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1969 VOLUME II

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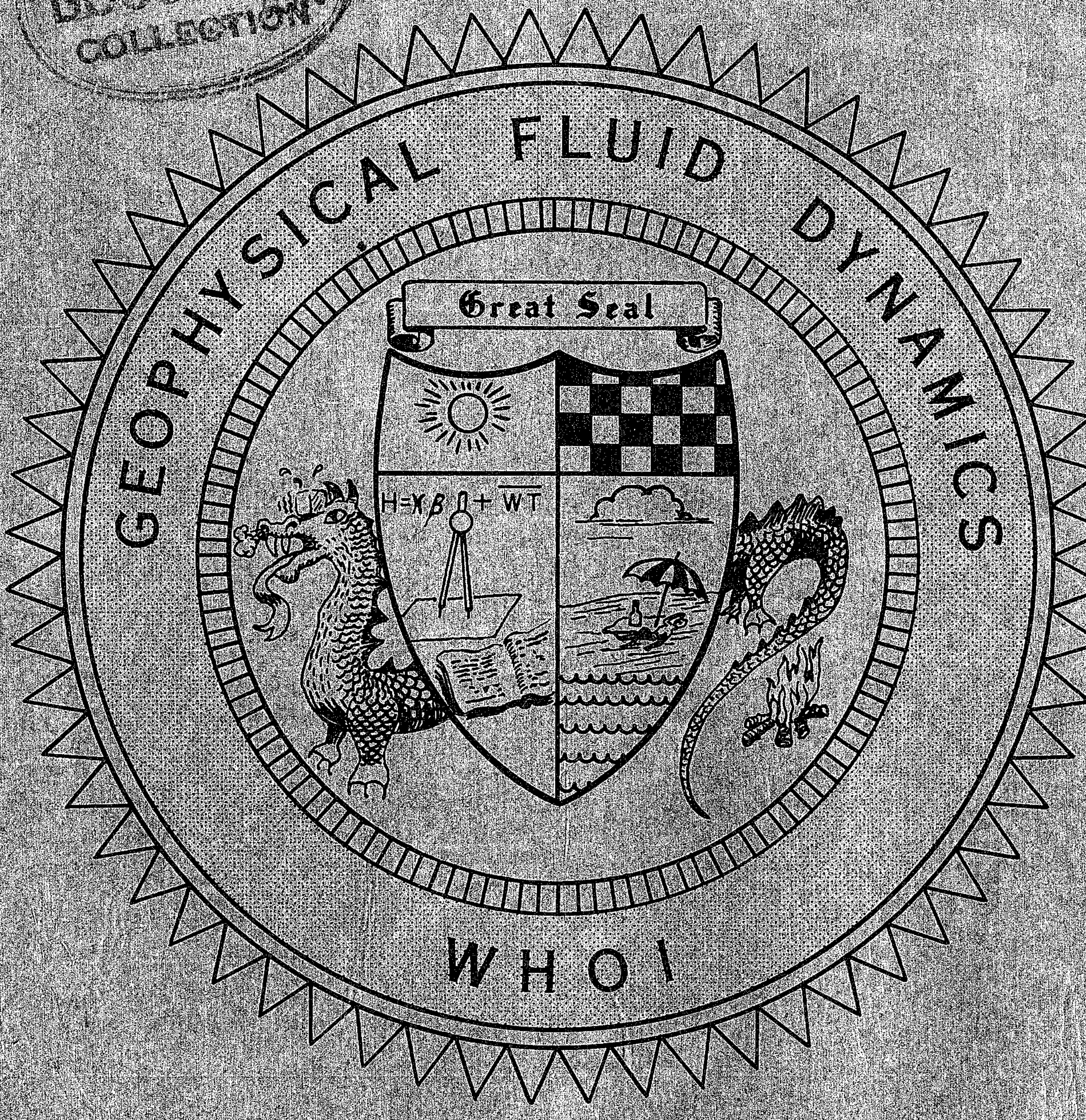

FELLOWSHIP LECTURES

Notes on the 1969

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

Reference No. 69-41

Contents of the Volumes

1969

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Editors' Preface

This volume contains the manuscripts of research lectures by the Fellows in the summer program. The staff guided the selection of topics with several goals in mind. One goal was to isolate that part of a problem which might prove to be tractable in an effort of eight weeks or less. The more important goal was to find "open-ended" problems which would continue to challenge the Fellow after his return to the university. Our success in so guiding a Fellow exhibits considerable fluctuation. This year only two of the manuscripts neatly compass a finite problem. The remainder describe first thoughts on rather vast enterprises.

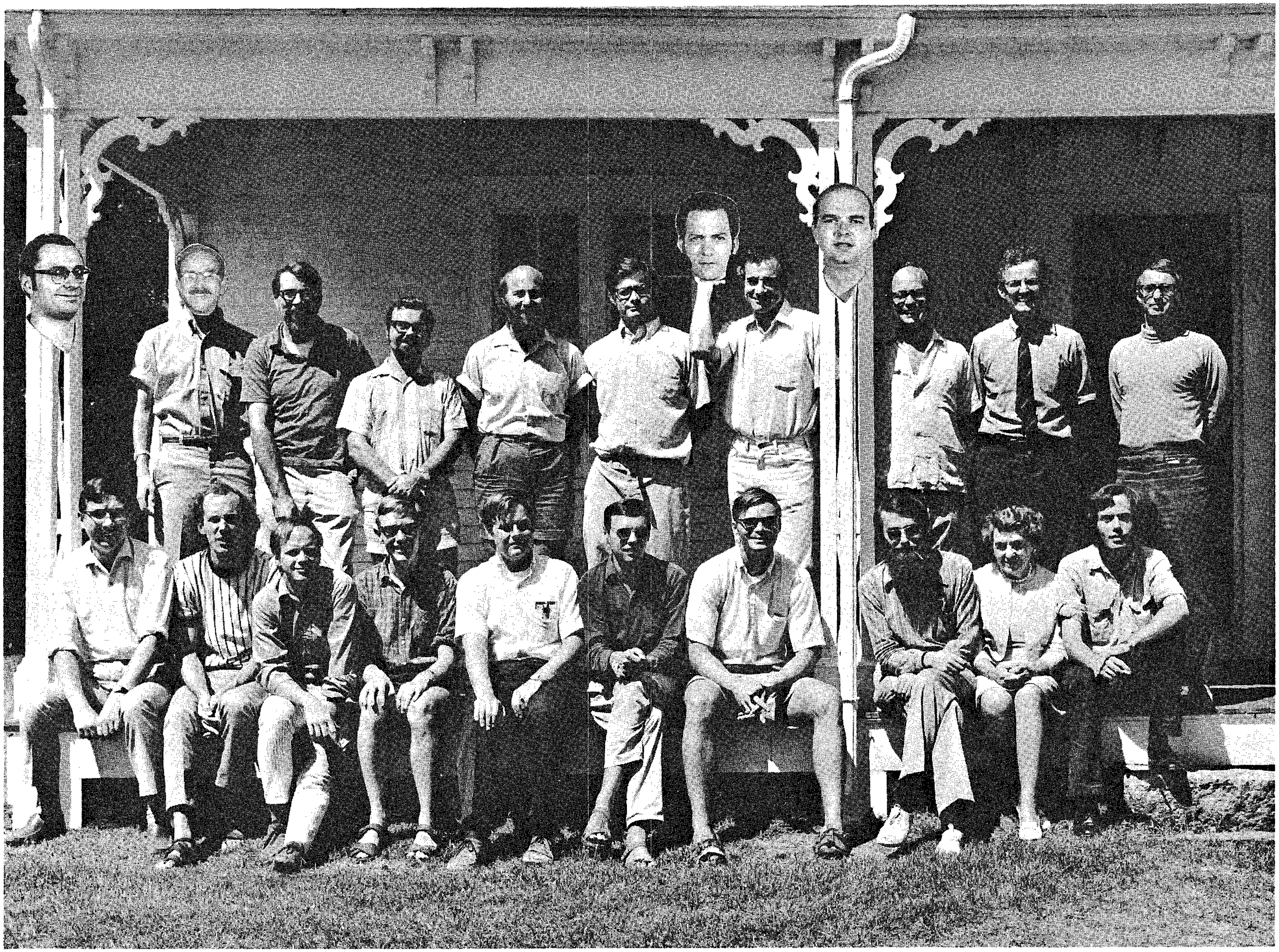
The degree of direction by a senior participant varied a great deal. In a few cases, there were frequent conferences and discussions about fruitful avenues of approach. In other cases, contact was limited to expression of interest and encouragement. The efforts cover a wide spectrum in originality also. Some of the reports represent a competent extension of research presented in the seminars - others are highly original contributions which are being prepared for publication.

Because of time limitations it was not possible for the manuscripts to be edited and reworked. They may contain errors the responsibility for which must rest on the shoulders of the participant-author. It must be emphasized that this volume in no way represents a collection of reports of completed and polished work.

We who took part in this eleventh summer of G.F.D. are grateful to the National Science Foundation for its continuing support of the program.

Mary C. Thayer

Willem V.R. Malkus



Standing (left to right): Buschi (posted), Stern, Prendergast, Gough, Keller, Kraichnan, Spiegel (being supported by Veronis), Toomre (pillared), Harrison, Backus, Malkus.
Seated: Zahn, Defouw, Gans, McKee, Trasco, Perdang, Barker, Auré, Thayer, Mészáros.

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Fellows' Lectures

OVERSTABLE DAMPING IN A STELLAR SEMI-CONVECTIVE ZONE

Jean-Luc Auré

Abstract

We show that, with a two-zone model and by an asymptotic treatment of the linearized equations of perturbation, the overstability due to the region stratified in mean molecular weight is totally damped by the upper radiative region in a B star departing from the main sequence. Perhaps, however, such an overstability is possible in the layer of gravitational sorting of some white dwarfs, for higher g-modes.

I. Introduction

A local treatment by Kato (1966) has shown the possibility of a mixing by overstability in a medium stratified in mean molecular weight. Kato concluded that the temperature gradient in a semi-convective zone was reduced to its adiabatic value by the mixing

$$\nabla_T = \nabla_{Ta} \qquad (1)$$

value adopted by Ledoux (1947) and Schwarzschild and Härm (1958), rather than taking the marginal convective value used by Stothers (1963) and Sakashita and Hayashi (1961):

$$\nabla_T = \nabla_{Ta} + \frac{\beta}{4-3\beta}\nabla_\mu \qquad (2)$$

where

$$\nabla_T = \left(\frac{\partial \operatorname{Log} T}{\partial \operatorname{Log} p}\right)_{\text{star}}$$

$$\nabla_{Ta} = \left(\frac{\partial \operatorname{Log} T}{\partial \operatorname{Log} p}\right)_{\text{adiabatic}}$$

$$\nabla_\mu = \left(\frac{\partial \operatorname{Log} \mu}{\partial \operatorname{Log} p}\right)_{\text{star}}$$

and β is the quotient of the gaseous pressure on the total pressure. In a recent paper (1969), Gabriel calculates the condition of overstability in a two-zone model. Unlike Kato, he concludes that his model is overstable only if

$$0 \leq \frac{\nabla_\rho - \nabla_{\rho a}}{\nabla_\mu} \leq \varepsilon \tag{3}$$

where

$$\nabla_\rho = \left(\frac{\partial \text{Log}\, \rho}{\partial \text{Log}\, p}\right)_{\text{star}}$$

$$\nabla_{\rho a} = \left(\frac{\partial \text{Log}\, \rho}{\partial \text{Log}\, p}\right)_{\text{adiabatic}}$$

and ε is very small (typically $\varepsilon \simeq 0.0002$). However, because the semi-convective region cannot be separated from the rest of the star, Kato's local analysis is not sufficient to conclude on this problem. On the other side, Gabriel's calculation of the damping seems irrelevant enough (arbitrary introduction of a singular point in the eigen function, and non-introduction of this singularity in the divergence of the flux).

In this work, we calculate the overstable damping in two-zone models. We show that, as we could expect, the radiative damping in a lower zone is quite negligible, and in an upper zone it is very strongly stabilizing. Our purpose is to show that the temperature gradient chosen by Kato, as well as Gabriel's gradient, are not right for building models of stars with semi-convective zones. We shall see that the overstability cannot occur in a B star departing from the main sequence.

II. Equations of perturbations

In our work, we shall use two-zone models formed by a radiative zone ($\nabla_\mu = 0$), in which the perturbations are damped, and a semi-convective zone ($\nabla_\mu \neq 0$) which excites the overstability. We have to state precisely that, in this work, the expression "semi-convective zone" is understood as a zone with $\nabla_\mu \neq 0$ and where the temperature gradient is between the adiabatic value and the marginal convective value, as several other authors have done. We make no assumption *a priori* on the properties of the semi-convection, and particularly on the way of transfer of the energy in this region. We only assume that the perturbations of flux are perturbations of radiative transfer.

The models are plane and parallel. The gravity and the flux of the heat are constant everywhere at the equilibrium. We shall neglect the perturbations of gravity. The z-axis is along the gravity and turned to the upper bound. The medium is a compressible perfect gas. To simplify, we assume the radiative pressure negligible before the gas pressure; that does not change the results very much, and allows us to take polytropic models (if $\beta \neq 1$, β has to be variable in the semi-convective zone, and then no more can we find solutions analytically). According to this assumption, we choose:

$$\nabla_{P_a} = 0.6 \tag{4}$$

$$\nabla_{T_a} = 0.4 \tag{5}$$

in both zones.

The radiative zone is a standard polytropic model. The polytropic index n = 3.

$$n = \frac{\nabla_\rho}{1-\nabla_\rho} \tag{6}$$

$$H_p = H_{p_0} + (z - z_0)(1-\nabla_\rho) \tag{7}$$

where H_p is the height scale of pressure. In the semi-convective zone, we assume:

$$\nabla_\mu > 0 \tag{8}$$

$$\nabla_T = \nabla_{T_a} + \varepsilon\, \nabla_\mu \tag{9}$$

$$\nabla_\rho = \nabla_{\rho_a} + (1-\varepsilon)\nabla_\rho \tag{10}$$

with $\quad 0 \leqslant \varepsilon \leqslant 1$

Afterwards, we shall assume $\nabla_\rho \neq 1$ (which introduces a mathematical singular case, and has no physical importance for our purpose).

We write the perturbed quantities

$$X = X_0 + X' e^{i\sigma t} e^{i(k_x x + k_y y)} \tag{11}$$

$$k^2 = k_x^2 + k_y^2 \tag{12}$$

where X_0 is the equilibrium value, and X' the only function of z.
In Eulerian coordinates, the linearized equations of perturbations are:

Conservation of mass:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\delta r} + \frac{\rho'}{\rho_0} + \delta_z \frac{\delta \operatorname{Log} \rho_0}{\delta z} = 0 \tag{13}$$

Conservation of momentum:

$$\sigma^2 \underset{\sim}{\delta r} = \frac{1}{\rho_0} \underset{\sim}{\nabla}\, p' + \underset{\sim}{g}\, \frac{\rho'}{\rho_0} \tag{14}$$

Radiative transfer:

$$\underset{\sim}{\nabla} \frac{4}{3} a T_0^3 T' = -\frac{\bar{K}_0 \rho_0}{c} \underset{\sim}{F}' - \left(\frac{\bar{K}\rho}{c}\right)' \underset{\sim}{F}_0 \tag{15}$$

Conservation of energy:

$$\left(\frac{p'}{p_0} + \delta_z \frac{\partial \log p_0}{\partial z}\right) - \Gamma_1 \left(\frac{\rho'}{\rho_0} + \delta_z \frac{\partial \log \rho_0}{\partial z}\right) = \frac{-(\Gamma_3 - 1)}{i\sigma p_0} \underset{\sim}{\nabla} \cdot \underset{\sim}{F}' \tag{16}$$

where the second part is corresponding to the non-adiabatic effects.

Conservation of the mean molecular weight:

$$\frac{\mu'}{\mu_0} = -\delta_z \frac{\partial \log \mu_0}{\partial z} \tag{17}$$

(The typical time of diffusion is very much longer than the period of the perturbation in the stars.)

Equation of state:

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0} - \frac{\mu'}{\mu_0} \tag{18}$$

p = pressure

ρ = density

T = temperature

μ = mean molecular weight

$\delta \underset{\sim}{r}$ = displacement

δ_z = vertical component of $\delta \underset{\sim}{r}$

g = gravity

a = constant of radiative pressure

$\bar{K}$ = mean coefficient of opacity

$\underset{\sim}{F}$ = flux of heat

c = light velocity

$$\Gamma_1 = \frac{1}{\nabla_{P_a}}$$

$$\Gamma_3 = 1 + \frac{\nabla_{T_a}}{\nabla_{P_a}}$$

Introducing the new functions:

$$A = \frac{F'_z}{F_{z_0}}, \quad B = \frac{p'}{p_0}, \quad C = \frac{T'}{T_0}, \quad D = \frac{\delta z}{H_{p_0}}$$

and the new variable: $\varpi = \operatorname{Log} p_0$

with $\frac{\partial}{\partial \varpi}$ written $\dot{}$

We obtain the following system:

$$\dot{D} = (\nabla_\mu - 1) D + \left(1 - \frac{k_g^2 H_{p_0}}{\sigma^2}\right) B - C \tag{19}$$

$$\dot{B} = \left(\nabla_\mu - \frac{\sigma^2 H_{p_0}}{g}\right) D - C \tag{20}$$

$$\dot{C} = (1 + K_\mu) \nabla_\mu \nabla_T D + (1 + K_p) \nabla_T B + (K_T - 5) \nabla_T C + A \nabla_T \tag{21}$$

$$\dot{A} = - \frac{i \sigma p_0 H_{p_0}}{F_{z_0} \nabla_{T_a}} \left[(\nabla_T - \nabla_{T_a}) D + (1 - \nabla_{P_a}) B - C \right] + \frac{k^2 H_{p_0}^2}{\nabla_T} C \tag{22}$$

where $K_\mu = \frac{\partial \operatorname{Log} \bar{K}_0}{\partial \operatorname{Log} \mu_0}$, $K_p = \frac{\partial \operatorname{Log} \bar{K}_0}{\partial \operatorname{Log} p_0}$, $K_T = \frac{\partial \operatorname{Log} \bar{K}_0}{\partial \operatorname{Log} T_0}$

K_μ, K_p and K_T are assumed constant in each zone.

Everywhere we assume H_p, p, ρ and $T \neq 0$. According to this assumption and as we shall make an asymptotic study ($\sigma \longrightarrow 0$) in which the number of modes becomes very large, the conditions at the upper and inner bounds will have a negligible influence on the eigenfunctions. The only conditions which we take into account will be the conditions of continuity of the eigenfunctions at the boundary between the two zones. (Let us remember that the equilibrium values of the physical parameters p_0, ρ_0, μ_0, T_0, $\underset{\sim}{F_0}$ are continuous at this bound.) In fact, we shall see that it suffices to connect the exponential envelopes of the eigenfunctions.

III. Inner zones - Quasi-adiabatic approximation

Three variable coefficients appear in the previous system:

$$V_1 = \frac{k^2 g H_{p_0}}{\sigma^2}$$

$$V_2 = k^2 H_{p_0}^2$$

$$V_3 = \frac{\sigma p_0 H_{p_0}}{F_{z0}}$$

with $$\frac{\sigma^2 H_p}{g} = -\frac{V_2}{V_1}$$

When $\sigma \longrightarrow 0$ we can separate a star into three parts corresponding to different approximations in our system. The constant coefficients are of the order of unity. Then we shall compare V_1, V_2, V_3 at unity.

	Region I	Region II	Region III	
outer	———————	———————	———————→ p	inner
	$V_1, V_2, V_3 < 1$	$V_1 > 1$	$V_1, V_3 > 1$	
		$V_2, V_3 \lesssim 1$	$V_2 \lesssim 1$	

$V_2 = k^2 H_p^2$ is everywhere smaller than one, for the first non-radial modes, (P_2 or P_4). In the inner region of a star, V_1 is almost equal to one for the fundamental mode. In our asymptotic treatment, we always take σ small enough for keeping $V_1 > 1$ in the regions II and III. On the other side, in the inner zone, p is very large and $V_3 > 1$. In region II, p is small enough so that $V_3 < 1$. In region I, H_p is very small and $V_1 < 1$. The limits of this region are strongly dependent on σ . When σ is tending to zero, the limit I - II is going to the surface and the limit II - III is moving to the center of the star. In the regions II and III, the eigenfunctions are very sensitive at the non-radial character of the perturbation. In region I, they are less sensitive.

In regions I and II, where the pressure is weak and the optical thickness is small, the time of relaxation in temperature is smaller than the period of the pulsation. During its motion, each layer will be tending to be in thermal equilibrium with the neighbouring layers. Then we shall be able to use an Eulerian quasi-isothermal approximation to determine the eigenfunctions. In region III, $V_3 \gg 1$, and consequently the quantity between the brackets in Eq. (22) (non-adiabatic term) is very small. Then we shall be able to approximate the true eigenfunctions by the adiabatic eigenfunctions.

In the astrophysical cases, the semi-convective zone is most often situated in the region III. Therefore we shall study the damping in this region first.

Adiabatically, the system (19) - (22) becomes:

$$\dot{D} = D\left(\nabla_\rho - \nabla_{\rho_a} - 1\right) + B\left(\nabla_{\rho_a} - V_1\right) \tag{23}$$

$$\dot{B} = D\left[\left(\nabla_\rho - \nabla_{\rho_a}\right) - \frac{V_2}{V_1}\right] + B\left(\nabla_{\rho_a} - 1\right) \tag{24}$$

As $V_1 \gg 1$ we can neglect $\nabla_{\rho a}$ before V_1. Then we obtain the second order equation:

$$\ddot{D} + \dot{D} + D\left(\left(\nabla_\rho - \nabla_{\rho a}\right)V_1 - k^2 H_p^2 + \left(\nabla_\rho - \nabla_{\rho a}\right)\left(\nabla_{\rho a} - \nabla_\rho + 1\right)\right) = 0 \qquad (25)$$

If $(\nabla_\rho - \nabla_{\rho a})$ is very small, the eigenfunctions are exponential and are almost the eigenfunctions of the convective neutrality. This case is not interesting for our purpose, for then, as we shall see it, its contribution to the excitation of the overstability is very weak.

If $(\nabla_\rho - \nabla_{\rho a})$ is not very small, the previous approximation obliges us to neglect $k^2 H_p^2$ and $(\nabla_{\rho a} + 1 - \nabla_\rho)(\nabla_\rho - \nabla_{\rho a})$ before $V_1(\nabla_\rho - \nabla_{\rho a})$. Then we obtain:

$$\ddot{D} + \dot{D} + D\left(\nabla_\rho - \nabla_{\rho a}\right)V_1 = 0 \qquad (26)$$

Introducing:

$$V_1\left(\nabla_\rho - \nabla_{\rho a}\right) = \eta^2 \qquad (27)$$

$$x = \int_{\varpi_0}^{\varpi} \eta \, d\varpi \qquad (28)$$

$$y = \left[\exp\left(\left(\int_{\varpi_0}^{\varpi} d\varpi\right) - \eta\right)\right]^{1/2} D \qquad (29)$$

The equation (27) becomes:

$$\frac{d^2 y}{d x^2} + (1 + \theta)\, y = 0 \qquad (30)$$

$$\theta = -\frac{1}{2}\frac{\nabla_\rho}{\eta^2}\left(1 - \frac{\nabla_\rho}{2}\right) \qquad (31)$$

Following our previous approximations we have to neglect θ before 1. We shall obtain the solution:

$$y = K' \cos(x + \psi) \qquad (32)$$

which can be put under the following form:

$$D = K \exp(\alpha \varpi) \cos\left(\frac{2}{(1-\nabla_p)} \sqrt{\frac{(\nabla_p - \nabla_{p_a}) k^2 g H_{p_0}}{\sigma^2}} + \varphi\right) \tag{33}$$

with

$$K = K'\left(\frac{k^2 g\,(\nabla_p - \nabla_{p_a})}{\sigma^2}\right)^{-1/4} \left(\frac{H_{p_0 *}}{p_{0*}(1-\nabla_p)}\right)^{-1/4} \tag{34}$$

and

$$\alpha = \frac{(\nabla_p - 3)}{4} \tag{35}$$

where the quantities with the index * are taken at the frontier of the two zones. Of course, the eigenfunctions are formally similar in both zones, but the different coefficients are not the same and we shall have to connect both solutions. Then we can write the adiabatic expression of C and B:

$$C = D\,(\nabla_T - \nabla_{T_a}) \tag{36}$$

$$B \simeq K_1 \exp(\alpha' \varpi) \sin\left(\frac{2}{(1-\nabla_p)}\,\eta + \varphi\right) \tag{37}$$

with

$$K_1 = -K\,\frac{(\nabla_p - \nabla_{p_a})}{\eta} \left(\frac{H_{p_0}}{H_{p_0 *}}\right)^{1/2} \tag{38}$$

$$\alpha' = \alpha + \frac{\nabla_p - 1}{2} \tag{39}$$

Therefore, we can calculate the divergence of the perturbed flux, (the divergence of the unperturbed flux being null).

$$(\underset{\sim}{\nabla} \cdot \underset{\sim}{F})' = \frac{F_{3_0}}{H_{p_0}\nabla_T}\left[-\ddot{C} + \dot{D}(1+\kappa_\mu)\nabla_\mu \nabla_T + \dot{B}(1+\kappa_p)\nabla_T + \dot{C}(\kappa_T - 5)\nabla_T + k^2 H_p^2\, C\right] \tag{40}$$

Only keeping the terms of the highest order in η :

$$\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{F}'\right) = \frac{F_{z0}}{H_{p0}\nabla_T}\,\eta^2\left(\nabla_T - \nabla_{Ta}\right)D \tag{41}$$

We still must write the conditions of continuity of eigenfunctions at the frontier.

$$D_{Rad}\left(z_*\right) = D_{s-c}\left(z_*\right) \tag{42}$$

$$\dot{D}_{Rad}\left(z_*\right) - \dot{D}_{s-c}\left(z_*\right) = D\left(z_*\right)\left(\nabla_{\rho Rad} - \nabla_{\rho s-c}\right) \tag{43}$$

We obtain a system of transcendental equations, difficult to solve analytically. However, as we study the asymptotic behaviour of the integral of damping or of excitation, it seems a good approximation to connect the envelopes of the eigenfunctions.

$$K_{Rad} = K_{s-c}\,\exp\left(\left(\alpha_{s-c} - \alpha_{Rad}\right)\varpi_*\right) \tag{44}$$

The connection of the phases φ , which would be corresponding to the second condition, is not important for we shall take a mean value for the oscillating terms (in cosines).

IV. Integral of excitation - Damping by the lower layers

In the non-adiabatic problem the time-dependence of the perturbations has the form $\exp(-\sigma' t + i\sigma t)$, where σ' is given by (Ledoux and Wahaven 1958):

$$\sigma' = \frac{J}{2\sigma^2 I} \tag{45}$$

with

$$I = \int_V \rho\,\underset{\sim}{\delta r}\cdot\underset{\sim}{\delta r}\,dv \tag{46}$$

$$J = \int_V \frac{\delta T}{T}\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{F}\right)'dv \tag{47}$$

where δT is the Lagrangian perturbation of the temperature:

$$\frac{\delta T}{T} = -\nabla_{Ta}\, D \tag{48}$$

If we assume that the quasi-adiabatic approximation is relevant in the whole model:

$$J = \int_{z_1}^{z_2} - K^2 \eta^2 (\nabla_T - \nabla_{Ta}) \frac{F_{z0} \nabla_{Ta}}{H_p \nabla_T} \exp(2 \alpha \varpi) \cos^2\left(\frac{2\eta}{(1-\nabla_p)} + \varphi\right) dz \tag{49}$$

where z_1 and z_2 are the lower and the upper bounds of the model, z_* being the frontier between both zones. We can separate J into its radiative part and its semi-convective part. The asymptotic assumption allows us to do this for $\cos^2$ by its average value:

$$\overline{\cos^2\left(\frac{2\eta}{1-\nabla_p} + \varphi\right)} = \frac{1}{2} \tag{50}$$

Only one of the signs of J is interesting for us:

if $J > 0$, the motion is damped and the model is stable;

if $J < 0$, the model is overstable.

In this part, we assume the lower zone is radiative, and the upper semi-convective. Taking into account the integral

$$L = \int_{z_a}^{z_b} \exp(2 \alpha \varpi)\, dz \tag{51}$$

$$L = \frac{2 H_{po*}}{(\nabla_p + 1)} \left[x_b^{-\frac{\nabla_p+1}{2}} - x_a^{-\frac{\nabla_p+1}{2}} \right] \tag{52}$$

where $x = p/p_*$, and putting down

$$K_{Rad}^2 H_{po*} = 1$$

we can write the condition of overstability:

$$\left(\nabla_{\rho_1}-\nabla_{\rho_a}\right)\frac{\left(\nabla_{T_1}-\nabla_{T_a}\right)\nabla_{T_a}}{\left(\nabla_{\rho_1}+1\right)\nabla_{T_1}}\left[1-x_1^{-\frac{1+\nabla_{\rho_1}}{2}}\right] <$$

$$< \frac{\nabla_\mu^2\,\varepsilon\,(1-\varepsilon)\,\nabla_{T_a}\left[x_2^{-\frac{\nabla_{\rho_a}+1}{2}-\frac{1-\varepsilon}{2}\nabla_\mu}-1\right]}{\left(\nabla_{\rho_a}+(1-\varepsilon)\nabla_\mu+1\right)\left(\nabla_{T_a}+\varepsilon\,\nabla_\mu\right)} \tag{53}$$

where the index 1 distinguishes the radiative zone.

We can overestimate the first part by taking $\chi_1 \longrightarrow \infty$. Introducing into the second part:

$$y = \frac{\mu_2}{\mu_*} \tag{54}$$

with $\chi_2 = y^{1/\nabla_\mu}$

Typically, y can have two values:

- y = 0.4 (transition Helium - Hydrogen, for instance in a B star with a large semi-convective zone);
- y = 0.25 (transition Metals - Hydrogen, for instance in the gravitational sorting zone of the white dwarfs).

