varying amplitudes of its fundamental and subharmonic components, might then supply a stable solution for the region where none is yet known.

It can be seen that this analysis depends crucially on the water being shallow, since this is necessary to make $2\omega$, and $\omega_a$ very nearly the same. Without this, the amplitude of the fundamental solution would be much larger than that of the subharmonic. Yet the wave amplitudes must also be so small that there is no substantial change in form from the linear case. This latter condition makes even qualitative verification difficult. This is true because to reduce the amplitudes of the waves it is necessary to reduce the forcing, which amounts to reducing $R$. But this may shrink the frequency range over which the fundamental solution is unstable below the precision of the wave number frequency control. For the apparatus of Fig.1 $\frac{2\omega_i - \omega_j}{\omega_i} = \pm 3\%$, but it was not possible to change the frequency by less than $1\%$.

It is remarkable that this analysis exhibits a resonant interaction in a forced system when none exists in the corresponding free system. Surface gravity waves, after all, have no internal resonances before fourth order. The forcing, by introducing other pairs of frequency and wave number into the competition, substantially increase the possibilities for resonance. It is also important that the free system has near-resonance due to the flatness of the dispersion relation.

There are some obvious directions for extension or modification of this work. First, the amplitude equations might be investigated numerically. This would be particularly interesting for the region of parameters where all of the critical points have been found to be unstable. Second, some similar analysis should be attempted on a revised model wherein $p(x,t) = 0$ and where $\frac{\partial^2 \phi}{\partial t^2} |_{x=0} = a_{\omega} \omega_a \omega \hat{t}$. This method of forcing is much more physical. Third, a careful experimental program could be undertaken to provide a basis for quantitative comparison of this last modification.

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STABILITY OF A ROTATING BOUNDARY FLOW

Tor Gammelsrød

1. Introduction

Inertial oscillations are often observed in the ocean, also at great depths (Webster, 1968). In the laboratory studies by Tatro and Mollo-Christensen (1966) and Green and Mollo-Christensen (1970) it is found that Ekman layer instabilities and the interior flow are coupled by inertial waves. It is also found that Ekman layers become unstable at Reynolds number of order 100. Therefore one would expect all atmospheric and oceanic Ekman layers to be unstable, and the observed inertial oscillations in the ocean may be related to instability of the upper or lower boundary layer. A theoretical study on instability of Ekman layers was made by Lilly (1966).

In this lecture the stability of a boundary layer with a constant shear is studied. The velocity in the boundary layer is two-dimensional and decreasing linearly with depth.

2. Formulation

Let us consider an infinite deep ocean with constant density. Our basic state will be a velocity in the x-direction monotonically decreasing with depth (Fig.1). This current is maintained by a body-force $F_x(z)$ in the x-direction and balanced by the Coriolis force.

Fig.1 General model
If we assume that the fluid is inviscid the equations for conservation of mass and momentum are:

\[
\begin{align*}
\rho u_t + \rho u u_x + \rho v u_y + \rho w u_z - \frac{\partial p}{\partial x} &= -\rho \frac{\partial \rho}{\partial x} \\
\rho v_t + \rho u v_x + \rho v v_y + \rho w v_z + f u &= -\frac{\partial p}{\partial y} + F_y(\eta) \\
\rho w_t + \rho u w_x + \rho v w_y + \rho w w_z + f w &= -\frac{\partial p}{\partial z} - p^* \\
\end{align*}
\]

(1)

The notation is as usual except for the pressure \( p^* \). Because \( \rho = \text{constant} \) let us define \( \rho = \rho^*/\rho \).

Our basic state is

\[
\begin{align*}
\int u &= F_y(\eta) \\
\frac{\partial p}{\partial z} &= -q
\end{align*}
\]

Let us now perturbate this velocity profile and let the perturbations be independent of \( x \); the perturbations may be considered as rolls (Langmuir - cells) with their axes parallel to the current.

If we neglect terms which contain squares of the perturbation quantities the equations (1) yield

\[
\begin{align*}
u_t + \frac{f u}{\rho^*} w - \frac{f v}{\rho^*} &= 0 \\
v_t + \frac{f u}{\rho^*} u &= -p_y \\
w_t &= -p^*_x \\
\end{align*}
\]

(2)

Assuming solutions of the type \( \phi = \phi(\eta) e^{i(y + \omega t)} \) we get when we eliminate \( u, v \) and \( p \) the following equation for the vertical velocity:

\[
\begin{align*}
\hat{W}(\eta) + \frac{i \omega \hat{U}}{\omega^2 - \omega^2 - \hat{U}} - \hat{W}(\eta) &= 0
\end{align*}
\]

(3)

When \( U(\eta) \) is a linear function Eq. (3) becomes an ordinary differential equation with constant coefficients which may be solved easily. Let us therefore consider this case in detail.

3. Broken line profile

a. Equations

Our basic current as shown in Fig. 2, \( U = a(\eta + h) \) in Region I, where \( a = \frac{du}{dz} \) is a constant. This current is balanced between the
Coriolis force and the body-force, which now may be interpreted as a parabolic stress field
\[ \tau = \frac{\partial u}{\partial t} \]
where \( \tau \) is the stress. In Region II we have \( u \equiv 0 \)

In Region I Eq. (3) reduces to
\[ \frac{\partial^2 \hat{W}_x}{\partial z^2} + \frac{\omega^2 x}{\beta - \omega^2} \hat{W}_x = 0 \]  
which has a general solution of the form
\[ \hat{W}_x = Ae^{-\sqrt{\frac{\omega^2 x}{\beta - \omega^2}} z} + Be^{\sqrt{\frac{\omega^2 x}{\beta - \omega^2}} z} \]
where
\[ \beta = \frac{\omega^2}{\beta - \omega^2} \]  
and
\[ \alpha = \frac{\omega^2}{\beta - \omega^2} \]  

A and B are constants. In Region II Eq. (3) reduces to
\[ \frac{\partial^2 \hat{W}_x}{\partial z^2} + \frac{\omega^2 x}{\beta - \omega^2} \hat{W}_x = 0 \]
and a general solution is
\[ \hat{W}_x = Ce^{-i\sqrt{\beta} z} + De^{i\sqrt{\beta} z} \]

We are looking for wave-solutions, so let us assume \( \omega < \frac{1}{\beta} \).

b. Boundary conditions

When \( z \to -\infty \) it is necessary to apply a radiation boundary condition. In Region II we may write the equations of motions and conservation of mass as follows:
\[ \frac{\partial^2 \hat{W}_x}{\partial z^2} + \frac{\omega^2 x}{\beta - \omega^2} \hat{W}_x = 0 \]
\[ \nabla \cdot \hat{v} = 0 \]
If we multiply Eq. (10) with \( \nabla \) and use Eq. (11) we get the following energy equation
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \nabla \cdot \hat{v} \right) = -\nabla \cdot (\hat{v} \rho) \]
Averaging this equation in the horizontal (-) and integrating it in
Fig. 3 The relation between $\chi$ and $\omega$ for the neutral case. Unstable regions are also shown.

Fig. 4 The relation between $\Delta$ and $\omega$ for the neutral case. The unstable region, corresponding to the unstable regions in Fig. 3, is confined to $\Delta \leq l$. 
Fig. 5 Here is shown the relation between $\chi$ and $\xi$ for $n = 1$ and $n = 2$. For the solid part of the curves we may find the corresponding frequency at the solid part of the $(\xi, \omega)$ curve, and for the dashed part of the $(\chi, \xi)$ curve we use the dashed part of the frequency curve.
At any finite time the disturbances have not reached \( z = -\infty \) such that \( (Wp)_{-\infty} = 0 \), we want the energy to propagate downwards. Therefore

\[
\frac{\partial}{\partial t} \int_{0}^{\infty} \left( \frac{1}{2} \frac{\partial \psi}{\partial z} \right) dz' = - (Wp)_{z} + (Wp)_{-\infty} > 0
\]

which yields

\[
(Wp)_{z} < 0
\] (12)

that is \( W \) and \( p \) have to be negatively correlated. The third of Eqs. (2) gives

\[
i \omega W = - p_x. \quad \text{We have} \quad p \sim e^{\pm i \sqrt{\gamma^2 - 4r}} \quad \text{or} \quad p_x \sim e^{\pm i \sqrt{r}}
\]

To satisfy (12) the + sign has to be chosen. Therefore \( C = 0 \) and Eq. (9) reduces to

\[
\hat{W}_{\parallel} = D e^{i \sqrt{\gamma}}
\] (13)

At the interface \( z = -h \) the kinematic boundary condition is

\[
\hat{W}_{I} - \hat{W}_{II} = 0
\] (14)

and the dynamic boundary condition is

\[
\frac{\partial p_{z}}{\partial y} - \frac{\partial p_{x}}{\partial y} = 0
\] (15)

Using Eqs. (2), Eq. (15) may be transformed to yield

\[
\frac{d \hat{w}_{y}}{d \bar{z}} - \frac{d \hat{w}_{z}}{d \bar{z}} = \frac{a \beta i \xi}{f^{2} - \omega^{2}} \hat{W}_{z}
\] (16)

Eq. (13) yields

\[
\Delta \hat{W}_{z} = i \sqrt{r} \hat{W}_{z}
\]

Therefore, combining (14) and (16) we obtain

\[
\frac{d \hat{w}_{z}}{d \bar{z}} + (q - i \sqrt{r}) \hat{W}_{z} = 0 \quad (\text{at} \ z = -h)
\] (17)

At the surface we use

\[
\hat{W}_{I} = 0 \quad (\text{at} \ z = 0)
\] (18)

as boundary condition.

c. Eigenfunctions

Equation (5) with boundary conditions (17) and (18) now yields

\[
q + \sqrt{q^2 - 4r} - 2i \sqrt{r} = \left\{ q - \sqrt{q^2 - 4r} - 2i \sqrt{r} \right\} e^{q\sqrt{q^2 - 4r} h}
\]

or
where we have introduced the non-dimensional variables

\[
\chi = \frac{U_o}{f} \quad \Omega = \frac{\Omega_o}{f} \quad \lambda = \frac{\lambda_o}{f}
\]

where \( U_o \) is the velocity at the surface.

Equation (19) is of the form

\[
p + Q = (p - Q)e^{i\frac{\chi}{1 - \Omega^2} Q} \quad (20)
\]

where

\[
p = \lambda - 2 \Omega \sqrt{1 - \Omega^2} \quad (21)
\]

and

\[
Q = \sqrt{\lambda^2 + 4 \Omega^2 (1 - \Omega^2)} \quad (22)
\]

d. Neutral solution

Let us now assume \( \Omega \) real and \( \Omega < 1 \). Eq. (20) then implies

\[
\frac{\chi}{1 - \Omega^2} Q = n \pi, \quad n = 0, 1, 2, \ldots
\]

\[
p - Q = \pm (p - Q) \quad (23)
\]

However, if the + sign is chosen this implies \( Q = 0 \) and this violates our assumptions on \( \Omega \). Therefore we get

\[
\frac{\chi}{1 - \Omega^2} Q = (2n-1) \pi, \quad n = 1, 2, \ldots
\]

and

\[
p = 0 \quad (24)
\]

Equation (21) therefore yields

\[
\lambda = 2 \Omega \sqrt{1 - \Omega^2} \quad (25a)
\]

or, in dimensional form

\[
\frac{\lambda}{f} = 2 \frac{\Omega_o}{f} \sqrt{1 - (\frac{\omega_o}{f})^2} \quad (25b)
\]

Our solutions are of the form \( e^{i(k_y y + \omega t)} \), that is, waves propagating downwards for \( m > 0 \) and \( \omega > 0 \). Now also \( \lambda > 0 \) and we see from (25b) that for \( \omega > 0 \) we have \( \lambda > 0 \), and the waves are to propagate in the opposite direction of the vorticity of the mean field

\[
\nabla \times \vec{\omega} = \vec{\alpha} \cdot \vec{J}
\]

where \( \vec{J} \) is a unit vector along the y-axis.
From Eq. (25a) we obtain

\[ \Omega^2 = \frac{1}{2} \left( 1 \pm \sqrt{1 - \beta^2} \right) \]  

(26)

and we see that a necessary condition for a neutral solution (\( \Omega \) real) is

\[ A \leq 1 \]

or in dimensional form

\[ \frac{U_e}{h f} \leq 1 \]

This may also be shown explicitly (see Appendix). From Eq. (23) we get

\[ \frac{1}{1 - \Omega^2} \sqrt{\beta^2 + 4 \beta \Omega^2 (1 - \Omega^2)} = (2n - 1) \pi \]  

(27)

Substituting for \( \Omega \) using the + sign in Eq. (26) yields

\[ \frac{2 \sqrt{\pi}}{n} \kappa = (2n - 1) \sqrt{1 - \beta^2} \]  

(28a)

and - sign yields

\[ \frac{2 \sqrt{\pi}}{n} \kappa = (2n - 1) \beta^3 \left[ 1 - \frac{1}{\sqrt{1 - \beta^2}} \right]^{3/2} \]  

(28b)

If we substitute for \( \beta \) from Eq. (25a) in Eq. (27) we get

\[ \frac{2 \sqrt{\pi}}{n} \kappa = \frac{1}{\Omega} \sqrt{1 - \Omega^2} (2n - 1) \]  

(29)

The graph of (29) is shown in Fig. 3 and of Eq. (25a) in Fig. 4 while the relations (28) are shown in Fig. 5, together with (25a).

e. Unstable regions

We found that

\[ \omega_\Pi \sim e^{i \gamma' t'} e^{i \frac{x_\Omega}{2 \Omega_\Pi} \gamma} \]

where we have used the non-dimensional coordinates \( y' = y/\ell, \quad z' = z/\ell, \quad t' = t \ell' \). Therefore \( \omega_\Pi \) is exponentially increasing with time when \( \Omega_{\Pi} \leq 0 \). Let us therefore write

\[ \Omega = \Omega_o - i \varepsilon \]

where \( \varepsilon = \varepsilon_\Pi + i \varepsilon_\Omega \) and \( \varepsilon_\Omega > 0 \) is the unstable case. Differentiating Eq. (20) yields

\[ \sigma \left( \frac{\rho + \Omega}{\rho - \Omega} \right) = \left( e^{i \frac{X_o}{Q_o}} Q \right) \sigma \left( \frac{i \chi}{1 - \Omega^2} Q \right) \]  

(30)

We found for \( \Omega_o = \Omega_\Pi \) (real) that

\[ \rho_o = 0 \]

or

\[ \lambda_o = \left[ 4 \Omega_\Pi^2 \left( 1 - \Omega_\Pi^2 \right)^{1/2} \right] \]

and

\[ e^{i \frac{X_o}{Q_o} \Omega_o} = -1 \]

or

\[ \frac{X_o}{1 - \Omega_\Pi^2} Q_o = (2n - 1) \pi \]

and we may also obtain

\[ Q_o = \left[ 8 \Omega_\Pi \left( 1 - \Omega_\Pi^2 \right)^{1/3} \right] \]
Equation (30) now implies

\[ -2 \frac{\delta P}{Q_0} = i \left[ \delta K \frac{Q_0}{1-\Omega^2} + x_0 \delta \left( \frac{Q}{1-\Omega^2} \right) \right] \]

or

\[ \delta K = -\frac{x_0}{Q_0} \left[ \frac{2i \delta P}{Q_0} + x_0 \delta \left( \frac{Q}{1-\Omega^2} \right) \right] \]  \hspace{1cm} (31)

We have

\[ \rho = \alpha_0 - 2 \Omega \frac{1}{1-\Omega^2} \]

\[ = \alpha_0 - 2 \frac{\Omega_0 - i \varepsilon}{(1-\Omega^2)^2} \]

such that

\[ \rho = \rho_0 - 2 \frac{\Omega_0 - i \varepsilon}{(1-\Omega^2)^2} \]

In the same way we get

\[ Q \frac{Q_0}{1-\Omega^2} = \frac{Q_0}{1-\Omega^2} \left[ 1 + \frac{4 \Omega_0 \varepsilon}{Q_0} \right] \]

\[ = \frac{Q_0}{1-\Omega^2} \left[ 1 + \frac{4 \Omega_0 \varepsilon}{Q_0} \right] \]

such that

\[ Q \frac{Q_0}{1-\Omega^2} = \frac{Q_0}{1-\Omega^2} \left[ 1 + \frac{4 \Omega_0 \varepsilon}{Q_0} \right] \]

which gives

Equation (31) now implies

\[ \delta K = \frac{-x_0}{Q_0} \left[ \frac{2i \delta P}{Q_0} + x_0 \delta \left( \frac{Q}{1-\Omega^2} \right) \right] \]

which yields

\[ \delta K = \frac{x_0 \Omega_0 (2 \Omega_0^2 + 1)}{(1-\Omega^2)(2 \Omega_0^2 - 1)^2} \varepsilon \]  \hspace{1cm} (32)

Substituting \( \varepsilon = \varepsilon_0 + i \varepsilon_1 \) yields two equations for \( \delta K \) and \( \varepsilon_1 \) if we consider \( \varepsilon \) as our free variable. The imaginary part of (32) gives

\[ \varepsilon_1 = \frac{x_0 \Omega_0 (2 \Omega_0^2 + 1)}{(1-\Omega^2)(2 \Omega_0^2 - 1)} \varepsilon_0 \]  \hspace{1cm} (33)

From (33) we conclude that for \( \Omega_0 > \frac{1}{\sqrt{2}} \) we have \( \varepsilon_1 > 0 \) when \( \varepsilon_0 > 0 \) and for \( \Omega_0 < \frac{1}{\sqrt{2}} \) we have \( \varepsilon_1 < 0 \) when \( \varepsilon_0 < 0 \). Therefore, when we have instability \( (\varepsilon_1 > 0) \) the phase speed increases when \( \Omega_0 > \frac{1}{\sqrt{2}} \) and decreases when \( \Omega_0 < \frac{1}{\sqrt{2}} \).