If y is given, the overstability is only dependent on two parameters: ε and ∇_μ. Introducing:

$$f\left(\varepsilon,\nabla_\mu\right) = g\left(\varepsilon,\nabla_\mu\right)\cdot h\left(\varepsilon,\nabla_\mu\right) \tag{55}$$

$$g\left(\varepsilon,\nabla_\mu\right) = \frac{\nabla_\mu^2\,\varepsilon\,(1-\varepsilon)\,\nabla_{T_a}}{\left(\nabla_{\rho_a}+(1-\varepsilon)\nabla_\mu+1\right)\left(\nabla_{T_a}+\varepsilon\,\nabla_\mu\right)} \tag{56}$$

$$h\left(\varepsilon,\nabla_\mu\right) = y^{-\frac{(1-\varepsilon)}{2}-\frac{0.8}{\nabla_\mu}}-1 \tag{57}$$

$f(\varepsilon,\nabla_\mu)$ is a function which characterizes the excitation in a semi-convective zone of lower bound given. We have tabulated $f(\varepsilon,\nabla_\mu)$ for y = 0.25 and y = 0.4. The function is small depending on y, and increases slightly when y decreases. These results are resumed in the two following figures:

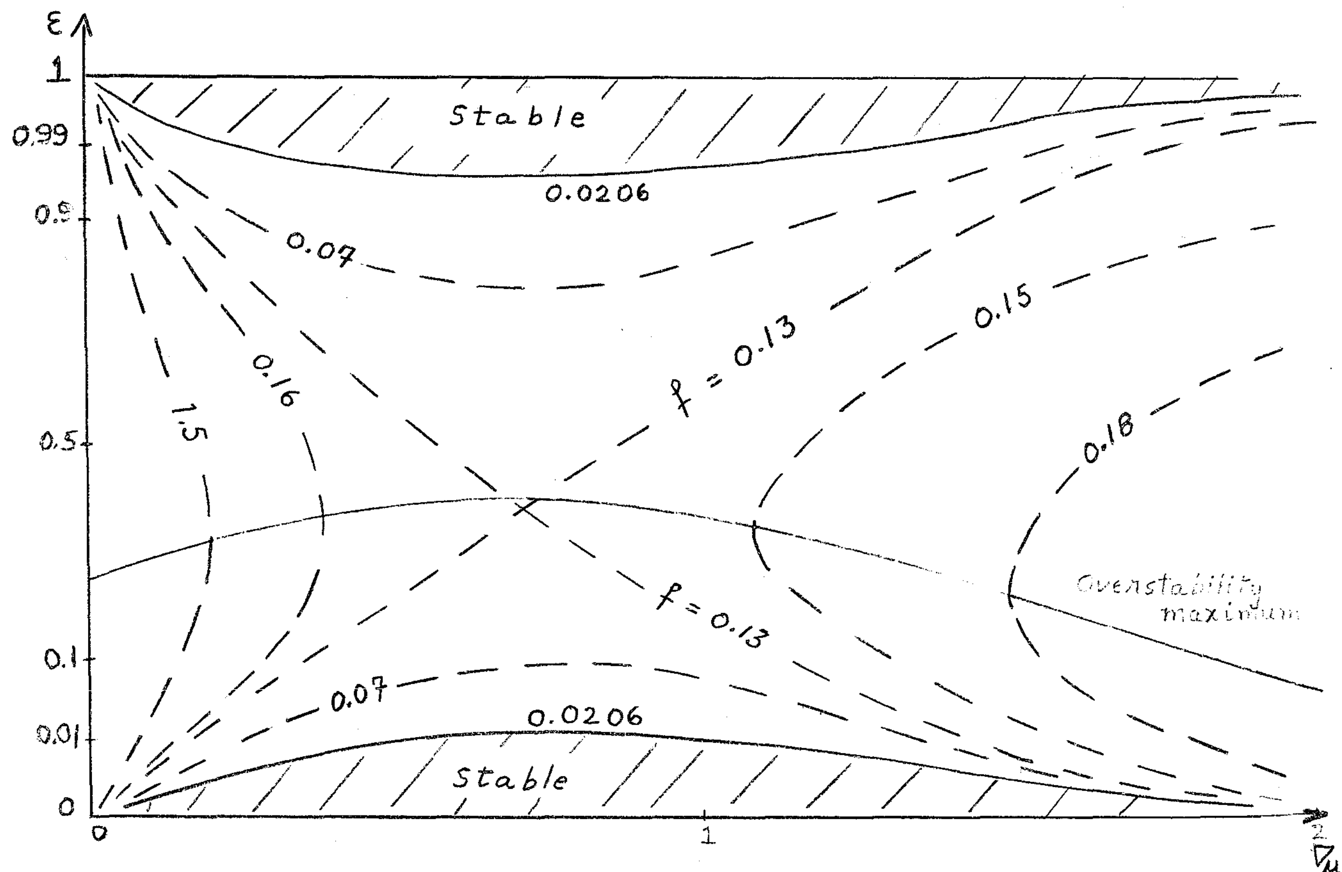

Fig. 1: Curves of isovalue of $f(\varepsilon, \nabla_\mu)$ for y = 0.4

In fact, the astrophysical cases correspond to $0.1 < \nabla_\mu < 2$ ($\nabla_\mu \simeq 0.25$ in a B giant star, $\nabla_\mu \simeq 1.5$ in a white dwarf). This result clearly shows that the damping by the inner layers is almost negligible, as long as the quasi-adiabatic approximation is relevant in the semi-convective zone. We could expect this result a priori from the general aspect of the eigenfunction. But this case is interesting, for it makes conspicuous the behaviour of the excitation for the interesting values of ∇_μ. Especially, let us note the big difference between our results and the previous results

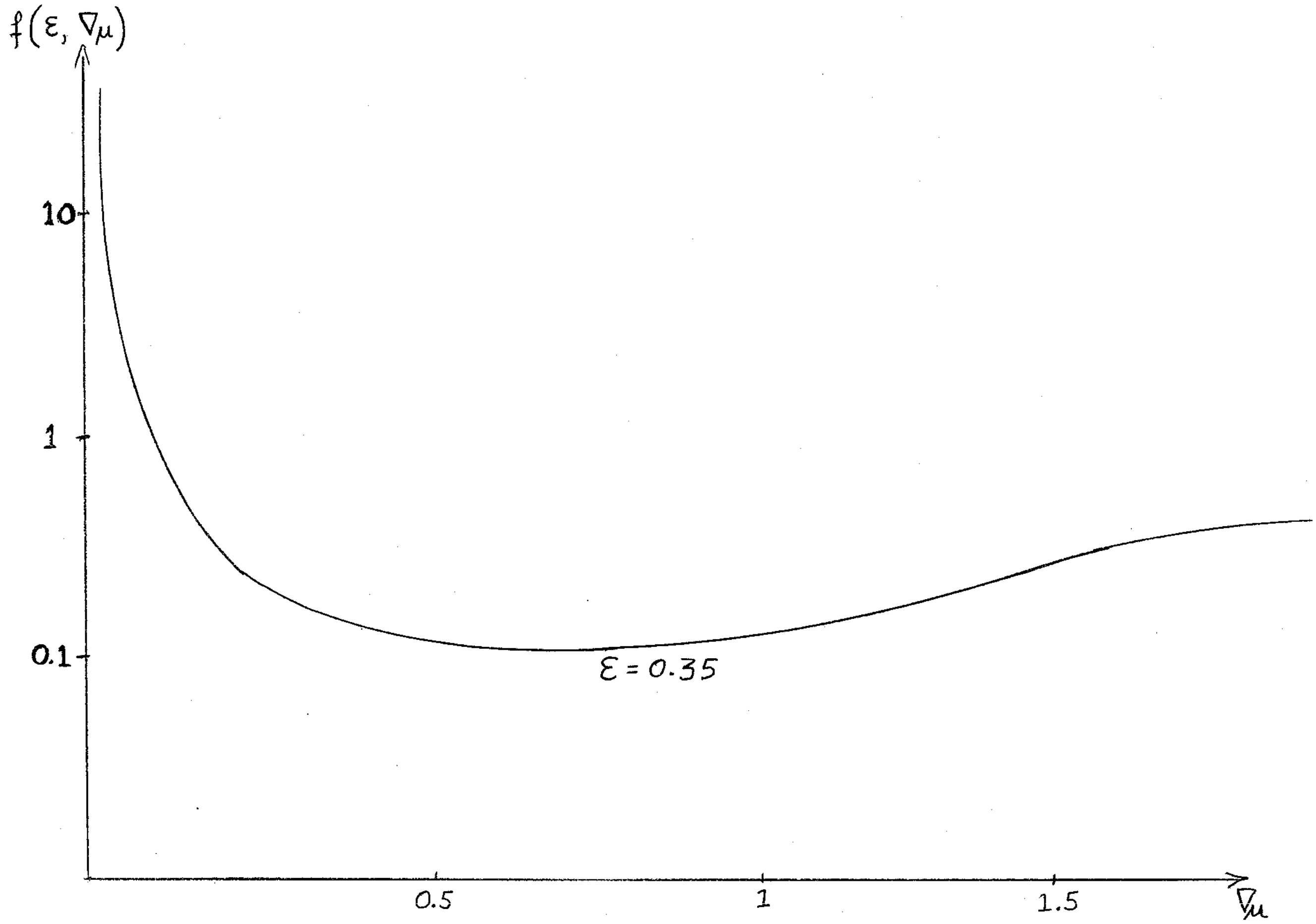

Fig.2: Curve of largest excitation

of other authors. If the model were overstable, and if we adopted the theory of mixing by the overstability, then we should have to take for temperature gradient, the gradient corresponding to the largest overstability. Therefore in this case, the gradient would be not $\nabla_T \simeq \nabla_{T_a}$ as Kato chooses it, not $\nabla_T \simeq \nabla_{T_a} + \nabla_\mu$ as Gabriel has calculated, but $\nabla = \nabla_{T_a} + \varepsilon \nabla_\mu$ with about $\varepsilon \simeq 0.3$ for the interesting values of ∇_μ.

We can observe also that a semi-convective layer, as thin as we want, will be able to be excited for the good values of ε, (if y stays constant) when the damping is only produced by the lower zones.

V. Damping by the upper layer

Generally, however, the most important damping is produced by the upper layers. The condition of overstability (Eq. 53) becomes:

$$\left(\nabla_{\rho_1}-\nabla_{\rho_a}\right)\frac{\left(\nabla_{T_1}-\nabla_{T_a}\right)\nabla_{T_a}}{\left(\nabla_{\rho_1}+1\right)\nabla_{T_1}}\left[x_1^{-\frac{\nabla_{\rho_1}+1}{2}}-1\right]<g\left(\varepsilon,\nabla_\mu\right)\left[1-y^{-\frac{1-\varepsilon}{2}-\frac{0.8}{\nabla_\mu}}\right] \tag{58}$$

where z_1 is now the upper bound and z_2 the lower. The radiative layer is upper, and the semi-convective layer lower. Then, we see immediately that the excitation is limited, whereas the damping is quickly growing when x_1 decreases. However, when the pressure is decreasing, the quasi-adiabatic approximation becomes improper. When the coefficient $V_3 << 1$ we have previously shown that we should have to use a new approximation.

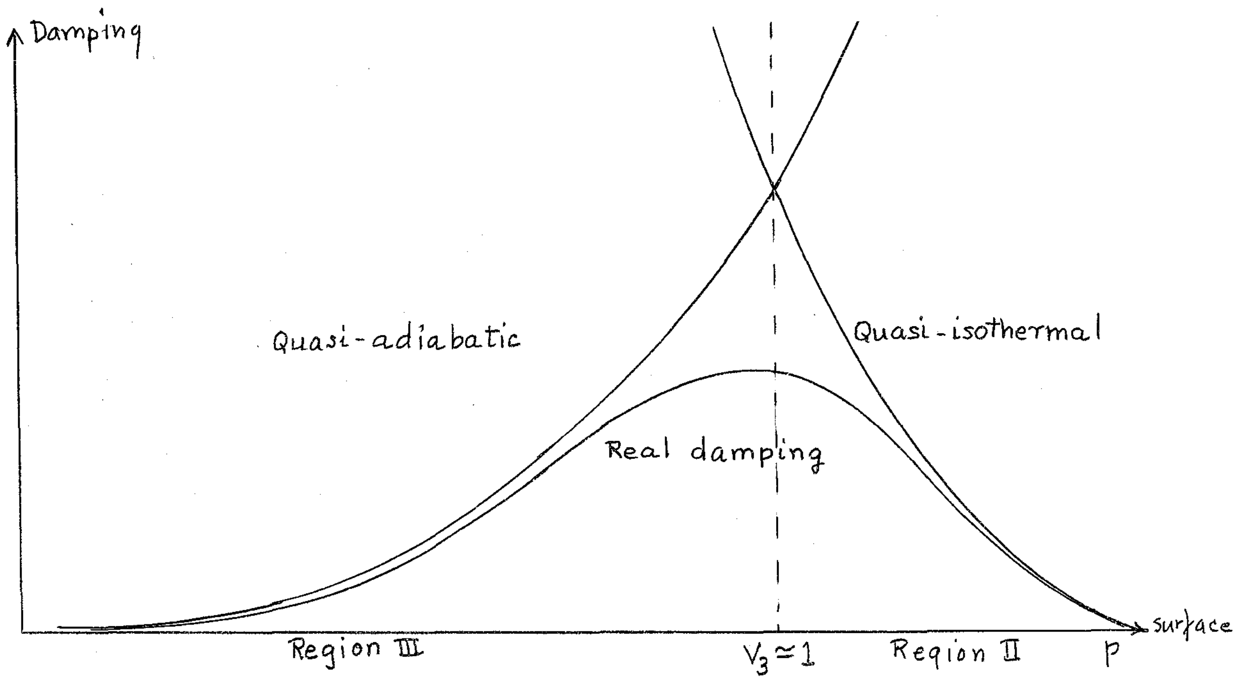

Fig.3 : Real damping in a star

In the most external layers, a quasi-isothermal approximation allows us to calculate the radiative damping. Similarly we shall be able to estimate the excitation in these layers. The real damping has a maximum value, between the fields of validity of both approximations. We shall estimate the situation of this maximum by the equality of the quasi-adiabatic and quasi-isothermal dampings. We shall estimate the integral of damping by cutting the quasi-adiabatic integral at this value. It is a very rough approximation, but it seems sufficient for our purpose, and it is better than an arbitrary cut. Especially, we shall have the frequency-dependence of the damping which we could not obtain in any other way.

According to the quasi-isothermal assumption the equations of perturbation become:

$$\dot{D} = -D - \frac{k^2 g H_{po}}{\sigma^2} B \tag{59}$$

$$\dot{B} = -\frac{\sigma^2 H_{po}}{g} D \tag{60}$$

Taking into account $V_2 \ll 1$ in this region, we obtain the eigenfunction

$$D = K_1 p_o^{-1} + K_2 H_{po} \tag{61}$$

In the external zones, the first term gives the most important contribution to the damping. We shall use the approximate eigen value:

$$D = K x^{-1} \tag{62}$$

Then we obtain

$$(\underset{\sim}{\nabla} \cdot \underset{\sim}{F}') = i\sigma \frac{(\nabla_T - \nabla_{Ta})}{\nabla_{Ta}} K p_{o*} \tag{63}$$

$$\frac{\delta T}{T} = C_i + C_R - D\nabla_T \tag{64}$$

$$C_i = \int A_i \nabla_T \, d\varpi = -\frac{i\sigma}{F_3} \frac{(\nabla_T - \nabla_{Ta})}{\nabla_{Ta}} \frac{\nabla_T H_{po} K p_{o*}}{(1-\nabla_\rho)^2} \tag{65}$$

$$\left(\frac{dJ}{dv}\right)_{quasi\text{-}isothermal} = \frac{\sigma^2(\nabla_T - \nabla_{Ta})^2}{F_3 \nabla_{Ta}^2} \frac{\nabla_T H_p K^2 p_{o*}^2}{(1-\nabla_p)^2} \tag{66}$$

We want to compare this expression with that obtained from Eq.(49). Then we have to connect the two eigenfunctions, choosing to give to them the same values at the point where the two dampings are equal:

$$\overline{D}_{ad} = \frac{1}{\sqrt{2}} K_{ad} \exp(\alpha \varpi) \tag{67}$$

$$D_{is} = K_{is} \chi^{-1} \tag{68}$$

$$K_{is} = \frac{1}{\sqrt{2}} K_{ad} \chi^{-1-\alpha} \tag{69}$$

Then the two dampings are equal when:

$$\chi^{-\frac{1+\nabla_p}{2}} = \frac{\sigma^2 p_{o*}}{k F_{3o}} \frac{\nabla_T}{\nabla_{Ta}(1-\nabla_p)} \sqrt{\frac{H_{p o*}}{\nabla_{Ta}\, g}} \tag{70}$$

There would be other ways to obtain an estimation of the upper limit of the damping. For instance, we should be able to stop the integration when the coefficient V_3 is equal to one; then

$$\chi^{(2-\nabla_p)} = \frac{F_3 \nabla_{Ta}}{\sigma\, p_{o*} H_{p o*}} \tag{71}$$

Or we can estimate $(\nabla \cdot F)'$ by taking into account that V_3 is small, but using the adiabatic eigenfunctions. We obtain:

$$\chi = \frac{k^2 g F_{3o} \nabla_{Ta}}{\sigma^3 p_{o*} \nabla_T} \sqrt{(\nabla_p - \nabla_{pa})\nabla_{Ta}} \tag{72}$$

These miscellaneous estimations do not give very different results in our problem. However they show the size of our approximations. We shall use the first determination which seems to be the less rough. We note the limit of integration of the damping is strongly dependent on the frequency. When $\sigma \longrightarrow 0$, $X_{limit} \longrightarrow \infty$. However, in fact, we are limited to the low values of σ , if we want to keep a physical sense in our study. But it is surprising to ascertain the relative value of radiative damping before the excitation decreases with σ , unlike other cases. That is due to the fact that the excitation and the damping are the same mechanism, and are growing together when $\sigma \longrightarrow 0$, while the limit of integration of damping is decreasing.

The expression (58) becomes

$$\frac{(\nabla_{\rho_1}-\nabla_{\rho_a})(\nabla_{T_1}-\nabla_{T_a})\nabla_{T_a}}{(\nabla_{\rho_1}+1)\nabla_{T_1}}\left[\frac{\sigma^2 p_{o*}}{k F_{3^o}}\frac{\nabla_T}{\nabla_{T_a}(1-\nabla_\rho)}\sqrt{\frac{H_{p o*}}{\nabla_{T_a} g}}-1\right] < f'(\varepsilon,\nabla_\mu,y) \tag{73}$$

where $f'(\varepsilon,\nabla_\mu,y)$ is the second part of (58)

$$f'(\varepsilon,\nabla_\mu,y) = g(\varepsilon,\nabla_\mu)\, h'(\varepsilon,\nabla_\mu,y) \tag{74}$$

with $h'(\varepsilon,\nabla_\mu,y)$ being everywhere smaller than one. We can overestimate the excitation by replacing f' by $g(\varepsilon,\nabla_\mu)$, which has previously been tabulated. We see immediately that with the numerical data of our problem, if the quantity between the brackets is much larger than unity, the overstability always will be damped by the upper radiative zone. That means that if the semi-convective zone is not situated near the maximum of the damping, it will not be able to excite the overstability. With our numerical values, the quantity between the brackets has to be less than 0.3 (for a B giant star) and 4.5 (for a white dwarf) to obtain the overstability. We shall discuss these numerical results in the following part.

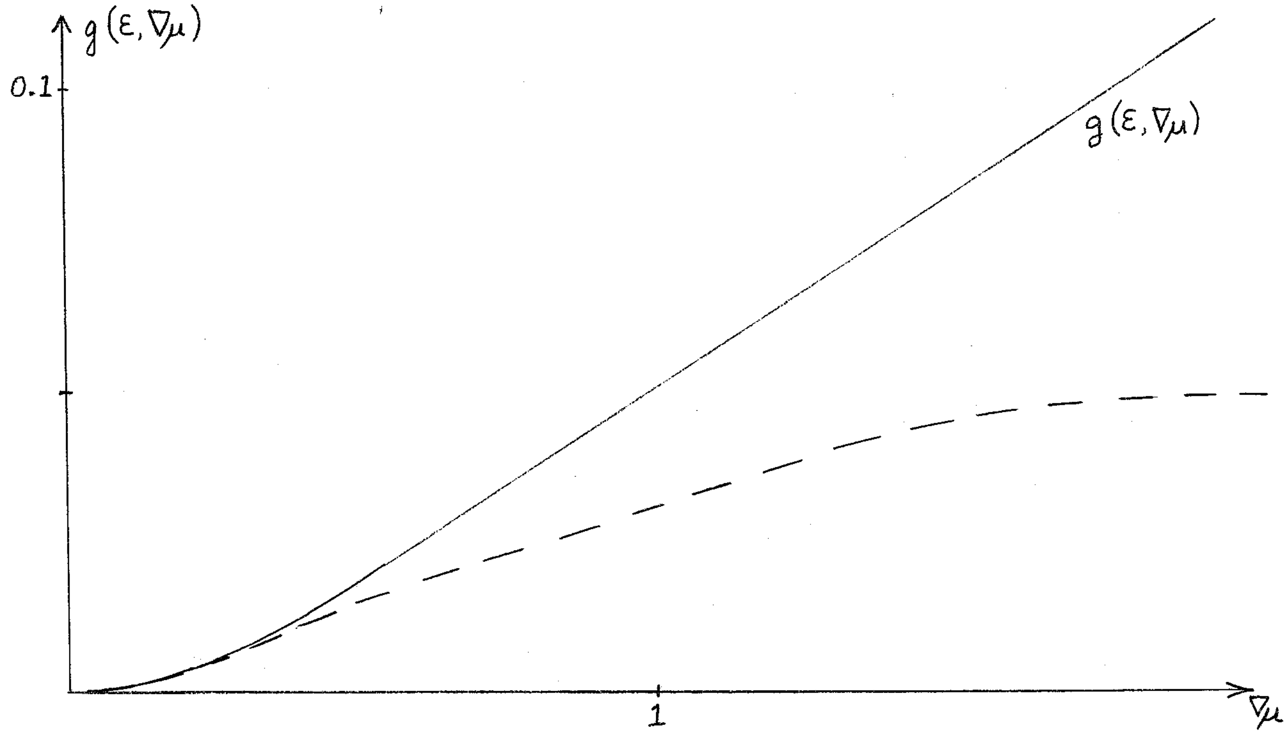

Fig. 4. Maximum curve of $g(\varepsilon, \nabla_\mu)$.
(The dotted curve is an evaluation of the maximum real excitation $f'(\varepsilon, \nabla_\mu, y)$ for y = 2.5)

Before that, let us estimate the behaviour of the excitation in the external layers.

In the same way the quasi-adiabatic approximation is irrelevant in the external layers for the radiative damping, it is not applicable to the external semi-convective zones. It is interesting to examine separately this case from the former, for, if in the inner region, the eigenfunctions are very similar in both zones, in the outer region the behaviour of the perturbations is very different in each case.

That is a consequence of the behaviour of the Eulerian perturbation of the density:

$$\frac{\rho'}{\rho_0} = \frac{p'}{p_0} - \frac{T'}{T_0} = B - C \tag{75}$$

in a radiative layer. Now B and C are negligible before D, and consequently we should be able to obtain the approximate eigenfunction from the equation (13) which becomes:

$$\frac{\partial \delta_z}{\partial z} = - \delta_z \frac{\partial \operatorname{Log} \rho_o}{\partial z} \tag{76}$$

for the tangential term ($\simeq V_1 B$) becomes negligible in the external regions. The eigenfunction (which is the same as (62)) has an exponential behaviour in space in these regions.

On the contrary, in a zone where $\nabla_\mu \neq 0$, the behaviour of ρ'/ρ_o is given by μ'/μ_o which is of the same order as the displacement. Studying this case in region II of the star ($V_3 < 1$ and $V_1 > 1$), we obtain a good approximation of the eigenfunctions, which then have an oscillatory character in space. The system Eq. (19) - (22) shows that if we expand the eigenfunctions in series of σ , and if we keep only the first term of each function:

$$B_{real} \sim \sigma \, D_{real} \tag{77}$$

$$C_{real} \sim \sigma \, D_{real} \tag{78}$$

$$A_{imag} \sim \sigma^2 \, D_{real} \qquad \text{etc.} \tag{79}$$

Then, in first approximation we can calculate the Eulerian isothermal eigenfunctions (C = 0). Then we obtain:

$$\ddot{D} + \dot{D}\left(\nabla_\rho - \nabla_\mu\right) + \frac{k^2 g H_p \nabla_\mu}{\sigma^2} D = 0 \tag{80}$$

Using the same method as for the calculations of the quasi-adiabatic eigenfunctions:

$$D = K \rho_o^{-1/2} \mu_o^{1/2} \xi^{-1/2} \cos\left(\frac{2\xi}{(1-\nabla_\rho)} + \varphi\right) \tag{81}$$

where

$$\xi^2 = \frac{k_g^2 H_{p_o} \nabla_\mu}{\sigma^2} \tag{82}$$

(if $\nabla_T = \nabla_{Ta}, \xi = \eta$ and we find again the quasi-adiabatic eigenfunction). C will be given by Eq. (21), where we only keep the most important term:

$$C \simeq K(1-k_\mu) \nabla_\mu \nabla_T p_o^{-1/2} \mu_o^{1/2} \xi^{-3/2} \sin\left(\frac{2\xi}{1-\nabla_p} + \varphi\right) \tag{83}$$

very different from C in the external radiative zone.

If we take into account the perturbation of the flux, the eigenfunctions become complex. We obtain:

$$\left(\underset{\sim}{\nabla} \cdot \underset{\sim}{F}\right)'_{imag} = \frac{\sigma p_o}{\nabla_{Ta}} \left(\nabla_T - \nabla_{Ta}\right) D_{Real} \tag{84}$$

$$\left(\underset{\sim}{\nabla} \cdot \underset{\sim}{F}\right)'_{Real} = \frac{\sigma p_o}{\nabla T_a} C_{imag} \tag{85}$$

$$\left(\frac{\delta T}{T}\right)_{Real} = D_{Real} \nabla_T \tag{86}$$

$$\left(\frac{\delta T}{T}\right)_{imag} = \frac{\sigma p H_p (\nabla_T - \nabla_{Ta})}{F_z \nabla_{Ta} \xi^2} D_{Real} \tag{87}$$

Then we obtain:

$$J = \frac{\sigma^2 (\nabla_T - \nabla_{Ta})}{F_z \nabla_{Ta}} K^2 \left(\frac{p_{o*}^2 \mu_{o*} H_{po*}}{\xi_{o*}^3 p_o}\right) \frac{\left[1 - x^{\left(\frac{5}{2} - \frac{\nabla_p}{2} + \frac{\nabla_\mu}{2}\right)}\right]}{(5 - \nabla_p + \nabla_\mu)} \tag{88}$$

if the lower limit $x = 1$. Then we see the excitation strongly decreasing with the pressure, and also with σ (as $\sigma^5 K^2$). After connecting the eigenfunction with that of the radiative zone, we obtain:

$$\frac{\frac{\partial J_{Rad}}{\partial z}}{\frac{\partial J_{s-c}}{\partial z}} = \frac{(\nabla_T - \nabla_{Ta})_{Rad} \nabla_{T_{Rad}} (\nabla_p - \nabla_{pa})_{Rad}}{(\nabla_T - \nabla_{Ta})_{s-c} \nabla_{Ta} (1 - \nabla_{p_{Rad}})^2} \xi^2 \tag{89}$$

which shows, for a given pressure, the mechanism of excitation is very much weaker than the radiative damping in the external zone (when $\sigma \to 0$), because the oscillatory aspect of the eigenfunction in the zone with $\nabla_\mu \neq 0$.

These results allow limitation of the quasi-adiabatic expression of the excitation, if the top of the semi-convective zone is in the quasi-isothermal region. Using the same method as in the radiative case, we see that we stop the integration when:

$$\chi = \frac{F_3 \nabla_{Ta} k^2 g}{\sigma^3 p_0} \sqrt{\frac{(\nabla_\rho \nabla_{\rho a}) \nabla_\mu}{\nabla_T}} \tag{90}$$

This value is a little different from that found for the radiative damping. Especially, this limit is in σ^{-3} (σ^{-2} in the radiative zone), i.e. when σ is decreasing, the excitation decreases more quickly than the damping in the external layers.

VI. Some astrophysical applications

The previous study has shown that generally the strong damping due to the upper radiative layers prevents the overstability. Equation (53) is conspicuous in that the damping, as the excitation, varies as a negative power of the pressure (that is due to the behaviour of the eigenfunction) as far as the quasi-adiabatic approximation is relevant. The only case where the overstability will be able to appear will happen when the semi-convective zone is situated at the maximum of the real damping (cf. Fig.3).

If we apply our results to the B stars departing from the main sequence, in which a semi-convective zone has grown between the convective case and the radiative envelope, we obtain:

$$\text{Excitation}_{max} \simeq 7 \cdot 10^{-3} \text{ in our unities}$$

(The excitation is almost independent from the chosen model.)

$$\text{Damping}_1 \simeq 1.4 \ 10^{15} \cdot \sigma^{-2}$$

for the model of 28.2 $M_\odot$ by Schwarzschild and Härm (we have chosen the model in which the semi-convective zone is the most developed).

$$\text{Damping}_2 \simeq 1.7 \ 10^{14} \cdot \sigma^{-2}$$

$$\text{Damping}_3 \simeq 8 \ 10^{13} \cdot \sigma^{-2}$$

$$\text{Damping}_4 \simeq 1.1 \ 10^{13} \cdot \sigma^{-2}$$

The models 2, 3, 4, respectively, have 62.7, 121.1 and 218.3 solar masses. For these stars, the frequency of the fundamental mode is about 4 10^{-4} 5^{-1}. It should be physically unreasonable to choose a g-mode higher than the thousandth. In the better assumption the damping is not less than 180 (24000 for the model of 28.2 $M_\odot$). That clearly shows that the overstability never appears in these stars. The assumption of mixing by overstability, done by Kato, seems then to have no more sense. Likewise the explanation of the instability of β Canis Majoris stars has to be forsaken in the same way.

It can be interesting to state precisely why our results are so different from those of Gabriel. We have introduced no singular point in our system. Gabriel has introduced one singular point at the bottom of his semi-convective zone, for his eigenfunction. But he does not use the same expression for his eigenvalue and for his divergence of flux and then maintains in the integral J a singular point which has not a real existence.

Another application of our study concerns the layer of gravitational sorting of the white dwarfs. This zone is situated at the bottom of the atmosphere of the white dwarfs, and a very strong gradient of molecular weight exists in it. Following our knowledge of white dwarfs, it is possible that, during their evolution, a frontier between a convective zone and a radiative zone crosses the layer of sorting which would become the zone of excitation. On the other side the gravitational sorting happens near the surface of the star, and we can hope that the radiative damping by the upper layers is not too large. Unhappily, our present knowledge of the atmosphere of white dwarfs is not sufficient to allow us to calculate a relevant result. Besides, if the zone of gravitational sorting is situated near the maximum of damping, neither the quasi-adiabatic approximation, nor the quasi-isothermal, is utilizable.

VII. Conclusion

This study allowed us to eliminate the assumption of mixing by overstability in a semi-convective zone, and also the assumption of this overstability in β Canis Majoris stars. However, that does not exclude the assumption of destabilization by finite amplitude instability described by Veronis. On the other hand, perhaps it is possible that an overstability is able to exist in white dwarfs. Our study does not allow us to tell more.