The real part of Eq. (32) gives

\[ \delta K = \frac{-x_0 \Omega_0 (2 \Omega_0^2 + 1)}{(1-\Omega^2)(2 \Omega_0^2 - 1)} \varepsilon_0 \]

\[ \varepsilon_0 = \frac{-x_0 \Omega_0 (2 \Omega_0^2 + 1)}{2 \Omega_0^2 - 1} \]  \hspace{1cm} (34)
This means that we have instability $\varepsilon_\eta > 0$ for $\delta \kappa < 0$ when $\Omega_o > \sqrt{\frac{\kappa}{\beta}}$ and $\varepsilon_\eta > 0$ for $\delta \kappa > 0$ when $\Omega_o < \sqrt{\frac{\kappa}{\beta}}$ as indicated in Fig. 3. The corresponding unstable region in the $\kappa-\Omega_o$ plane is shown in Fig. 4. From Eq. (34) we may further conclude that for $\Omega_o = 0$ and $\Omega_o = \sqrt{\frac{\kappa}{\beta}}$ we have $\varepsilon_\eta = 0$, and that for higher modes ($\eta$ bigger) the slower growth rates.

Discussion,

For a given depth $h$ and Coriolis parameter $f$ we see from Fig. 5 that the boundary layer becomes unstable for an arbitrary small $U_\eta$. Waves with horizontal wave numbers near zero become unstable first and the corresponding frequency is slight below $f$.

Example: for $\beta = 0.5$ we find for the lowest mode ($n = 1$) that a wave number $\chi = 0.35 \frac{1}{h} 2^{-\frac{k}{2}}$ will be unstable and oscillate with a frequency approximately $3\%$ below $f$. There is also an unstable wave number $\chi = 5 \frac{1}{h} 2^{-\frac{k}{2}}$ with much lower frequency ($74\%$ below $f$). All waves with wave numbers below $\chi$ and above $\chi$ will be unstable, and their corresponding frequencies will be between $f$ and $\frac{0.05}{10^6} f$ for the long waves and between 0 and $\frac{2h}{10^6} f$ for the short waves. For higher modes also other wavelengths will be unstable. They will have the same frequencies as the lowest mode waves but slower growth rates.

We may also interpret this model in another and maybe more interesting way. At $t = 0$ let $U_\eta$ be given, and let $h$ be small such that $\frac{U_\eta}{f} > 1$. As the time goes the boundary layer will grow and after a while the equality $\frac{U_\eta}{h f} = 1$ is satisfied. Then the boundary layer will be unstable and start radiating inertial oscillations with equal horizontal and vertical wave number $\chi = \frac{\pi}{h} 2^{-\frac{k}{2}}$ corresponding to a wavelength $\lambda = 4\sqrt{\frac{\pi}{\beta}} h$, and with frequency near $30\%$ below $f$. If $h$ continues growing both the unstable wavelength bands and the frequency band broaden.

As upper critical limit for the depth of the boundary layer is also found by Pollard, Rhines and Thompson although their model is quite different.
Appendix

If we in Eq. (3) substitute
\[ W = e^{-z \omega} \psi \]
we get
\[ \psi'' + \left\{ \frac{1}{2} \left[ \frac{4U''}{(z^2 - \omega^2)} \right]^2 + \frac{i(U'' + \frac{w^2}{\omega^2})}{(z^2 - \omega^2)} \} \psi = 0 \]  
(A2)

When \( z \to -\infty \) \( U' = U'' = 0 \) and the solution for great depths is
\[ \psi \sim e^{i \omega U \frac{z}{z^2 - \omega^2}} \]

Now, if we multiply (A2) with the complex conjugate \( \psi^* \) and use
\[ \psi^* \frac{d^2 \psi}{dz^2} = \frac{d}{dz} \left( \psi^* \frac{d \psi}{dz} \right) - \frac{d \psi^*}{dz} \frac{d \psi}{dz} \]
we obtain
\[ \frac{d}{dz} (\psi^* \frac{d \psi}{dz}) - \frac{d \psi^*}{dz} \frac{d \psi}{dz} + \psi \psi'^* \left[ \frac{1}{2} \left( \frac{4U''}{z^2 - \omega^2} \right)^2 + \frac{1}{2} \frac{i(U'' + \frac{w^2}{\omega^2})}{(z^2 - \omega^2)} \right] = 0 \]  
(A3)

The imaginary part of this equation yields
\[ -\text{Im} \left\{ \frac{d}{dz} (\psi^* \frac{d \psi}{dz}) \right\} + \psi^* \psi' \frac{2U''}{z^2 - \omega^2} = 0 \]  
(A4)

For \( z = 0 \) we have \( \psi(0) = 0 = \psi'(0) \). Therefore, integrating (A4) from a great depth to the surface gives
\[ -\text{Im} \left\{ \psi^* \left( \frac{d \psi}{dz} \right) \right\} + \int_{z_\infty}^{0} \frac{\psi^* \psi' 2U''}{2(z^2 - \omega^2)} dz = 0 \]

For great depths we have
\[ \frac{d \psi}{dz} = \frac{\lambda}{(z^2 - \omega^2)^\frac{1}{2}} \psi \]
such that
\[ -\frac{\lambda}{(z^2 - \omega^2)^\frac{1}{2}} \psi^* \psi' + \int_{z_\infty}^{0} \frac{\psi^* \psi' 2U''}{2(z^2 - \omega^2)} dz = 0 \]
or
\[ \int_{z_\infty}^{0} \frac{w \psi^*}{w_\infty} \frac{U''}{2(z^2 - \omega^2)} dz = \frac{2 \omega}{\lambda} \left[ 1 - (\frac{\omega}{\lambda})^2 \right] = F(\omega) \]

Now \( F(\omega) = 1 \) such that
\[ \int_{z_\infty}^{0} \frac{w \psi^*}{w_\infty} \frac{U''}{2(z^2 - \omega^2)} dz \leq 1 \]
(A5)

For the broken line profile we have \( U' \neq 0 \) only for \( z = -h \). Therefore
\[ \int_{z_\infty}^{0} \frac{w \psi^*}{w_\infty} \frac{U''}{2(z^2 - \omega^2)} dz = \frac{w_\infty}{w} \frac{h}{h} \leq 1 \]
and a necessary condition for a neutral solution is
\[ \frac{U'}{h} \leq 1 \].
Acknowledgements

I would like to express my gratitude to Dr. Melvin Stern for his suggestion of the problem and extensive guidance in the study.

References


OBSERVATION OF A BOUNDARY JET IN A ROTATING ANNULLUS

Ryuji Kimura

1. Introduction

We are concerned here with an idealized fluid configuration which may be stated as follows: The basic state is a homogeneous viscous fluid layer bounded by a vertical wall (x-z plane) and a sloping bottom at $z = \infty y + z_0$ with a boundary current along the coast (x-direction) (Fig 1). This fluid configuration is motivated by the coastal region of the ocean with a western boundary current.

![Fig. 1](image-url)
The depth of the fluid layer varies in the radial direction. We define the reference depth (H) as the depth just below the inner radius of the ring and this is fixed at 4.6 cm. The Ekman number \( E = \frac{3}{\pi H^2} \) is 4.73 \times 10^{-4}, hence \( E^2 = 2.17 \times 10^{-2} \) and \( E^4 = 0.147 \). We call the above system the basic system (B.S.). Besides the experiments with B.S., the following variation experiments were performed.

1) The same as B.S., but the container has a flat bottom or a sloping bottom with the sign of the gradient reversed:

\[
\frac{dh}{dr} = +0.42.
\]

2) The same as B.S., but the system has a jet or a shear layer at the middle part of the fluid layer.

3) The same as variation 1, but H is 9.2 cm.

All the observations were performed with steady values of the external parameters. The flow is visualized by introducing a droplet of 1% solution of NaOH (except in the case of the measurement of the velocity profile in which case the dye is produced by electrolysis of working fluid).

3. Formation of the basic current

Our system is an example of a free shear layer maintained by the transport of zonal momentum from the upper Ekman layer. A linear theory (see p.97 of Greenspan, 1969) applied in our situation tells us that velocity in the interior region (u), correct to order \( E^4 \) is given by

\[
u = \begin{cases} 
\frac{1}{2} (1 - e^{-my} - e^{-m \sinh my}) & 0 \leq y \leq 1 \\
\frac{1}{2} (\cosh m - 1) e^{-my} & 1 \leq y 
\end{cases}
\]

where u is the nondimensional velocity normalized by the velocity of the driving ring (U), \( y \) is the nondimensional distance normalized by the width of the ring (\( A = 1.85 \) cm) and \( m = \left( \frac{4}{E} \right) \frac{\omega}{H} \). In this calculation we neglect the effect of the circular geometry and the sloping bottom. The equation (1) predicts both the amplitude of the induced flow and the shape of the velocity profile.
In Fig. 3 we compare the observed velocity profile (in B.S.) with the theoretical prediction expressed in (1). In this figure the velocity is normalized by the maximum velocity ($U_{\text{max}}$). The observed profiles in the figure were obtained from photographs as shown in Fig. 4.

Notice that the observed profiles show dependence on the Rossby number: 1) The position of the maximum velocity shifts toward the inner region with increase of the Rossby number. 2) In the negative Rossby region (where the ring rotates slower than the basic system) the width of the jet increases as the Rossby number increases. Comparing these results with the theoretical profile we notice that the observed profiles shift toward the side wall. Veronis and Yang (1972) calculated velocity profiles of a boundary jet including the effect of advective terms, but the correction due to the nonlinear effect shows the opposite sense to Fig. 3. The author supposes that this discrepancy may be caused by the centrifugal effects in the upper Ekman layer.

The observed amplitude of the basic flow in the negative Rossby region shows no systematic dependence on Rossby number. The amplitudes are compared to the theoretical values obtained from (1) in Table 1.
The observed amplitude of the basic flow in the positive Rossby region shows a definite dependence on the Rossby number. In the Rossby region between 0.1 and 0.2 the relation can be expressed as

\[
\frac{[\omega_b]}{f_0} = 0.49 - 0.7 \text{Ro},
\]

This may be caused by the frictional dissipation of the momentum at the side wall, because the momentum from the upper Ekman layer is transported downward at the outer radius of the ring in the positive Rossby region, so that dissipation at the side wall is expected to be larger than that in the negative Rossby region.

### Table 1.

<table>
<thead>
<tr>
<th>a/H</th>
<th>Obs.</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.402</td>
<td>0.42</td>
<td>0.40</td>
</tr>
<tr>
<td>0.201</td>
<td>0.37</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Fig. 4
4. Stability of the boundary jet

4.1 Onset of instability

Stability of the boundary jet was observed 15 min. after mixing of the fluid layer. When the Rossby number is small, the initial disturbance settles down in the first five minutes: Eddies produced by the initial disturbance are deformed by the shear current and disappear. When the Rossby number exceeds some value, however, a row of eddies is formed in the inside of the ring \((r \approx r_{\text{ring}})\) and eddies move in the shear flow like bearings between two plates which move with different speeds. We call this situation "unstable". The unstable eddy has a vertical axis and is almost two-dimensional (a vertical roll).

In a very few cases the fluid showed a peculiar behavior in which the flow remains axisymmetric even if the Rossby number is greater than the critical value. But the flow field in the interior region is different from the stable case: If we put a droplet of dye near the ring, the dye describes a spiral toward the center instead of describing a circle. The author could not find whether this is intrinsic to instability or is caused by stratification of the fluid layer.

4.2 Critical Rossby number for instability

The critical Rossby number \((R_c)\) in B.S. is compared with that in the variation 1) in Fig. 5. The thick segments of the line in Fig. 1 show the region of \(R_o\) in which the critical Rossby number exists. The sign on the segments is the sign of Rossby number. Notice that \(R_c\) is either near 0.11 or near 0.18 according to the combination of the flow direction and sloping condition.

![Fig. 5](image)

Figure 6 shows the similar results for the variation 3. This result suggests that the increase of the fluid depth stabilizes the flow.
Figure 7 shows the similar results for variation 2. This result suggests that the boundary jet is more stable than the free jet or the shear layer.

4.3 Discussion of the dynamic process

Two basic mechanisms for instability are possible for our system: inertial instability and dynamic instability.

The condition

$$\frac{d}{dr} (\text{angular momentum}) > 0$$

(2)

gives the sufficient condition for inertial instability. The Rayleigh criterion for dynamic instability (in rotating system) can be stated that

$$\frac{d}{dr} (\text{potential vorticity})$$

(3)

does not change sign for stability. By applying (1) to (2) and (3) we can estimate the lower limit of $R_c$ for both kinds of instability. The results for B.S. are as follows:

for $Ro > 0$, \[ R_c\text{inertial} = 0.116 \]
\[ R_c\text{dynamic} = 0.005; \]

for $Ro < 0$, \[ R_c\text{inertial} = 0.228 \]
\[ R_c\text{dynamic} = 0.01. \]

Applying this result to Fig. 5 we find that $R_c$ obtained at the negative $Ro$ region in G.S. is not caused by the inertial instability. Since the features of the instability in other cases are almost the same as $R_o < 0$ (B.S.) case, we can estimate that the inertial instability is not essential for the mechanism of instability observed in our experiment. The
effect of sloping bottoms expected from (3) is consistent with Fig.5. Notice in Fig.5 that we have almost the same value of $R_c(\sim 0.18)$ in both the positive and negative regions, when the sloping bottom favors stabilization. This result suggests that the circular geometry of our fluid system does not significantly affect the instability of the basic flow, so that we can expect that our results can be well-explained by the theory which neglects the effect of circular geometry. In addition, Fig.5 tells us that the sloping bottom gives no essential effect for instability when it does not favor stabilization.

For a more sophisticated discussion on the stability properties of the boundary jet, it is necessary to apply the linear theory by Busse (1968) to our system. But this is outside the scope of the present study.

5. Eddy field in the unstable regime

5.1 Stationary eddy fields and time-dependent eddy fields.

It was found that our fluid system allows two types of eddy field in the unstable situation: One is the stationary eddy field in which a row of eddies moves along the shear flow without changing form. The detailed structure will be described in the next section. The other is the flow field in which eddies have a finite lifetime and the configuration of eddies is disorganized.

Table 2 summarizes the situations for these two types of eddy fields. Notice that the time-dependent eddy field does not appear in the free jet. Comparing Table 2 with Fig.5, we find that the time-dependent eddy field appears only when the bottom slope favors stabilization.
Fig. 8.
5.2 Flow pattern of the stationary eddy field

The observed wave numbers (k normalized by a) in the stationary eddy field (and also in the time-dependent field in some cases) are shown in Fig.9. These results show that the fluid has no strict preference for any particular wave number. Considering that the initial condition for these data is the independent random disturbance with finite amplitude, the wave number in the stationary state seems to be affected by the initial condition.

However, it may be said that

1) The wave number for the free jet or the shear layer is smaller than that for the boundary jet;

2) The wave number decreases with increase of Ro. The decreasing rate in the flat bottom case (the time-dependent field, so that the wave number is not so clearly defined) is larger than the sloping bottom case.
The observations of the phase velocity of the eddies were performed by measuring the travel time of an eddy between the known distances. The accuracy of the data was not so good, but we could get a rough idea for the phase velocity of the eddies as shown in Fig.10.

\[ \omega_{\text{pr}} \]

\[ \omega_{\text{pr(max)}} \]

\( \omega_{\text{pr(max)}} = \text{max. angular velocity of the basic flow.} \)

\( \text{B.J. (F.B.,+Ro)} = \text{boundary jet in the flat bottom in the positive Rossby number region.} \)

6. Tentative conclusion

1) The leading mechanism for instability is not the inertial type, but the dynamic instability.

2) The critical Rossby number for instability is much affected by the sloping bottom, when the sloping bottom favors stabilization. When the sloping bottom does not favor stabilization, however, the critical Rossby number in the sloping bottom case is similar to that in the flat bottom case.

3) The stability property of the eddy fluid in a single row may obey the following rule:

\[ \frac{\text{dh}}{\text{dy}} : \text{gradient of the fluid depth} \]

\[ \frac{\text{d}U}{\text{d}y} : \text{basic shear in which eddies are located.} \]

4) The eddy field in the free jet (in which we have double rows of eddies) is always stabilized.
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References

Greenspan, H. P. 1969 "The theory of rotating fluids".

EXCITATION OF LOW MODE INTERNAL WAVES
BY A TURBULENT BOUNDARY LAYER

Erik L. Petersen

1. Introduction

The existence of internal gravity waves of small amplitude in a Boussinesq inviscid, adiabatic fluid has been the subject for many theoretical studies.

The aim of this paper has been to understand the mechanism by which a turbulent layer can generate waves in a surrounding essentially nonturbulent medium.

This mechanism has been studied by A. A. Townsend in a series of papers (1965, 1966, and 1968). To overcome the considerable difficulties of the effect of the variation of the horizontal mean wind vector with
Townsend (1967) uses an approximation by which the internal waves are considered as an assembly of wave packets of limited spatial extent which are radiated by the turbulent atmospheric boundary layer and the approximation is one of large wave numbers.