Acknowledgments

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PERIODIC MOTIONS IN A SPHERICAL SHELL

Terrance G. Barker

The region between two concentric spheres of radii a and b, rotating together at a frequency Ω , are filled with a homogeneous, incompressible, inviscid fluid. We wish to compute unforced, periodic motion of the fluid in that region. Several investigators have studied motions of infinitesimal amplitude (neglect of advective terms) in enclosed rotating regions. For the spherical shell, as Stewartson and Rickard (1968) point out, the linearized governing equation for the pressure is hyperbolic in space, while the boundary conditions are of the generalized Dirichlet type (a linear relation between the pressure and its gradient on the boundary) - see Greenspan (1965) for the exact form. This problem is not well-posed in a mathematical sense, so we are not aided by general theorems on the existence of solutions. Greenspan (1965) has obtained the result, applicable to all contained rotation motion, that the eigenfrequencies ω are in the range

$$|\omega| < 2\Omega .$$

For some special geometries (e.g. completely filled sphere and ellipsoid, and coaxial cylinder) the equations and boundary conditions are separable under suitable coordinate transformations. For the spherical shell, no such transformation has been found. Malkus (1968) has found a set of solutions which have zero radial velocity and which satisfy the full non-linear problem. By neglecting radial motions, Longuet-Higgins (1964) computed global solutions to the linearized equations (essentially the Laplace tidal equations). In a later paper (1965), he relaxed the condition that the outer boundary be fixed and studied the effects of the free surface on the solution. Using the Laplace tidal equations (for lack of a better name) as a starting point, Stewartson and Rickard (1969) wrote the solutions as coefficients of a series expansion in a gap width parameter

$$\eta = \frac{a-b}{a+b}$$

The higher-order terms possessed singularities on the circles $\cos\theta = \frac{\omega}{2\Omega}$. At these points, the characteristics of the governing equations were tangent to the inner boundary. The following study was motivated by the feeling that the singular behaviour of Stewartson and Rickard's results were due to the neglect of advective terms.

We begin by writing the equations of conservation of momentum and mass for a homogeneous, incompressible, inviscid fluid. In spherical polar coordinates (r , θ , φ), these are (in the rotating frame)

$$u_t + (\underset{\sim}{q}\cdot\nabla)u + \frac{1}{r}(u^2+w^2) - 2\Omega\sin\theta W + p_r = 0 \tag{1}$$

$$v_t + (\underset{\sim}{q}\cdot\nabla)v + \frac{1}{r}\left(uv - w^2 \operatorname{ctn}\theta\right) - 2\Omega\cos\theta W + \frac{1}{r}p_\theta = 0 \tag{2}$$

$$W_t + (\underset{\sim}{q} \cdot \nabla) w + \frac{1}{r} w (u + v \operatorname{ctn} \theta) + 2\Omega (u \sin\theta + v \cos\theta) + \frac{1}{r \sin\theta} p_\varphi = 0 \qquad (3)$$

$$\frac{1}{r} (r^2 u)_r + \frac{1}{\sin\theta} \left[(v \sin\theta)_\theta + w_\varphi \right] = 0 \qquad (4)$$

where the velocity $\underset{\sim}{q} = (u\hat{r} + v\hat{\theta} + w\hat{\varphi})$. θ is the co-latitude and φ is the azimuthal angle. The pressure p includes the gravitational and centrifugal body forces. Subscripts indicate differentiation.

The variables are now scaled. Since a starting point will be the solution for a very thin shell, our expansion parameter in thickness will be

$$\eta = \frac{a-b}{a+b} .$$

$$\begin{aligned} &\bar{t} = \frac{1}{\omega} t \\ &u = \eta^2 U ; \qquad v = V ; \qquad w = W \\ &p = r P ; \qquad r = c(1+\eta R) ; \qquad c = \frac{a+b}{2} \end{aligned} \qquad (5)$$

The choice of η^2, rather than η^1, in scaling the radial velocity u, is made in order that the continuity equation is two-dimensional (does not contain U). In these new coordinates, equations (1-4) are

$$\eta^3 \omega U_t + \frac{\eta}{c} (\underset{\sim}{Q} \cdot \nabla) \eta^2 U - \frac{\eta}{c(1+\eta R)} (\eta^4 U^2 + W^2) - 2\Omega \sin\theta \, W \eta + \eta P + (1+\eta R) P_R = 0 \qquad (6)$$

$$\omega V_t + \frac{1}{c} (\underset{\sim}{Q} \cdot \nabla) V + \frac{1}{c} (1+\eta R)^{-1} (\eta^2 U V - W^2 \operatorname{ctn} \theta) - 2\Omega \cos\theta \, W + P_\theta = 0 \qquad (7)$$

$$\omega W_t + \frac{1}{c} (\underset{\sim}{Q} \cdot \nabla) W + \frac{1}{c} (1+\eta R)^{-1} W (\eta^2 U + V \operatorname{ctn} \theta) + 2\Omega (\eta^2 U \sin\theta + V \cos\theta) + \frac{1}{\sin\theta} P_\varphi = 0 \qquad (8)$$

$$\eta (1+\eta R) U_R + 2\eta^2 U + \frac{1}{\sin\theta} \left[(V \sin\theta)_\theta + W_\varphi \right] = 0 \qquad (9)$$

where $$(\underset{\sim}{Q}\cdot\nabla) = \eta U \frac{\partial}{\partial R} + (1+\eta R)^{-1} V \frac{\partial}{\partial\theta} + \frac{W}{\sin\theta}\frac{\partial}{\partial\varphi}$$
$$\underset{\sim}{Q} = (\eta^2 U, V, W)$$

Write the dependent variables and the frequency ω as functions of η ;

$$X = X(\underset{\sim}{r}, t, \eta) ,$$
$$\omega = \omega(\eta) ,$$

where X is one of U , V , W , P . Express X and ω as Taylor series in η about the point $\eta = 0$:

$$X = \sum_{j=0}^{N} \eta^j \left[\frac{\partial^j}{\partial\eta^j} X\right]_{\eta=0}$$
$$\omega = \sum_{j=0}^{N} \eta^j \left[\frac{\partial^j}{\partial\eta^j}\omega\right]_{\eta=0} \tag{10}$$

We follow the procedure shown by Millman and Keller (1969) to compute the coefficients of the series.

Consider first the problem for infinitesimal gap width. Set $\eta = 0$ in equations (6-9). An inner expansion in terms of an amplitude parameter ϵ is needed. Write

$$X(\eta=0) = X(\eta=0, \epsilon)$$

and construct a Taylor series as in (10) for ϵ . Differentiate with respect to ϵ the equations (6-9) with $\eta = 0$ and set $\epsilon = 0$. The state about which perturbations in amplitude are made is one of rigid body rotation - $U = V = W = P = 0$. The result is

$$\dot{P}_R = 0 \tag{11}$$
$$\omega \dot{V}_t - 2\Omega\cos\theta\,\dot{W} + \dot{P}_\theta = 0 \tag{12}$$
$$\omega \dot{W}_t + 2\Omega\cos\theta\,\dot{V} + \frac{1}{\sin\theta}\dot{P}_\varphi = 0 \tag{13}$$
$$(\dot{V}\sin\theta)_\theta + \dot{W}_\varphi = 0 \tag{14}$$

where $\frac{\partial}{\partial \epsilon}(\ ) = (\dot{\ })$. The two-dimensional form of the continuity equation (14) lets us write a stream function ψ, defined by

$$\dot{V} = \frac{1}{\sin\theta}\dot{\psi}_\theta \qquad \dot{W} = -\dot{\psi}_\varphi \tag{15}$$

Eliminating P from (12) and (13), we have

$$\frac{\partial}{\partial t}\nabla_h^2 \dot{\psi} + 2\Omega\,\dot{\psi}_\varphi = 0 \tag{16}$$

where
$$\nabla_h^2 = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$
is the horizontal Laplacian operator.

This equation is studied extensively by Longuet-Higgins (1965) and its solutions are

$$\dot{\psi}_{nm} = P_n^m(\cos\theta)\exp\left[im\left(\varphi + \frac{2\Omega}{n(n+1)}t\right)\right] \tag{17}$$

Differentiate (6-9) twice with respect to ϵ to find the first contribution of the advective terms. Set $\epsilon = \eta = 0$. Only the horizontal momentum equations (12,13) change, and are

$$\omega\ddot{V}_t - 2\Omega\cos\theta\,\ddot{W} + \ddot{P}_\theta = \frac{-4}{c}\left[\left(\dot{V}\frac{\partial}{\partial\theta} + \frac{\dot{W}}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\dot{V} - \dot{W}^2\operatorname{ctn}\theta\right] \tag{18}$$

$$\omega\ddot{W}_t + 2\Omega\cos\theta\,\ddot{V} + \frac{1}{\sin\theta}\ddot{P}_\varphi = \frac{-4}{c}\left[\left(\dot{V}\frac{\partial}{\partial\theta} + \frac{\dot{W}}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\dot{W} - \dot{W}\dot{V}\operatorname{ctn}\theta\right] \tag{19}$$

The term in brackets on the right-hand side of (18) is equivalent to (in terms of the stream function)

$$\frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^2\theta}\dot{\psi}_\varphi^2 + \dot{\psi}_\theta^2\right) + \dot{\psi}_\theta\nabla_h^2\dot{\psi} \tag{20}$$

and the bracketed term is (19) to

$$\frac{1}{2}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin^2\theta}\dot{\psi}_\varphi^2 + \dot{\psi}_\theta^2\right) + \frac{1}{\sin\theta}\dot{\psi}_\varphi\nabla_h^2\dot{\psi}\ . \tag{21}$$

But according to (16) and (17)

$$\nabla_h^2 \dot{\psi} = -\frac{2\Omega m}{\omega}\dot{\psi} .$$

Then, upon defining

$$P' = \frac{1}{2}\left(\dot{\psi}_\varphi^2 \csc^2\theta + \dot{\psi}_\theta^2\right) - \frac{\Omega m}{\omega}\dot{\psi}^2$$

and setting

$$P'' = \ddot{P} - P',$$

the form of equations (11-14) remains unchanged, P'' replacing $\ddot{P}$. Thus, the stream function $\dot{\psi}_{nm}$ is a solution to the non-linear problem. The advective terms only add to the dynamic pressure.

We now increase the gap width, i.e. consider terms of first order in η . Differentiate (6-9) once with respect to η and set $\eta = 0$, and find (using $\frac{\partial}{\partial\eta}(\;\;) = (\overset{*}{\;\;})$)

$$\overset{*}{P}_R = H(\theta,\varphi) \tag{22}$$

$$\overset{*}{\omega} V_t + \omega \overset{*}{V}_t + \frac{1}{c}\left(V\frac{\partial}{\partial\theta} + \frac{W}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\overset{*}{V} + \frac{1}{c}\left[U\frac{\partial}{\partial R} - R\left(V\frac{\partial}{\partial\theta} + \frac{W}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]V +$$

$$+\frac{1}{c}\left(\overset{*}{V}\frac{\partial}{\partial\theta} + \frac{\overset{*}{W}}{\sin\theta}\frac{\partial}{\partial\theta}\right)V + \frac{1}{c}RW^2\operatorname{ctn}\theta - \frac{1}{c}2W\overset{*}{W}\operatorname{ctn}\theta - 2\Omega\cos\theta\,\overset{*}{W} + \overset{*}{P}_\theta = 0 \tag{23}$$

$$\omega\overset{*}{W}_t + \overset{*}{\omega}W_t + \frac{1}{c}\left(V\frac{\partial}{\partial\theta} + \frac{W}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\overset{*}{W} + \frac{1}{c}\left[U\frac{\partial}{\partial R} - R\left(V\frac{\partial}{\partial\theta} + \frac{W}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]W +$$

$$+\frac{1}{c}W\overset{*}{V}\operatorname{ctn}\theta + \frac{1}{c}V\operatorname{ctn}\theta\left(\overset{*}{W} - WR\right) + 2\Omega\overset{*}{V}\cos\theta + \frac{1}{\sin\theta}\overset{*}{P}_\varphi = 0 \tag{24}$$

$$U_R + \frac{1}{\sin\theta}\left[(\overset{*}{V}\sin\theta)_\theta + \overset{*}{W}_\varphi\right] = 0 \tag{25}$$

where

$$H(\theta,\varphi) = -P + W^2 + 2\Omega\sin\theta\,W$$

is a known function of the solutions for infinitesimal gap width. Observe that the continuity equation (25) indicates that we no longer have radial non-divergence. U appears nowhere else in set (22-25) and the continuity equation must be thought of as an equation for U.

The radial momentum equation (22) can be integrated immediately.

$$\overset{*}{P} = H(\theta,\varphi)R + f(\theta,\varphi)$$

f is as yet unknown. Let

$$f' = f + \frac{1}{c}\left(W\overset{*}{W} + V\overset{*}{V}\right)$$

and

$$H' = H + \frac{\Omega m}{\omega}\psi^2 + \frac{1}{2}\left|\underset{\sim}{Q}\right|^2$$

We do this to remove from the equations the terms that can be incorporated into the pressure. Equations (23 and 24) become

$$\omega\overset{*}{V}_t + \frac{W}{c\sin\theta}\left[\overset{*}{V}_\varphi - \left(\overset{*}{W}\sin\theta\right)_\theta\right] - \left(\alpha\Psi + 2\Omega\cos\theta\right)\overset{*}{W} + f'_\theta = -H'_\theta R - \overset{*}{\omega}V_t \tag{26}$$

$$\omega\overset{*}{W}_t + \frac{V}{c\sin\theta}\left[\overset{*}{V}_\varphi - \left(\overset{*}{\omega}\sin\theta\right)_\theta\right] + \left(\alpha\Psi + 2\Omega\cos\theta\right)\overset{*}{V} + \frac{1}{\sin\theta}f'_\varphi = -\frac{1}{\sin\theta}H'_\varphi R - \overset{*}{\omega}W_t\,. \tag{27}$$

These equations cannot be attacked by linearizing in amplitude as was done previously. This is because the linear solutions contain unacceptable singularities and the iterations would merely carry them through to higher order. Towards solving (25-27), let

$$\begin{aligned} \overset{*}{V} &= a_1(\theta,\varphi)R + b_1(\theta,\varphi) \\ \overset{*}{W} &= a_2(\theta,\varphi)R + b_2(\theta,\varphi) \\ U_R &= a_3 R + b_3 \end{aligned} \tag{28}$$

where

$$a_3 = -\frac{1}{\sin\theta}\left[\left(a_1\sin\theta\right)_\theta + a_{2\varphi}\right]$$

$$b_3 = -\frac{1}{\sin\theta}\left[\left(b_1\sin\theta\right)_\theta + b_{2\varphi}\right]$$

Using the boundary conditions

$$U(R=-1) = U(R=+1) = 0\,,$$

we have $\quad b_3 = 0, \quad a_3 = \text{const.}$

The functions b_1, b_2 satisfy the same equations as U and V (0^{th} order in η) and these solutions have been presented. The equations for the a's are more difficult. They satisfy

$$\left(B - \frac{V}{c\sin\theta}\frac{\partial}{\partial\varphi}\right)a_1 + \left(\omega\frac{\partial}{\partial t} + \frac{V}{c\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\right)a_2 = -\frac{1}{\sin\theta}H_\varphi' \tag{29}$$

$$\left(\omega\frac{\partial}{\partial t} + \frac{W}{c\sin\theta}\frac{\partial}{\partial\varphi}\right)a_1 + \left(-B - \frac{W}{c\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\right)a_2 = -H_\theta' \tag{30}$$

where

$$B = \frac{2m\Omega}{\omega}\psi + 2\Omega\cos\theta$$

This is a linear second order system with periodic coefficients. Time did not permit further investigation of these equations.

Some further information about the problem may be obtained from the method of characteristics. We would like to know whether the inclusion of finite amplitude effects produce any changes in the characteristics described in the linear problem by Stewartson and Rickard (1962). In that case, the characteristic curves are tangent to the inner sphere at the circle $\cos\theta = \omega/2\Omega$.

Now, write the dependent variable in the original equations (1-4) as functions of a characteristic parameter, ξ ,

$$u, v, w, \text{ or } p = f(\xi, \Delta, t).$$

Along a characteristic, across which there is an abrupt change in the velocity-pressure field, derivatives with respect to ξ will be large very near the characteristic. Then, in that neighborhood, we may write

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} \tag{31}$$

where χ is one of Λ , θ , φ and t . Writing equations (1-4) in matrix form, and using (31), we see that the characteristics satisfy

$$(\nabla\xi)^2\left(\frac{D}{Dt}\xi\right)^2 = 0$$

However, it is the term $(\nabla\xi)^2 = 0$ that gives rise to the characteristics of Stewartson and Rickard. It appears, then, that the finite amplitude terms do not change their features. (Note: these results were obtained at a late stage in the investigation, and have not been thoroughly checked.)

One might conclude from the preceding that singular behaviour is a feature of inviscid, incompressible shellular flow. However, Malkus (1968) has found well-behaved solutions (as noted earlier) to the finite amplitude, arbitrary width problem. It is not clear what role they play. Solving equations (29) and (30) should shed some light on this point.

The Laplace tidal equations are often defenced on the basis that density stratification inhibits vertical motion sufficiently to justify neglect of u. It may be that reconstructing the problem with a variable density field included may lead to tractable results.

However, we still do not know the answer for a constant density, inviscid fluid.

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PENETRATIVE CONVECTION IN A ROTATING FLUID

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The object of this study is to discover techniques appropriate to the analysis of penetrative convection in a rotating stratified fluid contained between two plane parallel boundaries. The phenomenon of penetrative convection occurs when an inversion or unstable density stratification is set up near the lower boundary of an otherwise stably stratified fluid. When convection occurs in the unstable region it tends to penetrate into the stable region above. One system which exhibits this phenomenon consists of a layer of water with boundary temperatures of $0^{\circ}C$ at the lower boundary and $> 4^{\circ}C$ at the upper boundary. The resulting temperature and density profiles for the stable regime (before convection sets in) is shown in Fig.1.

The maximum density ρ_0 occurs at a height d where the temperature T_0 is $4^{\circ}C$. This system has been studied in detail in the non-rotating case[2,4]. In this paper we consider the case in which we assume that the

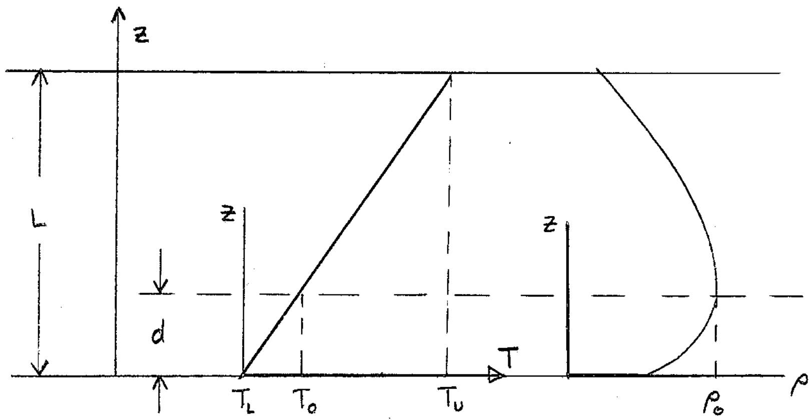

Fig.1

system described above is rotating about the z-axis with constant angular velocity Ω .

The Boussinesq equations for this case are the following:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} + 2\Omega \hat{k} \times \underline{v} = -\frac{1}{\rho_0} \nabla \tilde{P} - g \frac{\rho}{\rho_0} \hat{k} + \nu \nabla^2 \underline{v} \tag{1}$$

$$\nabla \cdot \underline{v} = 0 \tag{2}$$

$$\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T = \kappa \nabla^2 T \tag{3}$$

$$\rho = \rho_0 \left[1 - \alpha (T - T_0)^2 \right] \tag{4}$$

where $\hat{k}$ is a unit vector in the z-direction, κ and ν are the thermal and viscous diffusivities respectively, and ρ_0 is the maximum density of water at the level where the temperature is $T_0 = 4^{\circ}C$. Equation (4) is an adequate approximation to the equation of state in the neighborhood of the $4^{\circ}C$ point.

Let $\overline{Q}(z)$ be the horizontal average of a quantity Q and let $\langle Q \rangle$ be the overall (horizontal and vertical) average of the quantity Q. $\overline{Q}$ is a function of z alone in the non-convective regime. In particular, let

$$T = \overline{T}(z) + \theta(x,y,z,t) \quad \text{where} \quad \overline{\theta} = 0. \tag{5}$$

If we take the horizontal average of (3) taking account of (2) and integrate the result with respect to z we obtain:

$$-K\,\frac{d\overline{T}}{dz} + \overline{wT} = H = \text{constant} \tag{6}$$

where H is the heat flux and $w = \underset{\sim}{v}\cdot\hat{k}$.

Upon taking the vertical average of (6) we obtain:

$$-K\,\frac{\Delta T}{d} + \langle wT\rangle = H \tag{7}$$

where $\Delta T = T_0 - T_L = T_0 = 4^{\circ}C$ and d is the height at which $T = T_0$. If we now subtract (6) and (7) and integrate the result with respect to z we obtain:

$$\overline{T} - T_0 = -\Delta T + \beta z + \frac{1}{K}\int_0^z \overline{wT}\,dz \tag{8}$$

where $$\beta \equiv \frac{\Delta T}{d} - \frac{\langle wT\rangle}{K} \tag{9}$$

Now using definition (5) we may write the equation of state (4) as follows:

$$\frac{\rho}{\rho_0} = 1 - \alpha\left(\overline{T}-T_0\right)^2 - 2\alpha\,\theta\left(\overline{T}-T_0\right) - \alpha\,\theta^2 \tag{10}$$

where $\overline{T} - T_0$ is given by (8) and (9). Then if we set:

$$\tilde{p} = p - g\rho_0\int_0^z\left[1-\alpha\left(\overline{T}-T_0\right)^2\right]dz \tag{11}$$

Eq.(1) becomes:

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v} + 2\Omega\hat{k}\times\underset{\sim}{v} = -\frac{1}{\rho_0}\nabla\tilde{p} + \left[2g\alpha\left(\overline{T}-T_0\right)\theta + g\alpha\,\theta^2\right]\hat{k} + \nu\nabla^2\underset{\sim}{v}. \tag{12}$$

Finally, by differentiating (6) and subtracting the result from (3) taking account of definition (5) we obtain:

$$\frac{\partial\theta}{\partial t} + w\,\frac{d\overline{T}}{dz} - K\nabla^2\theta = -\underset{\sim}{v}\cdot\nabla\theta + \frac{d}{dz}\left(\overline{wT}\right) \tag{13}$$

Equations (8), (12) and (13) are the relevant mathematical equations in our study. They may be reduced by standard procedures to the following set:

$$\frac{\partial \zeta}{\partial t} - 2\Omega \frac{\partial w}{\partial z} - \nu \nabla^2 \zeta = -I \tag{14}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w + 2\Omega \frac{\partial \zeta}{\partial z} = 2g\alpha\left(\overline{\overline{T}} - T_0\right)\nabla_1^2 \theta + g\alpha \nabla_1^2 \theta^2 + J \tag{15}$$

$$\frac{\partial \theta}{\partial t} + w \frac{d\overline{\overline{T}}}{dz} - \kappa \nabla^2 \theta = -h \tag{16}$$

$$\overline{\overline{T}} - T_0 = -\Delta T + \beta z + \frac{1}{\kappa}\int_0^z \overline{wT}\, dz \tag{17}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\nabla_1^2 u + \frac{\partial^2 w}{\partial x \partial z}\right) + 2\Omega \frac{\partial^2 w}{\partial y \partial z} = M \tag{18}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\nabla_1^2 v + \frac{\partial^2 w}{\partial y \partial z}\right) - 2\Omega \frac{\partial^2 w}{\partial x \partial z} = N \tag{19}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{20}$$

$$\zeta = \underline{\omega} \cdot \hat{k}, \quad \underline{\omega} = \nabla \times \underline{v} \tag{21}$$

and I, J, h, M, N are non-linear terms given by:

$$I \equiv \frac{\partial}{\partial x}\left(\underline{v} \cdot \nabla v\right) - \frac{\partial}{\partial y}\left(\underline{v} \cdot \nabla u\right) \tag{22}$$

$$J \equiv \frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\left(\underline{v} \cdot \nabla u\right) + \frac{\partial}{\partial y}\left(\underline{v} \cdot \nabla v\right)\right] - \nabla_1^2\left(\underline{v} \cdot \nabla w\right) \tag{23}$$

$$h \equiv \underline{v} \cdot \nabla \theta - \frac{d}{dz}\left(\overline{wT}\right) \tag{24}$$

$$M \equiv \frac{\partial^2}{\partial x \partial y}\left(\underline{v} \cdot \nabla v\right) - \frac{\partial^2}{\partial y^2}\left(\underline{v} \cdot \nabla u\right) \tag{25}$$

$$N = \frac{\partial^2}{\partial x \partial y}\left(\underline{v} \cdot \nabla u\right) - \frac{\partial^2}{\partial x^2}\left(\underline{v} \cdot \nabla v\right) \tag{26}$$

where u and v are the x and y-components of $\underline{v}$ respectively. If we assume for convenience that the boundaries are free we obtain the following boundary conditions:

$$\left.\begin{aligned} &w = 0 \\ &\frac{\partial^2 w}{\partial z^2} = 0 \\ &\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \\ &\theta = 0 \\ &\frac{\partial \zeta_0}{\partial z} = 0 \end{aligned}\right\} \quad at \ z = 0, L$$

B.C. (1)
B.C. (2)
B.C. (3)
B.C. (4)
B.C. (5)

Next we introduce the following parameters:

$$\sigma \equiv \frac{\nu}{K} = \text{Prandtl No.} \qquad \text{P (1)}$$

$$\lambda \equiv \frac{L}{d} \qquad \text{P (2)}$$

$$R \equiv 2\lambda^4 \frac{g\alpha(\Delta T)^2}{K\nu} d^3 = 2\lambda^4 \times \text{Rayleigh No.} \qquad \text{P (3)}$$

$$\mathcal{T} \equiv \frac{4\Omega^2 L^4}{\nu^2} = \text{Taylor No.} \qquad \text{P (4)}$$

where L is the height of the water layer.

A convenient scaling is given by:

$$\underline{r} = L\underline{r}^* \qquad \text{S (1)}$$

$$t = \frac{L^2}{K} t^* \qquad \text{S (2)}$$

$$\underline{v} = \frac{K}{L}\underline{v}^* \qquad \text{S (3)}$$

$$\theta = \frac{\nu K}{2L^3 g\alpha \Delta T}\theta^* \qquad \text{S (4)}$$

If we introduce the above scaling and parameters into Eqs.(14) through (19) we obtain finally:

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\zeta - \mathcal{T}^{1/2}\frac{\partial w}{\partial z} = -\frac{1}{\sigma}K \qquad (27)$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w + \mathcal{T}^{1/2}\frac{\partial \zeta}{\partial z} = \frac{\overline{T}-T_0}{\Delta T}\nabla_1^2\theta + \frac{\lambda}{2R}\nabla_1^2\theta^2 + \frac{1}{\sigma}L \qquad (28)$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta + \left[R + \overline{wT} - \langle wT\rangle\right] w = -h \qquad (29)$$

$$\frac{\overline{T}-T_0}{\Delta T} = -1 + \lambda\left[1 - \frac{1}{R}\langle wT\rangle\right]z + \frac{\lambda}{R}\int_0^z \overline{wT}\, dz \tag{30}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\nabla_1^2 u + \frac{\partial^2 w}{\partial x \partial z}\right) + \mathcal{T}^{1/2}\frac{\partial^2 w}{\partial y \partial z} = \frac{1}{\sigma}M \tag{31}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\nabla_1^2 v + \frac{\partial^2 w}{\partial y \partial z}\right) - \mathcal{T}^{1/2}\frac{\partial^2 w}{\partial x \partial z} = \frac{1}{\sigma}N \tag{32}$$

where we have dropped the asterisks. The boundaries are then at z = 0 and z = 1.

We shall employ the method of Malkus and Veronis (1958) and expand the velocity and temperature fluctuations in a power series of small amplitude factor ε. We also expand the parameter R in a power series:

$$\left.\begin{aligned} \underline{v} &= \varepsilon \underline{v}_0 + \varepsilon^2 \underline{v}_1 + \dots \\ \theta &= \varepsilon\theta_0 + \varepsilon^2\theta_1 + \dots \\ R &= R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \dots \end{aligned}\right\} \tag{33}$$

If we now substitute (33) into equations (27) through (32) and equate the coefficients of each power of ε to zero we obtain a series of ordered sets of equations in v_i , θ_i , and R_i .

Linear analysis.

If we look at the first order equations (found by equating the coefficients of ε^1 to zero) we find that they are linear in the fluctuation quantities. They are:

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\zeta_0 - \mathcal{T}^{1/2}\frac{\partial w_0}{\partial z} = 0 \tag{34}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w_0 + \mathcal{T}^{1/2}\frac{\partial \zeta_0}{\partial z} = -(1-\lambda z)\nabla_1^2\theta_0 \tag{35}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta_0 + R_0 w_0 = 0 \tag{36}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\nabla_1^2 u_0 + \frac{\partial^2 w_0}{\partial x \partial z}\right) - \mathcal{T}^{1/2}\frac{\partial^2 w_0}{\partial y \partial z} = 0 \tag{37}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\nabla_1^2 v + \frac{\partial^2 w_0}{\partial y \partial z}\right) - T^{1/2}\frac{\partial w_0}{\partial x \partial z} = 0 \tag{38}$$

The first order boundary conditions are:

$$\left.\begin{aligned} & w_0 = 0 && \text{B.C.(1)} \\ & \frac{\partial^2 w_0}{\partial^2 z^2} = 0 && \text{B.C.(2)} \\ & \frac{\partial u_0}{\partial z} = \frac{\partial v_0}{\partial z} = 0 && \text{B.C.(3)} \\ & \theta_0 = 0 && \text{B.C.(4)} \\ & \frac{\partial \zeta_0}{\partial z} = 0 && \text{B.C.(5)} \end{aligned}\right\} \quad \text{at } z = 0,\ 1$$

Any disturbance that satisfies Eqs.(34) through (38) and the boundary conditions can be expressed uniquely in terms of a complete set of orthogonal modes at which one may be written:

$$\begin{pmatrix} \zeta_0 \\ w_0 \\ \theta_0 \end{pmatrix} = \begin{pmatrix} Z(z) \\ W(z) \\ \Theta(z) \end{pmatrix} f(x,y)e^{st} \tag{39}$$

$$\nabla_1^2 f(x,y) = -a^2 f(x,y) \tag{40}$$

where a is a characteristic value corresponding to the mode, and represents the scaled horizontal wave number of the fluctuations. Substitution of (39) and (40) into equations (34), (35) and (36) and boundary conditions (1), (2), (4) and (5) gives:

$$\left[\frac{s}{\sigma} - (D^2 - a^2)\right] Z - T^{1/2} DW = 0 \tag{41}$$

$$\left[\frac{s}{\sigma} - (D^2 - a^2)\right](D^2 - a^2)W + T^{1/2} DZ = a^2(1 - \lambda z)\Theta \tag{42}$$

$$\left[s - (D^2 - a^2)\right]\Theta + R_0 W = 0 \tag{43}$$

and the boundary conditions:

$$\left.\begin{aligned} W &= 0 \\ D^2W &= 0 \\ \Theta &= 0 \\ DZ &= 0 \end{aligned}\right\} \quad \text{at } Z = 0,\ 1 \qquad \begin{aligned} &\text{B.C.(1)} \\ &\text{B.C.(2)} \\ &\text{B.C.(4)} \\ &\text{B.C.(5)} \end{aligned}$$

The object is to solve for the eigenfunctions Z(z;a), W(z;a) and Θ (z;a) in terms of the wave number a and to obtain the characteristic value equation for a in terms of R_0 and T. We restrict our attention to the marginal states corresponding to $Re(s)=0$. In general the marginally stable states fall into two physically distinct classes corresponding to $Im(s)=0$ and $Im(s)\neq 0$. In the first case we have $s = 0$ and we have what is known as exchange of stabilities. This means that instability sets in as stationary convection with cellular structure. In the second case $s = s_1$, where $s_1 \equiv Im(s) \neq 0$, which means that instability sets in as time-dependent oscillations which grow exponentially. This is known as overstability. In this paper we examine two methods which, taken together, will yield sufficient information regarding the behavior of the system when instability sets in.