Although in this paper we are merely concerned with an atmosphere without a mean wind shear and the generation of low-mode internal waves with wavelength large compared with the variation of the Brunt-Väisälä frequency $N$, it is interesting to ask how much of an answer the wave-packed approximation can give to the following question: Is it possible to specify a boundary layer with a turbulent kinetic energy spectrum $K$ and an overlaying nonturbulent atmosphere with $N$ and $\mathcal{U}$ varying with height so that the combination of $k$, $N$ and $\mathcal{U}$ causes in some specified sense a maximum influence on the boundary layer and/or the atmosphere above?

$N$ is defined as

$$N^2 = \frac{g}{\rho \mathcal{P}}$$

(1.1)

where $g$ is acceleration due to gravity and $\rho$ is the mean potential density.

To see what we mean by 'a maximum influence on the boundary layer' we can cite a rather provocative conclusion from Townsend (1968):

"Radiation fluxes of energy and momentum from the boundary layer are sufficient to modify its structure considerably, and it is possible that, in neutral conditions, the earth's boundary layer is nearly as thick as is dynamically possible in the stably-stratified environment. In other words, the radiation of internal waves restricts the layer thickness and the effects are comparable with those of the Coriolis forces". As we have not been fully able to understand the cited paper we will desist from discussing it here.

An important effect caused by the radiation is the creation of patches of clear-air turbulence by breaking of internal waves in the atmosphere above the boundary layer. Other important effects can be caused in the upper atmosphere through the ability of the troposphere to transport energy from the energetic boundary layer to the stratosphere and ionosphere through the leak of energy upwards in the low mode internal waves. (E. Gossard and Munk (1954), E. Gossard (1962).

\footnote{For example.}
It is not possible to find the above effect using the wave-packed approximation, because to this approximation, as we will see later, the wave energy will either be totally absorbed or totally reflected at certain critical levels. J.R. Booker and F.P. Bretherton (1966) find that instead of total absorption the waves pass the critical level but are attenuated by a factor $\exp[-2\text{i}(R_+)^{1/2}]$ where $R$ is the Richardson number.

2. The Wave Packed Approximation

In order to understand the difficulties which arise when one seeks an answer to the question posed in section 1, we will use some results from F.P. Bretherton (1966).

Making the following assumptions: the motion is inviscid, airdynamic, Boussinesq and non-rotating. He shows that the vertical velocity $w$ satisfies the linearized equation

$$\frac{D^2 w}{Dt^2} + \nabla w - \frac{D}{Dt}(u \frac{\partial w_y}{\partial x} + v \frac{\partial w_x}{\partial y} + N^2 (w_x + w_y)) = 0$$

(2.1)

where $D = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$ and $u(z) = (u(z), V(z))$.

An approximate solution to this equation can be found using the WKB approximation. Here the solution is in the form

$$w = R \left[ (w + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots) e^{i \varepsilon \phi} \right]$$

(2.2)

where $\varepsilon$ defines a time variable $\tau = \varepsilon t$. The lowest order solution in $\varepsilon$ gives the frequency relation

$$\omega = k \cdot U + N \left( \frac{k_x^2 + k_y^2}{m^2} \right)^{1/2}$$

(2.3)

for plane internal gravity waves

$$w = \omega \sin (k_x x + k_y y + m z - \omega t)$$

where $k = (k_x, k_y)$ is the horizontal wave number vector, $m$ the vertical wave number and $\omega = \omega - k \cdot U$ is the local wave frequency as measured moving with the fluid at height $z$. This solution which is only locally valid shows that a wave packet of dimensions small compared with the scale of vertical inhomogeneity moves in a direction inclined to the horizontal by an angle $\phi = \cos^{-1}(w)/N$. The group velocity is equal

$$G = \left| \frac{N}{k} \sin \phi \cos \phi = \frac{\omega}{|k|} \sin \phi \right|$$

and the vertical group velocity

$$w_g = G \cos \phi = \frac{\omega}{|k|} \sin \phi \cos \phi.$$
Satisfying Eq. (2.1) to the next order in $\varepsilon$ leads to a kind of wave action conservation principle which states that for a wave packet, the total energy of the packet is proportional to $\omega_L$. This principle combined with the kinematic conservation principle which says that for a wave group the horizontal wave number ($|k|$) and absolute frequency ($\omega$) are unchanged as the wave group propagates, enables us to find an approximation to the variations of the wave amplitude with height.

The energy density of the wave motion is equal to $\frac{N^2a^2}{\omega_L}$ where $a$ is the vertical displacement of a fluid particle from its equilibrium level. The total energy of a wave packet as it moves will be $\sim N^2a^2 \omega$. The quantity $\frac{N^2a^2 \omega}{\omega_L}$ is then conserved as the packet moves and so is

$$\frac{(N^2a^2 \omega_L |k|)}{\omega_L} = a^2 \omega_L (N^2-a^2)^2 = \text{constant}$$

Using from above

$$\phi = \frac{\omega_L |k| \sin \phi \cos \phi}{\omega_L / n \text{and so}}$$

$$\sin \phi = N^{-1}N^2-a^2$$

where 'o' indicates values at the top of the boundary layer. This is the same variation A. A. Townsend (1968) finds and it is the basis for the discussions through his paper.

It is seen that to this approximation, the amplitude becomes infinite where $\omega_L$ becomes either zero or equal to the local value of $N$. From Eq. (2.3) it follows that $\omega_L \to m \Rightarrow m \to \infty$ and that $\omega_L \to 0 \Rightarrow m \to \infty$ because $|k|$ is constant. This indicates reflection conditions for $\omega_L = N$ and accumulation conditions for $\omega_L = 0$.

At a certain height wave energy is confined to wave numbers which have not satisfied either the condition for reflection or that for accumulation at any lower level and we would expect the contribution of wave energy at that height to be large for wave numbers either undergoing reflection or being accumulated there.

Because of the difficulties we have seen arising in this section we have found it tempting to attack the problem of the generating of
internal waves along other lines using in essence the principles applied by O. M. Phillips (1966) to explain the generating of surface waves by the wind.

3. The Model Atmosphere

For the purpose of the present discussion a simple model will be adapted which retains some features of a real atmosphere and allows a fairly simple discussion.

A discontinuity is assumed between two nearly adiabatic layers. Two cases will be considered (a) where the discontinuity is abrupt and (b) where it is extending over a range \( \xi \). The height of the turbulent boundary layer is \( H \) and the mean horizontal wind is assumed constant with height (except very near the ground in the boundary layer). The height \( z \) is measured upwards from the middle of the temperature inversion and the vertical velocities are assumed to vanish at a height \( h_1 \).

Two situations in the atmosphere to which our model could pertain are: 1) where the inversion is at the top of the Ekman layer and the turbulent boundary layer is of the order of 100 meters caused by mechanically induced turbulence due to a very large wind shear near the ground; this again caused by a speed-up of the whole layer as a result of the sharp decreasing of the downwards momentum-transport before sun-down. (The forces governing the speed-up are the pressure and coriolis forces, of course.) 2) where the turbulent layer is the whole Ekman layer and the inversion could be on a frontal slope.
At the ocean (where the boundary layer extends downwards from \( h_1 \)) is recognized as a situation often occurring near the mouth of some fjords where there is a layer of fresh over salt water (The Dead Water Phenomena in Norwegian fjords). Case (b) resembles a situation which frequently arises in tropical and subtropical waters when the upper layer of the ocean has been well mixed as a result of a storm at the action of the trade winds. The thermocline is then fairly abrupt and separates water masses above and below, each of which is almost homogeneous (O. M. Phillips (1966) p.166).

4. Model (a): Assumptions and Theory

Let us first consider model (a) without a turbulent layer.

Assuming: potential flow and vertical disturbance at the interface of the form 
\[
F = \eta \exp\left[ i (k_x x - \omega t) \right],
\]
and using the requirements of vanishing vertical velocities at the boundaries, continuity of vertical velocity and pressure through the interface it is easily shown (Lamb 1932, p.371)

\[
\frac{C_0^2}{C} = \frac{g h}{4 c^2 \theta^2} \left( \frac{1}{\cosh^2(k h_1) + \frac{1}{\cosh^2(k h_2)}} \right)
\]

(4.1)

and if \( \omega_0 \) is the vertical velocity at the interface then the vertical velocities vary upwards and downwards respectively as

\[
\begin{align*}
\omega_1 &= \omega_0 \frac{\sinh k h}{\sinh k h_1} \\
\omega_2 &= \omega_0 \frac{\sinh k h}{\sinh k h_2}
\end{align*}
\]

(4.2)

At time \( t = 0 \) a turbulent boundary layer of height \( H \) is introduced. The motion in the system will still have the boundary condition \( \omega = 0 \) at \( z = -h_2 \), but at the same time the generated wave motion will have to satisfy conditions imposed by the movements at the top of the boundary layer. As the inertial conditions, we assume a stationary homogeneous field of turbulence. After some time we expect to have significant vertical velocity amplitudes outside the boundary layer caused by the excitation. The wave movement will extend through the whole system including the boundary layer and possibly be changing the form of the turbulent velocity spectrum.
Let us hypothesize that the amplitudes of the vertical velocities have this form at the top of the boundary layer:

\[
W_{z= h + H} = W_0 \sinh k (h_0 + (-h_0 + H)) + W_T
\]  

(4.3)

where we have used Eq. (4.2).

\( W_T \) is defined as the difference between the total velocity and the wave-induced velocity. At \( t = 0 \), \( W_0 = 0 \) and \( W_T \) is the velocity induced by the turbulent eddies in the boundary layer. As \( W_0 \) increases the boundary layer is distorted by the straining induced by the internal waves and an increasingly part of \( W_T \) will have some phase relation to \( W_0 \). Whether this effect will stabilize or lead to a faster growth of the waves is unknown at present. Here the strain will be assumed small and then \( W_T \) to be independent of \( W_0 \).

We will further assume \( W_T \) to be a stationary second-order stochastic process. \( W_T \) shall then possess the following properties:

1) Mean-zero: for all \((x, t)\) \( E\{W_T(x, t)\} = 0 \)

2) Second-order: for all \((x, t)\) \( \text{VAR}\{W_T(x, t)\} < \infty \)

3) Covariance-stationarity: \( \text{COV}\{W_T(x, t), W_T(x + \epsilon, t + \tau)\} = \text{function}(\epsilon, \tau) \)

A number of statistical relationships can be established for \( W_T(x, t) \) based on these three assumptions. A fundamental result is the Wiener-Khinchine theorem, or rather an extension of the theorem established by Cramer (1940). Let us consider only correlations to the same time; the theorem may then be stated as follows:

To the homogeneous \( W_T(x, t) \) \((x = (x, y))\) there can be assigned a random process, \( z(k, t) \) with orthogonal increments, so that for each fixed \( x \) we have the spectral representation

\[
w_T(k, t) = \mathbb{E} \{ \mathbb{E}\{z(k, t)\} \}
\]

\( z(k, t) \) is a process with orthogonal increments in the sense that \( \mathbb{E}\{z(k, t)\} = 0 \) for any two-dimensional \( \Delta \) interval and

\[
dz(k, t) \cdot dz'(k', t') = \begin{cases} 0 & \text{for any two disjoint two-dimensional intervals } \Delta \Delta \frac{1}{2} \text{ and } \frac{1}{2} \Delta \Delta \frac{1}{2} \\ dF(k, t) & \text{otherwise} \end{cases}
\]

where \( F(k, t) \) is the spectral distribution function.

It follows that the correlation function can be represented as
and if $\mathcal{R}(r,t)$ falls off sufficiently rapidly at infinity, this Fourier-Stieltjes integral can be written as an ordinary Fourier integral of the spectral density function

$$\phi(k,t) = \frac{\partial^2 F(k,t)}{\partial k \partial t}$$

and we have the Fourier transform pairs

$$&&$$

In the following we will assume all the functions where it is convenient to possess the necessary properties that enable us to treat them as second-order homogeneous functions.

5. Model (a) Calculations

**Region 1**

The potential density is constant $\equiv \phi_1$. The flow is irrotational $\nabla^2 \phi_1 = 0$ where $\phi_1$ is the velocity potential. The boundary condition at $z = h_1 : w(z=h_1) = 0$ gives

\[ \phi_1(r,z,t) = \int \frac{e^{ikr}}{k} dC(k,t) \cosh k(z-h_1) \]  

(5.1)

The boundary condition at $z = 0$:

The displacements of the interface have the form
and we have \(-\frac{\partial \phi}{\partial z}\bigg|_{z=0} - \frac{2\pi}{at}\)

further, the pressure is continuous through the interface.

Region (2)

The potential density is constant \(\phi_z\) and the flow is irrotational \(\nabla^2 \phi = 0\) which gives

\[
\int e^{ikr} d[A(k_1,t) e^{kz} + B(k,t) e^{-kz}] = \frac{i k^4}{16} d[A(k_1,t) + B(k,t)]
\]  

(5.3)

As previously argued we will have the velocity

\[
w = w_0 \frac{\sinh(kh)}{\cosh(kh_z)} \quad \text{at} \quad z = -h_x + H
\]

where

\[
w_0 = \int e^{ikr} dW_0(k,t) \]

\[
w_\tau = \int e^{ikr} dW_\tau(k,t)
\]

and

\[
-\frac{\partial \phi}{\partial z} = \int e^{ikr} \left[dW_\tau(k,t) + dW_0(k,t) \frac{\sinh(kH)}{\cosh(kh_z)}\right]
\]  

(5.4)

\[
\text{at} \quad z = H
\]

\[
\frac{\partial \phi}{\partial z} = \frac{\partial \tau}{\partial t}
\]  

(5.5)

and the continuity of pressures

\[
\phi_1 \left[\frac{\partial \phi_1}{\partial t} - g \frac{\partial z}{\partial t}\right] = \phi_2 \left[\frac{\partial \phi_2}{\partial t} - g \frac{\partial z}{\partial t}\right]
\]  

(5.6)

From Eq. (5.1):

\[-k \sin h (-k h_z) \frac{dC}{d\eta} \frac{d\eta'}{k \sinh(k h_z)} = \frac{d\eta'}{dC}\]

(5.7)

From Eq. (5.3) and (5.4)

\[-k \left[dA e^{k(-h_x+H)} - dB e^{-k(-h_x+H)}\right] = dW_\tau + \frac{\sinh(kH)}{\cosh(kh_z)} dW_0
\]

(5.8)

From Eq. (5.5)

\[-k [dA - dB] = d\eta'
\]

\[dA = dB - \frac{d\eta'}{k}\]

into Eq. (5.8) yields
From Eq. (5.6)

\[ -k \left[ (dB - dx')e^{-k(h_x + H)} - db e^{-k(h_x + H)} \right] = dW_t + \frac{\sinh(kh)}{\sinh(kh_x)} dW_o \]

\[ dB = \frac{dW_t - dx' e^{-k(h_x + H)}}{k \sinh(k(h_x - H))} + \frac{\sinh(kh)}{2k \sinh(k(h_x - H)) \sinh(kh_x)} dW_o \]  (5.10)

\[ dA = \frac{dW_t - dx' e^{-k(h_x + H)}}{2k \sinh(k(h_x - H))} + \frac{\sinh(kh)}{2k \sinh(k(h_x - H)) \sinh(kh_x)} dW_o \]  (5.11)

Using Eq. (5.12), (5.1) and (5.3)

\[ \rho (\cosh kh, dC') - \rho_2 (dA' + dB') = \frac{q}{d} \eta [\rho_1 - \rho_2] \]

which with Eq. (5.7), (5.11) and (5.12) becomes

\[ \rho (\cosh kh, dC') - \rho_2 (dA' + dB') = \frac{q}{d} \eta [\rho_1 - \rho_2] \]

and using \( dW_o = d\eta' \)

\[ d\eta' \left[ \rho_1 \cosh kh_1 + \rho_2 \cosh h_x h_x + \frac{\sinh(kh)}{\sinh(kh_x)} \right] = \frac{q}{d} \eta [\rho_1 - \rho_2] + \frac{\rho_2 dW_o}{k \sinh(k(h_x - H))} \]

and finally

\[ d\eta'' + \omega^2 d\eta = m dW_t \]  (5.13)

where

\[ \omega_t = \frac{q k (\rho_1 - \rho_2)}{\rho_1 \cosh kh_1 + \rho_2 \cosh h_x h_x + \frac{\sinh(kh)}{\sinh(kh_x)} \sinh kh_x} \]  (5.14)

\[ m = \frac{\rho_2}{(\rho_1 \cosh kh_1 + \rho_2 \cosh h_x h_x + \frac{\sinh(kh)}{\sinh(kh_x)} \sinh kh_x)} \]  (5.15)

We see that \( \omega_t \to \infty \) from Eq. (4.1) as \( H \to 0 \).

Before we proceed with the differential equation, let us look at case (b).
6. Model (b)

The vertical motion in the 'thermocline layer' is governed by Eq. (2.1). In the absence of a mean shear this equation can be approximated by

$$\frac{\partial^2}{\partial z^2} (\nabla^2 w) + N^2 \frac{\partial^2 w}{\partial z^2} = 0$$

(6.1)

(see O.M. Phillips (1966) p. 162).

$w$ is as before

$$w = \int e^{ik\cdot r} dW(k, t)$$

The contribution to $w$ from wave number $k$ is

$$w_k = e^{ik\cdot r} dW(k, t)$$

which substituted into Eq. (6.1) gives

$$\frac{\partial^2 (dW)}{\partial t^2} - \frac{\partial^2 (dW)}{\partial z^2} = 0$$

(6.2)

(which is simply the well-known equation

$$\frac{\partial^2 (dW)}{\partial t^2} + \left( \frac{N^2}{\partial z^2} \right) k^2 (dW) = 0$$

if we put

$$dW(z, t, k) = dW(z, k) e^{-i\omega t}$$

In the lowest mode of internal wave motion the vertical velocity changes little across a sharp thermocline and as before

$$dW_1 = dW_1\bigg|_{z=0}$$

The only difference between this example and the previous one is that instead of the pressure condition of the interface we here can find the change in $\frac{\partial (dW)}{\partial z}$ over the thermocline range $\varepsilon$ by integrating Eq. (6.2) over

$$\frac{\partial}{\partial t} \left( \frac{\partial (dW(0))}{\partial z} - \frac{\partial (dW(0))}{\partial z} \right) = \frac{1}{\varepsilon} \left( \frac{\partial^2 (dW)}{\partial t^2} + N^2 (dW) \right) d\varepsilon$$

$$= k^2 \left[ \frac{\partial^2 (dW(0))}{\partial t^2} - \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial z} d\varepsilon \right]$$

$$= k^2 \left[ \frac{1}{\varepsilon} \frac{\partial^2 (dW(0))}{\partial t^2} + dW(0) \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial z} \right]$$

(6.3)

where $\rho > 0$ is the difference in density below and above the thermocline.

Using the results from the first example, we have for the velocity in region (1)

$$\Phi_{1, k} = dC \cos \kappa (z-h_1) e^{ik\cdot r}$$

$$dW(k, z, t) e^{ik\cdot r} = -\frac{\partial \Phi_{1, k}}{\partial z} = -k dC \sin \kappa (z-h_1) e^{ik\cdot r}$$

$$\frac{\partial (dW)}{\partial z} = -k^2 dC \cos \kappa (z-h_1)$$
\[
\frac{d (dW)}{dz} \bigg|_{z=0} = -k \eta \cosh (k h)
\]

**Region 2**

\[
\phi_{2,k} = (dA e^{kz} + dB e^{-kz}) e^{ikx}
\]

\[
dWe^{i\eta r} = -\frac{\partial \Phi_{2,k}}{\partial z} = -k (dA e^{kz} - dB e^{-kz}) e^{ikx}
\]

\[
\frac{d (dW)}{dz} \bigg|_{z=0} = -k^2 (dA e^{kz} + dB e^{-kz})
\]

which gives

\[
\frac{d (dW)_{0+}}{dz} = -k \left[ \frac{2dW - 2\eta \cosh k(h_z - H)}{2 k \sinh k(h_z - H)} \right] = \frac{\sinh (k H) d \eta}{k \sinh (h_z - H) \sinh (k h_z)}
\]

Substituting into Eq. (6.3) yields

\[
-k \left[ \frac{\sinh (k H)}{\sinh (h_z - H) \sinh (k h_z)} \right] \frac{dW}{\sinh (k h_z)} = \frac{dW}{k \sinh (k h_z)}
\]

\[
= k^2 \left[ \frac{d \eta''}{dz} + \frac{d \eta'}{dz} \frac{d \eta'}{dz} \frac{\frac{d \rho}{dz}}{\rho} \right]
\]

Integrating this equation with respect to time gives

\[
d \eta'' + \omega_T^2 d \eta = m dW_T
\]

where

\[
\omega_T^2 = k g \frac{d \rho}{dz} \left[ \coth(k h) + \coth(k(h_z - H)) + k \varepsilon - \frac{\sinh(k h)}{\sinh(k h_z) \sinh(k h)} \right]^{-1}
\]

\[
m = \left[ \frac{\sinh(k h_z) \sinh(k h)}{\sinh(k h_z) \sinh(k h_z)} \right]^{-1}
\]

As we would expect this is a differential equation of the same form as before. We notice that for \( H \to 0 \)

\[
\omega_T^2 \to k g \frac{d \rho}{dz} \left[ \coth(k h) + \coth(k h_z) + k \varepsilon \right]^{-1}
\]

the same result as obtained by O.M. Phillips (1966).
7. The Differential Equations

The solution to the differential equation

\[ d\eta'' + \omega_T^2 d\eta = m dW_T' \]

with \( d\eta = d\eta' = 0 \) for \( t = 0 \) is

\[ d\eta = \frac{i}{\omega_T} \int_0^t m dW_T' (t') \sin \omega_T (t-t') dt' \]

(see (f.eks.) Courant 1968, p. 449)

and so

\[ \psi(x,t) = \frac{m^2}{\omega_T^3} \int_0^t \Omega(k,t-t') \sin \omega_T (t-t') \sin \omega_T (t-t') dt' dt'' \]

Here \( \psi(x,t) \) is the time-dependent spectrum of the vertical displacement at the interface and \( \Omega (k,t-t') \) represents the spectrum of the vertical accelerations with time delay \( t'-t'' = t' \). Because of the assumption of stationarity \( \Omega (k,t-t') \) is independent of the time origin \( t' \) and the double integral can be reduced using the transformation

\[ t = t' - t'' \]

\[ t' = t \quad \text{with the Jacobian} = 1 \]

We have

\[ \psi(x,t) = \frac{m^2}{\omega_T^3} \int_0^t \Omega(k,t-t') \sin \omega_T (t-t') \sin \omega_T (t+t-t') dt' \]

\[ = \frac{m^2}{2\omega_T^3} \int_0^t \Omega(k,t) dt \left[ \cos \omega_T (t) - \cos \omega_T (t-t') + 2 \sin \omega_T (t-t') + \frac{\sin \omega_T (t-t')}{2\omega_T} \right] dt \]

\[ = \frac{m^2}{2\omega_T^3} \int_0^t \Omega(k,t) \cos \omega_T (t) dt \]

\[ + \frac{m^2}{2\omega_T^3} \int_0^t \Omega(k,t) \sin \omega_T (t) dt \]

\[ - \frac{m^2}{2\omega_T^3} \int_0^t \Omega(k,t) \sin \omega_T (t) dt \]
When \( t \) becomes large compared to the integral scale of the process, we can use the integration limits \( \pm \infty \) instead of \( \pm t \). The second term is zero because \( \Omega(k,T) \) is an even function of \( T \). The third term can be written as

\[
- \frac{1}{2\omega_T} \int_{-\infty}^{\infty} \Omega(k,T) \cos \omega_T T \, dt + \int_{-\infty}^{\infty} \Omega(k,T) \sin \omega_T T \, dt
\]

and we have

\[
\psi(k,t) = -\frac{m^2(2\omega_T - \sin 2\omega_T t)}{2\omega_T^2} \int_{-\infty}^{\infty} \Omega(k,T) \cos \omega_T T \, dt
\]

or

\[
\psi(k,t) = -\frac{m^2}{\omega_T^2} t \int_{-\infty}^{\infty} \Omega(k,T) \cos \omega_T T \, dt
\]

Introducing the Fourier-transform pairs

\[
\Pi(k,n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega(k,T) e^{int} \, dt
\]

\[
\Omega(k,T) = \int_{-\infty}^{\infty} \Pi(k,n) e^{-int} \, dn
\]

we get \( \Pi(k,n) \): the wave-number frequency spectrum of the vertical accelerations at the top of the boundary layer.

From

\[
\int_{-\infty}^{\infty} \Omega(k,T) \cos \omega_T T \, dt = 2\pi \Pi(k,\omega_T)
\]

where the value of \( \Pi(k,n) \) is to be taken as the frequency \( n = \omega_T \) and \( \omega_T \) is given by Eq. (5.14) we get

\[
\psi(k,t) = \frac{\omega_T}{\omega_T^2} \int_{-\infty}^{\infty} \Pi(k,\omega_T) \, d\omega_T
\]

From this equation it follows that to the approximations done, the spectrum of the displacements at the interface increases linearly with time under the influence of the turbulent accelerations at the top of the boundary layer.

The mean potential energy density for the wave motion is

\[
\frac{1}{2} \delta p g \bar{\eta}^2
\]

and hence the total energy density \( E_{\text{tot}} = \delta p g \bar{\eta}^2 \) so that

\[
E_{\text{tot}} = \delta p g \bar{\eta}^2 = \delta p g \left( \psi(k,t) \right) \, dk
\]

\[
= \frac{2\delta p m^2}{\omega_T^2} \int \Pi(k,\omega_T) \, dk
\]
and

\[
\frac{\partial E_{\text{ext}}}{\partial t} = 2\sigma \rho g m^2 \int k (k, w) d\kappa
\]

which gives us the energy flux through the top of the boundary layer.

It would be interesting to relate the rate of energy loss from the boundary layer by radiation of internal waves to the rate of production as well as to the turbulent kinematic energy in the layer. This requires a relation between \( \int (k, w) \) and \( (5, h) d\kappa dn \) and then between the latter and the kinetic energy spectrum for the boundary layer. The kinetic energy spectrum is a spectrum of velocities and integrates to the mean kinetic turbulent energy. It is a spectrum of the time derivative of the vertical velocity. To estimate \( T(k, n) \) from the spectrum of vertical velocities, \( K_{\nu r} \), one way would be to use Taylor's Hypothesis which says that when the turbulence level is low enough, the evolution in spatial pattern of a lump of fluid during its transit past a fixed point may be so slight that the pattern is effectively 'frozen' during passage. Then the changes at the point with time are due only to spatial nonuniformities being convected past the point at mean wind speed. \( (v_c) \)

\[
\begin{align*}
\langle w(z+t) w(z+\xi, t+\tau) \rangle &= R_w(z, \tau) = \int K_{\nu r}(k, n) e^{i(k\xi-n\tau)} d\kappa dn \\
\frac{\partial \langle w(z+t) \rangle}{\partial t} &= \frac{\partial R_w(z, \tau)}{\partial t} = \int \frac{\partial}{\partial \tau} K_{\nu r}(k, n) e^{i(k\xi-n\tau)} d\kappa dn \\
\frac{\partial \langle w(z+t) \rangle}{\partial z} &= <v_c \frac{\partial}{\partial x} (w(z,t)) > - <v_c \frac{\partial}{\partial x} (w(z+\xi, t+\tau)) >
\end{align*}
\]

From Eq. (7.2)

\[
- \int v_c \frac{\partial}{\partial t} R_w(z, \tau) = v_c^2 \int \left(-k^2 K_{\nu r}(k, n) e^{i(k\xi-n\tau)} \right) d\kappa dn
\]

which gives

\[
\Pi = v_c^2 k^2 K_{\nu r}.
\]

If we know \( K_{\nu r} \) at the top of the boundary layer and the energy production in the whole boundary layer we would then be able to estimate the effect of radiation on the energy balance in the boundary layer.

Unfortunately the assumption of the whole top of the boundary layer moving as a rigid plate with the constant convection velocity \( v_c \) relative
to the upper fluid is not a very realistic one especially not in this case where we are interested in small wave-number turbulence. If it were true we would have the very simple dispersion relation for the boundary layer:
\[ n = \pm k \cdot V_c. \]
In \( \Pi(k, n) \) there would for one specific \( k \) only correspond two frequencies \( n = \pm k \cdot V_c \); but in general for every \( k \) there will exist a wide range of frequencies.

However let us suppose that there exists a frequency \( n_0 \) for which \( \Pi(k, n) \) is greatest for a particular wave number \( k \) and let us define a convection velocity in the direction of the mean wind by \( n_0 = \pm k \cdot V_c(k) \).

\( V_c(k) \) is in general a function of \( k \). From Eq. (7.1)
\[ \psi(k, t) = \frac{2 \pi m t}{\omega^2} \Pi(k, \omega). \]
where the relation between \( k \) and \( \omega \) is given by Eq. (5.14). Because \( \Pi(k, n) \) is maximum for \( n = n_0 \) we will expect \( \psi(k, t) \) to be maximum for \( \omega = n_0 / k \), i.e. the polar wave spectrum \( \psi(k, \omega^2) \) will have a directional maximum for
\[ \omega = \frac{k V_c \cos \alpha_m}{\omega}. \]

Note the \( V_c = V_c \left| K \cdot \omega_m \right| \), and only for an isotropic \( \Pi(k, n) \) will the directional maximum be bi-modal. Eq. (7.3) can be written \( \omega = \frac{\omega_m}{V_c} \) where \( C \) is the phase speed. This is a kind of a resonance condition for the system. If the projection of the convection speed vector on the phase speed vector equals the phase speed vector we will expect this will be a direction where waves with significant amplitude will develop with time.

All the descriptions have been given in a coordinate system moving with the wind speed \( U \) of the upper layer. In this coordinate system the wave crests will lie perpendicular to the wind direction if \( C = V_c \) and parallel to the wind direction if \( V_c \gg C \). To get the phase speed vector as seen from the ground \( U \) has to be added to \( C \).

8. Conclusions

It has been shown for a simple atmosphere model where a discontinuity is separating two adiabatic layers, that the turbulent boundary layer at the bottom can generate waves in the atmosphere and these waves grow linearly with time. Further, it was shown that there exists a kind of resonance condition which will favour the growth of waves in certain
directions; the growth also being dependent on the form of the kinetic energy spectrum in the turbulent boundary layer.

More information can possibly be gained from this first simple model. An obvious extension of this study is to relax the upper boundary condition for the upper region and let $N(z)^2$ vary, perhaps as in the former-mentioned papers by Gossard and Munk (1954) and Gossard (1962). Another tempting study is the caustic one as mentioned in section 1.

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References


A THEORY OF BOUDINAGE

Ronald B. Smith

The term "boudinage", as used in structural geology, refers to the nearly periodic pinching or breaking of one rock layer surrounded by rock of different physical properties (see Fig. 1). Boudinage is found commonly in sedimentary beds such as sandstone or limestone in shale or dolomite in calcite, and in hard rock situations such as a quartz vein in gneiss (de Sitter, 1964). In all cases the boudinaged layer is believed to be more competent (i.e., more resistant to deformation) than the surrounding rock. Ramberg (1955) has shown convincingly how compression perpendicular to the layers can produce tensile stress in the competent layer. Unexplained, however, is the mechanism which determines the spacing of the break points. This writer believes Ramberg's result, that the ratio of boudin length to bed thickness \( r = \lambda/H \) can have any value depending on the applied compressive stress and rock strength, may be in conflict with observation. It seems in fact, that observed values of 'r' cluster strongly about a value like 2, in spite of the wide variety of rock types, temperature and pressures and applied loads associated with boudinage.

![Fig. 1 Boudinage](image)

The objective of this paper is to find a mechanism which will select a particular scale for the breaking and pinching.

The approach used herein is to consider the deformation of the rock, imposed by large scale tectonic forces acting over millions of years, as flowage governed by a stress-rate of strain relation. And deformation can be thought of as part elastic (i.e., reversible) part fracture and part flow. Over a sufficiently long period of time, called the rheidity by Cary (1953), the flow deformation will dominate the others. The material may then be called a rheid. The rheidity for rocks will vary from
a few weeks to ten thousand years but for times longer than this the flow model is appropriate. We shall further assume that the stress-rate of strain relationship is isotropic and, up to some critical stress, linear (i.e. Newtonian).

Consider a layer cake geometry where two thick beds surround a thin bed of higher viscosity. Let this whole system be squeezed perpendicularly to the beds. The less viscous beds will tend to flow out horizontally and in doing so will impart a tensile stress \((-x^2)\) in the mare viscous bed, through a shearing stress at the interfaces. The competent layer cannot remain of constant thickness and pinching will tend to occur at \(x = 0\). Subsequent pinching may occur at \(x = \pm L/2, \gamma = 1, 2, 3, \ldots\). This model of Ramberg does not agree with the hypothesis that the boudins tend to form on the scale of the competent bed thickness.

Look instead at motions occurring on the scale of the competent bed thickness \((H)\) which is assumed to be much smaller than the scale of the applied compressive load. This situation is shown in Fig.2.

![Fig.2 Local geometry and basic flow](image)

Presumably the choice of break of pinch points must be determined by existing inhomogeneities, possible slight variations in the layer thickness \((H)\) caused when the rocks were formed.
These irregularities could interact with the mean tensile stress or the mean flow to intensify the stresses locally. In what follows we will investigate the latter of these effects.

We introduce a perturbation velocity field which will be represented as a separable stream function

$$\Psi(x, y) = \Phi(y) e^{i\alpha(x - ct)}$$

and the perturbation interface deflection as

$$h(x) = \eta e^{i\alpha(x - ct)}$$

Inertia plays no role in this flow (Reynolds number $\sim 10^{-30}$) so in each layer

$$\nabla^2 \Psi = 0 \quad \alpha \Phi'' - 2\alpha \Phi' \Phi + \alpha^2 \Phi = 0$$

At an interface between layers of different viscosity we require continuity of:

- normal velocity $\Phi = \Phi_s$ \hspace{1cm} (4a)
- tangential velocity $\Phi' + S\eta = \Phi_s' + S_s \eta$ \hspace{1cm} (4b)
- tangential stress $\Phi'' + \alpha \Phi' = m [\Phi'' + \alpha \Phi_s']$ \hspace{1cm} (4c)
- normal stress $\Phi'' + 3 \alpha \Phi' = m [\Phi'' + 3 \alpha \Phi_s']$ \hspace{1cm} (4d)

where $S_s = -\frac{\partial \eta}{\partial y}$, $S_s = -\frac{\partial \eta}{\partial y}$ evaluated at $Z = \pm H/2$ (U, $\eta y$) is the basic flow) and $m = \alpha \Phi_s'$ is the viscosity ratio. The kinematic condition at the interface is

$$\left( -\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) h = -\Psi_x$$

or

$$\eta = \frac{\Phi}{C'}$$

where $C' = C - U$ is the phase speed measured with respect to the velocity at the interface. Conditions (4) and (5) are applied at the mean position at the interface $Z = \pm H/2$ and valid only for small interface deflection. A more complete description of these interface conditions is given by Yih (1967). In addition to Eqs. (3), (4) and (5) we shall require the disturbance to be bounded at large $|y|$. 