Method 1

We begin with Eqs.(41) through (43) with $s \neq 0$. Let $F \equiv D^2 - a^2$. Then elimination of W and Z in these equations leads to the following equation for Θ :

$$(s-F)\left(\frac{s}{\sigma}-F\right)^2 F\Theta + T(s-F)D^2\Theta + R_0 a^2\left(\frac{s}{\sigma}-F\right)(1-\lambda z)\Theta = 0 \tag{44}$$

It is customary in stability problems to introduce the following definitions:

Let

$$\left.\begin{aligned} a^2 &\equiv \pi^2\alpha^2 \\ \pi z &\equiv \xi \\ s &\equiv \pi^2 p \\ R_o &= \pi^4\alpha^4 R_o' \\ \mathcal{T} &= \pi^4\alpha^4 T' \\ D &\equiv \frac{d}{d\xi} \\ F &\equiv \frac{d^2}{d\xi^2} - \alpha^2 \end{aligned}\right\} \tag{45}$$

Then (44) becomes:

$$(p-F)\left(\frac{p}{\sigma}-F\right)^2 F\Theta + \alpha^4 T'(p-F)D^2\Theta + R_o'\alpha^6\left(\frac{p}{\sigma}-F\right)\left(1-\frac{\lambda}{\pi}\xi\right)\Theta = 0 \tag{46}$$

The appropriate boundary conditions for this equation may be deduced from (41) through (43) and B.C.(1) through B.C.(5). They are:

$$\Theta = D^2\Theta = D^4\Theta = D^6\Theta = 0 \qquad at\ \xi = 0, \pi. \qquad \text{B.C.(6)}$$

The method now consists essentially of taking advantage of B.C.(6) to expand Θ in a Fourier sine series:

$$\Theta = \sum_{1}^{\infty} A_m \sin m\xi \tag{47}$$

Substitution of (47) into (46) yields (after some computation) the following infinite set of equations for the coefficients A_n:

$$\sum_{m=1}^{\infty}\left\{ f(m)\delta_{mr} + \frac{1}{2}\, g(m,r)\left[(-1)^{m+r}-1\right]\right\} A_m = 0 \qquad r = 1, 2, \ldots \tag{48}$$

where

$$f(m) = 1 + \frac{m^2\alpha^4 T'}{b_m\left(\frac{p}{\sigma}+b_m\right)^2} - \frac{\alpha^6\left(1-\frac{\lambda}{2}\right)R_o'}{b_m(p+b_m)\left(\frac{p}{\sigma}+b_m\right)}, \quad b_m \equiv \alpha^2 + m^2 \tag{49}$$

and

$$g(m,r) = \frac{\lambda}{\pi^2}\ \frac{\alpha^6 R_o'}{b_r(p+b_r)\left(\frac{p}{\sigma}+b_r\right)}\ \frac{8mr}{(r^2-m^2)^2}, \quad b_r \equiv \alpha^2 + r^2 \tag{50}$$

In order for these equations to have non-trivial solutions we must have:

$$\left.\begin{aligned} &\det\{h(n,r)\}=0 \\ \text{where}\quad &h(n,r)\equiv f(n)\,\delta_{nr}+\tfrac{1}{2}g(n,r)\left[(-1)^{n+r}-1\right]. \end{aligned}\right\} \tag{51}$$

This is the desired characteristic equation which must be solved. The subsequent procedure is suggested by the observation that the matrix elements $h(n,r)$ decrease in magnitude as n and r increase.

We first approximate $\det\{h(n,r)\}$ by the $|X|$ determinant $h(1,1)$. Thus we set

$$\det\{h(n,r)\}\cong h(1,1)=0 \tag{52}$$

This gives us a cubic equation in p :

$$p^3+Ap^2+Bp+C=0 \tag{53}$$

where

$$\left.\begin{aligned} A &= b_1(2\sigma+1) \\ B &= b_1^2\sigma(2+\sigma)+\frac{\alpha^4 T'\sigma^2}{b_1}-\frac{\alpha^6(1-\frac{\lambda}{2})R_0'\sigma}{b_1} \\ C &= \sigma^2\left[b_1^3+\alpha^4T'-\alpha^6\left(1-\frac{\lambda}{2}\right)R_0'\right] \end{aligned}\right\} \tag{54}$$

We then take a given solution p_0 of (53) and set $p=p_0+p_1$ where $p_1 \ll p_0$ in the next approximation:

$$\det\{h(n,r)\}\cong\det\begin{pmatrix} h(1,1) & h(1,2) \\ h(2,1) & h(2,2)\end{pmatrix}=0 \tag{55}$$

and reduce this to a cubic equation in p_1 . We solve this and repeat the procedure above for the next approximation and so on. We can also examine the criteria for the two kinds of instability discussed above at each stage of the approximation and obtain successive approximations to the characteristic value equation:

$$a=a(R_0,T) \tag{56}$$

for the case of exchange of stabilities.

Method 2

In this method we seek asymptotic solutions in closed form with respect to the large parameters R_0, assuming that T is comparable to R_0 for the case of convective instability $(s=0)$. We again begin with Eqs. (41) through (43) with $s=0$ and eliminate W and Z in these equations. After some manipulation we obtain the equation:

$$(D^2-a^2)\left[(D^2-a^2)^3\Theta+\mathcal{T}D^2\Theta+a^2R_0(1-\lambda z)\Theta\right]=0 \tag{57}$$

with the boundary conditions:

$$\Theta=D^2\Theta=D^4\Theta=D^6\Theta=0 \quad at \quad z=0,1 \qquad \text{B.C.(6)}$$

Let $G\equiv(D^2-a^2)^3\Theta+\mathcal{T}D^2\Theta+a^2R_0(1-\lambda z)\Theta.$ (58)

Then (57) and B.C. (6) yield:

$$\left.\begin{array}{l}(D^2-a^2)G=0\\ \text{with } G=0 \text{ at } z=0,1\end{array}\right\} \tag{59}$$

This implies that $G\equiv 0$. Thus we must find asymptotic solutions for:

$$(D^2-a^2)^3\Theta+\mathcal{T}D^2\Theta+a^2R_0(1-\lambda z)\Theta=0 \tag{60}$$

with the boundary conditions:

$$\Theta=D^2\Theta=D^4\Theta=0 \quad at \quad z=0,1. \qquad \text{B.C.(7)}$$

We now let:

$$k^6=a^2R_0 \tag{61}$$

and assume that $R_0\sim\mathcal{T}\sim a^4$.

We also set:

$$b\equiv\frac{\mathcal{T}}{3a^4} \quad and \quad c\equiv\frac{R_0}{3a^4}. \tag{62}$$

The object is to find solutions of the form

$$\Theta=u(z)e^{ik\phi(z)} \tag{63}$$

correct to the highest powers of k.

By substituting (63) in (60) and equating coefficients of the highest powers of k to zero we obtain the following six solutions after some computation:

$$\Theta_i^{\pm} = u_i^{\pm}\, e^{ik\phi_i^{\pm}} \qquad i = 1,2,3 \tag{64}$$

where

$$\phi_i^{\pm} = \mp \frac{ia}{k} \int_{z^*}^{z} \left[n_i(z') + 1 \right]^{1/2} dz' \tag{65}$$

and

$$u_i^{\pm} = C\left[(n_i)^{(4-b)} (n_i + 1)^{1+b} \right]^{-1/4} \exp \frac{b}{4n_i} \tag{66}$$

where C is an arbitrary constant and z^* is an arbitrary point between 0 and 1 and $n_i\,(i=1,2,3)$ are the solutions of the cubic equation:

$$\eta^3 + 3b\eta + 2A = 0 \tag{67}$$

where

$$2A = -1 + 3\left[c(1-\lambda z) + b \right]. \tag{68}$$

These solutions are given as follows:

$$\eta_1 = -(S - R) \tag{69}$$

$$n_2 = \tfrac{1}{2}(S-R) + \tfrac{1}{2} i\sqrt{3}\,(S+R) \tag{70}$$

$$n_3 = \tfrac{1}{2}(S-R) - \tfrac{1}{2} i\sqrt{3}\,(S+R) \tag{71}$$

where

$$S \equiv \left[A + (b^3 + A^2)^{1/2} \right]^{1/3} \tag{72}$$

$$R \equiv \left[-A + (b^3 + A^2)^{1/2} \right]^{1/3} \tag{73}$$

Examination of (64), (65) and (66) reveals that the solutions are in general singular at $\eta_i = 0$ and at the turning points $\eta_i = -1$. By examining the solutions n_2 and n_3 we find that η_1 and η_2 are 0 or -1 only if b = 0 and A = 0. We exclude these singularities and turning points by restricting our attention to the case $b > 0$. We also find that $\eta_1 = 0$ only if A = 0. This is equivalent to the condition:

$$z = \frac{1}{\lambda} + \frac{1}{\lambda c}\left(b - \frac{1}{3} \right). \tag{74}$$

In terms of parameters, this means that $\eta_1 = 0$ in the fluid only if

$$-1 \le \frac{1}{c}\left(b - \frac{1}{3} \right) \le \lambda - 1. \tag{75}$$

We exclude this case by restricting our attention further to Taylor numbers such that $\frac{1}{c}\left(b-\frac{1}{3}\right)$ lies outside this range (cf. Fig.2 below for the case $\lambda = 2$).

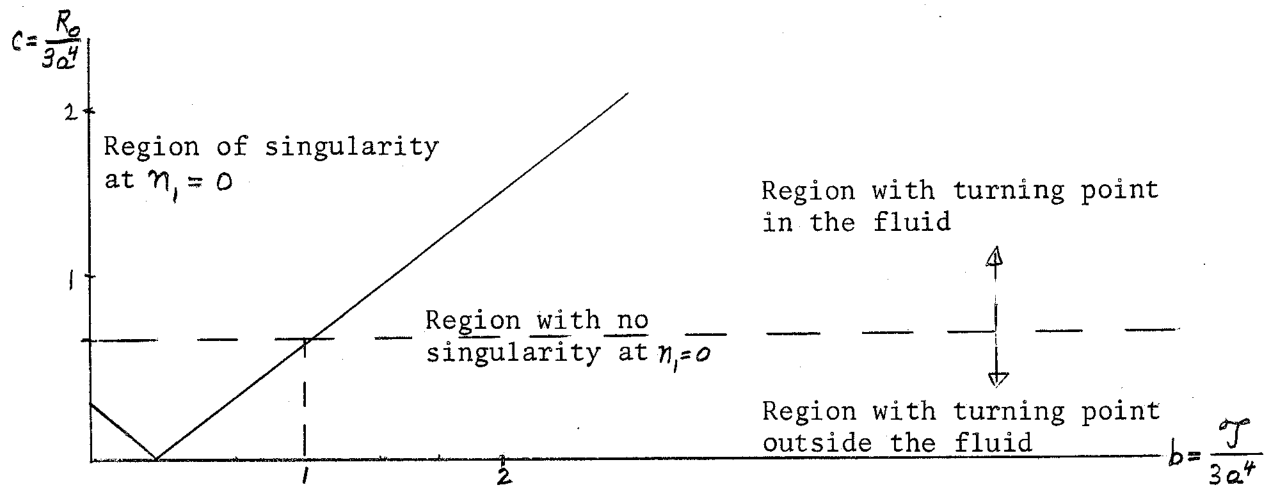

Fig.2

We are left with the expected singularity at the turning point $\eta_1 = -1$ which corresponds to the point z_o given by

$$\lambda z_o = 1 - \frac{2}{3c} . \qquad (76)$$

We see from (76) that the solutions $\Theta_1^{\pm}$ have a turning point in the fluid only if $c > \frac{2}{3}$. We also note that the position of the turning point is independent of the Taylor number. Since we have restricted our attention to cases where the singularity at $\eta_1 = 0$ does not occur in the fluid and (as we shall see) to cases where $c > \frac{2}{3}$, we find that $A > 0$ which implies that $\eta_1 < 0$ for all points in the fluid. Then by examining (65) we find that the solutions $\Theta_1^{\pm}$ have exponential character when $-\eta_1 < 1$ or $\lambda z > 1 - \frac{2}{3c}$ and oscillatory character when $-\eta_1 > 1$ or $\lambda z < 1 - \frac{2}{3c}$. A graph of z_o vs. c and a typical profile of the solution are given in Fig. 3 for the case $\lambda = 2$.

As we have noted above, the solutions $\Theta_1^{\pm}$ have no turning point in the fluid if $c < \frac{2}{3}$. In this case it can be shown that we cannot satisfy the boundary conditions with the solution $\Theta_1^{\pm}$ $(i = 1, 2, 3)$. Thus

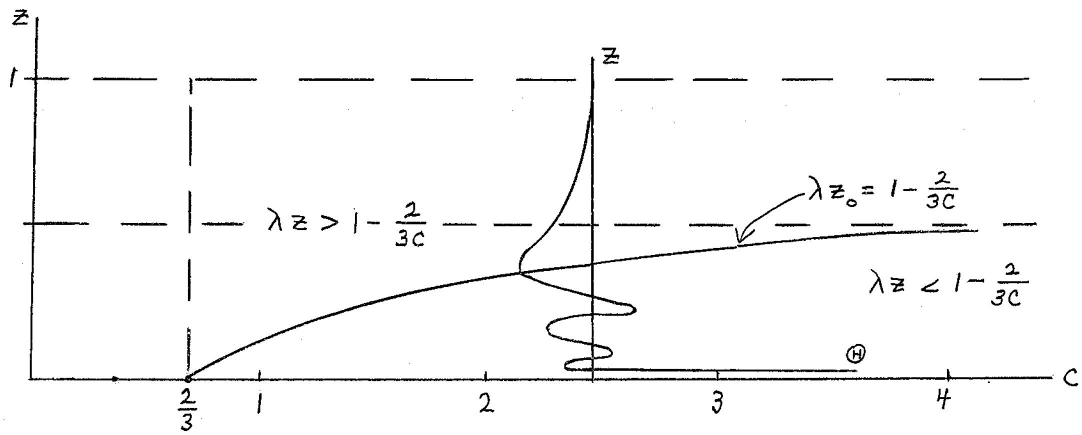

Fig.3

no solution to the problem exists for $c < \frac{2}{3}$ and this gives us immediately a lower bound for R_o:

$$R_o \geq 2a^4 \tag{77}$$

We assume in the sequel that $c > \frac{2}{3}$.

Now the solutions $\Theta_1^{\pm}$ are not valid near the turning points. Instead of employing the usual procedure of matching solutions in the two regions below and above the turning point by appropriate connection formulae it turns out to be more convenient to obtain uniformly valid asymptotic solutions which reduce to $\Theta_1^{\pm}$ away from the turning points. We proceed as follows:

Let $\mu = -\eta_1$ and $q = 1-\mu$.

Then $\Theta_1^{\pm}$ may be written as follows:

$$\Theta_1^{\pm} = \left[\mu^{4-b}\begin{pmatrix}+q\\-q\end{pmatrix}^{1+b}\right]^{-1/4} exp\left[\pm a\int_{z^*}^{z}\sqrt{q}\,dz - \frac{b}{4\mu}\right] \tag{78}$$

where we choose +q in the region above the turning point and -q in the region below the turning point $\mu = 1$ $(q = 0)$. We now look for a second order equation which has the same asymptotic solution as (60). It can easily be verified that such an equation is given by:

$$D^2\Theta + fD\Theta - a^2 q\Theta + \left[\frac{1}{2}Df + \frac{1}{4}f^2\right]\Theta = 0 \tag{79}$$

where

$$f = -\left[\frac{2}{\mu} - \frac{b}{2\mu^2 q}\right]\frac{c\lambda}{\mu^2+b} \tag{80}$$

If we now set

$$\Theta = V\left[\mu^{4-b}\begin{pmatrix}+q\\-q\end{pmatrix}^{b}\right]^{-1/4} e^{-b/4\mu} \equiv V e^{-\frac{1}{2}\int f\,dz} \tag{81}$$

and substitute in (79) we obtain:

$$D^2V - a^2 q V = 0 \tag{82}$$

Two independent uniformly valid asymptotic solutions of (82) can be found and they are given by:

$$V_0 = (D\Psi)^{-1/2} A_i\left[a^{2/3}\Psi\right] \tag{83}$$

$$V_2 = (D\Psi)^{-1/2} B_i\left[a^{2/3}\Psi\right] \tag{84}$$

where

$$\Psi(z) = \begin{cases} \left[\frac{3}{2}\int_{z_0}^{z}\sqrt{q'}\,dz'\right]^{2/3} & \text{for } z > z_0 \\ -\left[\frac{3}{2}\int_{z}^{z_0}\sqrt{-q'}\,dz'\right]^{2/3} & \text{for } z < z_0 \end{cases} \tag{85}$$

and A_i and B_i are the Airy functions. The solutions V_0 and V_2 are given approximately by

$$V_0 \sim \begin{cases} q^{-1/4}\exp\left[-a\int_{z_0}^{z}\sqrt{q'}\,dz'\right] & q > 0 \quad z > z_0 \\ [-q]^{-1/4}\sin\left[a\int_{z}^{z_0}\sqrt{-q'}\,dz' + \frac{\pi}{4}\right] & q < 0 \quad z < z_0 \end{cases} \tag{86}$$

and

$$V_2 \sim \begin{cases} q^{-1/4}\exp\left[a\int_{z_0}^{z}\sqrt{q'}\,dz'\right] & q > 0 \quad z > z_0 \\ [-q]^{-1/4}\cos\left[a\int_{z}^{z_0}\sqrt{-q'}\,dz' + \frac{\pi}{4}\right] & q < 0 \quad z < z_0 \end{cases} \tag{87}$$

away from the turning points.

Thus we can replace the solutions (78) by the solutions:

$$\Theta_0 = V_0(z)\left[\mu^{4-b}\begin{pmatrix}+q\\-q\end{pmatrix}^b\right]^{-1/4} \exp\left(-\frac{b}{4\mu}\right) \tag{88}$$

$$\Theta_2 = V_2(z)\left[\mu^{4-b}\begin{pmatrix}+q\\-q\end{pmatrix}^b\right]^{-1/4} \exp\left(-\frac{b}{4\mu}\right) \tag{89}$$

Of the six solutions Θ_0 , Θ_2 , $\Theta^{\pm}_{2,3}$, the latter four have no turning points in the fluid and one can show that they cannot be used to satisfy the boundary conditions. Thus we must satisfy the boundary conditions B.C.(2) with a solution of the form:

$$\Theta = A_0\Theta_0 + A_2\Theta_2 \tag{90}$$

Since $\Theta(0) = \Theta(1) = 0$ we obtain the condition for a non-trivial solution:

$$\left.\begin{aligned} &T_i\left[a^{2/3}\psi(0)\right] = T_i\left[a^{2/3}\psi(1)\right] \\ \text{where}\quad &T_i \equiv \frac{A_i}{B_i} \end{aligned}\right\} \tag{91}$$

which is the desired characteristic value equation.

$$a = a(R_0, \mathcal{T}). \tag{92}$$

Solving for A_2 in terms of A_0 we obtain:

$$A_2 = -A_0 T_i\left[a^{2/3}\psi(0)\right] \tag{93}$$

Then the solution:

$$\left.\begin{aligned} &\Theta = \Theta_0 + \gamma\,\Theta_2 \\ \text{where}\quad &\gamma = -T_i\left[a^{2/3}\psi(0)\right] \end{aligned}\right\} \tag{94}$$

can be shown to satisfy the boundary conditions $D^2\Theta = 0$ and $D^4\Theta = 0$ asymptotically.

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INSTABILITY OF THE SOLAR CHROMOSPHERE

Richard J. Defouw

The kinetic temperature of the solar chromosphere is an increasing function of height so that one would expect this layer of the atmosphere to be extremely stable. However, observations indicate that jets of gas called spicules are continually ejected from the chromosphere. The problem is therefore to explain the existence of these jets in a temperature inversion.

The solution which I propose is based on the concept of thermal instability. If we define the heat loss function $\mathcal{L}$ as the energy lost minus the energy gained per gram per second, Parker (1953) showed that thermal instability results when

$$\frac{\partial \mathcal{L}}{\partial T} < 0. \qquad (1)$$

Weymann (1960) and Field (1965) pointed out that density (ρ) variations will accompany changes in temperature (T) owing to the tendency for pressure equilibrium to be maintained. The resulting criterion for thermal instability is

$$\mathcal{L}' \equiv \frac{\partial \mathcal{L}}{\partial T} - \frac{\rho}{T}\frac{\partial \mathcal{L}}{\partial \rho} < 0. \qquad (2)$$

If we subject a thermally unstable medium to a gravitational field, buoyancy forces occasioned by the density perturbations will result in convective motions. An elementary parcel calculation shows that the growth rate of the vertical velocity of a parcel is

$$n = -\frac{\mathcal{L}'}{2c_p} \pm \left[\frac{\mathcal{L}'^2}{4c_p^2} - \frac{g}{T}\left(\beta - \beta_{ad}\right)\right]^{1/2}, \qquad (3)$$

where g is the gravity, β is the temperature gradient, and β_{ad} is the adiabatic temperature gradient. We see from (3) that a sufficient condition for instability in a gravitational field is that (2) be satisfied. However, if there is a sufficiently steep temperature inversion, (3) indicates overstability.

Unfortunately, the criterion for thermal instability is, in general, not as simple as inequality (2). By considering a partially ionized hydrogen gas assumed to be optically thin in the Lyman continuum, I have shown that the correct condition for thermal instability of a uniform medium is

$$\left(\mathcal{L}_x - \frac{\rho}{1+x}\mathcal{L}_\rho\right)\left(\ell_T - \frac{\rho}{T}\ell_\rho\right) - \left(\mathcal{L}_T - \frac{\rho}{T}\mathcal{L}_\rho\right)\left(\ell_x - \frac{\rho}{1+x}\ell_\rho\right) < 0\,, \tag{4}$$

where $\mathcal{L}$ is the heat loss function defined above and ℓ is the number of ionizations minus the number of recombinations per gram per second. The subscripts denote partial derivatives and x is the fraction of atoms which are ionized. The main simplification used in obtaining (4) is the use of a model hydrogen atom possessing the ground state and the continuum of a real hydrogen atom but with no excited bound states. If we assume that ionization is collisional and recombination is radiative (the free-bound radiation being the only radiation from the system), we find that inequality (4) is satisfied only when $T > 17500\ ^{o}K$.

When this treatment of thermal instability is incorporated into standard linear convective stability theory, we find results similar to those expressed in (3), namely, that thermal instability can lead to instability in a gravitational field, even if there is a temperature inversion. In a sufficiently steep temperature inversion we find overstability as before. However, the following modification is introduced: even when

(4) is satisfied, there is stability for certain ranges of ρ and $\beta - \beta_{ad}$. In the solar chromosphere these ranges are not encountered and we therefore expect instability.

In regions of magnetic field, a sufficient condition for monotonic instability (as opposed to overstability) is that (4) be satisfied. Since spicules are known to occur in magnetic fields, we apparently have a possible explanation of these jets.

VIOLENT TRANSITION IN A ROTATIONALLY-CONSTRAINED FLUID

Roger F. Gans

1. Introduction

When a container of length L and diameter D, filled with a fluid of small viscosity, is rotated about its symmetry axis and precessed about the normal to that axis, its behavior is observed to be a strong function of the ratio L/D (for constant viscosity) and a weak function of viscosity (for constant L/D). In the approximate range, $.5 < L/D < .95$, the most spectacular observation is a violent transition from a quasilaminar, quasisteady flow, whose steady characteristics are predictable from a linear theory, to an ordered turbulence which appears to possess mean circulation and which is dissipating vastly more energy than the laminar flow.

One can measure the dissipation by measuring the change in the torque required to maintain the basic rotation state. Figure 1 shows a schematic plot of torque vs. precession rate.

The two critical values A and B are observed to be functions of L/D. In the range of interest they are both increasing functions of

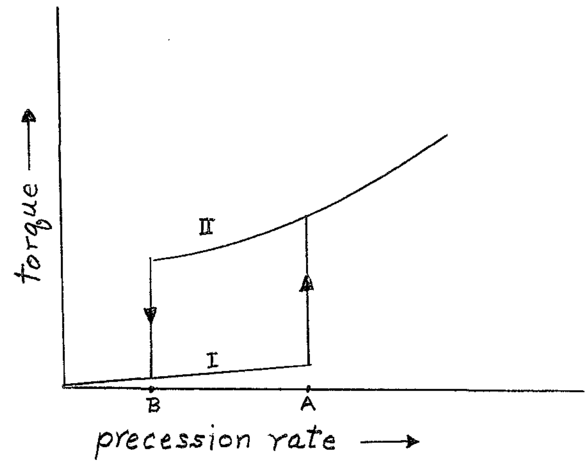

Fig.1: Observed torque vs. precession rate

1 - L/D, and they are <u>different</u> functions. The flow domain I has a strong resemblance to that predictable by linear theory; the flow domain II is not understood.

In this paper the point of view that this transition can serve as a model for other possible geophysical transitions is adopted, and this provides a geophysical motivation for attacking a problem which is interesting in its own right.

The observations lead me to seek a mathematical analysis in which viscosity is unimportant and which will lead to a forced transition. A successful conclusion to such an analysis would be the prediction of the up-transition point A as a function of L/D. Thus a relation between Ω , the precession rate, and $\langle q, q\rangle^{1/2}$, the magnitude of the departure of the fluid from solid corotation with the container, will be sought, of the form

$$\Omega = C_1\langle q,q\rangle^{1/2} - C_2\langle q,q\rangle^{n/2} \qquad C_1, C_2 > 0;\ n > 1 \tag{1.1}$$

Figure 2 shows a schematic of such a relation. As Ω increases beyond A there must be a transition to a flow state not on the diagram, and so one can identify the point A in Fig.2 with the point A in Fig.1.

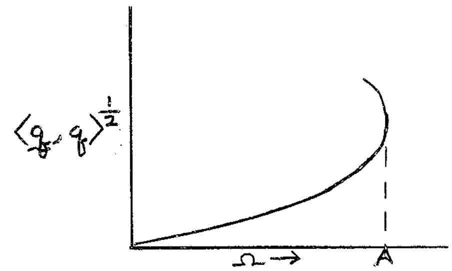

Fig. 2: Schematic amplitude curve

The problem is complicated by the existence of resonant modes which can be driven by the precession. For

$$\frac{L}{D} = \frac{\sqrt{3}}{k_\ell} \frac{2n+1}{2} \pi \qquad n = 0, 1, 2, \cdots \tag{1.2}$$

the amplitude of the response, $\langle q_r, q_r \rangle^{1/2}$, computed on the basis of a linear, inviscid theory, is infinite. Here $\{k_\ell\}$ satisfies

$$k_\ell J_1'(k_\ell) + 2 J_1(k_\ell) = 0 \tag{1.3}$$

where $J_1(k_\ell)$ is the first order Bessel function of the first kind and the prime denotes differentiation. The set $\{k_\ell\}$ forms an infinite monotone increasing sequence asymptotic to the zeroes of $J_0\{k_\ell\}$, and the right-hand side of (1.2) can be made arbitrarily close to any L/D by proper choice of n and l. This means that in any physical system the possibility of resonance must be taken into account, and in the analysis to follow it will be supposed that there is some pair ℓ, n for which (1.2) is satisfied.

The analysis will be carried out using a modified perturbation technique based on the method illustrated by Millman and Keller (1969). Both $\underline{q}$ and Ω will be supposed to possess expansions in powers of ϵ, which will be identified with the nondimensional resonance response, this being the natural parameter in the problem. It will then be shown, after proceeding to terms of $O(\epsilon^3)$, that for k_ℓ sufficiently large, the nonresonant response is dominant, and, that by going to third order in the nonresonant response one can arrive at Eq.(1.1), with $\langle q_r, q_r \rangle^{1/2}$ being the nonresonant magnitude, n being 3 and C_2 an integral of the resonant mode times non-resonant nonlinear terms.

The computation of C_2 is very tedious and summertime was too short for completion of this computation, however the nature of the explicit dependence on L/D is presented.

The method depends on the effects of viscosity being small compared to the nonlinear effects. A complete discussion of this point is beyond the scope of this paper, but will be part of a fuller exposition of this problem to appear later. The only place in the current analysis where this causes any trouble is the double-periodic (in the azimuthal direction) flow, which can be written as a function of the nonresonant response. If one substitutes the nonresonant response given by the inviscid analysis into the function, the doubly-periodic flow cannot be made to satisfy its boundary condition. However, if the velocities put into the function satisfy no-slip boundary conditions, then the double-periodic flow will satisfy its boundary condition. Since the velocities put into the function differ from the true velocities only near the boundaries, the interior representation given of the doubly-periodic response is good. It is only necessary in integral, and so this is acceptable.

The remainder of this paper is divided into three sections, presenting the formulation, the calculations as far as they have been carried, and some discussion of the results and the experiments.

2. Formulation

Consider a cylinder of length L and diameter D rotating about its symmetry axis with angular velocity $\omega \underset{\sim}{k}$ and precessing about an axis at right angles to the symmetry axis with angular velocity $\Omega \underset{\sim}{i}$, filled with an incompressible, inviscid fluid. If one nondimensionalizes as

$$\underset{\sim}{r} = D/2\, \underset{\sim}{r}' ; \quad \underset{\sim}{v} = \omega D/2\, \underset{\sim}{v}' ; \quad \Omega = \omega \Omega' ; \quad P + \tfrac{1}{2}(\underset{\sim}{\Omega}^2 \times \underset{\sim}{r})^2 = \omega^2 \frac{D^2}{4} V \tag{2.1}$$

the equations, in a coordinate system rotating with $\Omega \underset{\sim}{i}$, are

$$\begin{gathered} \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} + 2\Omega \underset{\sim}{i} \times \underset{\sim}{v} + \nabla V = 0 \\ \nabla \underset{\sim}{v} = 0 \\ \underset{\sim}{v} \cdot \underset{\sim}{n} = 0 \quad \text{on } \Sigma \end{gathered} \tag{2.2}$$

where the primes have been dropped, Σ is the container surface and $\underset{\sim}{n}$ a unit normal to Σ. The vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the x, y and z directions, and $\hat{\underset{\sim}{\varpi}}$ and $\hat{\underset{\sim}{\varphi}}$ are unit vectors in the ϖ and φ directions, where ϖ and φ are defined by

$$x = \varpi \cos \varphi \qquad y = \varpi \sin \varphi \, .$$

Solid corotation can be separated out by putting $\underset{\sim}{v} = \underset{\sim}{k} \times \underset{\sim}{r} + \underset{\sim}{q}$ and $V = \frac{1}{2}(\underset{\sim}{k} \times \underset{\sim}{r})^2 + Q$. The set (2.2) can then be separated into linear and nonlinear terms driven by an inhomogeneous term $2\Omega \underset{\sim}{i} \times (\underset{\sim}{k} \times \underset{\sim}{r})$. Symbolically one may write

$$L(Q) + N(QQ, \Omega Q) = \Omega f \tag{2.3}$$

where L is a linear operator, N a quadratic operator and f a known function.

This problem is to be solved using a modified perturbation analysis in which $Q = Q(\epsilon)$ and $\Omega = \Omega(\epsilon)$. The appropriate initial solution is

$Q(0) = 0 = \Omega(0)$. The sequence of problems making up the modified perturbation analysis is generated by successively differentiating (2.3) with respect to ϵ and setting $\epsilon = 0$. A quantity which has been differentiated n times with respect to ϵ and evaluated at $\epsilon = 0$ will be indicated by putting an n over the symbol. For example

$$\left\{\frac{\partial^n}{\partial \epsilon^n} Q\right\}_{\epsilon=0} = \overset{n}{Q} .$$

The initial differentiation produces the problem

$$L(\overset{1}{Q}) = \overset{1}{\Omega} f \tag{2.4}$$

There is no contribution from N because it is quadratic. There exists a function Φ such that

$$\begin{aligned} L(\Phi) &= 0 \\ \langle \Phi, \Phi \rangle &= 1 \\ \langle \Phi, f \rangle &\neq 0 \end{aligned} \tag{2.5}$$

Because of the last property the solvability condition for (2.4) determines $\overset{1}{\Omega} = 0$ and $\overset{1}{Q} = A\Phi$. One can define

$$\epsilon = \langle \Phi, Q \rangle \tag{2.6}$$

and by differentiating with respect to ϵ ,

$$1 = \langle \Phi, A\Phi \rangle = A \tag{2.7}$$

show that A = 1.

Differentiating a second time produces the second order problem

$$L(\overset{2}{Q}) + \overset{2}{N}(\overset{1}{Q}\,\overset{1}{Q}) = \overset{2}{\Omega} f \tag{2.8}$$

It can be shown that $\langle \Phi, \overset{2}{N}(\overset{1}{Q}\,\overset{1}{Q}) \rangle = 0$ and so $\overset{2}{\Omega} = 0$. Because of the definition (2.6) $\overset{2}{Q}$ has no resonant (homogeneous) part and the operator can be inverted to give

$$\overset{2}{Q} = -L^{-1}\left\{\overset{2}{N}(Q\,Q)\right\} \tag{2.9}$$

The third differentiation produces the problem

$$L(\overset{3}{Q}) + \overset{3}{N}(\overset{1}{Q}\overset{2}{Q}) = \overset{3}{\Omega} f \tag{2.10}$$

and $\langle \Phi, \overset{3}{N}(\overset{1}{Q}\overset{2}{Q}) \rangle \neq 0$, so the solvability condition determines

$$\overset{3}{\Omega} = \frac{\langle \Phi, \overset{3}{N}(\overset{1}{Q}\overset{2}{Q}) \rangle}{\langle \Phi, f \rangle} \tag{2.11}$$

and for (2.11) satisfied the operator L in (2.10) can be inverted, and one can write $\overset{3}{Q}$ in terms of a set of orthonormal functions Ψ_n as

$$\overset{3}{Q} = \sum_n \langle \Psi_n, \left[f \overset{3}{\Omega} - \overset{3}{N}(\overset{1}{Q}\overset{2}{Q}) \right] \rangle \Psi_n \tag{2.12}$$

The functions Ψ_n clearly have the property that $\langle \Phi, \Psi_n \rangle = 0$.

Making use of the definition of ϵ and the expressions for $\overset{3}{\Omega}$, $\overset{1}{Q}$, $\overset{2}{Q}$, and $\overset{3}{Q}$ allows one to write Q correct to $O(\overset{3}{\epsilon})$ as

$$\begin{aligned} Q = {} & \langle \Phi, Q \rangle \Phi + \frac{1}{2!} L^{-1}\{\overset{2}{N}(\overset{1}{Q}\overset{1}{Q})\} \langle \Phi, Q \rangle^2 + \\ & + \frac{1}{3!} \sum_n \langle \Phi, \overset{3}{N}(\overset{1}{Q}\overset{2}{Q}) \rangle \frac{\langle \Phi_n, f \rangle}{\langle \Phi, f \rangle} \langle \Phi, Q \rangle^3 \Psi_n - \\ & - \frac{1}{3!} \sum_n \langle \Psi_n, \overset{3}{N}(\overset{1}{Q}\overset{2}{Q}) \rangle \langle \Phi, Q \rangle^3 \Psi_n + O(\epsilon^4) \end{aligned} \tag{2.13}$$

and, if the projection of the nonresonant mode on f is much larger than the projection of the resonant mode on f, it is possible for the nonresonant part of $\overset{3}{Q}\epsilon^3$ to be larger than $\overset{1}{Q}\epsilon$. The condition for this may be written directly from (2.13) as

$$\frac{\langle \Phi, f \rangle}{\langle \Psi, f \rangle} < \langle \Phi, \overset{3}{N}(\overset{1}{Q}\overset{2}{Q}) \rangle \langle \Phi, Q \rangle^2 \tag{2.14}$$

It is easy to show that the left-hand side of (2.14) goes to zero as the resonant wave number goes to infinity. It is not obvious, however, that (2.14) can be satisfied. It will be stated without proof that for any precession rate Ω one can find some resonant wave number K such that (2.14) is satisfied for resonant wave numbers $k_\ell > K$. (It has been shown (Gans, 1969) that for precession rates small enough so that viscosity is important, resonance is unimportant for resonant wave numbers $k_\ell > (\omega D^2/4\nu)^{1/7}$, where ν is the kinematic viscosity.)

To compute the modification of Ω required by (1.1) one needs to find the largest $\overset{n}{N}(QQ, \Omega Q)$ such that $\langle \Phi, \overset{n}{N} \rangle \neq 0$. If (2.14) is satisfied this will be that part of $\overset{9}{N}$ due to the self-interaction of $\overset{3}{Q}$ and $\overset{3}{\Omega}$. The modification of Ω is $\overset{9}{\Omega}$ and this particular component is

$$\Omega^+_9 = \frac{\langle \Phi, \overset{9}{N}(\overset{3}{Q}\overset{6}{Q}, \Omega \overset{6}{Q}) \rangle}{\langle \Phi, f \rangle} \tag{2.15}$$

where

$$\overset{6}{Q} = -L^{-1}\{\overset{6}{N}(\overset{3}{Q}\overset{3}{Q}, \overset{3}{Q}\overset{3}{\Omega})\} \tag{2.16}$$

in direct analogy to (2.9).

In the following section calculations leading to the determination of $\overset{9}{\Omega}{}^+$ will be pursued. It is a three-step process which may be summarized as follows:

$$\begin{aligned} &1) \quad \overset{3}{Q} = \overset{3}{\Omega} \langle \Psi, f \rangle \Psi \\ &2) \quad \overset{6}{Q} = -L^{-1}\{\overset{6}{N}(\overset{3}{Q}\overset{3}{Q}, \overset{3}{\Omega}\overset{3}{Q})\} \\ &3) \quad \overset{9}{\Omega}{}^+ = \frac{\langle \Phi, \overset{9}{N}(\overset{3}{Q}\overset{6}{Q}, \overset{3}{\Omega}\overset{6}{Q}) \rangle}{\langle \Phi, f \rangle} \end{aligned} \tag{2.17}$$

When one has arrived at step 3 and wishes an explicit result for $\overset{9}{\Omega}{}^+$ it is permissable to replace $\overset{3}{\Omega}$ by $3!\epsilon^{-3}\Omega$ and so one does not need to work out all the nonlinear homogeneous terms.

3. Calculations

In this section $\overset{3}{Q}$ is given explicitly. The series for $\overset{3}{Q}$ is truncated after the first term and $\overset{6}{Q}$ is calculated using the truncated set. The axisymmetric part of $\overset{6}{Q}$ is given explicitly and the doubly-periodic part in terms of indefinite integrals over the radial coordinate ϖ . Finally an expression for $\overset{9}{\Omega}$ is given in terms of indefinite integrals over combinations involving $\overset{3}{Q}$ and $\overset{6}{Q}$.

The general equations for Q and q are obtained from (2.2) by direct substitution, and are

$$\underset{\sim}{k}\times\underset{\sim}{r}\cdot\nabla\underset{\sim}{q}+\underset{\sim}{q}\cdot\nabla(\underset{\sim}{k}\times\underset{\sim}{r})+\nabla Q+\underset{\sim}{q}\cdot\nabla\underset{\sim}{q}+2\Omega\underset{\sim}{i}k\,\underset{\sim}{q}=-2\Omega\underset{\sim}{i}\,k\times(\underset{\sim}{k}\times\underset{\sim}{r})$$

$$\nabla\cdot\underset{\sim}{q}=0$$

$$\underset{\sim}{q}\cdot\underset{\sim}{n}=0 \quad \text{on } \Sigma \tag{3.1}$$

One can seek solutions proportional to $e^{im\varphi}$, and, for $m\neq 0,2$ one can re-write (3.1) as

$$\Delta_m P-\frac{4}{m^2}\,{}_mP_{zz}=-\nabla\cdot{}_m\underset{\sim}{N}+\frac{4}{m^2}\underset{\sim}{k}\cdot{}_m\underset{\sim}{N}_z+\frac{2i}{m}\underset{\sim}{k}\cdot\nabla\times{}_m\underset{\sim}{N}$$

$${}_mP_{\varpi}+\frac{2}{\varpi}\,{}_mP=-\hat{\underset{\sim}{\varpi}}\,{}_m\underset{\sim}{N}+\frac{2i}{m}\hat{\underset{\sim}{\varphi}}\,{}_m\underset{\sim}{N} \quad \text{on } \varpi=1$$

$${}_mP_z=-\underset{\sim}{k}\cdot{}_m\underset{\sim}{N}-\Omega\varpi\delta_{m1} \quad \text{on } z=\pm L/D \tag{3.2}$$

where δ_{m1} is the Kronecker δ , and

$${}_mP=\frac{1}{2\pi}\int_0^{2\pi}e^{im\varphi}\,Q\,d\varphi$$

$${}_m\underset{\sim}{N}=\frac{1}{2\pi}\int_0^{2\pi}e^{-im\varphi}\left\{\underset{\sim}{q}\cdot\nabla\underset{\sim}{q}+2\Omega\underset{\sim}{i}\times\underset{\sim}{q}\right\}d\varphi \tag{3.3}$$

The components of q can be defined in a similar way as

$${}_mu=\frac{1}{2\pi}\int_0^{2\pi}e^{-im\varphi}\,\underset{\sim}{q}\cdot\hat{\underset{\sim}{\varpi}}\,d\varphi$$

$${}_mv=\frac{1}{2\pi}\int_0^{2\pi}e^{-im\varphi}\,\underset{\sim}{q}\cdot\hat{\underset{\sim}{\varphi}}\,d\varphi$$

$${}_mw=\frac{1}{2\pi}\int_0^{2\pi}e^{-im\varphi}\,\underset{\sim}{q}\cdot\underset{\sim}{k}\,d\varphi \tag{3.4}$$

and written in terms of ${}_mP$ and ${}_mN$ as

$$ {}_m\mathcal{U} = \frac{-im}{4-m^2}\left\{ {}_mP_{\varpi} + \frac{2}{\varpi}\, {}_mP + \hat{\underset{\sim}{\varpi}} \cdot {}_m\underset{\sim}{N} - \frac{2i}{m}\, \hat{\underset{\sim}{\varphi}} \cdot {}_m\underset{\sim}{N} \right\} $$

$$ {}_m\mathcal{V} = \frac{1}{4-m^2}\left\{ 2\, {}_mP_{\varpi} + \frac{m^2}{\varpi}\, {}_mP + 2\hat{\underset{\sim}{\varpi}}\, {}_m\underset{\sim}{N} - im\hat{\underset{\sim}{\varphi}} \cdot {}_m\underset{\sim}{N} \right\} $$

$$ {}_m\mathcal{W} = \frac{i}{m}\left\{ {}_mP_z + \underset{\sim}{k} + {}_m\underset{\sim}{N} + \Omega\varpi\, \delta_{m1} \right\} \quad m \neq 0,2 \tag{3.5} $$

The expansion parameter ϵ and the spatial coordinate φ are independent and so one can differentiate (3.2) - (3.5) with respect to ϵ without changing any of the definitions, hence all the definitions also apply to the quantities at any order.

Step one solutions:

Step one requires the finding of $\overset{3}{Q}$. Because of the nonresonant condition $\overset{3}{Q}$ can be expressed in terms of ${}_1\overset{3}{P}$ to the required accuracy, and, ${}_1\overset{3}{N}$ is negligible. The appropriate equations are

$$ \begin{aligned} &\Delta\, {}_1\overset{3}{P} - 4\, {}_1\overset{3}{P}_{zz} = 0 \\ &{}_1\overset{3}{P}_{\varpi} + \frac{2}{\varpi}\, {}_1\overset{3}{P} = 0 \qquad \text{on } \varpi = 1 \\ &\quad {}_1\overset{3}{P}_z = -\Omega\varpi \qquad \text{on } z = \pm L/D \end{aligned} \tag{3.6} $$

The orthonormal functions $\left\{ {}_1\Psi_\ell \right\}$ appropriate to (3.6) are

$$ \left\{ {}_1\Psi_\ell \right\} = C(k_\ell)\, J_1(k_\ell \varpi) \sin \frac{k_1}{\sqrt{3}} z \tag{3.7} $$

where

$$ C(k_\ell) = \frac{k_\ell}{J_1(k_\ell)(3+k_\ell^2)^{1/2}\sqrt{\pi}\left[\frac{L}{D} - \frac{\sqrt{3}}{2} \sin\left(\frac{2k_\ell}{\sqrt{3}} \frac{L}{D} \right) \right]} $$

and

$$ k_\ell J_1'(k_\ell) + 2 J_1(k_\ell) = 0 $$

If the resonance condition (1.2) is satisfied, then there will be some $\ell = L$ such that the third of (3.6) cannot be satisfied. The resonant term is thus a member of the set (3.7) and Φ is

$$\Phi = C(k_L)\, J_1(k_L \varpi) \sin \frac{k_L}{\sqrt{3}} z\, e^{i\varphi} \tag{3.8}$$

The parameter k_L will be called the resonant wave number.

Now one can express ${}_1\overset{3}{P}$ in a sum over $\ell \neq L$ of the functions $\{{}_1\Psi_\ell\}$ and then compute the velocity components using (3.5). This representation will be truncated after ${}_1\Psi_1$ in order that one can form higher order nonlinear terms which will fit on the paper. This does not introduce a serious error as

$$\langle {}_1\Psi_1, \varpi \rangle \sim 10 \langle {}_1\Psi_2, \varpi \rangle \tag{3.9}$$

Step two axisymmetric solutions:

As noted above m = 0 is a special case and one must work with (3.1) directly. The nonlinear term

$$\begin{aligned} {}_0\overset{6}{\underset{\sim}{N}} &= \frac{1}{2\pi} \int_0^{2\pi} \left[\overset{3}{q} \cdot \nabla \overset{3}{q} + 2\overset{3}{\Omega}\, i \times \overset{3}{\underset{\sim}{q}} \right] d\varphi \\ &= ({}_1\overset{3}{u}, {}_1\overset{3}{v}, {}_1\overset{3}{w}) \cdot \nabla ({}_1\overset{3}{u}{}^*, {}_1\overset{3}{v}{}^*, {}_1\overset{3}{w}{}^*) + ({}_1\overset{3}{u}{}^*, {}_1\overset{3}{v}{}^*, {}_1\overset{3}{w}{}^*) \cdot \nabla ({}_1\overset{3}{u}, {}_1\overset{3}{v}, {}_1\overset{3}{w}) + \\ &\quad + \overset{3}{\Omega}(1, -i, 0) \times ({}_1\overset{3}{u}{}^*, {}_1\overset{3}{v}{}^*, {}_1\overset{3}{w}{}^*) + \overset{3}{\Omega}(1, i, 0) \times ({}_1\overset{3}{u}, {}_1\overset{3}{v}, {}_1\overset{3}{w}) \end{aligned} \tag{3.10}$$

and (3.1) can be written

$$\begin{aligned} -2\, {}_0\overset{6}{v} + {}_0\overset{6}{P}_\varpi &= \hat{\underset{\sim}{\varpi}} \cdot {}_0\overset{6}{\underset{\sim}{N}} \\ {}_0\overset{6}{u} &= 0 \\ {}_0\overset{6}{P}_z &= \underset{\sim}{k} \cdot {}_0\overset{6}{\underset{\sim}{N}} \\ {}_0\overset{6}{w}_z &= 0 \end{aligned} \tag{3.11}$$

as $\hat{\underset{\sim}{\varphi}} \cdot {}_0\overset{6}{\underset{\sim}{N}} = 0$. The axisymmetric solution has only an azimuthal component given by

$${}_0\overset{6}{v} = \frac{1}{2} \int \underset{\sim}{k} \cdot {}_0\overset{6}{\underset{\sim}{N}}_\varpi \, dz - \frac{1}{2} \hat{\underset{\sim}{\varpi}} \cdot {}_0\overset{6}{\underset{\sim}{N}} \tag{3.12}$$

There is an arbitrary function of ϖ associated with the indefinite integral, and there is no boundary condition available to determine it.

It is chosen so that ${}_0\overset{6}{v}$ vanishes on the curve where $\underset{\sim}{k} \times \underset{\sim}{r} + \overset{3}{\underset{\sim}{q}}$ vanishes. This makes the "retrograde" rectified motion occur with respect to the actual axis of no motion rather than the original undisplaced axis of no motion $\varpi = 0$. Doing this gives

$$
\begin{aligned}
{}_0\overset{6}{v} = -\overset{3}{\Omega}{}^2 \Big\{ & \frac{A^2}{4}\left[\frac{4}{\varpi^3}\Psi^2 - \frac{8}{\varpi^2}\Psi\Psi_\varpi + \frac{4}{\varpi}\Psi_\varpi^2 - 16\,\frac{k^2}{\varpi}\Psi^2 \cdot 2k^2\Psi\Psi_\varpi\right] + \\
& + \frac{3A^2}{2k}\left[\frac{4}{\varpi^2}\Psi_z - \frac{4}{\varpi}\Psi_{\varpi z} - 3k^2\Psi_z\right] - \frac{3}{2}A^2\left[\Psi_z\Psi_{\varpi z} + \frac{2}{\varpi}\Psi_z^2\right] + \\
& + \frac{1}{2}A\varpi\left(\Psi_{\varpi z} + \frac{2}{\varpi}\Psi_z\right) + g(\varpi)\Big\}
\end{aligned}
\tag{3.13}
$$

where the indices of ${}_1\Psi_1$ and k_1 have been suppressed and

$$A = \frac{\sqrt{3}\, J_1(k_1)}{k_1^3 \cos\left(\frac{k_1}{\sqrt{3}}\,\frac{L}{D}\right)}$$

Step two doubly-periodic solutions:

Again it is necessary to work directly with the equations (3.1). The nonlinear term is

$$ {}_2\overset{6}{\underset{\sim}{N}} = \frac{1}{2\pi}\int_0^{2\pi} e^{-2i\varphi}\left[\overset{3}{\underset{\sim}{q}} \cdot \nabla \overset{3}{\underset{\sim}{q}} + 2\overset{3}{\Omega}\,\underset{\sim}{i} \times \overset{3}{\underset{\sim}{q}}\right] d\varphi \tag{3.14}$$

and the equations are

$$
\begin{aligned}
2i\,{}_2\overset{6}{u} - 2\,{}_2\overset{6}{v} + {}_2\overset{6}{P}_\varpi &= -\hat{\underset{\sim}{\varpi}} \cdot {}_2\overset{6}{\underset{\sim}{N}} \\
2i\,{}_2\overset{6}{v} + 2\,{}_2\overset{6}{u} + \frac{2i}{\varpi}\,{}_2\overset{6}{P} &= -\hat{\underset{\sim}{\varphi}} \cdot {}_2\overset{6}{\underset{\sim}{N}} \\
2i\,{}_2\overset{6}{w}_z + {}_2\overset{6}{P}_z &= -\underset{\sim}{k} \cdot {}_2\overset{6}{\underset{\sim}{N}} \\
\frac{1}{\varpi}\left(\varpi\,{}_2\overset{6}{u}\right)_\varpi + \frac{2i}{\varpi}\,{}_2\overset{6}{v} + {}_2\overset{6}{w}_z &= 0
\end{aligned}
\tag{3.15}
$$

Viewed as inhomogeneous equations for ${}_2\overset{6}{u}$ and ${}_2\overset{6}{v}$ the first and second equations differ by a factor of i, and here is a solvability condition for ${}_2\overset{6}{P}$, namely

$$ {}_2\overset{6}{P}_\varpi + \frac{2}{\varpi}\,{}_2\overset{6}{P} = -\hat{\underset{\sim}{\varpi}} \cdot {}_2\overset{6}{\underset{\sim}{N}} + i\,\hat{\underset{\sim}{\varphi}} \cdot {}_2\overset{6}{\underset{\sim}{N}} \tag{3.16}$$

This can be integrated directly for $\varpi^2\,{}_2\overset{6}{P}$, to give

$$\overset{6}{{}_2P} = -\frac{1}{\varpi^2}\int \varpi^2\left[{}_2\overset{6}{\underset{\sim}{N}}\cdot\hat{\underset{\sim}{\varpi}} - i\,{}_2\overset{6}{\underset{\sim}{N}}\cdot\hat{\underset{\sim}{\varphi}}\right] d\varpi \tag{3.17}$$

and the third of (3.15) gives ${}_2\overset{6}{w}$ as

$$\begin{aligned}{}_2\overset{6}{w} &= -\frac{1}{2\varpi^2}\int \varpi^2\left[{}_2\overset{6}{\underset{\sim}{N}}\cdot\hat{\underset{\sim}{\varpi}} - i\,{}_2\overset{6}{\underset{\sim}{N}}\cdot\hat{\underset{\sim}{\varphi}}\right]_z d\varpi + \frac{1}{2}\,{}_2\overset{6}{\underset{\sim}{N}}\cdot\underset{\sim}{k} \\ &= \frac{iA^2\overset{3}{\Omega}{}^2}{2}\left\{8\Psi_\varpi\Psi_{\varpi z} + 7k^2\Psi\Psi_z - \frac{3}{\varpi^2}\Psi\Psi_z + \frac{5}{\varpi}(1-k^2\varpi^2)\Psi_z\Psi_\varpi\right\} - \\ &\quad - \frac{iA\overset{3}{\Omega}{}^2}{2}\left\{12\Psi_\varpi\cdot\frac{12}{\varpi}\Psi\cdot\varpi k^2\Psi\right\}\end{aligned} \tag{3.18}$$

The fourth equation of the set (3.15) can then be rewritten as an equation for ${}_2\overset{6}{u}$, namely

$${}_2\overset{6}{u}_\varpi - \frac{1}{\varpi}\,{}_2\overset{6}{u} + \frac{1}{\varpi}\hat{\underset{\sim}{\varphi}}\cdot{}_2\overset{6}{\underset{\sim}{N}} + \frac{2i}{\varpi^2}\,{}_2\overset{6}{P} - \frac{i}{2}\,{}_2\overset{6}{P}_{zz} - \frac{i}{2}\underset{\sim}{k}\cdot{}_2\overset{6}{\underset{\sim}{N}}_z = 0 \tag{3.19}$$

which can be integrated for $\frac{1}{\varpi}\,{}_2\overset{6}{u}$ to give

$${}_2\overset{6}{u} = \varpi\int\frac{1}{\varpi}\left\{\frac{i}{2}\underset{\sim}{k}\cdot{}_2\overset{6}{\underset{\sim}{N}}_z + \frac{i}{2}\,{}_2\overset{6}{P}_{zz} - \frac{2i}{\varpi^2}\,{}_2\overset{6}{P} - \frac{1}{\varpi}\hat{\underset{\sim}{\varphi}}\cdot{}_2\overset{6}{\underset{\sim}{N}}\right\} d\varpi \tag{3.20}$$

and, given ${}_2\overset{6}{u}$ we can find ${}_2\overset{6}{v}$ by substitution.

An exact integral for the right-hand side of (3.20) has not yet been found. The various functions can be approximated and an approximate value of the indefinite integral can be found. This has no place here and will be performed in a later, more complete exposition of the work begun here. For what has been begun here it is enough to give the z-dependence of the step-two solutions. These are

$$\begin{aligned}{}_0\overset{6}{v} &= {}^1_0b(\varpi)\sin^2\frac{k}{\sqrt{3}}z + {}^2_0b(\varpi)\cos\frac{k}{\sqrt{3}}z + {}^3_0b(\varpi) \\ {}_2\overset{6}{u} &= {}^1_2a(\varpi)\sin^2\frac{k}{\sqrt{3}}z + {}^2_2a(\varpi)\cos\frac{k}{\sqrt{3}}z + {}^3_2a(\varpi) \\ {}_2\overset{6}{v} &= {}^1_2b(\varpi)\sin^2\frac{k}{\sqrt{3}}z + {}^2_2b(\varpi)\cos\frac{k}{\sqrt{3}}z + {}^3_2b(\varpi) \\ {}_2\overset{6}{w} &= {}^1_2c(\varpi)\sin\frac{k}{\sqrt{3}}z\cos\frac{k}{\sqrt{3}}z + {}^2_2c(\varpi)\sin\frac{k}{\sqrt{3}}z\end{aligned} \tag{3.21}$$

where the functions of ϖ are in principle known.

Step three problem:

The part of the step three problem which is of major interest is that for which $m = 1$. One can use the representation (3.2) as the basic model, and

$$ {}_1\overset{9}{\mathbf{N}} = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\varphi}\left\{2\overset{3}{\Omega}\,\mathbf{i}\times\overset{3}{\mathbf{q}}_6 + \overset{3}{\mathbf{q}}\cdot\nabla\overset{6}{\mathbf{q}} + \overset{6}{\mathbf{q}}\cdot\nabla\overset{3}{\mathbf{q}}\right\}d\varphi \qquad (3.22)$$

The solvability condition can be derived by multiplication by Φ^* and integration by parts using the boundary conditions (e.g.,Greenspan, 1964), and, after some manipulations one arrives at an equation for ${}_1\overset{9}{\Omega}^{\dagger}$, <u>viz.</u>,

$$ 3i\,\overset{9}{\Omega} + \int_0^1 e^{-i\varphi}\Phi\,\varpi^2\,d\varpi = 3i\int_0^1 {}_1\overset{9}{\mathbf{N}}\cdot\mathbf{k}\,e^{-i\varphi}\Phi\,\varpi\,d\varpi + \int_{-L/D}^{L/D} e^{-i\varphi}\Phi\left[2i\,\hat{\boldsymbol{\varphi}}\cdot{}_1\overset{9}{\mathbf{N}} - \hat{\boldsymbol{\varpi}}\cdot{}_1\overset{9}{\mathbf{N}}\right]dz + \int_0^1\!\!\int_{-L/D}^{L/D} e^{-i\varphi}\Phi\left\{\frac{1}{\varpi}\left[\varpi\left(\hat{\boldsymbol{\varpi}}\cdot{}_1\overset{9}{\mathbf{N}} - 2i\,{}_1\overset{9}{\mathbf{N}}\cdot\hat{\boldsymbol{\varphi}}\right)\right]_{\varpi} + \frac{1}{\varpi}\left(\hat{\boldsymbol{\varphi}}\cdot{}_1\overset{9}{\mathbf{N}} - 2i\,\hat{\boldsymbol{\varpi}}\cdot{}_1\overset{9}{\mathbf{N}}\right) - 3\mathbf{k}\cdot{}_1\overset{9}{\mathbf{N}}_z\right\}\varpi\,d\varpi\,dz \qquad (3.23)$$

Time does not permit an explicit solution of (3.23). One can, however, take advantage of the representation (3.21) to exhibit at least the form of the L/D dependence. Doing this gives a four-term expression for $\overset{9}{\Omega}^{\dagger}$, namely

$$\overset{9}{\Omega}^{\dagger} = \sum_{n=0}^{3} A_n(k_L)\cos^n \frac{k}{\sqrt{3}}\frac{L}{D} \qquad (3.24)$$

and the dependence on L/D occurs both explicitly in the cosine terms and through k_L in the coefficients.

4. <u>Discussion</u>

The novel feature of this work is the interaction of resonant and non-resonant responses. The linear response, and, in fact, the entire response up to the transition point, is dominated by the non-resonant velocity, and yet the transition point is a resonance phenomenon. If there were no resonant mode Φ then there could be no transition.

Figure 3 shows experimentally determined transition points plotted

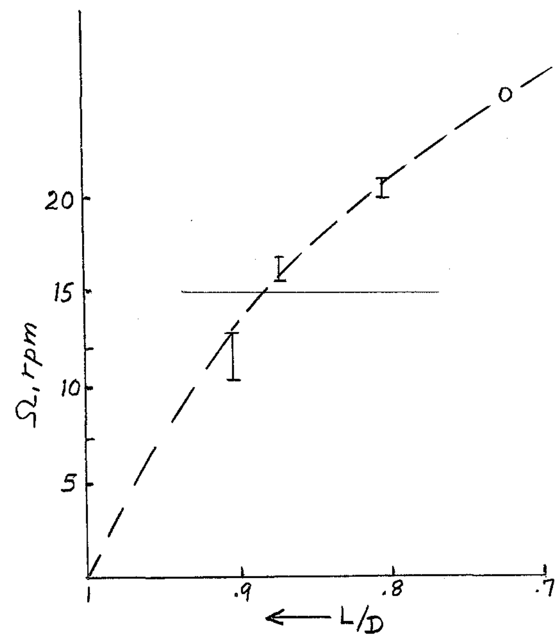

Fig.3: Transition point vs. L/D - ω = 900 rpm.

against L/D. Unfortunately (3.24) predicts too complicated a dependence on L/D for comparison without finding the coefficients $A_n(k_L)$.

Experiments were also done with glycerine-water mixtures to investigate the question of the dependence of the transition phenomenon on viscosity. The effect is small. At an L/D of .87 the transition point changed by a factor of 2 for a change in viscosity of a factor of 1000. This is a tentative observation since the character of the flow before and after transition is quite different at these high viscosities and the transition was observed by the author watching the change in character of the flow, rather than measuring the torque. This observational technique works with water since the author was able to reproduce data originally recorded at U.C.L.A. using torque observations, at Walsh Cottage using visual observations.

There is a further experimental observation which the analysis in this paper does not touch, but which is interesting. In a range of precession rate between the linear region and the transition point the flow exhibits time dependence in the form of vortices. These can be observed by putting small, slightly buoyant particles into the fluid. Figure 4 shows schematically the position of these particles in the linear range of precession.

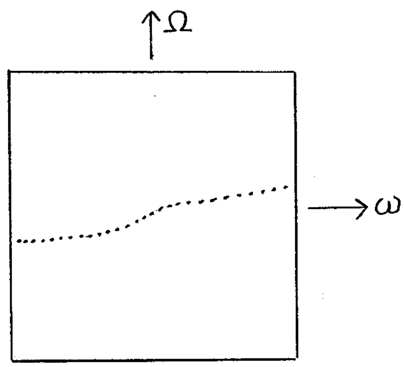

Fig.4: The linear response.

The vortices are thrown off by the central curve, which represents the locus of $\overset{0}{\underset{\sim}{q}} + \overset{3}{\underset{\sim}{q}} = 0$, move to a position about halfway out, stay for tens of seconds, which is hundreds of rotation periods, and then collapse back into the central curve to be regenerated. At any given time there may be up to four vortices present.