This system represents an eigenvalue problem where $C'$ must be chosen to satisfy all the specifications of the problem. The eigen-
value and the effect of the basic flow do not appear in the interior equations but only in the interface condition (4b). This condition can be rewritten as

$$\phi' - \phi_2 = \frac{\delta}{H_2} \phi_1$$

where \( \delta = \frac{S_2 - S_1}{C L} \)

is the nondimensional eigenvalue. The biharmonic equation (Eq. 3) has the general solution

$$\phi_2(y) = k_4 e^{-\alpha y} + k_5 y e^{-\alpha y} + m_1 e^{\alpha y} + n_1 y e^{\alpha y}$$

in each layer \( i = 1, 2, 3 \). With the use of the condition at \( y = \infty \)

$$m_1 = n_1 = k_3 = \ell_3 = 0$$

and the system reduces to eight equations and eight unknown coefficients. Temporarily we will focus attention on the antisymmetric part of the total solution so set

$$\phi(y) = -\phi(-y)$$

whereupon

$$k_1 = -m_1, \quad k_3 = -m_3$$
$$\ell_1 = n_3, \quad \ell_2 = n_2$$

When Eq. (6) is now substituted into Eq. (4), it is found that the determinant of a certain 4x4 matrix must vanish for a solution to exist. This requires a choice of the eigenvalue

$$\delta = -m \left[ \sinh^2 \beta (-m^2 + \beta + m) + \sinh \beta \cosh \beta (1 + m^2) + \cosh^2 \beta (m^2 - \beta + m) \right]$$

where \( \beta = 2 \frac{H_2}{L} \).

The solution corresponding to this choice represents a secondary flow (see Fig. 3) driven by the interaction of an interface displacement and the basic flow. This flow is forced by a concentrated sheet of vorticity at \( \varphi = \pm \frac{H_2}{L} \) (see Eq. (4b)) associated with the discontinuity in the basic shear caused by the variation in viscosity. At any instant this secondary flow is uniquely determined by the shape of the interface. It is in fact the flow that minimizes the dissipation rate for a given forcing. Alteration of the velocity field with time is due solely to changes in the interface shape. The nature of the effect of the secondary flow back on the interface may be seen from the fact that
C is real. The pattern of vertical velocity at \( z = \pm \frac{H}{2} \) is phase shifted by \( \pi/2 \) with respect to the interface shape so the interface deflection does not grow or decay but simply moves to the right, partly convected and partly propagated.

There are also secondary flows caused by symmetric interface deflections and by deflections on one interface that are uncorrelated with the other. These likewise produce no growth as they are phase shifted with respect to the deflection that cause them and uncorrelated with all other components of the deflection.

We must conclude that the growth of boudins must be caused by something not included in the foregoing, either 1) effects of the non-newtonian behavior of the rocks; 2) effects of finite interface displacement; 3) some other aspect of the mean flow, perhaps interaction of an interface displacement with the basic tensile stress.

Consider for example a "soft" material which either fails at some critical stress or smoothly decreases its resistance to deformation with increasing stress. The deformation rate might then increase rapidly at points where the stress field associated with the secondary flow adds to the basic tensile stress in the competent layer. As we have seen, each sinusoidal component of the total interface deflection drives a secondary flow. The amplitude of the velocity and the corresponding stress field will depend strongly on the wave number even if all the wave numbers have equal interface displacement. Large wave numbers have small amplitudes because the large gradients lead to large dissipation. Small wave numbers also have large dissipation as the region of dissipation is large. A measure of this behavior is
amplitude of perturbation stress field

as derived from (5b) and (6). This function has a maximum at \( \beta_m \sim 1.5 \) where \( \beta_m \) is almost independent of the viscosity ratio. The corresponding wavelength \( \lambda_H \sim 2 \) is the wavelength of (other things being equal) the largest periodic stress. The breaking of pinching will tend to occur with this spacing.

Stated another way, the function \( |\beta_j| \) maps a white spectrum of interface displacement into a peaked spectrum of tensile stress and thus selects a scale for the breaking or pinching.

Conclusion

A mechanism for selecting the length of boudins has been described. Unlike Ramberg's result, this special length is independent of the rate of tectonic deformation, average rock strength and only weakly dependent on the viscosity ratio. The calculated length (\( \lambda \approx 2H \)) agrees (\( \pm 50\% \)) with most observed boudins.

There are several problems however. First, only the Newtonian case has been calculated explicitly and the disturbances do not grow in this model. Thus actual growth of boudinage has yet to be demonstrated mathematically. This might be accomplished by treating the above solution as the first term solution to an asymptotic expansion in the strain rate. The second order problem, which will include forcing from the first order, will bring in the effects of non-Newtonian behavior and hopefully produce growth. Second, no suggestion has been made that other models might not be equally successful at determining a boudin length scale. The agreement between observation and theory, therefore, cannot be used as evidence that the present model is correct.

References

A MULTI-TIMING APPROACH TO THE LINEAR SPIN-UP PROBLEM
Jean-Pierre St-Maurice

1. Introduction

The main adjustment of a rotating fluid to a slight change in the rotation rate of its boundaries is basically understood. Greenspan and Howard (1963) have shown that the fluid is spun up in a time of order $t \sim E^{-1}$, where $E$ is the Ekman number $\nu/\Omega L^2$ and assumed small; $\nu$ is the coefficient of kinematic viscosity, $\Omega$ the angular velocity and $L$ the characteristic length scale parallel to the axis of rotation. The characteristic time scale found for the spin up is much shorter than the ordinary diffusion time scale. This happens because Ekman suction pumps fluid from the inviscid interior into the boundary layer. This forces angular momentum to be carried toward the axis of rotation and spins up the interior.

Later on, Holton (1965) and then Walin and Sakurai (1969), and Buzyna and Veronis (1971) used the basic time scale found by Greenspan and Howard (1963) to analyse the spin up of a continuously stratified fluid. The response of the fluid now had a horizontal wave number dependence. In that case, when the fluid moves toward the horizontal boundaries, a pressure gradient is set up in the horizontal by the perturbed density field. Angular momentum is then carried toward the axis of rotation and this "spins up" the fluid (Buzyna and Veronis, 1971). The equilibrium profile of the "spun-up" fluid now has a vertical dependence. The fluid will achieve solid body rotation only through ordinary diffusive processes (Buzyna and Veronis, 1971).

In the two problems just considered we can distinguish three physically very different time scales operating when $E$ is very small. There is first an order $\Omega^{-1}$ scale during which the Ekman layer establishes itself. After a time of order $E^{1/2} \Omega^{-1}$ the fluid is "spun up". Finally a diffusive time scale of order $E \Omega L^2$ is also needed specially in the stratified problem.

It is thus of interest to study the linear spin up of both homogeneous and stratified fluids by means of a boundary layer analysis and
the use of a three-timing technique. In particular, in the stratified case no complete solution has been obtained. Greenspan and Howard (1963) have solved the homogeneous problem using a different technique. We will then show how their results can be obtained again, with some improvements, using a three-timing technique. This analysis will then provide a tool for the study of the more difficult stratified spin-up problem which is now being investigated.

2. Homogeneous spin up

2.1 Formulation

The equations used are those of a time-dependent viscous incompressible and homogeneous fluid in a coordinate system rotating about the z-axis with an angular velocity \( \Omega \). In this non-dimensionalized form and using cylindrical coordinates with no angular dependence, they read

\[
\begin{align*}
\frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - 2 \omega \mathbf{v} &= -\nabla p + \mathbf{E} \nabla^2 \mathbf{v} \\
\frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2 \mathbf{u} &= \mathbf{E} \nabla^2 \mathbf{v} \\
\frac{\partial}{\partial t} \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w} &= -\nabla p + \mathbf{E} \nabla^2 \mathbf{w}
\end{align*}
\]

where \( \epsilon = \mathbf{v} / \Omega \) is the Rossby number

\[
E = \frac{\partial}{\Omega L^2} \quad \text{is the Ekman number}
\]

\[
p = (p' - \rho \frac{\partial \mathbf{v}^*}{\partial t}) / (\rho L \Omega \mathbf{v}^*)
\]

with

\[
\begin{align*}
\mathbf{v}^* &= \text{pressure} \\
\rho &= \text{density} \\
\mathbf{v}^* &= \text{characteristic velocity}
\end{align*}
\]

\( \mathbf{u} \), \( \mathbf{v} \) and \( \mathbf{w} \) are the radial, tangential and vertical components of the velocity respectively.

At the boundaries the fluid is given a tangential velocity \( \mathbf{u} \Delta \Omega \) at time \( t = 0 \) and thereafter. \( \Delta \Omega \) is then chosen small enough so that the Rossby number \( \Delta \Omega / \epsilon \) is negligible.
The variables are next separated into interior variables and boundary variables. In the Ekman boundary layer

$$ \frac{d}{d\zeta} = \frac{1}{\zeta^2} \frac{d}{d\zeta} $$

where $\frac{d}{d\zeta}$ is of order one and $\zeta$ is defined by

$$ \zeta = (1 + \zeta) E^{-V_\infty} \text{ near } \zeta = \pm 1. $$

**Boundary layers**

In the Ekman layer the variables are written with a tilde, e.g., $\tilde{X}$. All the boundary layer variables must vanish when $\zeta \to \infty$.

We do not consider side boundary layers in this problem. They are the ordinary Stewardson $E^V$ layer and the $E^H$ layer. They set the radial and tangential flow equal to zero at the walls and absorb the flow from the Ekman layer to let it go vertically along the walls and be redistributed into the interior.

Following Greenspan and Howard (1963) $u$ and $v$ are taken proportional to the distance from the axis $u = w/r$ and $v = w/r$.

This also implies $f = r^n$.

Finally the variables are ordered in powers of $E^{1/2}$ i.e.

$$ X = \sum_{n=0}^{\infty} X_n (E^{1/2})^n $$

The three time scales will be noted $\tau$, $\tau$ and $\tau$ respectively, with

$$ \tau = E^\frac{1}{2} t \text{ and } \tau = E^2 t. $$

The equations themselves are then ordered in powers of $E^{1/2}$ both for the interior and the Ekman layers. The system to solve is then the following!

**INTERIOR**

<table>
<thead>
<tr>
<th>$n = 0$</th>
<th>$n = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_t - 2 V_e = -2 \mu_e$</td>
<td>$w_t + \mu_e = -2 \mu_e$</td>
</tr>
<tr>
<td>$V_e r + 2 \mu_e = 0$</td>
<td>$V_e r + \mu_e + 2 \mu_e = 0$</td>
</tr>
<tr>
<td>$w_t = -\mu_e$</td>
<td>$w_t + w_e = -\mu_e$</td>
</tr>
<tr>
<td>$2 \mu_e + w_e = 0$</td>
<td>$2 \mu_e + w_e = 0$</td>
</tr>
</tbody>
</table>
n = 2
\[
\begin{align*}
\mu_z t + \mu_1 t + \mu_0 t - 2 \mu_1^2 &= -2 \mu_1 + \nabla^2 \mu_0 \\
\nu_z t + \nu_1 t + \nu_0 t + 2 \mu_1 &= \nabla^2 \nu_0 \\
\omega_z t + \omega_1 t + \omega_0 t &= - \beta_2 + \nabla^2 \omega_0 \\
2 \mu_1 + \omega_2 z &= 0
\end{align*}
\] (2.3a) (2.3b) (2.3c) (2.3d)

Boundary

\begin{align*}
\bar{p}_{0z} &= 0 & \bar{\mu}_{0z} - 2 \bar{\nu}_0 &= \bar{\beta}_{0zz} \\
\bar{w}_{0z} &= 0 & \bar{\nu}_{0z} + 2 \bar{\mu}_1 &= \bar{\nu}_{0zz} \\
\bar{\beta}_z &= 0 & 2 \bar{\mu}_1 &= \pm \bar{\omega}_{1z} \text{ near } z = \pm 1
\end{align*}
\] (2.4a) (2.4b) (2.4c)

Boundary conditions at \( z = \pm 1 \)

\begin{align*}
\bar{\nu}_z + \bar{\nu}_0 &= \begin{cases} 
1 & \text{ for } t = 0 \\
0 & \text{ for } t < 0 
\end{cases} \\
\bar{\mu}_m + \bar{\nu}_m &= 0 \\
\bar{\nu}_m + \bar{\nu}_m &= 0, \quad m \neq 0 \\
\bar{\omega}_m + \bar{\omega}_m &= 0
\end{align*}
\] (2.7a) (2.7b) (2.7c) (2.7d)

all variables = 0 at \( t = 0 \) (2.7e)

\[ \bar{x} \xrightarrow{t \to 0} 0 \] (2.7f)

The superscript "0" denotes the boundary value of a variable.
2.2 Solution

Since $\frac{\partial \omega}{\partial y} = 0$ from (2.4) and (2.7f), $\omega_0$ must be zero at $y = \pm 1$. Then $\omega_0$ can be a solution for (2.1) if

$$\omega_0 = \frac{2 \pi}{n} \sin(n \pi y) \left[ A^m_0(t, T) \sin 2t + B^m_0(t, T) \cos 2t \right]$$

(2.8)

However the conditions (2.7e) imply $A^m_0(0) = B^m_0(0) = 0$ and this solution is independent of the change in rotation of the boundaries at $t = 0$. Then it is shown in appendix I that $A_n = B_n = 0$.

The system (2.1) then reduces to the Taylor-Proudman theorem with

$$\mu_0 = \omega_0 = 0$$

$$v_0 = v_0 = 0$$

First order system

To form an equation for $v_0$ one must go to (2.2b). But first $\rho_i$ and $\omega_i$ must be found. The equation for $\omega_i$, that follows from (2.2) is

$$\omega_{i\theta} + 4 \omega_i = 0$$

(2.9)

The boundary conditions on $\omega_i$ come from the solution of (2.5), which is the transient Ekman layer. Hence it is found that

$$\mu_0 = (1 - \nu_0) \frac{1}{2} \int_0^t \frac{d \xi}{\sqrt{\pi \nu_0 \alpha}} \sin 2\xi$$

$$v_0 = (1 - \nu_0) \frac{1}{2} \int_0^t \frac{d \xi}{\sqrt{\pi \nu_0 \alpha}} \cos 2\xi$$

$$\omega_i = \pm 2 (1 - \nu_0) \int_0^t \frac{d \xi}{\sqrt{\pi \nu_0 \alpha}} e^{-2 \xi^2 / \nu_0} \tan \frac{\nu_0 \xi}{\sqrt{\nu_0 \alpha}} \tan \frac{\nu_0 \xi}{\sqrt{\nu_0 \alpha}}$$

(2.10a)

(2.10b)

(2.10c)

Then the solution of (2.9) that satisfies the boundary conditions given by (2.10c) and (2.7) is

$$\omega_i = 2 \pi (1 - \nu_0) \int_0^t \frac{d \xi}{\sqrt{\pi \nu_0 \alpha}} \frac{\sin 2\xi}{\sqrt{\nu_0 \alpha}}$$

(2.11)

From (2.2) we then have

$$v_{\theta t} = -v_{\theta t} + 2 (1 - \nu_0) \xi (2t)$$

(2.12)

where $\xi(2t) = \int_0^t \frac{d \xi}{\sqrt{\pi \nu_0 \alpha}}$ is a Fresnel integral.

A necessary condition for $v_\theta$ to be non-secular (i.e., for $v_\theta$ not to grow larger than $v_\theta$ after a certain time) is that $v_{\theta t} \xrightarrow{t \to \infty} 0$.

Then we obtain from

$$v_\theta = 2 - \varphi(t) e^{-t^2}, \quad \varphi(0) = 1$$

(2.13)
From this it follows that

\[
\begin{align*}
\nu_t &= g z e^{-T} S(z t) f(T) \\
\mu_t &= -c e^{-T} S(z t) \frac{1}{T} \\
\nu_i &= f(T) e^{-T} \left[ \int_{T}^{T} S(z \kappa) d\kappa \right] + A_i(T, T)
\end{align*}
\]

(2.14a) (2.14b) (2.14c)

where \( A_1(T) = 0 \).

**Second order system**

To find \( \frac{1}{T} f(T) \) we must go to the second order interior equation. We will form an equation for \( \tilde{\nu}^2 \) and need \( \tilde{\nu}_{z=0} \) as a boundary condition. But \( \tilde{\nu}_{z=0} \) is given from the second order Ekman layer.