The vortex generation process is very different from the violent transition process. Not only are the vortices a time-dependent phenomenon, but they are more dependent on viscosity and less on geometry than the transition phenomenon. Vortices have been observed for L/D = .87, which has a well-defined transition, and for L/D = 1, which is the primary resonance. An increase in viscosity of a factor of 100 is sufficient to prevent vortex formation. This leads one to believe that the vortices represent an interesting stability problem, perhaps related to the Ekman layer instability.

There are two things to be done in the future. One is to finish this work: put numbers into (3.24) and deal explicitly with the viscosity.

The other is to look at the analytic aspects of the vortices. At the moment they seem inaccessible, but one ought not to neglect spontaneous generation of vortices in a laboratory setting since these do have geophysical relevance. There is the meteorological question of tornados and hurricanes, and even the question of the Great Red Spot of Jupiter.

On that note of future hope this work ends.

I wish to thank Willem Malkus, and especially Joseph Keller, for helping me to translate my intuitive feelings about the ordering process in the light of resonance and nonresonance into what I hope is a coherent and consistent structure. If it is not, it is my fault; if it is, their help was indispensible during its construction.

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WAVES IN A ROTATING, STRATIFIED FLUID OF VARIABLE DEPTH

William D. McKee

Introduction

This work is concerned with the propagation of waves in a stratified rotating fluid of variable depth. It deals with the modification of waves by a slowly varying bottom rather than with waves actually produced by the combined effects of rotation and bottom topography (Veronis, 1966; Rhines, 1969).

Previous analyses of the problem have been based on a perturbation approach or the use of characteristics. In the former (Cox and Landstrom, 1962) it is assumed that the depth is everywhere close to some meaningfully defined average depth and the analysis proceeds via Fourier transformation. The latter approach (Magaard, 1962) is based upon the hyperbolic nature of the governing differential equation but requires special assumptions regarding the density field in order to obtain manageable characteristics.

Here, we explore a third avenue of attack: namely an asymptotic analysis based on a "small wavelength" assumption. The basic idea is quite simple: if the wavelength is much smaller than all other horizontal length scales, we expect the waves to behave locally as if they were in water of constant depth. This defines a local dispersion relation and we trace the progress of wavefronts by means of rays. The wave amplitudes are found by solving a certain ordinary differential equation along the rays.

The theory is a straightforward generalization of the work of Keller and Mow (1969) for the non-rotating case, although an interesting new effect appears when the direction of propagation is oblique to the depth contours. Non-linear effects and viscosity are ignored but the theory makes no special assumptions about the density structure or the form of the bottom topography. The Boussinesq approximation is not made, nor is the motion assumed hydrostatic.

If the Boussinesq approximation is made it is possible to obtain "exact" solutions for waves in an infinite wedge if (1) the system is not rotating, or (2) the waves are normally incident upon the "beach". It is shown that the asymptotic theory agrees with these exact solutions near the apex of the wedge provided the angle of the wedge is small.

It is also possible to apply the theory to obtain the eigenfrequencies of trapped waves in various geometries much as Shen, Meyer and Keller (1968) have used Keller's (1958) asymptotic theory for waves in a non-rotating, unstratified ocean for this purpose.

Basic Equations

As is customary, we take the z-axis vertically up and the x-, y-axes horizontal. The rotation vector is (0, 0, ½f) and the equation of the bottom is $z = -h(x,y)$. (It is possible, with some increase in algebraic complexity, to treat the case of oblique rotation.) The basic state is one of no motion with a density field $\rho_0(z)$ and a free surface $z=0$. We consider small time-periodic disturbances for which the velocity is $(u,v,w)e^{i\omega t}$. The density and pressure changes are $\rho e^{i\omega t}$ and $p e^{i\omega t}$ respectively and the equation of the free surface is $z = \zeta(x,y)e^{i\omega t}$.

Neglecting diffusion, non-linearities and viscosity the basic equations are

$$\rho_0(i\omega u - fv) = -p_x , \tag{1}$$

$$\rho_0(i\omega v + fu) = -p_y , \tag{2}$$

$$\rho_0 i\omega w = -p_z - \rho g , \tag{3}$$

$$u_x + v_y + w_z = 0 \tag{4}$$

$$i\omega\rho + w\rho_{0z} = 0 \tag{5}$$

From these we find

$$u = \frac{1}{\rho_0(\omega^2 - f^2)}(i\omega p_x + f p_y) , \tag{6}$$

$$v = \frac{1}{\rho_0(\omega^2 - f^2)} \left(i\omega p_y - f p_x \right) , \tag{7}$$

$$w = i\omega p_z \left(\omega^2 \rho_0 + g \rho_{0z} \right)^{-1} . \tag{8}$$

The linearized free surface conditions are

$$i\omega \zeta(x,y) = w(x,y,0) ,$$

$$p(x,y,0) = g\rho_0(0)\zeta(x,y) ,$$

which imply

$$\zeta(x,y) = \frac{p(x,y,0)}{g\rho_0(0)} , \tag{9}$$

$$\rho = -p_z \rho_{0z} \left(\omega^2 \rho_0 + g\rho_{0z} \right)^{-1} \tag{10}$$

The bottom boundary condition is

$$w = u h_x + v h_y \text{ at } z = -h(x,y) . \tag{11}$$

Eliminating all other variables in favour of the pressure, we find

$$\nabla^2 p + \rho_0(\omega^2 - f^2) \left[p_z \left(g\rho_{0z} + \omega^2 \rho_0 \right)^{-1} \right]_z = 0 , \tag{12}$$

$$p_z = \left(\frac{\rho_{0z}(0)}{\rho_0(0)} + \frac{\omega^2}{g} \right) p \text{ at } z = 0 , \tag{13}$$

$$(\omega^2 - f^2) p_z = -(\omega^2 + g\rho_{0z}/\rho_0)(\underline{\nabla} h \cdot \underline{\nabla} p - i\gamma \underline{k} \cdot \underline{\nabla} h \times \nabla p) \text{ at } z = -h(x,y) , \tag{14}$$

where $\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right)$, $\nabla^2 = \underline{\nabla} \cdot \underline{\nabla}$, $\underline{k} = (0,0,1)$ and $\gamma = f/\omega$.

Usually, the problem is formulated in terms of the vertical velocity w. If the bottom is flat, the two approaches are equally convenient. However with variable topography, the bottom boundary condition (11) involves u, v and w and it is more straightforward to work with the pressure and use (14) than to use w and perform various integrals to obtain u and v. We shall see an example of this when we treat the problem of a wedge. Should any of the boundaries of our flow domain be vertical walls, we require

$$i\omega p_n + f p_\tau = 0 \tag{15}$$

on such a wall, where (n,τ) are the normal and tangential directions at the wall, measured in the same right-handed sense as (x,y).

Equations (12), (13), (14) (or their counterparts for w) have been extensively studied for regions of constant depth. (See Eckart, 1960; Phillips, 1966; Krauss, 1966.) We define

$$N^2(z) = -\frac{g\rho_{0z}}{\rho_0} \tag{16}$$

$$\delta = \frac{\omega^2 - N^2}{\omega^2 - f^2} = -c^{-2} \, . \tag{17}$$

If $\omega^2 > N^2$ and $\omega^2 > f^2$ then there exists a freely propagating surface mode. If $\delta > 0$ at all depths, this is the only freely propagating wave, (so that there are not any waves for $\omega < f$ and $\omega < N$). If $\delta < 0$ for some range of z, there are also freely propagating internal wave modes. The analysis for these may be simplified by assuming the upper surface to be rigid. Equation (13) is then replaced by

$$p_z = 0 \quad \text{on } z = 0 \tag{18}$$

We note that $\delta < 0$ requires $f^2 < \omega^2 < N^2$, or $N^2 < \omega^2 < f^2$. Only the first of these is oceanographically relevant. We also note that $\delta < 0$ implies (12) to be hyperbolic in space.

<u>Solutions in a Wedge</u>.

Recently, Wunsch (1968), (1969) has considered internal waves in an infinite wedge with a rigid top and bottom (Fig.1). Assuming an exponential density distribution, making the Boussinesq approximation, and neglecting rotation, he obtained exact solutions for normal incidence and approximate solutions for oblique incidence (the approximation being based on the slope of the wedge being small). Here, we reexamine his problem

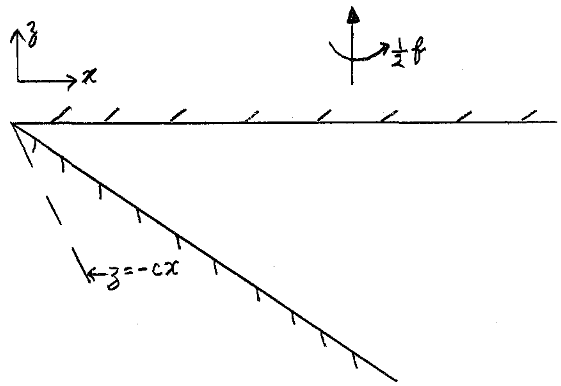

Fig.1

and find exact solutions for all angles of incidence if f = 0, and for normal incidence if f ≠ 0. A factor e^{ily} is assumed throughout.

If the density is $\rho_0(z) = \rho_{00}\exp(-\frac{N^2 z}{g})$, the quantity δ in (17) is a constant. Putting $\delta = -c^{-2}$ with $c>0$ and making the Boussinesq approximation we reduce (12) to

$$\nabla^2 p - c^2 p_{zz} = 0 \tag{19}$$

With $p = \hat{p}(x,z)e^{ily}$ we see that the characteristics of (19) are the lines $z = \pm cx$. We assume the bottom to be given by $z = -\varepsilon x$ with $0 < \varepsilon < c$. The only possible singularities are then at the origin and at infinity.

Following Wunsch, we introduce the "pseudo-polar" coordinates $r = (x^2 - \frac{z^2}{c^2})^{1/2}$ and $\phi = \arctan\frac{iz}{cx}$. Equation (19) then has the solutions $\mathcal{C}_\nu(lr)e^{i\nu\phi}$ where $\mathcal{C}_\nu$ is a modified Bessel function I_ν, $I_{-\nu}$ or K_ν. If $l=0$ these are replaced by $r^{\pm\nu}e^{i\nu\phi}$.

Because of (18) we take the solution

$$p = \mathcal{C}_\nu(lr)\cos(\nu\phi) = \mathcal{C}_\nu\left[l\left(x^2 - \frac{z^2}{c^2}\right)^{1/2}\right]\cos\left[\frac{1}{2}i\nu\ln\left(\frac{cx-z}{cx+z}\right)\right]e^{ily}. \tag{20}$$

Then we find

$$e^{-ily}p_z = \frac{-lz}{c^2 r}\mathcal{C}_\nu'(lr)\cos\nu\phi + \frac{i\nu cx}{c^2r^2}\mathcal{C}_\nu(lr)\sin\nu\phi$$

$$e^{-ily}p_x = \frac{lx}{r}\mathcal{C}_\nu'(lr)\cos\nu\phi - \frac{i\nu cz}{c^2r^2}\mathcal{C}_\nu(lr)\sin\nu\phi$$

Clearly, $p_z = 0$ at $z = 0$. At $z = -\varepsilon x$, (14) becomes

$$-c^2p_z + \varepsilon p_x - i\gamma\varepsilon p_y = 0 \quad at \quad z = -\varepsilon x$$

This requires:

$$\left\{-\frac{i\nu}{x} c \sin\left[\frac{1}{2} i\nu \ln\left(\frac{c+\varepsilon}{c-\varepsilon}\right)\right] + \ell \gamma \varepsilon \cos\left[\frac{1}{2} i\nu \ln\left(\frac{c+\varepsilon}{c-\varepsilon}\right)\right]\right\} = 0 \tag{21}$$

Hence if $\gamma\ell \neq 0$ we cannot obtain an exact solution in this way. If $\gamma\ell = 0$ (i.e. no rotation, or normal incidence) the solution is

$$\frac{1}{2} i\nu \ln\left(\frac{c+\varepsilon}{c-\varepsilon}\right) = m\pi, \tag{22}$$

where m is an integer. If ε is small, we might expect (22) to provide an approximate solution when $\gamma\ell \neq 0$.

Wunsch (1969) gave the solution $w = e^{i\ell y} \mathcal{C}_\nu(\ell r) \sin \nu\phi$ but was unable to apply the boundary condition (14) exactly. He gave (22) as an approximate solution for $\varepsilon \ll 1$. We have shown (20) and (22) to provide an exact solution for the case he considered, namely f = 0, provided ε is less than the critical slope c.

Except for m = 0, the index ν given by (22) is pure imaginary. The solutions $I_{\pm\nu}$ are unbounded at infinity whilst the K_ν are exponentially small there. Now $I_\nu(\psi) \sim (\frac{1}{2}\psi)^\nu / \Gamma(\nu+1)$ as $\psi \to 0$ if ν is not a negative integer. Hence, for ν pure imaginary both I_ν and $I_{-\nu}$ are bounded at the origin. Since $K_\nu(\psi) = \frac{1}{2}\pi \dfrac{I_{-\nu}(\psi) - I_\nu(\psi)}{\sin \nu\pi}$, K_ν is also bounded at the origin (in direct contrast to the case $\mathrm{Re}(\nu) \neq 0$). The solution given by (20) and (22) with $\mathcal{C}_\nu = K_\nu$ is therefore bounded everywhere and exponentially damped at infinity. It represents an "edge wave". However it has infinitely many oscillations in any neighbourhood of the origin. The corresponding solution for $\ell = 0$ is bounded but still oscillatory at infinity.

It should be remarked that the assumed exponential density distribution is quite unrealistic at large depths (as is the Boussinesq approximation). Nevertheless, the solutions might be expected to have some validity near the apex of the wedge.* In particular the K_ν solution decays exponentially at infinity and so the unrealistic nature of the model might not be of importance for these modes.

The Asymptotic Theory.

The fact that exact solutions are available (even when the Boussinesq approximation is made) only for very special bottom profiles forces us to seek other methods. Here we will develop an asymptotic method valid when the wavelength is small compared with all other horizontal length scales. The analysis parallels that of Keller and Mow (1969) for the non-rotating case.

If ℓ_0 is a typical horizontal length scale we non-dimensionalize as follows:

$$x^* = x/\ell_0,\ y^* = y/\ell_0,\ z^* = \omega^2 z/g,\ h^* = \frac{\omega^2 h}{g},\ \beta = \frac{\omega^2 \ell_0}{g} \tag{23}$$

The basic equations (12), (13), (14) become:

$$\beta^2\left[p'' - \frac{(\delta\rho_0)'}{\delta\rho_0}p'\right] + \delta\nabla^2 p = 0 \tag{24}$$

$$p' - \delta(1-\gamma^2)p = 0 \text{ on } z = 0 \tag{25}$$

$$\beta^2 p' + \delta\left[\underline{\nabla} h\cdot\underline{\nabla} p - i\gamma\underline{k}\cdot\underline{\nabla} h\times\underline{\nabla} p\right] = 0 \text{ on } z = -h(x,y), \tag{26}$$

where asterisks have been dropped, a dash denotes $\frac{\partial}{\partial z}$ and ∂, γ are as defined previously. Without any loss of generality at all we may write the pressure field as:

$$p(x,y,z,\beta) = A(x,y,z,\beta)\,\phi(z;x,y)\,e^{i\beta S(x,y)}, \tag{27}$$

where ϕ is a solution of the eigenvalue problem:

$$\left(\phi'' - \frac{(\partial\rho_0)'}{(\partial\rho_0)}\phi'\right) - n^2(x,y)\delta\phi = 0, \tag{28}$$

$$\phi' - (1-\gamma^2)\delta\phi = 0 \text{ on } z = 0, \tag{29}$$

*i.e. where $\frac{N^2 h}{g} \ll 1$.

$$\phi' = 0 \text{ on } z = h(x,y). \tag{30}$$

At each point (x,y) we have an eigenvalue $n^2(x,y)$ and an eigenfunction $\phi(z;x,y)$ corresponding to wave propagation in water of uniform depth $h(x,y)$. Should we wish to make the Boussinesq approximation, the term in ϕ' is dropped from (28), and should we wish to have a rigid rather than a free upper surface the term in ϕ is dropped from (29). Equation (29) serves only to fix the dispersion relation and is not needed again. Hence the theory equally well applies to fluids with a rigid top. Note that ϕ and n^3 depend on (x,y) only because h does.

The crucial step in the analysis is that we pose, for large β, the asymptotic expansion:

$$A(x,y,z,\beta) = A_0(x,y) + \sum_1^{\infty} (i\beta)^{-m} A_m(x,y,z). \tag{31}$$

To lowest order, all the explicit variation with depth is contained in the eigenfunction ϕ. We assume that δ and γ are $O(1)$ quantities with respect to β.

Substituting (31) into (24), (25), (26) and using (28), (29), (30) we find by equating the coefficient of each power of β to zero:

$$\underline{\nabla} s \cdot \underline{\nabla} s = n^2(x,y), \tag{32}$$

$$\phi A_m'' - \left(\phi \frac{(\delta \rho_0)'}{\delta \rho_0} - 2\phi'\right) A_m' = \delta\left[2\underline{\nabla} s \cdot \underline{\nabla}(\phi A_{m-1}) + \phi A_{m-1} \nabla^2 s + \nabla^2(\phi A_{m-2})\right], \tag{33}$$

$$A_m' = 0 \text{ on } z = 0, \tag{34}$$

$$\phi A_m' = \delta A_{m-1} \phi\left[\underline{\nabla} s \cdot \underline{\nabla} h - i\gamma \underline{k} \cdot \underline{\nabla} h \times \underline{\nabla} s\right] + \delta\left[\underline{\nabla} h \cdot \underline{\nabla}(A_{m-2}\phi) - i\gamma \underline{k} \cdot \underline{\nabla} h \times \underline{\nabla}(A_{m-2}\phi)\right]. \tag{35}$$

We have defined $A_m \equiv 0$ for $m < 0$. Equations (33), (34), (35) are identically satisfied for $m = 0$ since A_0 is independent of z.

Equation (32) is the ei onal equation of geometrical optics and can be solved by means of rays. These are the orthogonal trajectories of the wavefronts $\mathcal{S}$ = constant and are the solutions of the ray equation

$$\frac{d}{d\sigma}\left(n\,\frac{d\underline{r}}{d\sigma}\right)=\underline{\nabla}\,n \tag{36}$$

where σ denotes arclength measured along the ray in the direction of propagation and $\underline{r}(\sigma)=(x(\sigma),y(\sigma))$ is a general point on the ray. Along a ray

$$\mathcal{S}\left[x(\sigma),y(\sigma)\right]=\mathcal{S}\left[x(\sigma_0),y(\sigma_0)\right]+\int_{\sigma_0}^{\sigma} n\left[x(\sigma'),y(\sigma')\right]\,d\sigma'. \tag{37}$$

We now follow the steps of Keller and Mow (1969) to solve the transport equations for the A_m. For simplicity we will only show the dependence of these quantities on z. Multiplying (33) by $\frac{\phi}{\delta\rho_0}$ and integrating we find, using (34):

$$\frac{\phi^2 A_m'(z)}{\delta\rho_0}=-\int_z^0\frac{\phi}{\rho_0}\left[2\underline{\nabla}\mathcal{S}\cdot\underline{\nabla}(\phi A_{m-1})+\phi A_{m-1}\nabla^2\mathcal{S}+\nabla^2(\phi A_{m-2})\right]dz\,. \tag{38}$$

We multiply this by $\frac{\delta\rho_0}{\phi^2}$ and integrate again to obtain

$$A_m(z)=A_m(-h)-\int_{-h}^{z}\frac{\delta\rho_0}{\phi^2}\int_{z'}^{0}\frac{\phi}{\rho_0}\left\{2\underline{\nabla}\mathcal{S}\cdot\nabla(\phi A_{m-1})+\phi A_{m-1}\nabla^2\mathcal{S}+\nabla^2(\phi A_{m-2})\right\}dz''\,dz' \tag{39}$$

From (39) and (35) we find

$$\frac{\phi^2 A_m(-h)}{\rho_0(-h)}\left(\underline{\nabla}\mathcal{S}\cdot\underline{\nabla}h-i\gamma\underline{k}\cdot\underline{\nabla}h\times\underline{\nabla}\mathcal{S}\right)+\frac{\phi(-h)}{\rho_0(-h)}\Big[\underline{\nabla}h\cdot\underline{\nabla}\{\phi(-h)A_{m-1}(-h)\}-$$

$$-i\gamma\underline{k}\cdot\underline{\nabla}h\times\underline{\nabla}\{\phi(-h)A_{m-1}(-h)\}\Big]=-\int_{-h}^{0}\frac{\phi}{\rho_0}\left[2\underline{\nabla}\mathcal{S}\cdot\underline{\nabla}(\phi A_m)+\phi A_m\nabla^2\mathcal{S}+\nabla^2(\phi A_{m-1})\right]dz \tag{40}$$

Substituting this into (39) we obtain

$$\frac{\phi^2(-h)}{\rho_0(-h)}A_m(-h)\left(\underline{\nabla}\mathcal{S}\cdot\underline{\nabla}h-i\gamma\underline{k}\cdot\underline{\nabla}h\times\underline{\nabla}\mathcal{S}\right)+\int_{-h}^{0}\frac{\phi}{\rho_0}\left\{2\underline{\nabla}\mathcal{S}\cdot\underline{\nabla}(\phi A_m(-h))+\phi A_m(-h)\nabla^2\mathcal{S}\right\}dz=B_{m-1}\,, \tag{41}$$

where $B_{m-1}(x,y)=-\frac{\phi(-h)}{\rho_0(-h)}\left\{\underline{\nabla}h\cdot\underline{\nabla}(\phi(-h)A_{m-1}(-h))-i\gamma\underline{k}\cdot\underline{\nabla}h\times\underline{\nabla}(\phi(-h)A_{m-1}(-h))\right\}-$

$$-\int_{-h}^{0}\frac{\phi}{\rho_0}\left\{2\underline{\nabla}\mathcal{S}\cdot\underline{\nabla}(\phi I_{m-1})+\phi I_{m-1}\nabla^2\mathcal{S}+\nabla^2(\phi A_{m-1})\right\}dz\,, \tag{42}$$

and $I_{m-1} = A_m(z) - A_m(-h)$ is given by (39).

Defining

$$\psi = \int_{-h}^{0} \frac{\phi^2}{\rho_0} dz, \tag{43}$$

we can reduce (42) to

$$2\psi \underline{\nabla} s \cdot \underline{\nabla} A_m(-h) + A_m(-h) \underline{\nabla} s \cdot \underline{\nabla} \psi + A_m(-h)\left[\psi \nabla^2 s - \frac{i\gamma\phi^2(-h)}{\rho_0(-h)} \underline{k} \cdot \underline{\nabla} h \times \underline{\nabla} s\right] = \psi^{-\frac{1}{2}} B_{m-1} . \tag{44}$$

Now, along a ray $\underline{\nabla} s \cdot \underline{\nabla} = n \frac{d}{d\sigma}$ so (44) becomes

$$2n \frac{d}{d\sigma}\left[A_m(-h)\psi^{1/2}\right] + \left[A_m(-h)\psi^{1/2}\right]\left[\nabla^2 s - \frac{i\phi^2(-h)\gamma}{\psi \rho_0(-h)} \underline{k} \cdot \underline{\nabla} h \times \underline{\nabla} s\right] = \psi^{-1/2} B_{m-1} \tag{45}$$

At a point $x(\sigma)$, $y(\sigma)$ on a ray we therefore have

$$A_m(-h,\sigma) = A_m(-h,\sigma_0) \frac{\psi^{1/2}(\sigma_0)}{\psi^{1/2}(\sigma)} G(\sigma,\sigma_0) + \frac{1}{2}\psi^{-1/2}(\sigma) \int_{\sigma_0}^{\sigma} G(\sigma,\sigma') \frac{\psi^{-1/2}(\sigma')}{n(\sigma')} B_{m-1}\, d\sigma' , \tag{46}$$

where σ_0 is an arbitrary point on the ray and

$$G(\sigma,\sigma') = \exp\left\{-\frac{1}{2}\int_{\sigma'}^{\sigma} \frac{\nabla^2 s - \frac{i\gamma\phi^2(-h)}{\psi \rho_0(-h)} \underline{k} \cdot \underline{\nabla} h \times \underline{\nabla} s}{n} d\sigma'\right\} \tag{47}$$

It can be shown (Luneberg, 1964) that

$$\exp\left\{-\frac{1}{2}\int_{\sigma_0}^{\sigma} \frac{\nabla^2 s}{n} d\sigma'\right\} = \left\{\frac{n(\sigma_0)}{n(\sigma)} \frac{da(\sigma_0)}{da(\sigma)}\right\}^{\frac{1}{2}} \tag{48}$$

where $\frac{da(\sigma)}{da(\sigma_0)}$ is the limit as the strip width $da(\sigma_0)$ tends to zero of the width of a narrow strip of rays at σ to the width of the same strip at σ_0 . Hence the solutions diverge as we approach a caustic because then $\frac{da(\sigma)}{da(\sigma_0)} \longrightarrow 0$.

The solution is now complete since Eqs.(39), (42), and (45) form a system for the successive determination of the A_m starting with $A_{-2} = A_{-1} = 0$. The most important term is A_0 which is given by

$$A_0(\sigma) = A_0(\sigma_0)\left[\frac{\psi(\sigma_0)\, n(\sigma_0)\, da(\sigma_0)}{\psi(\sigma)\, n(\sigma)\, da(\sigma)}\right]^{\frac{1}{2}} \exp\left\{\frac{i}{2}\int_{\sigma_0}^{\sigma} \frac{\gamma\phi^2(-h)\underline{k} \cdot \underline{\nabla} h \times \underline{\nabla} s}{n\psi\rho_0(-h)} d\sigma'\right\} \tag{49}$$

The exponential factor in (48) is a new effect due to the rotation. It arises from the fact that rotation introduces a new term into the bottom boundary condition (11). (We recall that this new term caused all the trouble in the wedge solutions discussed earlier.) The factor represents a slow phase change along the ray (slow because it is not multiplied by β) and disappears if the ray direction is parallel to the bottom slope as will happen, for example, in the case of waves normally incident upon a beach whose depth is a function only of distance from the shore.

Behaviour at Walls, Coastlines and Caustics.

At a vertical wall, we have seen that the boundary condition is (15). When a "wave" $p = A\phi e^{i\beta s_1}$ meets a vertical wall $\mathcal{C}$ there will (in general) be a "reflected wave" $p = B\phi e^{i\beta s_2}$. With the expansions

$$A \sim A_0(x,y) + \sum_1^\infty (i\beta)^{-m} A_m(x,y,z),$$
$$B \sim B_0(x,y) + \sum_1^\infty (i\beta)^{-m} B_m(x,y,z)$$

we find from (15), equating the term in β to zero

$$(i\omega s_{1_n} + f s_{1\tau})A_0 e^{i\beta s_1} = -(i\omega s_{2_n} + f s_{2\tau}) B_0 e^{i\beta s_2} \text{ on } \mathcal{C},$$

where n and τ are the normal and tangential directions introduced earlier. Thus we see that $s_1 = s_2$ on $\mathcal{C}$ and so $s_{1\tau} = s_{2\tau}$ on $\mathcal{C}$. Since $\underline{\nabla} s_1 \cdot \underline{\nabla} s_1 = \underline{\nabla} s_2 \cdot \underline{\nabla} s_2 = n^2$, this implies $s_{1_n} = -s_{2_n}$ on $\mathcal{C}$. (The + possibility implies $s_1 = s_2$ everywhere; unless of course $i\omega s_{1_n} + f s_{1\tau} = 0$ on $\mathcal{C}$, in which case no reflected wave is needed. Such a solution is the well-known Kelvin wave.) The amplitude of the reflected wave is therefore given by

$$B_0 = + A_0 \left[\frac{i\omega s_{1_n} + f s_{1\tau}}{i\omega s_{1_n} - f s_{1\tau}}\right]_{i\mathcal{C}}$$

At a caustic or a shoreline a ray will also be reflected in such a way that the phase $\mathcal{S}$ is continuous and the amplitude is changed from A_o to $A_o e^{-i\pi/2}$ (this is indicated by (49)).

An Example.

It is clear that the success or otherwise of the asymptotic method depends on our being able to solve the eikonal equation (32) for the phase function ϕ. For an arbitrary topography h(x,y) it is clearly not possible to obtain an analytic solution. We must use numerical methods for obtaining the rays and hence tracing the progress of the waves.

For purposes of illustration, it is convenient to consider a one-dimensional topography h(x). If the density is assumed to be exponential and the upper surface rigid it is possible to obtain very simple expressions for the quantities of interest.

In the non-rotating case, the theory has been worked out by Keller and Mow (1969). The solution for f $\neq$ 0 is similar and is quoted here. If (in dimensionless variables) the density is $\rho_o(z) = \rho_\infty e^{-\frac{N^2}{\omega^2} z}$ and h = h(x) then $c^2 = \frac{f^2-\omega^2}{\omega^2 - N^2}$. We find:

$$\phi = \frac{1}{\lambda_1} e^{\lambda_1(z+h)} - \frac{1}{\lambda_2} e^{\lambda_2(z+h)},$$

$$n = c\left[\left(\frac{m^2\pi^2}{h^2}\right) + \frac{N^4}{4\omega^4}\right]^{1/2},$$

where $\lambda_{1,2} = -\frac{N^2}{2\omega^2} \pm \frac{im\pi}{h}$. Furthermore,

$$\psi = \frac{-2h^3}{(m\pi)^2 \rho_\infty} e^{-\frac{N^2 h}{\omega^2}} \left[1 + \left(\frac{N^2 h}{2m\pi\omega^2}\right)^2\right]^{-1}$$

If $\mathcal{S}_y = \mu$ = constant,

$$\mathcal{S}_x^2 = c^2\left[\frac{m^2\pi^2}{h^2} + \frac{N^4}{4\omega^2} - \frac{\mu^2}{c^2}\right].$$

Now, if $\mathcal{S}_x^2 > 0$ the waves are progressive, otherwise they are exponentially damped. The different regions are separated by caustics at which $\mathcal{S}_x = 0$.

We will illustrate the theory by considering a wedge. Now, a wedge has no natural length scale. What then are we to use for our "typical length" ℓ_0 ? Our asymptotic theory requires some large parameter. We therefore consider wedges of small slope ε and define ℓ_0 implicitly by saying $\varepsilon\beta = O(1)$. Thus, if the dimensional wedge is defined by $z = 0$, $h = \varepsilon x$, the non-dimensional one will be $z = 0$, $h = \theta x$ where $0 = 0(1)$. We then find

$$\mathcal{S} = \pm\, kc\left[\left(x^2 + \frac{m^2\pi^2}{\theta k^2}\right)^{1/2} - \frac{m\pi}{\theta k}\ln\left\{\frac{\frac{m\pi}{\theta k} + \left(\frac{m^2\pi^2}{\theta^2 k^2} + x^2\right)^{1/2}}{x}\right\}\right] + \mu y$$

$$\text{where } k^2 = \frac{N^4}{4\omega^4} - \mu^2 \Big| \tau^2 .$$

For $x << 1$, $\mathcal{S} \simeq \pm \frac{m\pi c}{\theta}\ln x$. For $x >> 1$, $\mathcal{S} \simeq \pm kcx$. We therefore have progressive waves at infinity if $k^2 > 0$.

Along a ray, $\frac{dy}{dx} = -\mathcal{S}_y/\mathcal{S}_x$ and so $d\sigma = n(n^2-\mu^2)^{-1/2}dx$. Further $\nabla^2\mathcal{S} = \mathcal{S}_{xx} = \pm\frac{n n_x}{(n^2-\mu^2)^{1/2}}$.

Thus $\int_{\sigma_0}^{\sigma}\frac{\nabla^2\mathcal{S}}{n}d\sigma = \frac{1}{2}\ln\left[\frac{n^2(\sigma)-\mu^2}{n^2(\sigma_0)-\mu^2}\right]$ and so $\exp\left\{-\frac{1}{2}\int_{\sigma_0}^{\sigma}\frac{\nabla^2\mathcal{S}}{n}d\sigma\right\} = \left[\frac{n^2(\sigma_0)-\mu^2}{n^2(\sigma)-\mu^2}\right]^{1/4}$.

As a check, we observe that $\mu = 0 \Rightarrow \exp\left\{-\frac{1}{2}\int_{\sigma_0}^{\sigma}\frac{\nabla^2\mathcal{S}}{n}d\sigma\right\} = \left(\frac{n(\sigma_0)}{n(\sigma)}\right)^{1/2}$ which agrees with (48) since $\frac{da(\sigma)}{da(\sigma_0)} = 1$ for normal incidence.

To obtain the pressure, we have also to find $\exp\left\{\frac{i\theta f\mu}{2\omega}\int_{\sigma_0}^{\sigma}\frac{\phi^2(-h)}{n\psi\rho_0(-h)}d\sigma'\right\} = e^{i\phi}$. For small x this is just $\exp\left\{\frac{i\theta f\mu x}{m\pi\omega c^2}\right\}$.

Had we made the Boussinesq approximation, the above calculations would have been altered only by the fact that all terms in $\frac{N^2}{\omega^2}$ would have been missing. (In particular, k^2 would have been negative, implying that all solutions are either exponentially increasing or exponentially decreasing at infinity - in agreement with the exact solutions obtained earlier.)

For $x << 1$, the pressure becomes

$$p \sim \text{constant} \cos \frac{m\pi z}{\theta x} x^{\pm i\beta \frac{m\pi c}{\theta}} e^{i\beta\mu y} e^{i\phi}$$

irrespective of whether the Boussinesq approximation is made or not. Reverting to dimensional variables and putting $\theta = \beta\varepsilon$, $\ell = \beta\mu$ we find

$$p \sim \text{constant} \cos \frac{m\pi z}{\varepsilon x} x^{\pm \frac{im\pi c}{\varepsilon}} e^{i\ell y} e^{i\phi}$$

We wish to compare this with the exact solution obtained earlier for a Boussinesq fluid. Of course, we require the wedge to be shallow. Exact solutions were obtained for (1) f = 0 or (2) normal incidence. In both of these cases the term in $e^{i\phi}$ does not appear. (1) The exact solution is given by (20) and (22). Let us put $z/x = -\alpha$ where $0 \leq \alpha \leq \varepsilon << c$.

Then $\nu = \dfrac{2im\pi}{\ln\left(\frac{c+\varepsilon}{c-\varepsilon}\right)} \sim \dfrac{im\pi\varepsilon}{c}$ and $\ln\left(\dfrac{c+\alpha}{c-\alpha}\right) \sim \dfrac{\alpha}{c}$

Now, as $r \to 0$, $I_{\pm\nu}(r) \sim \dfrac{\left(\frac{1}{2}r\right)^{\pm\nu}}{\Gamma(1\pm\nu)}$ for all ν. Hence as $x \longrightarrow 0$, the solution (20) in dimensional variables behaves like

$$p \sim \text{constant}\ x^{\pm \frac{im\pi c}{\varepsilon}} \cos \frac{m\pi\alpha}{\varepsilon} e^{i\ell y}$$

in agreement with the exact solution.

(2) The exact solution is $\left(x^2 - \frac{z^2}{c^2}\right)^{\pm\nu/2} \cos\left(\frac{1}{2} i\nu \ln\left(\frac{cx-z}{cx+z}\right)\right)$ which again agrees if $\varepsilon << c$. Furthermore, if the water is homogeneous $N^2 = 0$. Then $k^2 = 0$ and $S = \pm \frac{m\pi c}{\varepsilon} \ln x$ for all x. The two solutions therefore agree, not only for small x, but for all x (provided of course $\varepsilon << c$). This is the case of inertia waves in a shallow wedge.

We have therefore seen that the asymptotic theory gives correct answers in a case where it is applicable, namely a shallow wedge. Furthermore, a basic deficiency of the Boussinesq approximation applied to this geometry has been observed - it predicts that all solutions are either exponentially big or small at infinity. The non-Boussinesq solutions permit a range of y wavenumbers for which this is not so. The trouble is, of course, due to the fact that if the depth goes to infinity, then so does the density, and so what are we to choose for our "mean density" in the Boussinesq approximation?

The situation may be improved by assuming the depth to be a linear function of x for $x < L_0$ and constant for $x > L_0$ and matching up the solutions at $x = L_0$. Here we make an alternative assumption, namely that the depth is given by

$$h = \frac{\varepsilon x}{1 + \varepsilon x / H_0} \quad \text{where} \quad \varepsilon << 1 .$$

For $x << 1$, $h \sim \varepsilon x$; for $x >> 1$, $h \sim H_0$. The solutions reduce to plane waves (or are exponentially damped) at infinity; for $x << 1$, they are identical with the wedge solutions. If $\frac{N^2 H_0}{\omega^2} << 1$ it is immaterial whether we make the Boussinesq approximation or not.

Wave Trapping.

Shen, Meyer and Keller (1968) have recently used the ingenious asymptotic method of Keller and Rubinow (1960) to determine the eigenfrequencies of waves trapped by bottom topography. Keller and Mow (1969) have indicated that its application to a stratified fluid is quite straightforward and routine. Likewise, it can be applied to a rotating stratified fluid. The crux of the argument is that the amplitude of a ray reflected from a caustic or shoreline is $e^{-i\pi/2}$ times the amplitude of the incident ray. We see from (49) that this is still true in the rotating case. Hence the results of Shen, Meyer and Keller (1968) can be applied to the rotating, stratified case as well.

Appendix A. A note on the Boussinesq approximation.

We consider a non-rotating cube of side π units filled with stratified fluid. With $\rho_0(z) = \rho_\infty \exp\left\{-\frac{N^2 z}{g}\right\}$ Eq.(12) becomes:

$$p_{zz} + \frac{N^2}{g} p_z + \left(1 - \frac{N^2}{\omega^2}\right) \nabla^2 p = 0, \tag{A1}$$

with the normal derivative of p vanishing on all faces of the cube. Since we are trying to solve a hyperbolic equation with elliptic boundary conditions, we expect trouble. Nevertheless, the problem has some physical meaning. The solution is:

$$p = A\left(\frac{1}{\lambda_1} e^{\lambda_1(z+\pi)} - \frac{1}{\lambda_2} e^{\lambda_2(z+\pi)}\right) \cos mx \cos ny$$

where $\lambda_{1,2}$ are the roots of $\lambda^2 + \frac{N^2}{g}\lambda - \left(1 - \frac{N^2}{\omega^2}\right)(m^2+n^2) = 0$. The boundary conditions imply

$$\left(1 - \frac{N^2}{\omega^2}\right)(m^2+n^2) = s^2 - \frac{N^4}{4g^2} \tag{A2}$$

where m, n, s are integers and $s \neq 0$.

Consider first the Boussinesq problem. The term $\frac{N^2}{g} p_z$ is dropped from (A1) and the term $\frac{N^4}{4g^2}$ disappears from (A2). If $\left(1 - \frac{N^2}{\omega^2}\right)$ is irrational, there is no solution. If it is rational, there is a solution. If $\left(1 - \frac{N^2}{\omega^2}\right)^{1/2}$ is rational there is also a solution. The possible frequencies are therefore dense in (-N,N). Furthermore, each possible mode is infinitely degenerate since if (m, n, s) is a solution then so is (qm, qn, qs) for any integer q. (There are also degeneracies due to the geometry e.g. if $m_1^2 + n_1^2 = m_2^2 + n_2^2$ then $\omega(m_1, n_1, s) = \omega(m_2, n_2, s)$, c.f. the vibrating membrane problem.)

Viscosity can of course be invoked to make the frequencies complex and damp out the higher modes but we shall show that part of the trouble is due to the Boussinesq assumption. We therefore return to the full

Eq.(A2). As before, the frequencies are dense in (-N,N) but the above-mentioned degeneracy has been removed. There may, however, still be degeneracies other than the "geometrical" ones referred to earlier if $\frac{N^4}{4g^2}$ is rational. It becomes a problem in number theory!

Appendix B. A Wave Trapping Problem.

Longuet-Higgins (1967; 1969) has shown that hydrostatic surface waves may be perfectly trapped around an island (submerged or not) only if the frequency of the waves ω is less than the inertial frequency f. Here "trapped" is used in the sense that there is a finite amount of energy associated with the wave motion.

Stern (private communication) has suggested that stratification might change this. In particular, it might be possible for there to be trapped waves around a submerged circular island in a two-layer ocean with $\omega > f$. If so, the island would be able to extract energy from the tides continually without it ever leaking away (in an inviscid model). Stern felt that such motions were most unlikely to exist, but that the point needed checking.

In order to investigate this problem we use the simplest possible model - viz. a hydrostatic two-layer model with a rigid top and a circular island. The situation is depicted in Fig.2. The displacement of the interface from its equilibrium position is denoted by $\zeta e^{i\omega t}$ and the undisturbed depths are H_1, H_2. All symbols have their previous meanings and a subscript $_1$ refers to the upper layer and a subscript $_2$ to the lower one. The equations of motion and continuity for the two layers are:

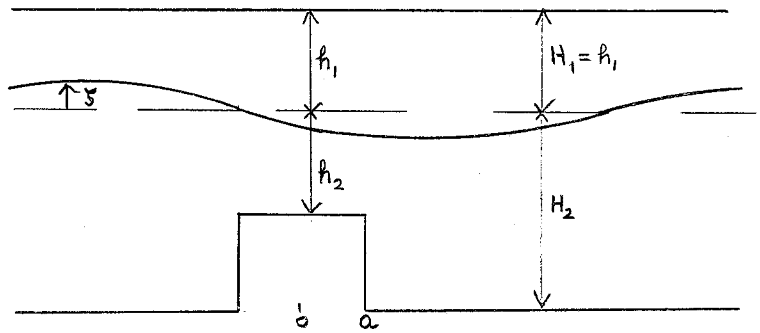

Fig.2

$$i\omega u_1 - f v_1 = -\frac{1}{\rho_1} p_x ,$$
$$i\omega v_1 + f u_1 = -\frac{1}{\rho_1} p_y ,$$
$$H_1 (u_{1x} + v_{1y}) = i\omega\zeta ,$$
$$i\omega u_2 - f v_2 = -\frac{1}{\rho_2} p_x - \frac{g\Delta\rho}{\rho_2}\zeta_x ,$$
$$i\omega v_2 + f u_2 = -\frac{1}{\rho_2} p_x - \frac{g\Delta\rho}{\rho_2}\zeta_y ,$$
$$H_2 (u_{2x} + v_{2y}) = -i\omega\zeta ,$$

where $\Delta\rho = \rho_2 - \rho_1 .$

From these, we find

$$\zeta = \frac{H_1}{\rho_1(\omega^2 - f^2)}\nabla^2 p = -\frac{H_2}{\rho_2(\omega^2 - f^2)}\left[\nabla^2 p + g\Delta\rho\nabla^2\zeta\right],$$
$$\nabla^2\zeta + \left(\frac{\omega^2 - f^2}{g\Delta\rho}\right)\left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2}\right)\zeta = 0 ,$$
$$\text{i.e.}\quad \nabla^2\left[\nabla^2 + \left(\frac{\omega^2 - f^2}{g\Delta\rho}\right)\left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2}\right)\right] p = 0.$$

For $r < a$, H_1, H_2 are to be replaced by h_1, h_2 respectively.

The motion therefore separates exactly into barotropic and baroclinic components. This is because the upper surface is rigid. Had we considered a free upper surface, separation would only have been possible to order $\frac{\Delta\rho}{\rho}$. Let us put $\left(\frac{\omega^2 - f^2}{g\Delta\rho}\right)\left(\frac{\rho_1}{h_1} + \frac{\rho_2}{h_2}\right) = K^2$. If $\omega > f$, $K > 0$. If there is to be finite energy outside $r = a$, we require the baroclinic component to be absent (if $\omega > f$). We therefore have $r > a$: $\nabla^2 p = 0$,

i.e. $p = A r^{-|m|} e^{im\theta}$ and hence $\zeta = 0$. $r < a: \nabla^2(\nabla^2 + K^2)p = 0$
i.e. $p = B r^{|m|} e^{im\theta} + C J_m(Kr) e^{im\theta}$.

At $r = a$ the pressure and the normal component of the mass transport must be continuous in each layer. Since we are using a hydrostatic model this implies

$$A a^{-|m|} = B a^{|m|} + C J_m(Ka)$$
$$-A a^{-|m|-1} = B|m| a^{|m|-1} + C K J_m'(Ka)$$
$$C J_m(Ka) = 0$$

Now $C = 0 \Rightarrow A = B = 0$. Hence $J_m(Ka) = 0$. The frequency is fixed by this relation

$$\omega^2 = f^2 + \frac{g \Delta\rho\, j_m^2}{a^2\left(\frac{\rho_1}{h_1} + \frac{\rho_2}{h_2}\right)}$$

where j_m is a zero of J_m. This relation is independent of H_2. It remains to see if there are any values of H_2 for which the final condition (continuous normal mass transport in the lower layer) is satisfied, i.e.

$$H_2\left(i\omega p_r + \frac{f}{a} p_\theta\right) = h_2\left(i\omega p_r + \frac{f}{a} p_\theta\right) + h_2 g \Delta\rho \left(i\omega \zeta_r + \frac{f}{a}\zeta_\theta\right) \text{ at } r = a$$

Substituting, we find that this implies

$$\frac{2 h_1 h_2}{\rho_1}\left(\frac{\rho_1}{h_1} + \frac{\rho_2}{h_2}\right)\omega = (H_2 - h_2)(-\omega + f \operatorname{sgn} m)$$

Thus $H_2 < 0$ and so no such motions are possible. Stern's conjecture has therefore been verified. Of course, "almost trapped" motions with $|\omega^2| > f^2$ may still exist with only a very slow leak of energy to infinity (c.f. Longuet-Higgins (1967)).

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ELECTROMAGNETIC TORQUES ON INTERSTELLAR CLOUDS

Pedro Mészáros

I. Introduction

The prevalent view on how stars are formed is that they are the result of the contraction into several stars of an originally extended, diffuse cloud of interstellar gas. The initial parameters of a cloud near the point of gravitational instability are reasonably well-known, and so are the final parameters of the resulting young stars. What happens in between, during the period of roughly 10^6 to 10^7 years that the contraction (or collapse) takes, is not exactly known. However one thing has been evident for some time now, and that is that angular momentum has to be lost in the process, in some way or other, if one is to satisfy observations on the final and initial stages.

Of the several mechanisms that have been proposed for getting rid of angular momentum, the two most promising are a) gravitational torques acting between the several protostars into which a contracting cloud sooner or later breaks up, and b) electromagnetic torques, either acting on the whole cloud before break-up occurs, or after the break-up acting on the individual protostars. The first mechanism has never been studied, mainly because very little is known about the break-up process, and similarly electromagnetic torques on the already broken-up photostars has been considered intractable. However, some estimates have been made on the transfer of angular momentum between the whole cloud and the exterior medium (Ebert et al., 1960; Mestel, 1965; Spitzer, 1968). These estimates have been done for highly idealized models, since an exact solution of the hydromagnetic equations is extremely complicated, but results indicate that this mechanism might dispose of the necessary amount of angular momentum in a few times 10^8 years. This would mean that the formation of stars would take at least $\sim 5 \times 10^8$ or so, which seems a rather large time.

In our present work, we intend to improve on these estimates, by taking more realistic models and by making use of some already-known solutions to similar problems (Bullard, 1949; Bullard and Gellman, 1954; Parker 1966), that had not been up to now utilized in the astrophysical context of star formation. It will be shown below that these new, improved estimates strongly suggest that magnetic spin-down can occur in a time of 10^6 years or less.

II. Delineation of the Problem.

We shall work on the hypothesis that magnetic spin-down occurs while the cloud has not yet broken up, and that the cloud is near the point of collapse, but has not begun it yet, or else that it is proceeding very slowly, so that changes in radius can be neglected.

A fairly realistic model of the situation consists of a spherical, highly conducting mass of gas, imbedded in a more tenuous, but also more conducting, gaseous medium, both of them pervaded by an initially uniform magnetic field, anchored at infinity. Reasonable initial conditions are hard to guess, but it seems that we can safely assume that initially the cloud rotates as a rigid body with angular velocity ω_0 , in the presence of a homogeneous, uniform field H_0 , and we shall assume that before this initial instant $t = 0$, no appreciable loss of angular momentum has been experienced by the cloud. Present ideas on how clouds themselves are formed, and how they reach the point of gravitational instability (Field, 1963; Mestel, 1965; Spitzer, 1968; Field, 1969) seem to justify this. So this is the model whose behaviour we ultimately would like to know, for which however a complete solution of the hydromagnetic equations would be required. We shall instead, proceed in several steps of increasing complexity, at each step utilizing a sub-model that emphasizes one or more characteristics of the problem, and finally treat in an approximate way the model itself. The parameters that enter the problem are:

1) Cloud's initial mass, radius and density:

$$M = 10^3 M_\odot \;, \quad R_o = 10^{19} cm \quad , \quad \rho_o = 0.48 \times 10^{-21} \frac{g}{cm^3}$$

2) Initial angular velocity (taken comparable to the differential rotation rate of the galaxy)

$$\omega_o = 10^{-15} sec^{-1}$$

3) Conductivity of the cloud $\sigma_1 = 10^{-11}$ (e.m.u)

of the exterior medium $\sigma_2 = 10^{-8}$ (e.m.u)

4) Magnetic field (initially homogeneous)

$$H_o = 10^{-6} \text{ gauss}$$

(Observations suggest that in dense clouds it might be higher, so we shall also explore the possibility that $H_o' = 5 \times 10^{-6}$ gauss.)

Additional assumptions are:

a) Leakage of field lines will be neglected (frozen-in fields) both inside and outside.

b) The cloud rotates as a rigid body. This simplifies things, but is difficult to justify, so we shall also explore what happens if we relax this restraint.

III. <u>Conducting Sphere Rotating in an Infinite Conducting Rigid Medium</u>.

In this sub-model, we take account of the fact that both the cloud and the exterior intercloud medium are conducting, i.e., currents can flow in both. We do not however take into account the fact that the exterior is gaseous, that is, we neglect the fact that angular momentum can be transmitted, through the field lines, to the material outside, since this is taken as infinite and rigid. Rotational energy is dissipated in the form of currents, which practically only flow within a thin magnetic skin layer at the edge of the sphere. This problem has been treated by Bullard (1949) and Bullard and Gellman (1954), in an exhaustive study aimed at geophysical problems.

Two minor inconveniences, which however are not serious, is that Bullard takes both interior and exterior conductivities as being equal, and that only steady-state solutions are used for the torques. A greater inconvenience is that the torques are calculated assuming ω = const., and not as a decreasing function of time as we would like to have it. We shall however circumvent that difficulty later in Section V, eliminating by the same token the problem that he considers only steady-state solutions. The basic equations are the usual Maxwell equations, neglecting displacement currents, which is astronomically justifiable. The electromagnetic fields are expanded in a series of poloidal and toroidal fields,

$$\vec{H} = \sum_{n,m=0}^{\infty} \left(\vec{S}_{nm} + \vec{T}_{nm} \right) \tag{1}$$

and then solutions are calculated for any kind of exterior (inducing) field. In particular, steady-state torques are calculated in the presence of a (stationary) exterior poloidal field $F\vec{S}_{nm}$ of any order, (where F is the strength of the field) by means of the expression

$$\tau = \iiint r^3 \sin^2\theta \left(I_\theta H_r - I_r H_\theta \right) d\theta\, d\phi\, dr \tag{2}$$

where integration is over the sphere. The result of interest is that

$$\tau = \tau_1 + \tau_2 + \tau_3 \tag{3}$$

where, if one calls b the radius of the sphere, and in the limit of large magnetic Reynolds number $R_m = 4\pi\sigma b^2\omega$,

$$\left.\begin{aligned}
\tau_1 &= \frac{m^{1/2} n(2n+1)(n+m)!\, b^2 F^2}{4(n+1)(n-1)!\,(2\pi\sigma\omega)^{1/2}} \\
\tau_2 &= \frac{n^2(n+2)(n-m+1)(n+m+1)!\, b^2 F^2}{4(n+1)(2n+3)(n-m)!\, m^{3/2}(2\pi\sigma\omega)^{1/2}} \\
\tau_3 &= \frac{(n-1)n(n+m)!\,(n^2-m^2)\, b^2 F^2}{4(2n-1)(n-m)!\, m^{3/2}(2\pi\sigma\omega)^{1/2}}
\end{aligned}\right\} \quad (4)$$

We are interested mainly in the case where $\vec{H}_0$ is homogeneous at infinity, and $\vec{\omega}$ is parallel or perpendicular to $\vec{H}_0$, i.e., in the $\vec{S}_{11}$ and the $\vec{S}_{10}$ case. For them,

$$\left.\begin{aligned}
\tau_{10} &= \frac{4}{75} R_m H_0^2 b^3 = 0.335\,\sigma\omega H_0^2 b^5 \\
\tau_{11} &= 1.69\, \frac{b^3 H_0^2}{\sqrt{R_m}} = 0.48\, \frac{b^2 H_0^2}{\sqrt{\sigma\omega}}
\end{aligned}\right\} \quad (5)$$

where for m = 0 we have taken twice the value in (4), due to certain terms that appear in the integration but are null when $m \neq 0$. A plot of these torques is

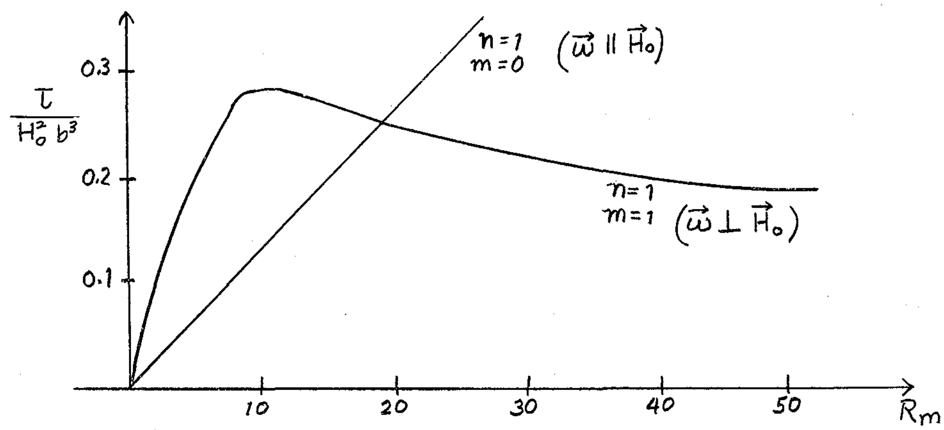

Fig.1

Since for our parameters the Reynolds number is of the order of 10^{14}, it is obvious that the 10 will be very strong, but the 11 very small. Assuming now that the expression for the torques does not change when ω is varied, and taking the moment of inertia of a homogeneous sphere as $\frac{2}{5}MR^2$, we see that the angular velocity must obey the equation

$$\frac{2}{5} M b^2 \frac{d\omega}{dt} = -\tau \tag{6}$$

When $\vec{H}_0 \perp \vec{\omega}$, taking a mean value of σ between σ_1 and σ_2 as $\sim 10^{-9}$, and noting that since we are interested in ω decreasing from $\omega_0 = 10^{-15}$ to say 10^{-17} we can take a mean $R_m \cong 10^{14} \cong$ const., then from (5) and (6)

$$\frac{d\omega}{dt} \cong -3.7 \times 10^{-7} \frac{H_0^2}{M}$$

$$\frac{\omega - \omega_0}{\omega_0} \cong -1 = -3.7 \times 10^{-7} \frac{H_0^2}{M\omega_0} t_{B\perp}$$

giving a perpendicular brake-down time

$$t_{B\perp} \sim 10^{40} \text{ sec} \tag{7}$$

obviously uninteresting. (Remember only steady-state fields were used.) However when $\vec{H}_0 \parallel \vec{\omega}_0$, again from (5) and (6)

$$\frac{1}{\omega} \frac{d\omega}{dt} = -0.84 \frac{\sigma H_0^2 b^3}{M}$$

the solution of which is

$$\omega = \omega_0 \exp\left(-0.84 \frac{\sigma H_0^2 b^3}{M} t\right)$$

In order to reduce ω by two orders of magnitude, the exponent must be roughly -5, so

$$t_{B\parallel} = \frac{5M}{0.84\sigma H_0^2 b^3} = \frac{5 \times 2 \times 10^{36}}{0.84 \times 10^{-9} \times 10^{-12} \times 10^{57}}$$

or

$$t_{B\parallel} = 1.2 \times 10^5 \text{ sec} \tag{8}$$

which is a couple of days. The model seems, indeed, to be much too good to be true, the reason being the rigid exterior that provides a strong foothold for the torques to act on the sphere. The result however is very suggestive, especially so since only steady-state fields were used, whereas transients, which are to be expected when one changes ω in a true scale smaller than the attenuation time, should be stronger by a factor $R_m^{1/2}$ as will be shown later. If the axis were oriented arbitrarily in respect to $\vec{\omega}_0$, since we can always decompose into $\parallel$ and $\perp$ components (Maxwell's equations being linear), we only have to replace in $t_{B\parallel}$ the factor H_0^2 by $H_0^2\cos^2\xi$, ξ being the angle formed by $\vec{H}_0$ with $\vec{\omega}_0$. Even in the extreme case of ξ = $89^o89'$ (i.e. $\vec{\omega}_0$ almost $\perp$ $\vec{H}_0$) we have $\cos^2\xi$ = $9\text{x}10^{-8}$ and $t_{B\parallel}$ = 10^{12} sec = $3\text{x}10^4$ yrs.

IV. Conducting Sphere in an Insulating Infinite Medium

This is our next sub-model, and the characteristic it reproduces to a certain point is the tenuous-gas behaviour of the exterior, in that the field lines outside are distorted out to a great distance, as they are entrained by the sphere into which they are frozen. The difference with the actual case is that these deformed lines will carry with them the exterior material, to which they are frozen, whereas they do not in the sub-model. This discrepancy is not very serious, since the outside gas is very tenuous. Where information is lost, however, is in the boundary conditions, since no currents can flow outside.

For the case $\vec{\omega}_0 \perp \vec{H}_0$ homogeneous at infinity, the initial value problem where the sphere has been accelerated instantly from rest up to ω_0 , remaining thereafter constant, has been treated analytically by Parker (1966), with the purpose of studying the reconnection of field lines, with geophysics in mind. That is, he calculates the magnetic fields for all space coordinates, time and R_m , by solving the Maxwell equations without displacement currents, but allowing this time for slowly decaying transients. In polar coordinates, if one takes $\phi = 0$ along the field at infinity, the result is

$$\left.\begin{aligned} H_r &= \frac{2}{r^2} S e^{i\phi} \sin\theta \\ H_\theta &= \frac{1}{r}\frac{\partial S}{\partial r} e^{i\phi}\cos\theta \\ H_\phi &= \frac{1}{r}\frac{\partial S}{\partial r} i\, e^{i\phi} \end{aligned}\right\} \tag{9}$$

where in terms of Bessel functions, S is

$$S = \frac{3}{2}\frac{H_o(b^3r)^{1/2}}{\sqrt{R_m}} i^{-3/2} \frac{J_{3/2}(i^{3/2}R_m^{1/2} r/b)}{J_{1/2}(i^{3/2}R_m^{1/2})} - \sum_{m=1}^{\infty} 3H_o i (b^3 r)^{1/2} \frac{exp(-i\omega t - n^2\pi^2\omega t/R_m)}{n^2\pi^2(i + n^2\pi^2/R_m)} \frac{J_{3/2}(n\pi r/b)}{J_{1/2}(n\pi)} \tag{10}$$

From here on we have to calculate the torque ourselves, for which we use

$$\tau_K = \iint_S \varepsilon_{ijk} x_i T_{jl} n_l \, dS = \frac{1}{4\pi}\iint_S (-H_r)(x_i H_j - x_j H_i)\, dS = -\frac{b^3}{4\pi}\iint_S H_r H_\phi \sin^2\theta \, d\theta \, d\phi \tag{11}$$

From (9) and (10), taking $r = b$ and using properties of Bessel functions for large arguments, we find after extensive manipulation

$$\left.\begin{aligned} H_r(b) &= 6H_o\left\{\frac{e^{-i\pi/4}}{2\sqrt{R_m}} + e^{-i\omega t}\sum_n \frac{e^{-\frac{n^2\pi^2\omega t}{R_m}}}{n^2\pi^2}\right\} e^{i\phi}\sin\theta \\ H_\theta(b) &= 3H_o\left\{\frac{1}{2} + \frac{e^{-i\pi/4}}{\sqrt{R_m}} - \frac{1}{2}e^{-i\omega t}\sum_n \frac{e^{-\frac{n^2\pi^2\omega t}{R_m}}}{n^2\pi^2}\right\} e^{i\phi}\cos\theta \\ H_\phi(b) &= 3H_o\left\{\frac{1}{2} + \frac{e^{-i\pi/4}}{\sqrt{R_m}} - \frac{1}{2}e^{-i\omega t}\sum_n \frac{e^{-\frac{n^2\pi^2\omega t}{R_m}}}{n^2\pi^2}\right\} i\, e^{i\phi} \end{aligned}\right\} \tag{12}$$

upon taking real parts and making the product, one gets (using just Σ for the sum)

$$H_r H_\phi = 18 H_o^2 \sin\theta \left\{\cos^2\phi \left[\frac{1}{4R_m} + \frac{\sin\omega t}{2\sqrt{2}R_m}\Sigma + \frac{\cos\omega t}{\sqrt{2}R_m}\Sigma + \cos\omega t \sin\omega t (\Sigma)^2\right] - \right.$$
$$\left. - \sin^2\phi\left[\frac{1}{4\sqrt{2R_m}} + \frac{1}{4R_m} + \frac{\cos\omega t}{4\sqrt{2R_m}}\Sigma + \frac{\sin\omega t}{2}\Sigma + \frac{\sin\omega t \cos\omega t}{2}(\Sigma)^2\right]\right\}$$

we can safely neglect the terms in $(\Sigma)^2$, and those in $\Sigma/\sqrt{R_m}$, and we are left with

$$H_r H_\phi = 18 H_o^2 \sin\theta \left\{\frac{\cos^2\theta}{4R_m} - \sin^2\phi\left[\frac{1}{4\sqrt{2R_m}} + \frac{1}{4R_m} + \frac{\sin\omega t}{2}\Sigma\right]\right\} \tag{13}$$

inserting this into (11) and integrating over angles, one finally obtains

$$\tau = \left[1.06\,\frac{b^3 H_0^2}{\sqrt{R_m}} - 1.97\,\frac{b^3 H_0^2}{R_m} + \cdots\right] + 3\,b^3 H_0^2 \sin\omega t \sum_n \frac{e^{-\frac{n^2\pi^2\omega t}{R_m}}}{n^2\pi^2} + \cdots \tag{14}$$

The term in brackets is the steady-state torque, and the rest is the transient torque. It is interesting to note that the first term in the series for the steady-state torque is of the same form, and only about 40% smaller, than what we obtained in (5) for $\vec{\omega} \perp \vec{H}$. That it would be smaller was to be expected, since in (5) we had a conducting, rigid exterior, with field lines practically locked-in. The next thing to notice is that the transient torque can be approximately written as

$$\tau_{tr} \cong \frac{1}{3}\, b^3 H_0^2 \sin\omega t\, e^{-\frac{10\omega t}{R_m}} \tag{15}$$

That is, the time scale of decay of the transients will be very large, of the order of 10^{28} sec. Furthermore the transient torques are larger than the steady-state torques by a factor $R_m^{1/2} \simeq 10^7$, so we can calculate spin down using only the transients. If we insert again (15) into (6) neglecting that ω_0 is not constant, we find that (since the exponent is very small)

$$\frac{d\omega}{dt} = -\frac{5}{6}\,\frac{H_0^2 b}{M}\sin\omega t\left(1 - \frac{10\omega}{R_m}t\right) \cong -\frac{5}{6}\,\frac{H_0^2 b}{M}\sin\omega t$$

Furthermore we are interested in time scales of $t \lesssim 10^{14}$sec., so $\omega t \ll 1$

$$\frac{d\omega}{dt} \cong -\frac{5}{6}\,\frac{H_0^2 b}{M}\left(\omega t - \frac{(\omega t)^3}{6} + \cdots\right) \cong -\frac{5}{6}\,\frac{H_0^2 b}{M}\,\omega t \tag{16}$$

whose solution is

$$\omega = \omega_0 \exp\left(-\frac{5}{12}\,\frac{H_0^2 b}{M}\,t^2\right) \tag{17}$$

For ω to decrease by $\sim 10^2$

$$\frac{\cancel{5}}{12}\,\frac{H_0^2 b}{M}\,t^2 \cong \cancel{5}$$

giving a spin-down time

$$t_{B_{tr}}^{\perp} = \left[\frac{12M}{H_0^2 b}\right]^{1/2} = \left[\frac{1.2\times 2\times 10^{37}}{10^{-12}\times 10^{19}}\right]^{1/2} = \left[2.4\times 10^{30}\right]^{1/2}$$

$$t_{B_{tr}}^{\perp} \cong 1.5\times 10^{15}\ \text{sec} = 5\times 10^{7}\ \text{yrs}. \tag{18}$$

Thus whereas the steady-state torques were useless in the $\vec{\omega} \perp \vec{H}_0$ case, the transient torques seem fairly effective.