With

\[
\tilde{\phi} = \tilde{\mu} + i \tilde{\nu}
\]

(2.6) becomes

\[
\hat{\phi}(iz) - (\tilde{\phi} + 2 i \tilde{\mu}) = \tilde{\phi} \quad \text{at } z = 2
\]

(2.15)

With the help of (2.11) and (2.15) and since

\[
\tilde{\omega}_{z=0} = 2 \Re e \left[ \int_{-\infty}^{\infty} \tilde{\phi} d\zeta \right] \quad \text{near } z = -2
\]

we have as a consequence of (2.15) - and with some algebraic manipulations -

\[
\tilde{\omega}_{z=0} = -i A_1(T, T) S(z t) + f(T) e^{-T} \left[ 1 - \cos z t - \sqrt{\frac{2}{\pi}} \cos z t \right]
\]

(2.16)

where \( J_0 \) is the zeroth order Bessel function that is bounded at the origin.

Using the asymptotic expansions for \( J_0(z t) \) and \( S(z t) \) for large \( t \) we find that as \( t \to \infty \)

\[
\tilde{\omega}_{z=0} \to -A_1(T, T) f(T) e^{-T} \left[ 1 - \cos z t - \sqrt{\frac{2}{\pi}} \cos z t \right]
\]

(2.17)

When \( t = O(z^2) \) the second order transport becomes more important than the first order one and the expansion breaks down. It follows from (2.3) that

\[
\nu_{z=0} = -c \tilde{\omega}_{z=0} \quad \text{as } z \to \infty
\]

(2.18)

Then we can find \( f(T) \) using (2.3b), i.e.

\[
\nu_{z=0} = -\nu_{z=0} - \nu_{z=0} + \tilde{\omega}_{z=0}
\]

For \( \nu_{z=0} \) to be as non-secular as possible we require \( \nu_{z=0} \) to grow as slowly as possible when \( t \to \infty \). But as \( t \to \infty \),

\[
\nu_{z=0} \to -\left( A_1(T) + A_2(T) \right) + c e^{-T} f(T) \cos z t - e^{-T} f(T) \sqrt{\frac{2}{\pi}} \cos z t
\]
The fastest growing terms come from

\[-(A_{1} + A_{2}) + e^{-T} \left[ \frac{1}{T} (T) + \frac{3}{T} + (T) \right] \]

since

\[ \int_{0}^{\frac{\pi}{2}} \cos 2 \alpha \, d\alpha = \frac{1}{2} \sqrt{\frac{\pi}{2}} \sin 2 T - \frac{\sqrt{2}}{4} \int_{T}^{\infty} \frac{\sin 2 \alpha}{T^{2}} \, d\alpha \]

Then

\[ A_{1} + A_{2} - e^{-T} \left( \frac{1}{T} (T) + \frac{3}{T} + (T) \right) = 0 \]

If \( T(\alpha) + \frac{3}{T} (T) \neq 0 \) then \( A_{1} \alpha T e^{-T} \) and \( \nu \) would be secular.

Then

\[ T(\alpha) = e^{-\frac{3\pi}{8} T} \]

\[ A_{1} = B_{1}(T) e^{-\frac{3\pi}{8} T}, \quad B_{1}(0) = 0 \]

(2.19)

(2.20)

It also follows from (2.3b), (2.18) and (2.19) that we have

\[ v_{x} = C \left[ \frac{1}{T} \int_{t}^{T} \sin 2 \alpha \, d\alpha - \frac{1}{T} \int_{t}^{T} \frac{\sin 2 \alpha}{T^{2}} \, d\alpha \right] \]

where \( B_{1}(0) = A_{1}(0) = 0 \)

(2.21)

We have that as \( t \to \infty \), \( v_{x} \) grows like \(-e^{-\frac{3\pi}{8} T} \sqrt{\frac{\pi}{2}} \sin 2 T \)

2.3 Discussion

By using (2.13), (2.14c) and (2.15), the swirl velocity in the interior is given to order \( E \) by

\[ v_{x} = \frac{1}{T} \left[ \int_{t}^{T} \sin 2 \alpha \, d\alpha - \frac{1}{T} \int_{t}^{T} \frac{\sin 2 \alpha}{T^{2}} \, d\alpha \right] + \frac{1}{T} \int_{t}^{T} \frac{\sin 2 \alpha}{T^{2}} \, d\alpha \]

(2.22)

Similarly with the help of (2.14), (2.16), (2.17) we obtain for the vertical velocity in the interior

\[ W_{1} = E \frac{1}{T} \left[ \frac{1}{T} \int_{t}^{T} \sin 2 \alpha \, d\alpha - \frac{1}{T} \int_{t}^{T} \frac{\sin 2 \alpha}{T^{2}} \, d\alpha \right] + E \frac{1}{T} \int_{t}^{T} \frac{\sin 2 \alpha}{T^{2}} \, d\alpha \]

(2.23)

The radial component can be obtained from (2.23) by the relation

\[ U_{1} = -\frac{V_{x}}{2} \]

(2.24)

Limiting cases

It is first noticed that for \( T \gg 1 \), (2.22) is in fact

\[ v_{x} = E \frac{1}{T} \left[ \frac{1}{T} \int_{t}^{T} \sin 2 \alpha \, d\alpha \right] \rightarrow E \frac{1}{T} \left( \frac{t}{T} - \frac{1}{T^{2}} \right) \]

(2.25)
This result agrees with Greenspan and Howard's (1963) using the same limit. It is understood that the second limit on the right of (2.25) is applied only when \( t \ll E^{-1} \).

The general result (2.22) is also similar to that of Greenspan and Howard (1963) when \( t \gg 1 \). (Notice that their relation (3.20a) is not valid when \( 2 \delta(2t) \neq 1 \). Instead the correct limit for small times is similar to (2.25a) of this paper.) We also have a higher order correction to that expression. In particular there is an undamped inertial wave present for all times, but of order \( E \) amplitude. We also notice that to order \( E^\nu \) and \( \epsilon \) there are initial motions that are damped like \( 1/\sqrt{E} \).

The transport terms (2.23) and (2.24) correspond to what is obtained by Greenspan and Howard (1963). Higher order corrections also contain an undamped oscillation.

Effects of the time scales

The inclusion of a time scale of order \( E^{-1} \) in the problem led to a new decaying factor in the exponential, i.e. a \( \epsilon^{-2\pi} \) dependence. This is a correction to the spin-up time and could be found without the inclusion of transient processes.

The short time scale on the other hand brings some small amplitude oscillatory motions that even grow, on a time scale of order \( E^{-1} \). For example the limit when \( t \to \infty \) of (2.23) leads to

\[
W_x = E^{k_2} e^{-\pi t/2} \left\{ \epsilon^{2\pi} \left[ 1 - \cos 2t - \sqrt{2} \cos 2t + \theta(t^{-\nu}) \right] \right\} + B(t) e^{-t/2} \left( 1 - \cos 2t \right) \text{ (2.26)}
\]

Thus when \( T \) becomes order 1 (\( t = 9(\epsilon^{-\nu}) \)) the term \( E^{k_2} \epsilon^{2\pi} \cos 2t \) and the expansion breaks down. This is perhaps not too surprising since the Ekman layer grows into the interior for a time of order \( E^{-1} \), as is seen from (2.10). But since the boundary layer variables are assumed to vanish into the interior for all times, the term \( E^{k_2} \epsilon^{2\pi} \cos 2t \) probably tries to compensate for that deficiency in the method.
3. Conclusion

It has been shown that the multi-timing analysis gives the same basic results that had been obtained in previous works. We also found a correction to the spin-up time for times of order $E^{-1}$.

For the homogeneous spin up some residual oscillatory motions exist at all times. Although some oscillations decay like $1/\sqrt{E}$, some grow in a time of order $E^{-1/2}$ to an amplitude of order $E^{1/2}$ presumably to allow for the growth of the boundary layer into the interior.

Some work is presently under way to apply this method to the linear spin up of a stratified fluid.

Acknowledgment

I wish to thank Dr. George Veronis for his help in the numerous discussions we had. I am also grateful to the Geophysical Fluid Dynamics summer program for its support.

References


APPENDIX I

The velocity field can be broken into two parts since the problem is linear. At the plates themselves there are no inertial oscillations, The oscillatory part of the complete solution (interior + boundary layer) must then vanish at the plates. The part of the velocity field that doesn't oscillate is then what is linked to the motion of the plates.

For the oscillatory solution we then have

$$\rho^* = \psi^* = \psi^e = 0$$

where $\rho, \psi, \psi^e$ represent the total solution (interior + boundary layer).
These modes also have zero amplitude initially (cf. (2.8)).

We then show that the amplitude of the inertial modes will always be zero. Taking the equation of motion, dotting with \( \mathbf{V} \) and integrating over the whole cylinder we have

\[
\frac{\partial \mathbf{V}^2}{\partial t} = -\nabla P + \mathbf{V} \cdot \mathbf{V} \quad (12)
\]

where

\[
\mathbf{V} = \int \frac{d\mathbf{r}}{\rho} \int_{\mathbf{r}}^\mathbf{r} \mathbf{\nabla} \mathbf{z}
\]

But

\[
-\nabla P = \mathbf{V} \cdot (\nabla P) = \frac{1}{\rho} (\nabla \mu) + \frac{2}{3} \mathbf{V} \cdot \mathbf{V} = 0 \quad (13)
\]

using the incompressibility and (11)

Similarly

\[
\mathbf{V} \cdot \nabla \mathbf{V} = \frac{1}{\rho} (\nabla \mu \cdot \mathbf{V}_{\mu}) + \frac{2}{3} \mathbf{V} \cdot \mathbf{V} = (\mathbf{V}^2 + \mathbf{V}_{\mu}^2)
\]

using (11) again,

Then

\[
\frac{\partial \mathbf{V}^2}{\partial t} = -E (\mathbf{V}^2 + \mathbf{V}_{\mu}^2) < 0 \quad (15)
\]

But initially \( \mathbf{V}^2 \) is zero. Then from (15) we obtain \( \mathbf{V}^2 = 0 \) for all times, i.e. \( A_n = B_n = 0 \) in (2.8).

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EQUATORIAL WAVES OF THE TROPICAL ATMOSPHERE

J. B. Tupaz

I. Introduction

Approximately one-half of the earth's surface is in the tropics (between approximately 30°N and 30°S), consequently, most of the earth's solar energy is absorbed primarily by the oceans, and, somewhat less, by the lower troposphere and land masses of this region. Thus the tropical atmosphere and oceans play a very significant and dynamical role in the earth's energy balance and the redistribution of this energy to other parts of the globe via the tropical atmospheric circulation and oceanic current system.

It is the purpose of this paper to study a part of this tropical atmospheric circulation by way of the free and forced equatorial
planetary waves using a simple atmospheric model. These planetary waves are referred to sometimes as quasi-horizontal motions of which there are two fundamental types: (1) inertia-gravity waves that travel both eastward (westerly) and westward (easterly), and (2) the quasi-geostrophic or "so-called" Rossby wave that travels westward. Most of the atmospheric energy or weather of large scale motions is associated with this latter type of wave, the Rossby wave.

Spectral analyses of data from stations in the tropical Pacific (Wallace and Chang, 1969, and Chang et al., 1970) verified the existence of westward moving disturbances in the lower and middle tropical troposphere with wavelengths of 3000 to 5000 km and periods relative to the ground of four to five days. Holton (1970) interpreted theoretically these easterly waves to be forced equatorial Rossby waves.

Matsuno (1966) using a simple divergent incompressible barotropic model studied the free waves of the equatorial quasi-geostrophic motions on an Equatorial \( \eta \)-plane, and Lindzen (1967) studied both the free and forced (due to diabatic heating) planetary waves of a gaseous envelope on a rotating sphere using both equatorial and mid-latitude \( \beta \)-planes. This study differs from both Matsuno's (1966) and Lindzen's (1967) work in that a compressible adiabatic tropical atmosphere is used on an equatorial \( \beta \)-plane with forcing being introduced into the model via vertical motion at the earth's surface, \( z = 0 \), which is specified by a particular mountain shape or structure.

II. Formulation of Problem:

A. Atmospheric Model: a compressible, inviscid, adiabatic, tropical stratified atmosphere with a constant basic flow (\( \mathbf{U} \)) is used. This model is both in steady state and hydrostatic equilibrium. An Equatorial \( \beta \)-plane is used, where \( \beta = \frac{2 \Omega}{a} \) (a constant):

\[
\Omega \equiv \text{earth's rotation} \sim 2\pi \text{ rad/day} \\
\alpha \equiv \text{earth's radius} \sim 6000 \text{ km} \\
\beta \equiv \text{earth's Coriolis parameter}
\]

Log-pressure coordinates, after Holton (1972), are used, where

\[
Z = -H \ln \frac{P}{P_0}
\] and
characteristic vertical scale length $\sim 10$ km
reference pressure $\sim 1000$ nb
gas constant for dry air, and

\[
\frac{\partial}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

where

By using the log-"p" coordinates, two advantages are derived: (1) density is eliminated from the governing equations, and (2) the static stability is approximately constant in the tropical troposphere.

Boundary Conditions are such that

(i) the fields $(u, v, w, q)$ are bounded as $z^* \rightarrow \infty$.
(ii) the vertical motion at the surface $(z^*=0)$, $w(x,y,z^*)$ defines the kind of forcing used. For free waves, $w(x,y,0) = 0$, i.e. no forcing.
(iii) the fields $(u,v,w,q)$ decay or approach zero as $y \rightarrow \pm \infty$.

Note: hereafter, the asterisk in "z*" is dropped for convenience.

B. Linearized Governing Equations:

(1) $U \frac{\partial u}{\partial x} - \beta y v = - \frac{\partial \Phi}{\partial x}$

(2) $U \frac{\partial v}{\partial x} + \beta y u = - \frac{\partial \Phi}{\partial y}$

(3) $\frac{\partial \Phi}{\partial z^*} = \frac{RT}{H}$

(4) $\frac{\partial y}{\partial x} + \frac{\partial v}{\partial y} + \left( \frac{\partial}{\partial z^*} - \frac{1}{H} \right) w^* = 0$

(5) $U \frac{\partial T}{\partial x} + w^* \Gamma = \frac{\partial^2 z^*}{\partial p^2} = 0 \rightarrow \text{adiabatic}$

\[ \Gamma \equiv \text{Static Stability} \quad \Gamma = \frac{T}{\Theta} \frac{\partial \Theta}{\partial \Theta} \]

for a tropical troposphere, $\Gamma \approx 30^\circ \text{C/km}$. After combining Eqs. (3) and (5), we obtain

(6) $\frac{\partial^2 z^*}{\partial x^2} \left( \partial z^* \right) + w^* \Gamma = 0$, where $S^* \equiv \frac{R}{UH} \Gamma$

Next, we assume that solutions to Eqs. (1) - (6) are of the form:
After substituting the periodic assumed solutions of Eq. (7) into Eqs. (1) - (6), we obtain:

\[ (1): \quad ikVu - Byv + ik\varphi = 0 \]
\[ (2): \quad Byu + ikVu + \varphi_y = 0 \]
\[ (4): \quad iku + vy + (\frac{\partial}{\partial z} - \frac{1}{H})w = 0 \]
\[ (6): \quad Sw + ik\varphi_z = 0 \]

where the field \((u, v, w, \varphi)\) equations are given by

\[ (7): \quad v = \frac{ikByv - ik\varphi\partial_y}{(By + k^2u^2)} \]
\[ (8): \quad U\varphi_y - Byu = \frac{1}{ik}\left[By\varphi_y + Bv + k^2u\varphi\right] \]
\[ (9): \quad U\varphi_z - Byw = \frac{(B^2y^2 - k^2u^2)\varphi}{S} \]
\[ (10): \quad U\varphi_y - By\varphi = \frac{1}{k}(B^2y^2u - k^2u^2) \]

C. **Wave Equation:**

After solving Eqs. (1) - (6) in terms of the field "v", we obtain the wave equation:

\[ (11): \quad \left[ \frac{1}{Us} \left( \frac{\partial^2}{\partial z^2} - \frac{1}{H} \frac{\partial^2}{\partial z} \right) \right] v - \left[ \frac{1}{U} \left( k^2u - By^2 \right) \right] \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial z} - k^2 \right) v = 0 \]

Equation (11) is separable, hence, we define the linear operators \(L_z\) and \(N_y\) as indicated in Eq. (11), and we define "\(v_n\)" by

\[ (12): \quad v_n = \sum_n \psi_n (z) \varphi_n (y) \]
Thus, we obtain:

\begin{equation}
L^n_y \psi_n (z) \psi_n (y) - N^n_y \psi_n (z) \psi_n (y) = 0
\end{equation}

After dividing Eq. (13) by $\psi_n \psi_n$, we obtain:

\begin{equation}
\frac{1}{\psi_n} \frac{d}{dz} \psi_n (z) = \frac{1}{\psi_n} \frac{d}{dy} \psi_n (y) = - \frac{1}{h_n},
\end{equation}

where $h_n$ is a separation constant. For the free waves, $h_n$ is given by the type of atmosphere analyzed, i.e., Lindzen (1967) used $h = \frac{c_p}{c_f} \sim 10.5$ km for an isothermal atmosphere. For the forced waves, $h_n$ must be determined from the latitudinal frequency Eq. (24) for each of the meridional modes, i.e., $n = 0, 1, 2$.

D. Vertical Equation:

Utilizing the linear operator $\frac{d}{dz}$ in Eq. (14), we obtain the Vertical Equation from Eq. (11):

\begin{equation}
\frac{\partial^2 \psi_n}{\partial z^2} - \frac{1}{h} \frac{\partial \psi_n}{\partial z} + \frac{U \psi_n}{h} = 0
\end{equation}

where the roots of Eq. (14) are:

\begin{equation}
r = \frac{1}{2H} \pm \sqrt{\frac{1}{\psi_h^2} - \frac{U^2}{h}}, \quad \text{and}
\end{equation}

the vertical wave number $\lambda$ is defined by

\begin{equation}
\lambda = \sqrt{\frac{1}{\psi_h^2} - \frac{U^2}{h}}.
\end{equation}

Hence, the general solution to Eq. (15) is given by

\begin{equation}
\psi_n (z) = C_1 e^{\frac{\lambda z}{h}} + C_2 e^{-\frac{\lambda z}{h}}
\end{equation}

We see from Eqs. (17) and (18) that when the vertical wave number ($\lambda$) is real (i.e., $h > \left| \frac{4U^2h^2}{\psi} \right|$), we get exponentially growing and decaying solutions, however, the upper boundary condition is such that $\psi_n (z)$ is bounded for large $z$, thus $C_1 = 0$ and the general solution for $\lambda$ real becomes

\begin{equation}
\psi_n (z) = C_2 e^{\frac{\lambda z}{h}}.
\end{equation}

When $\lambda$ is imaginary (i.e., $0 < h < \left| \frac{4U^2h^2}{\psi} \right|$), we get vertically propagating waves. The Vertical Propagation criteria for both types of waves is summarized by the following:

(i) Vertically Propagating Internal Waves

$0 < h < \left| \frac{4U^2h^2}{\psi} \right|$ energy propagates vertically away from the level of excitation.
(ii) **Vertically Trapped Waves**

(i) \( h > |4USh^2| \) energy is trapped near the level of excitation

(ii) \( h < 0 \)

(iii) We can also define a critical equivalent velocity squared, 

"\( h_c \)" by \( h_c = |4USh^2| \)

E. **Latitudinal Equation**:

Using the linear operator \( \mathcal{N}_Y \) in Eq. (14), we obtain the Latitudinal Equation from Eq. (11):

\[
\frac{d^2V_n}{dy^2} + \left( \left( \frac{E}{U} - k^2 + \frac{k^4U^4}{h_n^2} \right) - \frac{E}{h_n^2} \right)V_n = 0
\]

Equation (20) is a Schrödinger equation for a harmonic oscillator which has solutions in terms of Hermite polynomials when the lateral boundary conditions are such that \( V_n(y) \to 0 \) for \( y \to \pm \infty \), where

\[
V_n(\xi) = e^{-\xi^2}H_n(\xi), \quad \text{and} \quad \xi = \frac{E}{h_n^2}y^2
\]

and the \( n^{th} \) order Hermite polynomials are defined by

\[
H_n(\xi) = e^{\xi^2}\left( \frac{d}{d\xi} \right)^n \left( e^{-\xi^2} \right) \text{with the recurrence formulas given by:}
\]

\[
\frac{dH_n(\xi)}{d\xi} = 2nH_{n-1}(\xi), \quad \text{and}
\]

\[
H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)
\]

The frequency equation is obtained from the Latitudinal Eq. (20) and is given by:

\[
(\frac{E}{U} - k^2 + \frac{k^4U^4}{h_n^2})\frac{h^2}{\beta} = 2n + 1, \quad \text{where} \quad n = 0, 1, 2.
\]

III. **Free Waves**:

For the free waves, \( U = U(h, k, n) \) hence the frequency equation becomes:

\[
U - \left\{ h + \frac{(2n+1)\beta h^2}{\beta} \right\} + \frac{h^2}{k^2} = 0
\]

Eigenvalues \( (U) \) of Eq. (25) are of two types: (a) two inertia-gravity waves \( (U_G) \), one propagating eastward and the other westward, and (b) one **Rossby** wave \( (U_R) \) which propagates westward. These eigenvalues can be approximated when for
Figure 1 depicts the lowest meridional modes of free waves (Inertia-Gravity and Rossby Waves) as a function of their frequency ($\Omega k$) versus longitudinal wave number, $k$.

![Diagram of Free Equatorial Waves]

Fig. 1 Free Equatorial Waves.
Since $\psi_n(y) = \psi_n(z) \psi_n(\tilde{y})$, the eigensolutions $(u_n, v_n, w_n)$ are defined in terms of $\psi_n$ and the associated Hermite polynomials. These eigensolutions are given by

\begin{align*}
\phi_n &= \frac{i \beta^k h_n}{k} \left[ a_n H_{n-1}(\xi) - a_{n+1} H_{n+1}(\xi) \right] e^{\frac{-z}{k}} \psi_n(\xi) e^{ikz} \\
\varphi_n &= -\frac{i \beta^k h_n}{k} \left[ a_n H_{n-1}(\xi) + a_{n+1} H_{n+1}(\xi) \right] e^{\frac{-z}{k}} \psi_n(\xi) e^{ikz} \\
\psi_n &= \frac{\beta^k h_n}{S} \left[ a_n H_{n-1}(\xi) - a_{n+1} H_{n+1}(\xi) \right] e^{\frac{-z}{k}} \frac{\partial \psi_n(\xi)}{\partial \xi} e^{ikz}
\end{align*}

where

\begin{align*}
a_1 &= \frac{\psi(h_n^2 - \psi)}{h_n^2 + \psi}, \quad a_2 = \frac{\psi}{2(h_n^2 + \psi)}
\end{align*}

Figure 2 depicts the latitudinal behavior of the fields $(u_n, v_n, \phi_n, w_n)$ for the lowest three meridional modes. One can see from Fig. 2 that
when \( n \) is even, then the \( u_n \) field is symmetric about the equator and the \( u_n, q_n \) and \( \omega_n \) fields are asymmetric about the equator. The opposite is true when \( n \) is odd.

IV. Forced Waves:
A. Meridional Modes of the \( h_n \) Parameter:

For the forced waves, the \( h_n \) must be determined for each meridional mode from the frequency Eq.(24), hence, we obtain:

\[
(30) \quad h_n = \frac{(2n+1)\pi \beta}{2(\beta-k^2 \nu)} \left[ 1 \pm \sqrt{1 - \frac{4k^2 \nu(\beta-k^2 \nu)}{(2n+1)^2 \beta^2}} \right]
\]

We observe that \( h_n = h_0 ( \nu, k, n ) \). Some important implications derived from Eq. (30) are:

(i) for meteorological use, \( \sqrt{h_n} \) is real, i.e.

\[
(31) \quad \frac{4k^2 \nu(\beta-k^2 \nu)}{(2n+1)^2 \beta^2} \leq 1, \quad \text{and}
\]

(ii) \( \sqrt{h_n} \) can not be negative, otherwise the eigensolutions \( (u_n, v_n, q_n, \omega_n) \) would violate the lateral boundary conditions, i.e. if \( h_n < 0 \), \( \nu \) becomes imaginary, and \( v_n(\xi) \) in Eq. (21) would grow without bound as \( \xi \) or \( y \rightarrow \pm \infty \).

Also some very interesting results were obtained from an analysis of the \( (+) \) and \( (-) \) roots of \( \sqrt{h_n} \), Eq.(30).

(i) For \( \nu > 0 \), an easterly Basic Flow, only the \( (-) \) root or \( \sqrt{h_n}^- \) is a valid solution (i.e. so \( h_n > 0 \)), representing a westerly inertia-gravity wave.

(ii) For \( \nu < 0 \), a westerly Basic Flow, two cases are possible:

(a) \( \sqrt{h_n}^- \) is a valid solution for the entire range of \( h_n \), which represents easterly inertia-gravity waves, and

(b) \( \sqrt{h_n}^+ \) is also a valid solution but only in the range of \( 0 < k^2 \nu < \beta \), where \( \sqrt{h_n}^+ \) represents the Rossby waves. \( \beta = k^2 \nu \) is a Rossby wave cut-off. Since meteorologically inequality (3) is normally \( \& \), then by using the Binomial theorem, a first order approximation to the \( (\pm) \) roots for \( \sqrt{h_n} \), Eq.(30), is obtained, i.e.

\[
(32) \quad \sqrt{h_n}^+ \approx \frac{(2n+1)\beta}{\sqrt{\xi} - k^2 \nu}
\]

\[
(33) \quad \sqrt{h_n}^- \approx k^2 \nu^+ \frac{2n+1}{\beta}
\]
Fig. 3 depicts the meridional modes which are possible for both types of forced waves, i.e., the variation of $\frac{p_n}{K^2V}$ (Rossby) and $\frac{1}{K^2V}$ (gravity) as functions of $K^2V$ for both $V>0$ and $V<0$. Fig. 3 shows that all meridional modes are theoretically possible for both the westerly ($V<0$) inertia-gravity and the easterly ($V>0$) inertia-gravity waves for the entire range of $K^2V$, however, the Rossby waves are only possible within the range of $0 < K^2U < \beta$. Again to determine which modes propagate vertically, we must determine whether the vertical wave number, $\lambda_n$, Eq. (17) is real or imaginary.

Fig. 3 Equatorial Forced Waves

B. Critical y-value $y_d$:

Our lateral boundary conditions are such that the eigen solutions must decay as $y \rightarrow \pm \infty$. The solutions of $V_n(y)$ in terms of Hermite polynomials satisfy this condition.

Going back to the Schrödinger equation (20), we determine the critical y-value $y_d$ that satisfies our lateral boundary conditions for both the Rossby and inertia-gravity waves, i.e.
We observe that if the coefficient of the second term of Eq. (20) is \( \leq 0 \), then \( V_n(y) \) decays exponentially away from the equator, and, hence, the lateral boundary conditions are satisfied. On the other hand, if this coefficient \( > 0 \), then \( V_n(y) \) does not decay away from the equator, so there must exist a critical value of "y" beyond which the solution would decay in order to have a valid solution for \( V_n(y) \). If we equate the coefficient of the second term of Eq. (20) to zero and let \( y = y_d \), we obtain the critical y-value "\( y_d \)":

\[
\frac{d^2V_n}{dy^2} + \left\{ \left( \frac{\partial}{\partial y} - \kappa^2 + \frac{k^2V_n}{h_n} \right) - \frac{\beta^2}{h_n} \right\} V_n = 0
\]

\[ (20) \]

We observe that if the coefficient of the second term of Eq. (20) is \( \leq 0 \), then \( V_n(y) \) decays exponentially away from the equator, and, hence, the lateral boundary conditions are satisfied. On the other hand, if this coefficient \( > 0 \), then \( V_n(y) \) does not decay away from the equator, so there must exist a critical value of "y" beyond which the solution would decay in order to have a valid solution for \( V_n(y) \). If we equate the coefficient of the second term of Eq. (20) to zero and let \( y = y_d \), we obtain the critical y-value "\( y_d \)":

\[
\frac{d^2y_d}{y^2} = \left( \frac{\beta}{U} - \kappa^2 + \frac{k^2V_n}{h_n} \right) \frac{h_n}{\beta^2} = \left( \frac{2n+1}{\beta} \right) V_n \]

\[ (34) \]

Thus

\[
\frac{d^2y_d}{y^2} = \left( \frac{2n+1}{\beta} \right) \frac{h_n}{\beta^2} < \frac{a}{y_p}, \text{ where}
\]

\[
y_p = \frac{a}{y_p}, \text{ (radius of earth)}
\]

Substituting the two first order approximated values of \( \sqrt{h_n} \), Eqs. (32) and (33), into Eq. (35), we obtain:

\[
y_d^2(\text{Rossby}) \approx \frac{(2n+1)^2}{\beta^2}
\]

\[ (36) \]

\[
y_d^2(\text{gravity}) \approx \frac{k^2V_n^2}{\beta^2}
\]

\[ (37) \]

We observe that "\( y_d \)" critical value of y, for gravity waves does not depend on "n", whereas, for large "n", Rossby wave solutions are a bad approximation using the Equatorial \( \beta \)-plane. We observe from Eq. (36) that Rossby solutions are valid for \( n \leq 2 \).

C. Vertically Propagating Forced Waves

We observed earlier that when \( \lambda_n \) is real, then the only valid solution of Eq. (18) is the exponentially decaying solution, however, when \( \lambda_n \) is imaginary, there are two possible vertically propagating solutions to Eq. (18). Since the forcing in our model takes place at the surface (\( z = 0 \)), then the only valid solution is the one that propagates vertically upward away from the surface, the level of excitation. Hence, in our model, for \( \lambda_n \) imaginary, \( C_1 = 0 \) of Eq. (18) when \( U > 0 \), and \( C_2 = 0 \) when \( U < 0 \).
D. Forcing Function:

Suppose we define our forcing function as

\[ \omega(x, \xi, 0) = \mathcal{U} \frac{\partial \mathcal{M}^\dagger}{\partial x} + i k \mathcal{U} \mathcal{M}(\xi) e^{ix} \]

where \( \mathcal{M}^\dagger(x, \xi) = \mathcal{M}(\xi) e^{ix} \) and \( \xi \) is the same "stretched" y-coordinate defined by Eq. (21).

Since the forcing in our model is introduced by the vertical motion \( \omega \) forced by the mountain shape at \( z = 0 \), then, there are two possible approaches for solving the forced problem. The first approach is the most vigorous: given a specific mountain structure, solve for the eigenfunctions, \( \psi_n^\dagger \) (i.e., \( \mathcal{U}_n \mathcal{V}_n \psi_n^\dagger \mathcal{W}_n \)), by inverting the matrix given by:

\[ \sum_n a_n \omega_n(x, \xi, 0) = \mathcal{M}(\xi), \]

hence we can obtain the eigensolutions due to our specified mountain forcing:

\[ \omega(x, \xi, z) = \sum_n \omega_n(x, \xi, z) \]

\[ \mathcal{U}(x, \xi, z) = \sum_n \omega_n(x, \xi, z) \text{ and so forth for the other eigensolutions.} \]

The procedure given by Eq. (39) is normally complicated and difficult to perform, so a second approach, though not as rigorous and complete as the first, is used. In this second approach, the mountain structure is found that forces a particular given eigenfunction, \( \psi_n^\dagger \), i.e.,

For \( \gamma = 0 \), we obtain the zeroth order meridional mode eigenfunction, \( \psi_0^\dagger \):

\[ \psi_0^\dagger = \psi_0(\xi) e^{-\frac{\xi}{\mathcal{U}}} \]

where

\[ \psi_0(\xi) = e^{i \lambda_0 (C_0 \xi e^{\lambda_0 \xi} + C_0 \xi e^{-\lambda_0 \xi})} \]

\[ \lambda_0 = \sqrt{\frac{\mathcal{U} \mathcal{V}}{\mathcal{U} \mathcal{V}^2} - \frac{\mathcal{U} \mathcal{V}}{\mathcal{U} \mathcal{V}^2}} \]

\( \psi_0^\dagger \) is obtained from Eq. (30). Using the forcing function defined by Eq. (38), we substitute Eqs. (40) - (42) into Eq. (28) for \( z = 0 \). For \( \mathcal{U} > 0 \) and \( \lambda_0 \) imaginary, \( \psi_0^\dagger = 0 \) in Eq. (28), hence we can solve for the zeroth meridional mode mountain structure \( \mathcal{M}_0(\xi) \). We obtain:

\[ \mathcal{M}_0(\xi) = \frac{i C_0 \mathcal{V} \mathcal{W}_0}{k S (\mathcal{U} + \lambda_0)} \left( \frac{\mathcal{W}_0}{\mathcal{U}} - \lambda_0 \right) e^{-\frac{\xi}{\mathcal{U}}} \]
For $U<0$, $\lambda_o$ imaginary, $\hat{M}_n(\xi)$ is given by

$$\hat{M}_n(\xi) = \frac{i C^\lambda \beta_{\lambda_o} \psi_h^4}{k_0 \psi_h^2 \psi_{n+\lambda_o}} (\frac{i}{2} + \lambda_o) \xi e^{\frac{i}{\xi}}$$

For $\lambda_o$ real, Eq. (43) is still the mountain structure solution, but with $\lambda_o$ now being real. Thus, we observe that the general form for the mountain structure, $\hat{M}_n(\xi)$, for $n=0$ is given by:

$$\hat{M}_n(\xi) = \left( \hat{M}_{\nu_0} + i \hat{M}_{\mu_0} \right) \xi e^{\frac{i}{\xi}}$$

Moreover, we observe that in order to realize the zeroth meridional mode eigenfunction $\nu_0$, the zeroth mode mountain structure $\hat{M}_0(\xi)$ must take the shape of the first meridional mode Hermite polynomial. This is depicted in Fig. 4.

![Fig. 4 Mountain Structure $\hat{M}_0(\xi)$ as a function of $\xi$.](image)

Furthermore, it is observed that for a given meridional mode eigenfunction $\nu_n$, the corresponding mountain forcing $\hat{M}_n(\xi)$ required to force each eigensolution $\nu_n$ takes the form of

$$\hat{M}_n(\xi) = \left( \hat{M}_{\nu_n} + i \hat{M}_{\mu_n} \right) \left[ \alpha_{n,1} \psi_{n+1}(\xi) - \alpha_{n,2} \psi_{n-1}(\xi) \right] e^{\frac{i}{\xi}}$$

where $\alpha_{n,1}$ and $\alpha_{n,2}$ are given by Eq. (27).

V. Summary and Conclusions

As did Matsuno (1966) and Lindzen (1967) in their respective models, we found, using a simple atmospheric model on an equatorial Beta-plane, that there are two basic types of quasi-horizontal motion in the equatorial area: (1) easterly and westerly inertia-gravity waves, and (2) the
easterly Rossby waves. As expected, the inertia-gravity waves were found generally in the higher frequency range as compared to the Rossby waves.

For the forced waves in our model, we found that an easterly basic current \((U < 0)\) flowing over a specified mountain shape can generate only westerly or eastward propagating inertia-gravity waves. Our results show that all meridional modes are theoretically possible when \(U < 0\). Of course, the specific modes that are permissible are dependent on the forcing.

For westerly basic flow \((U > 0)\), forced easterly inertia-gravity and Rossby waves can exist. We found that all meridional modes of easterly inertia-gravity waves are theoretically possible, however, we found that Rossby waves are possible only in the range of \(0 < k^2 \beta < U < \beta\). It should be noted that the Rossby waves are constrained to the lowest meridional modes \((n \leq 2)\), that is, Rossby waves are latitudinally trapped "close" to the equatorial area. Again the specific forcing determines which meridional modes are permissible for each type of wave.

We found that both the free and forced waves are horizontally or latitudinally trapped in the equatorial area, i.e., these waves do not propagate their energy latitudinally away from the equatorial area. This means for the forced problem that the generated waves do not propagate latitudinally away from their source region which in our model is the equatorial area.

We also found that some of the forced waves can propagate vertically upward away from the level of excitation which in our model is the earth's surface. When the vertical wave number, \(\lambda_n\), is imaginary, then vertical propagation is possible. The only valid solution in this case, however, is the one that propagates vertically upward away from the earth's surface, the level of excitation. The solution is dependent on whether the basic current \((U)\) is easterly or westerly. On the other hand, when \(\lambda_n\) is real, then the forced waves are vertically trapped near the earth's surface.

To determine the effects of a given mountain structure in our model, a summation of all meridional modal eigenfunctions is required for
the specific forcing function. This is generally a difficult task to accomplish, hence, we performed the reverse problem, that is, given a specific eigenfunction \( \lambda_{\eta} \), determine what mountain shape will force such a solution. We found that for a given N-S velocity eigenfunction \( \lambda_{\eta} \), the corresponding mountain function has the same basic meridional structure as the \( \lambda_{\eta} \) and \( \omega_{\eta} \) eigensolutions, i.e. if the \( \lambda_{\eta} \) eigenfunction is symmetric about the equator, then the corresponding mountain function is asymmetric about the equator. The opposite is true when \( \lambda_{\eta} \) is asymmetric about the equator. In our model, all the meridional modal field eigenfunctions as well as the modal mountain forcing are functions of Hermite polynomials, hence, they decay away from the equator for large \( v^\prime \).

It is obvious from satellite observations that the cumulonimbus pseudo-line activity along the Intertropical Convergence Zone (ITCZ) plays a very important role in the global circulation and energy transport from the tropics to higher latitudes. The ITCZ is located, in the mean, between \( 5^\circ-8^\circ \) of the equator, at least, throughout much of the Pacific and Atlantic Oceans. Our simple model used in this study can perhaps be modified to include periodic longitudinal forcing similar to the observed easterly Rossby waves that propagate along the ITCZ. This forcing could possibly take the form of vertical motion at the top of an Ekman layer as some authors have done in various models. There are various other modifications that might be used such as to include diabatic heating due to some latent heat function. In any case, the author feels that there is much potential for further study and research in the tropics using relatively simple models similar to the one used in this study.

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FORCED BAROCLINIC WAVES ON CONTINENTAL SHELVES

Dong-Ping Wang

1. Introduction

Oceanic observations on the continental shelves suggest the existence of energetic low-frequency oscillations, (the typical period is about five days), which can be consistently interpreted as the barotropic shelf waves (e.g. Cutchin and Smith, 1973). Although the detailed generation mechanism is not clear, (e.g. Buchwald and Adams, 1968), evidence does suggest that these oscillations are closely related to the "global five-day atmospheric pressure waves", (Huyer and Pattullo, 1972). The observed oscillations are essentially barotropic, this can be understood from the consideration of the geostrophic adjustment process (e.g. Veronis and Stommel, 1956). Since the baroclinic radius of deformation
on the continental shelf is of order 10 km, which is too small compared to the large scale atmospheric forcing, (with length scale of 1000 km), the direct baroclinic response in the ocean can not be significant. Therefore, one would expect that the barotropic oscillation will account for much of the energy put into the ocean at periods longer than a pendulum day. On the other hand, observations do suggest the existence of some low-frequency baroclinic motions (particularly, along the coast and the continental slope) among the overwhelming barotropic oscillations, (Düng, 1973). One possible explanation of these observed baroclinic motions is that the presence of coast and/or continental slope will induce the transformation of baratropic wave energy into baroclinic wave energy. The mechanism of energy transformation between barotropic and baroclinic motions due to the irregular bottom is well-known for the high frequency (above inertial frequency) oscillations (Rattray, 1960). Whether the similar mechanism is possible or not for the low-frequency oscillations will be examined in the following sections.

In paragraph 2, we discuss the energy transformation between quasi-geostrophic barotropic shelf waves and ageostrophic internal Kelvin waves due to the presence of the coast. And in paragraph 3, discussion will be on the energy transformation between barotropic shelf waves and bottom trapped waves.

2. Energy Transformation between Shelf Waves and Internal Kelvin Waves.

We consider here the low-frequency wave motion travelling along the continental shelf. The shelf geometry is sketched in Fig.1 with y-axis parallel to the coastline. The system is described by the linearized, Boussinesque, hydrostatic equations of motion as

\[
\begin{align*}
\frac{u}{t} - f v &= - \frac{1}{\rho_0} \frac{\partial p}{\partial x} \\
\frac{v}{t} + f u &= - \frac{1}{\rho_0} \frac{\partial p}{\partial z} \\
\rho &= - g \rho - \rho_0 \\
\rho_x + \frac{\partial \rho}{\partial z} &= 0 \\
\rho_y + \rho_z &= 0
\end{align*}
\]
And the boundary conditions are
\[ \begin{align*}
\omega &= 0 \quad \text{at} \quad z = 0 \\
\omega &= -u H_x \quad \text{at} \quad z = -H(x) \\
u &= 0 \quad \text{at} \quad x = 0 \\
u &= 0 \quad \text{at} \quad x = L. \quad (L \text{ is infinite for the open shelf})
\end{align*} \]

Expressing \( u, v, \omega \) in terms of \( \rho \), we have
\[ \begin{align*}
(\partial t^2 + f^2) u &= - (\rho_x + f \rho_y) \\
(\partial t^2 + f^2) v &= - (\rho_y - f \rho_x) \\
w &= - \frac{1}{N^2} \frac{\partial \rho}{\partial z}
\end{align*} \]

where \( N^2 = \frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \), and is assumed to be a constant for simplicity.

Substituting Eq. (2.b) into the continuity equation, we have
\[ (\partial t^2 + f^2)(u_x + v_y + w_z) = - \partial_z \left[ \rho_{xx} + \rho_{yy} + \frac{(\partial t^2 + f^2)}{N^2} \rho_{zz} \right] = 0 \]

This in turn will imply, for a periodic solution, that
\[ \rho_{xx} + \rho_{yy} + \frac{(\partial t^2 + f^2)}{N^2} \rho_{zz} = 0 \quad (2.c) \]

Looking for the wavelike solution \( \rho \sim e^{i \xi (y + C z)} \), where \( \xi \) is the wave number and \( C \) is the phase speed. Since we are only interested in the low-frequency oscillations, \( \frac{\partial}{\partial z} \) term can be neglected. We have then
with boundary conditions

\[ P_{zz} = 0 \quad \text{at} \quad z = 0 \]

\[ \left( \frac{C}{f} \right) P_x = -H(x) \quad \text{at} \quad z = -H(x) \]

\[ P_x = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L. \]

Equation (2.d) is, in general, not separable due to the irregular bottom. However, in the limit of small bottom slope, the linearized boundary condition (i.e. replace \( z = -H(x) \) by \( z = -H_s \)) then allows for the separation of horizontal and vertical modes. The non-dimensional form of Eq. (2.d) is

\[ \frac{P_{xx}}{L^2} + \frac{1}{N^2} P_{zz} = 0 \]

with boundary conditions

\[ \frac{R_o \cdot \frac{1}{G_s^2}}{P_{zz}} = -\delta (1 + R_o \partial_x) P_x \quad \text{at} \quad z = -1 \]

\[ P_x = 0 \quad \text{at} \quad z = 0 \quad \text{at} \quad x = 0, 1 \]

where

\[ G_s^2 = \frac{N^2 H_s^2}{f^2 L_s^4} \quad , \quad R_o = \frac{C}{f L_s} \]

\[ \delta = \frac{H_s}{L_s} \quad , \quad \delta \ll 1 \]

and horizontal and vertical length are scales to the width and depth of the channel, respectively, (See Fig. 4). In the quasi-geostrophic limit \( (R_o \ll 1) \), Eq. (2.e) has the solution (Rhines, 1970)

\[ \psi_n = \sin \pi \frac{x}{L} \cosh \kappa_n z \]

with the dispersion relation

\[ \delta - \frac{R_o \kappa_n \tanh \kappa_n}{G_s^2} = 0 \]

where

\[ \kappa_n = G_s^2 \sqrt{\pi^2 \delta + \delta^2} \]

For a typical continental shelf, the baroclinic radius of deformation \( \left( \frac{N H_s}{f} \right) \) is about 10 km. Hence \( \delta \ll G_s^2 \) is very small if \( L_s \) is taken to be the width of the shelf. (Since shelf wave is essentially trapped within the shelf break, there is only slight difference by putting the
wall at the shelf break or at infinity, (e.g. Buchwald and Adams, 1968). Because $G_s << 1$, for the 1st mode, ($n = 1$), we have
\begin{align}
p(x) &\approx 2^2 \sin \pi x \\
\sigma (s) &\approx \frac{\sigma_s l}{\pi^2 + \sigma_s^2}
\end{align}
which is the familiar barotropic shelf wave solution. Hence the "barotropic" shelf wave exists in a stratified ocean with any irregular bottom, as long as the shelf width is much larger than the baroclinic radius of deformation. (This can be seen more clearly by examining the governing equation (Eq. (2.e)). If $G_s << 1$, the balance is not possible unless $\omega_2 \sim P_2 \sim O(G_s) << 1$, or the motion is essentially barotropic.) On the other hand, Eq. (2.e) also has the solution of a baroclinic mode, even in the limit of $G_s << 1$, if the ageostrophic effect is taken into account, (Huppert and Stern, 1973). We then have the internal Kelvin wave solution
\begin{equation}
\sigma(n) = e^{-i\sigma_s n^2},
\end{equation}
whose dispersion relation is
\begin{equation}
\sigma(n) = \frac{\ell}{\pi^2 + \sigma_s^2}
\end{equation}
for the 1st mode.

In the limit of $G_s << 1$,
\begin{equation}
\sigma(n) = \frac{\ell G_s}{\pi^2}
\end{equation}
or, internal Kelvin wave is non-dispersive, while the dispersion relation for shelf wave is
\begin{equation}
\sigma(s) = \frac{\sigma_s l}{\pi^2 + \sigma_s^2}
\end{equation}
If $\sigma_s = G_s$, which is typical for the continental shelf, resonance occurs at $\ell = \ell_c$ where
\begin{equation}
\frac{G_s}{\pi} = \frac{\sigma_s}{\pi^2 + \sigma_s^2}
\end{equation}
and is sketched in Fig. 2.
Analogous to the study of interaction between surface and internal tides (Rattray, 1960), resonance can be interpreted as the optimum condition for the energy transformation from quasi-geostrophic shelf waves into ageostrophic internal Kelvin waves. However, near the resonance, there is a strong coupling between two modes, and the previous discussion, which is based on the idea that two modes are independent, no longer remains valid. The result of numerical examination for a two-layered ocean with sloping bottom (no restriction on the slope) is given in Fig. 3.

The role of resonance conditions for the interaction between barotropic Kelvin waves and internal Poincaré waves (Rattray's mechanism), and inter-
action between internal Kelvin waves and barotropic quasi-geostrophic waves (or, shelf waves), is clearly shown for this simplified "baroclinic" ocean. The essential features evidently agree with those suggested by the continuously stratified model with a small sloping bottom.

3. Energy Transformation between Shelf Waves and Bottom Trapped Waves. 

Now, we consider a more general case for the shelf wave. As has been shown in the previous section, if the shelf width is large compared to the baroclinic radius of deformation, shelf waves can be adequately described by the shallow water equations,

\[
\begin{align*}
\bar{u}_t + f \bar{v} &= -\frac{1}{\rho_0} \bar{p}_x, \\
\bar{v}_t + f \bar{u} &= -\frac{1}{\rho_0} \bar{p}_y, \\
\bar{u}_x + \bar{v}_y + \bar{w} &= 0,
\end{align*}
\]

with boundary conditions

- \( \bar{u} = 0 \) at \( x = 0 \) and \( x \to \infty \)
- \( \bar{w} = 0 \) at \( z = 0 \)
- \( \bar{w} = -\bar{u} H_x \) at \( z = -H(x) \).

Assuming wavelike solution,

\[ \sim e^{i (y + C_x)} , \]

Eq. (3.a) can be solved numerically for a given depth profile, and the eigenfunctions are sketched (in Fig. 4) for a hyperbolic tangent depth profile.
Looking for the solution of the "forced wave" which travels with the forcing shelf wave, \( P \sim \mathcal{E}^{s} \mathcal{E}^{t} \). Equation (2.a) is still valid in the interior with the change of lower boundary condition, (the down-stream velocity is assumed to be negligible compared to the phase speed).

\[
\omega = -\overline{u} H(x) \quad \text{at} \quad z = -H(x) \tag{3.a}
\]

where \( \overline{u} \) is the cross-stream velocity of shelf waves. Equation (3.a) is a linearized boundary condition, i.e., we neglect the contribution of the cross-stream velocity of the "forced wave" by assuming it to be small compared to that of the forcing wave.

From the lower boundary condition (Eq.3.a), it is clear that forcing can be important only at the place where the bottom vertical velocity is large, or, at the continental slope (Fig.1).

Furthermore, since forcing is concentrated on the continental slope, the forced wave is expected to be trapped along this rather narrow region and does not "see" the lateral boundaries. (The shelf width is much larger than the slope width.) Rewriting the non-dimensional governing equation (Eq.2.d) with different scaling, we have

\[
P_{xx} - \rho \beta P + \frac{1}{2} \mu \gamma P_{zz} = 0 \tag{3.b}
\]

where \( \mu = \frac{g H_{o}^{2}}{f^{2} L_{o}^{2}} \) and \( H_{o}, L_{o} \) are the height and width of the continental slope (Fig.4). And the boundary conditions are,

\[
\begin{align*}
\omega &= -\overline{u} H(x) \quad \text{at} \quad z = -H(x), \\
\omega &= 0 \quad \text{at} \quad z = 0, \\
P &= 0 \quad \text{as} \quad x \rightarrow \pm \infty
\end{align*}
\]

Unlike the previous case in which motions are confined essentially in the small sloping shelf region, the motion now is confined in a large sloping region. Consequently, \( \mu = 1 \) is of the order of unity for the typical continental slope. Equation (3.b) with boundary conditions (Eq.3.a) no longer can be handled by the separation of variables method. Hence, the Laplace equation is solved numerically with SOR (successive over-relaxation) method for the case \( \mu = 1 \), and the resulting vertical velocity distribution is given in Fig.5.
Fig. 5 Forcing prescribed along the boundary,

The e-folding distance for the trapped wave is about 0.6 $L_R$ (where $L_R$ is baroclinic radius of deformation) which is twice as large as one would expect for the case of rectangular geometry. The increase of penetration depth is due to the effects of inclined plane and upward displacement of the upper lid. This is clearly shown in Fig. 6 for the case where the upper lid is removed while the forcing still remains in the vertical plane. The penetration depth for this case is just between the two extremum cases mentioned before. From this calculation, one concludes that bottom trapped waves can be forced by the barotropic shelf waves along the continental slope, because the width of the continental slope is compatible with the baroclinic radius of deformation.
4. Conclusions and Comments

A simple theory which relates the generation of internal Kelvin waves (along the coast) and bottom trapped waves (along the continental slope) by the barotropic shelf waves gives some quite consistent pictures of the observed low frequency baroclinic oscillations which are unlikely to be directly excited by the atmospheric forcing. One interesting question raised from this study is the observation that energy transformation to the internal Kelvin waves is very likely to come from shelf waves with very small group velocity (Fig.3). This may be a very efficient mechanism for the dispersion of geostrophic energy from the source region.

In other words, considering a slowly (longer than pendulum day) input atmospheric forcing along the continental shelf, the geostrophic
energy will be converted into the quasi-geostrophic waves (shelf waves) and be trapped locally if the forced shelf waves have very small group velocity. The ocean can no longer reestablish the balanced state unless there is a leakage of the trapped quasi-geostrophic waves. This study suggests the possibility of energy transformation from "trapped" quasi-geostrophic waves into "leaky" ageostrophic waves, and hence the possibility to achieve the geostrophic balanced state. The energy transformation from quasi-geostrophic mode to ageostrophic internal mode suggested here is physically sound and very realistic, while previously it has not been exploited.

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References