V. Discussion of the Two Sub-models.

We have now studied the torques for the case of an exterior medium that is rigid and conducting, and for one that is rigid and nonconducting; in the first case for both $H_0 \parallel \omega$ and $H_0 \perp \omega$, steady states only, and in the second for $H_0 \perp \omega$, both steady states and transients. The difference between conducting and nonconducting exteriors does not seem to be significant, the latter being somewhat smaller. The nonconducting exterior seems to be physically nearer to the actual situation, and the torques found are lower limits, since in the actual case the exterior mass will be entrained with the mass, thus enhancing angular momentum transfer. Remains the problem of the initial condition (jump from 0 to ω_0) that Parker has used, and which is not equal to ours. We have initially ω_0, and at t = 0 it starts decreasing. For every change $\delta\omega$ in the frequency, one would expect a transient of roughly the same order of magnitude, and functional form as in (15), in particular, with a similar decay rate. Thus at time t we will have present all the transients excited between t = 0 and t = t, one could take something like

$$\langle \tau_{tr} \rangle \simeq \frac{1}{3} b^3 H_0^2 \frac{1}{\omega - \omega_0} \int_{\omega_0}^{\omega} \sin \omega' t \, e^{-\frac{10}{R_m}\omega' t} d\omega' \tag{18}$$

where t is a parameter. Making again the approximation of small ωt, this becomes

$$\langle \tau_{tr} \rangle \simeq \frac{1}{3} b^3 H_0^2 \left(\frac{\omega + \omega_0}{2} \right) t \tag{19}$$

There might be some discussion as to what the exact form of (18) is, but it seems fairly reasonable that it is not very far off, and that the torques present due to the deceleration process will be something resembling Eq.(19), at least in order of magnitude. Solving again Eq.(6) we have this time

$$\frac{d\omega}{dt} = -\frac{1}{15}\frac{bH_0^2}{M}(\omega+\omega_0)t = \frac{d(\omega+\omega_0)}{dt} = \frac{d\Omega}{dt}$$

$$\Omega = \Omega_0 \exp\left(-\frac{bH_0^2}{30M}t^2\right)$$

$$\omega = \omega_0\left(2e^{-\frac{bH_0^2t^2}{30M}} - 1\right) \tag{20}$$

and making $\omega = 10^{-2}\omega_0$ we obtain

$$t_{B_{tr}}^{\perp} \cong 2.1\times 10^{15}\,\text{sec} = 7\times 10^{7}\,\text{yrs.} \tag{21}$$

If H_0 had been $5\text{x}10^{-6}$ instead of 10^{-6}, the time would have been 10^7 yrs, and if also b had become somewhat smaller, even better.

The conclusion is that in this model (nonconducting exterior), which seems to be physically very close to the actual case, perpendicular torques can decelerate the cloud in times $\sim 10^7$ yrs, and parallel torques, judging from the example of the conducting exterior model, should be even better.

VI. Rigid Conducting Sphere in a Conducting, Gaseous Exterior.

We finally would like to see what can be gleaned from the full model, without solving outrightly the hydromagnetic equations. It has been pointed out by Ebert, von Hoerner and Temesváry (1960) that if one takes $\vec{\omega} \parallel \vec{H}$, as the sphere rotates the field the field lines will become twisted, and the kink in the field lines will travel outwards in both directions along the field with the Alfven speed V_A, which for us is $\sim 10^6$ cm sec^{-1}. (see Fig.2). The material outside will be accelerated

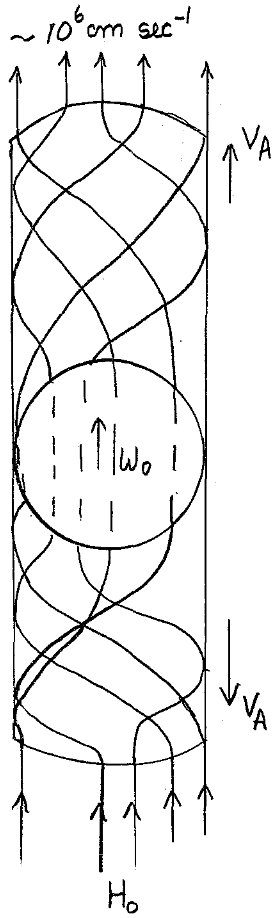

Fig.2

up to ω, and neglecting electromagnetic terms they find that

$$\frac{2}{5} M b^2 \frac{dw}{dt} = -\pi \rho_2 V_A b^4 \omega$$

gives a spin-down time

$$t_B \cong 10^9 \text{ yrs.}$$

However, the neglect of electromagnetic torques is a mistake (obviously the spin-down time obtained this way is too long), and we shall try to include it. The easiest way to do it would be to say that (neglecting now inertial terms, i.e., acceleration of outside material) all the loss of kinetic angular energy of the sphere is due to ohmic losses, which presumably occur only in a thin magnetic shear layer along the walls and the tops of the cylinder, of depth $\sim \frac{1}{\sqrt{\sigma_2 \omega}}$. The torque would then be given by

$$\tau = \frac{1}{\omega\sigma} \iiint I^2 dV \tag{22}$$

The currents might be estimated from

$$I \simeq \left| \frac{\nabla \wedge H}{4\pi} \right| \cong \frac{H_0}{\delta R 4\pi} = \frac{H_0\sqrt{\sigma\omega}}{4\pi} \tag{23}$$

and we will have

$$\frac{2}{5} M b^2 \frac{dw}{dt} = -\frac{1}{\sigma\omega}\left[\frac{H_0^2}{16\pi^2} \frac{2\pi}{\delta R^2} b^2 \delta R + \frac{H_0^2}{16\pi^2 \delta R^2} 4\pi b V_A t\, \delta R\right] = \frac{H_0^2}{8\pi} \frac{b^2}{\sqrt{\sigma\omega}}\left(1 + \frac{2V_A t}{b}\right)$$

Since we are interested in times of say 10^{14} sec or so, we can neglect the 1 and get, roughly

$$\frac{2}{3}\left(w^{3/2}-\omega_o^{3/2}\right) = -\frac{5\,H_o^2}{16\pi\sqrt{\sigma}\,M}\,\frac{V_A t^2}{b}$$

$$\text{or} \qquad t_B \simeq \sqrt{\frac{32\pi}{15}\,\sigma^{1/2}\,\frac{M w_o^{3/2} b}{H_o^2 V_A}} \sim 5\times 10^9\,\text{yrs.} \tag{24}$$

This is even worse than the deceleration obtained from inertial terms, and we are forced to the conclusion that electromagnetic effects are <u>not</u> confined by any means to simple ohmic dissipations. In fact, it has become obvious from the example of the two previous sub-models that

a) rotational energy goes also into stretching the field lines, giving terms like $(\vec{H}\cdot\nabla)\,\vec{H}$, and this should happen throughout the cylinder;

b) transient toroidal current systems must be set up by the deceleration of the sphere, and these presumably are not confined to a thin skin layer.

A treatment of these effects seems to be feasible, and it will be undertaken in the near future.

VII. Conclusion

The estimates presented above represent an improvement in previous ones, in that the models utilized are more realistic, and that torques have been calculated from actual solutions of the field equations in the presence of rotation. The expression for these torques is quite unlike the order of magnetic expressions that had been utilized up to now, the result being that (especially for the parallel torques) we are strongly led to believe that electromagnetic spin-down of rotating clouds <u>is</u> possible in times of $\sim 10^6$ yrs or less. The main qualitative difference with other estimates is that, whereas the prevalent view was that the available magnetic energy density $\frac{H_o^2}{8\pi}$ would be much too small compared to the rotational kinetic energy of the cloud to be able to stop it, we find that the magnetic energy is increased at the expense of the rotational energy, distorting the already existing field and setting up new fields and currents, and this transfer of energy is enough to slow down the clouds.

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THERMODYNAMIC AND STATISTICAL ASPECTS OF NON-ISOTROPIC CONVECTION

Jean Perdang

Abstract

In the first part of this paper the method of irreversible thermodynamics is applied to the problem of non-isotropic convection, and the most

general form of transport coefficients is derived. In the second part a statistical description of turbulent convection is given, using the Boussinesq approximation. This approach allows a specification of the corresponding heat transport coefficient, and reproduces the well-known $R^{1/3}$-law of the Nusselt number.

Introduction

In most of the astrophysical situations where convection is thought to be of importance, one hardly thinks of a complete hydrodynamic description of the convective processes. What is needed is a sufficiently simple mean description, that may be incorporated in a numerical program. This is the reason why, in the past, the only theories of astrophysical interest have been mixing-length formulations.

The aim of the present paper is to formulate a consistent scheme, in which the convective transport phenomena are described from a point of view of irreversible thermodynamics, and taking account of the symmetry properties of the problem involved. The approach adopted is somewhat similar to that of the description of thermal diffusion by J. Meixner (1941); as a full specification of the processes is too involved, we limit ourselves to a phenomenological description, in which, however, we try to take account of a maximum of information available from general principles.

Releasing the constraint of isotropy, we show that coupling effects arise between the different irreversible phenomena, which are excluded for usual gases: thus a concentration gradient leads to a contribution of the Reynolds stress tensor, and, conversely, a velocity gradient is coupled with a diffusion flux. If we assume an idealized isotropic convection state, our formal expressions for the fluxes coincide with those of the usual mixing-length theories.

In the second part of this paper an attempt is made to go beyond this first step towards a macroscopically useful description, adopting a statistical viewpoint to allow for a complete specification of the unknown parameters involved. Although in this paper this problem is treated in the framework of the Boussinesq approximation, it is likely that a generalization to situations of greater astrophysical interest is straightforward.

Part I. Themodynamic Formulation

1. Basic Equations

If we consider a gas G, made up of n species of particles, we picture the state of convection as the presence of an additional gas Q of "quasi-particles". Physically, the latter be conceived for instance as the excitation of collective modes in G; in the naive picture of the mixing-length formalism they would correspond to the turbulent eddies. Both systems G and Q give rise to a heterogeneous mixture M.

The most general hydrodynamic equations governing G may be written as follows:

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\underline{\rho} + \frac{\partial}{\partial \vec{r}}\cdot\left(\underline{\rho}\vec{v} + \underline{\vec{J}}\right) = \underline{\Gamma} && \text{(a)} \\ &\frac{\partial}{\partial t}\left(\rho\vec{v}\right) + \frac{\partial}{\partial \vec{r}}\cdot\left(\rho\vec{v}\vec{v} + \mathcal{P}\right) = \vec{F} && \text{(b)} \\ &\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2}+u\right)\right] + \frac{\partial}{\partial \vec{r}}\cdot\left[\rho\left(\frac{v^2}{2}+u\right)\vec{v} + \mathcal{P}\cdot\vec{v} + \vec{\mathcal{F}}\right] = \vec{F}\cdot\vec{v} + \varepsilon\rho && \text{(c)} \end{aligned}\right\} \quad (1.1)$$

with

$\underline{\rho} = (\rho_1, \rho_2, \cdots, \rho_n)$	set of partial densities
$\underline{\vec{J}} = (\vec{J}_1, \vec{J}_2, \cdots, \vec{J}_n)$	diffusion fluxes
$\underline{\Gamma} = (\Gamma_1, \Gamma_2, \cdots, \Gamma_n)$	partial rate of production of particles of each species

$\mathcal{P} = p E - \mathcal{T}$, the pressure-tensor, $\mathcal{T}$ the stress-tensor

u, internal energy per unit mass

$\vec{\mathcal{F}}$, heat flux

ε_ρ, rate of energy production per unit volume

The other symbols have their usual meaning.

With C. Truesdell (1957,a) we adopt the principle (principle 1) that the equations of the mean motion of a heterogeneous mixture are of the same form as those describing a simple medium (*).

We conclude from this principle that Eqs.(1.1) remain valid for the mixture M, with suitable definitions of the diffusion fluxes, the pressure tensor, as well as a suitable definition of the mean motion. As a provisional definition of the latter we may understand a mean over a time sufficiently long as compared to the time scale of interactions of the quasi-particles. A precise definition will be given in the statistical part of this paper.

We now establish a "caloric equation of state" for M, generalizing the line of reasoning of C. Truesdell (1957,b). We start from a postulate of existence of a caloric equation of state for each constituent of M, in the form

$$u_i = u_i(S_i, \rho, \underset{\sim}{\pi}) \qquad (1.3) \quad i = 1, 2 \dots n, \text{ and } Q$$

u_i, internal energies per unit mass, S_i entropy per unit mass. In this expression $\underset{\sim}{\pi}$ is a set of other parameters whose specific nature is not of interest here. The total internal energy, u, and entropy, S, per unit mass are then defined by

$$u = \sum_{i=1}^{n} u_i c_i + u_Q, \qquad S = \sum_{i=1}^{n} S_i c_i + S_Q \tag{1.4}$$

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* ... "le equazioni per il movimento medio coincidano con le equazioni che reggono il moto de un mezzo semplice."

where c_i represents the concentrations $c_i = \rho_i/\rho$. In usual thermodynamics one then introduces a caloric equation for the mixture M by

$$u = u(S, \underline{\rho}, \underset{\sim}{\Pi})$$

As is well-known, such an equation only exists provided all the components of the heterogeneous mixture have the same temperature, the latter being defined by

$$T_j = (\partial u/\partial S_j)_{\underline{\rho}, \underset{\sim}{\Pi}, S_1 \dots S_{j-1} S_{j+1} \dots S_n S_Q} \tag{1.5}$$

In the problem under consideration we have no reason to suppose that $T_a = T_i$, whereas we may assume $T_i = T_j$, $i,j = 1, \dots n$ (local thermal equilibrium between the components of G). Thus the relevant caloric equation of state for M takes the form

$$u = u_G(S_G, \underline{\rho}, \underset{\sim}{\Pi}) + u_Q(S_Q, \underline{\rho}, \underset{\sim}{\Pi}) \tag{1.6}$$

Solving (1.3) with respect to S_i, the second relation (1.4) similarly allows us to write

$$S = S_G(u_G, \underline{\rho}, \underset{\sim}{\Pi}) + S_Q(u_Q, \underline{\rho}, \underset{\sim}{\Pi}) \equiv S(u_G, u_Q, \underline{\rho}, \underset{\sim}{\Pi}) \tag{1.7}$$

One also notices that the temperatures may be defined equivalently by

$$T_j = (\partial u_j/\partial S)_{\underline{\rho}, \underset{\sim}{\Pi}, u_1 \dots u_{j-1} u_{j+1} \dots u_n u_Q} \tag{1.8}$$

Equation (1.7) directly leads to a generalized Gibbs equation: Assuming (1.7) valid along the line of mean motion (<u>principle 2</u>), the application of d/dt on this equation leads to

$$\frac{dS}{dt} = \left(\frac{1}{T_G}\frac{du_G}{dt} + \frac{1}{T_Q}\frac{du_Q}{dt}\right) - \rho^{1/2}\left(\frac{p_G}{T_G} + \frac{p_Q}{T_Q}\right)\frac{d\rho}{dt} - \left(\frac{\underline{\mu}_G}{T_G} + \frac{\underline{\mu}_Q}{T_Q}\right)\cdot\frac{d\underline{c}}{dt} - \left(\frac{\underset{\sim}{\beta}_G}{T_G} + \frac{\underset{\sim}{\beta}_Q}{T_Q}\right)\cdot\frac{d\underset{\sim}{\Pi}}{dt} \tag{1.9}$$

with the usual definitions of pressure and chemical potential

$$-p_i/\rho^2 T_i = (\partial S_i/\partial \rho)_{u, \underline{c}, \underset{\sim}{\Pi}} \quad ; \quad -\underline{\mu}_i/T_i = (\partial S_i/\partial \underline{c})_{u, \rho, \underset{\sim}{\Pi}} \tag{1.10}$$

and with the definition

$$-\underset{\sim}{\beta}_i/T_i = (\partial S_i/\partial \underset{\sim}{\Pi})_{u, \rho, \underline{c}}$$

Multiplying (1.9) by ρ , and taking account of (1.1) (interpreted as mean motion equations), we find

$$\frac{\partial}{\partial t}\rho S + \frac{\partial}{\partial \vec{r}}\cdot\left[\rho\vec{v}S + \vec{J}_S\right] = \sigma \tag{1.11}$$

where $\vec{J}_S$ and σ are the entropy flux and the entropy production respectively, given by

$$\begin{aligned}\vec{J}_S &= \left(\frac{\vec{\mathcal{F}}_G}{T_G}+\frac{\vec{\mathcal{F}}_Q}{T_Q}\right) - \left(\frac{\mu_G}{T_G}+\frac{\mu_Q}{T_Q}\right)\cdot\vec{J}\\ \sigma &= -\frac{\vec{\mathcal{F}}_G}{T_G^2}\cdot\frac{\partial}{\partial\vec{r}}T_G - \frac{\vec{\mathcal{F}}_Q}{T_Q^2}\cdot\frac{\partial}{\partial\vec{r}}T_Q - \underline{\vec{J}}\cdot\frac{\partial}{\partial\vec{r}}\left(\frac{\mu_G}{T_G}+\frac{\mu_Q}{T_Q}\right) + \left(\frac{\tau_Q}{T_Q}+\frac{\tau_G}{T_G}\right):\frac{\partial}{\partial\vec{r}}\vec{v} +\\ &\quad + \frac{(\varepsilon\rho)_G + E_{QG}}{T_G} + \frac{(\varepsilon\rho)_Q - E_{QG}}{T_G} - \rho\left(\frac{\underset{\sim}{\beta}_G}{T_G}+\frac{\underset{\sim}{\beta}_Q}{T_Q}\right)\cdot\frac{d\underset{\sim}{\pi}}{dt} - \left(\frac{\mu_G}{T_G}+\frac{\mu_Q}{T_Q}\right)\cdot\underline{r}\end{aligned} \tag{1.12}$$

where E_{QG} is the rate of energy conversion (per unit volume) of internal energy associated with the quasi-particles, into internal energy of the material gas.

Equations (1.12) coincide with the usual results of irreversible thermodynamics in the particular case of absence of quasi-particles at a different temperature, T_Q, from that of the gas, T_G.

To introduce the entropy flux $\vec{J}_S$ and the entropy production σ , we have separated the heat equation (obtained from (1.1c) by subtracting the kinetic energy in the usual fashion) into two parts

$$\left.\begin{aligned}\rho\frac{d}{dt}u_G - \frac{p_G}{\rho^2}\frac{d}{dt}\rho &= \tau_G : \frac{\partial}{\partial\vec{r}}\vec{v} - \frac{\partial}{\partial\vec{r}}\cdot\vec{\mathcal{F}}_G + (\varepsilon\rho)_G + E_{QG}\ (a)\\ \rho\frac{d}{dt}u_Q - \frac{p_Q}{\rho^2}\frac{d}{dt}\rho &= \tau_Q : \frac{\partial}{\partial\vec{r}}\vec{v} - \frac{\partial}{\partial\vec{r}}\cdot\vec{\mathcal{F}}_Q + (\varepsilon\rho)_Q - E_{QG}\ (b)\end{aligned}\right\} \tag{1.13}$$

such that (a) + (b) corresponds to the usual heat equation.

Up to this point, we have left the variables $\underset{\sim}{\pi}$ unspecified. As an important effect which we did not attempt to describe, we essentially think of the process of creation and destruction of quasi-particles, which may be symbolized by a chemical reaction of the type

$$Q \rightleftharpoons V$$

Q being a quasi-particle and V the "vacuum". If we associate an affinity A_Q and a degree of advancement ξ_Q to this process, the relevant term in the entropy production is written

$$-\rho \frac{A_Q}{T_Q} \frac{d}{dt} \xi_Q \tag{1.14}$$

This term allows for describing the effect of "convective relaxation phenomena", in the same fashion as the usual chemical term $\frac{\mu}{T} \cdot \underline{\Gamma}$ accounts for chemical relaxation processes (Meixner and Reik, 1959).

Let us mention that the diffusion fluxes $\vec{\underline{J}}$ and the reaction rates $\underline{\Gamma}$ as defined in (1.1a) are not independent, but satisfy

$$\sum_{i=1}^{n} \vec{J}_i = 0 \qquad \sum_{i=1}^{n} \Gamma_i = 0$$

where both conditions are expressions of mass conservation. To get independent diffusion fluxes, we eliminate $\vec{J}_n = \sum_{i=1}^{n-1} \vec{J}_i$, in which case we have

$$\vec{\underline{J}} \cdot \frac{\partial}{\partial \vec{r}} \left(\frac{\mu_G}{T_G} + \frac{\mu_Q}{T_Q} \right) = \sum_{i=1}^{n-1} \vec{J}_i \cdot \frac{\partial}{\partial \vec{r}} \left(\frac{\mu_{Gi} - \mu_{Gn}}{T_G} + \frac{\mu_{Qi} - \mu_{Qn}}{T_Q} \right)$$

A similar procedure allows us to get rid of the dependent reaction rates.

As usual, the entropy production now appears as a sum of independent products of "forces" and "fluxes"

$$\sigma = X \cdot J \quad \geqslant 0 \tag{1.15}$$

According to the second principle of thermodynamics (which is assumed to apply to a heterogeneous mixture of two gases at different temperatures (principle 3) σ is non-negative.

The thermodynamic fluxes and forces are defined by the following relations:

$$J = \left(\vec{\mathcal{F}}_G ,\ \vec{\mathcal{F}}_Q ,\ \vec{\underline{J}} ,\ \tau_G ,\ \tau_Q ,\ \underline{\Gamma} ,\ \rho \frac{d\xi}{dt} ,\ V_G ,\ V_Q \right)$$

$$X = \left(\frac{\partial}{\partial \vec{r}} \frac{1}{T_G} ,\ \frac{\partial}{\partial \vec{r}} \frac{1}{T_Q} ,\ \frac{\partial}{\partial \vec{r}} \left(\frac{\mu_G}{T_G} + \frac{\mu_Q}{T_Q} \right) ,\ \frac{1}{T_G} \frac{\partial}{\partial \vec{r}} \vec{v} ,\ \frac{1}{T_Q} \frac{\partial}{\partial \vec{r}} \vec{v} ,\ -\left(\frac{\mu_G}{T_G} + \frac{\mu_Q}{T_Q} \right) ,\ -\frac{A_Q}{T_Q} ,\ -\frac{\eta_G}{T_G} ,\ -\frac{\eta_Q}{T_Q} \right)$$

$$\text{where } \vec{\underline{J}} = \left(\vec{J}_1 , \vec{J}_2 , \cdots \vec{J}_{n-1} \right) ,\ \underline{\mu}_G = \left(\mu_1 - \mu_n , \mu_2 - \mu_n , \cdots \mu_{n-1} - \mu_n \right) \tag{1.16}$$

and similar definitions are used for $\underline{\Gamma}$ in order to have independent forces and fluxes. The energy generation rates are written in the following way

$$\frac{(\varepsilon\rho)_G + E_{QG}}{T_G} = -\frac{\eta_G}{T_G} V_G \quad ; \quad \frac{(\varepsilon\rho)_Q - E_{QG}}{T_Q} = -\frac{\eta_Q}{T_Q} V_Q \tag{1.17}$$

where V_G and V_Q are rates, and η_G and η_Q correspond to energies per unit mass.

Remarks.

Although up to this point our treatment is fairly general, it bears on one implicit assumption: Rigorously the laws of conservation of mass, momentum and energy (1.1) should be supplemented by an equation of conservation of angular momentum (see for instance, de Groot, 1961); if one splits the latter into an external and an internal part (a spin associated with the particles), the rate of variation of the internal angular momentum is proportional to the antisymmetric part of the stress tensor. This implies that a conservation of internal angular momentum is equivalent to a symmetric stress-tensor, $\mathcal{T}_G = \tilde{\mathcal{T}}_G$, a situation which is meaningful for usual gases. As far as the gas of quasi-particles is concerned, this situation may cease to be significant, as one easily may imagine a transfer leading to variations of angular momentum, from the mean flow to the convective motions. To describe this mechanism one separates $\mathcal{P}$ into its symmetric part $\frac{1}{2}(\mathcal{P}+\tilde{\mathcal{P}})$ and its antisymmetric part $\frac{1}{2}(\mathcal{P}-\tilde{\mathcal{P}})$, and writes the conservation law of total angular momentum. This leads to an additional term in the entropy production (see de Groot, 1961). In the formulation given here, we exclude this angular momentum mechanism (assumption 1), which implies, for the mean flow, that the $\mathcal{T}_Q$ stress-tensor, just as the $\mathcal{T}_G$ tensor is symmetric (*). A generalization to include this effect would be straightforward.

As a second point we note that the present formulation agrees with the usual technique of introducing certain mean quantities by writing

$$Q = \langle Q \rangle + q \tag{1.18}$$

Q being the velocity, the density, ... , and substituting into Eqs.(1.1) (for details in the case of absence of diffusion and explicit description of reactions, see Ledoux and Walraven, 1958). This leads to formal expressions of $\vec{\mathcal{F}}_Q$, τ_Q, p_Q, u_Q, as well as E_{QB}. The physical interpretation is however slightly different in both approaches: whereas in our description the convective irregular motions are considered as "microscopic effects" in the same sense as the individual motions of molecules, we only make use of one thermal energy equation (sum of (1.13a and b)) whereas in the usual technique one has to deal with an equation of convective energy and an equation of thermal energy of the particle gas.

2. Formal Constitutive Equations in the Frame of Irreversible Thermodynamics

To obtain expressions for the fluxes, we start from the following principles:

a) Brillouin's principle of equipresence: A quantity present as an independent variable in one constitutuve equation should be so present in all (principle 4).

b) Curie's symmetry principle: If certain causes give rise to certain effects, the symmetry elements of the causes must occur in the symmetry

(*) The possibility of an antisymmetric part in the turbulent stress-tensor has been stated explicitly by V.Bjerknes, J.Bjerknes, H.Solberg, T.Bergeron (1933) "Ob man bei der Turbulenzreibung diese drehende Spannung vernachlässigen kann, dürfte aber fraglich sein".

elements of the effects (principle 5). (Curie, 1908; a recent discussion is found in Finlayson and Scriven, 1969).

c) Linearity of the constitutive equations (principle 6).

d) Onsager's reciprocity relations (Onsager, 1931; Casimir, 1945) (principle 7).

Principles a) and b) supplemented by the requirement of dimensional invariance provide the most general basis for a non-linear treatment (in the thermodynamic sense) of transport phenomena induced by convection. We will, however, limit ourselves to the linear domain, in which an additional constraint is imposed by Onsager's reciprocity relations. The set of principles (a)-(d) defines the range of "irreversible thermodynamics".

According to principle (a) the most general form of the constitutive equations is provided by

$$\mathcal{J} = \mathcal{J}(\mathcal{X}) \quad \text{with} \quad 0 = \mathcal{J}(0) \tag{2.1}$$

where $\mathcal{J}$ and $\mathcal{X}$ are given by (1.16). Expanding (2.1) in a Taylor series one obtains

$$\mathcal{J} = L \cdot \mathcal{X} + \mathcal{X} \cdot \mathcal{M} \cdot \mathcal{X} + \dots \tag{2.2}$$

The linearity requirement (c), which is significant provided the second-order term in (2.2) is negligible (sufficiently small forces)(*) allows us to limit (2.2) to

$$\mathcal{J} = L \cdot \mathcal{X} \tag{2.3}$$

Substituting in (1.15) we have

(*)It is a well-known result that this assumption does not apply for the usual nuclear reactions of interest in the energy production in stars (independently of convection). For ionization and dissociation processes it is significant.

$$\sigma = \chi \cdot L \cdot \chi \equiv \chi \cdot \tfrac{1}{2}\left(L + \tilde{L}\right) \cdot \chi \geqslant 0 \qquad (2.4)$$

The latter condition implies that, provided all the forces are independent, the symmetrized matrix L of the phenomenological coefficients be non-negative.

Excluding the presence of magnetic fields, and the description of the system in a rotating reference frame, the Onsager reciprocity condition implies that

$$L_{ij} = \varepsilon_i L_{ji} \varepsilon_j \qquad \text{(no summation)} \qquad (2.5)$$

where $\varepsilon_i = 1$ if χ_i is invariant under time reflection, and $\varepsilon_i = -1$ if χ_i changes sign under time reflection. The only thermodynamic force in (1.16) displaying the latter property is the tensor $\partial / \partial \vec{\pi}\, \vec{v}$.

Before entering into the details of the constraints imposed by Curie's principle, we would like to emphasize that Luikov(1966) has proposed the following generalization of the phenomenological relation (2.3)

$$J = L \cdot \chi + D \cdot \frac{\partial}{\partial t} J$$

D being a diagonal matrix. This form has been suggested by Vernotte (1958) in the particular case of diffusion, to take account of the possibility of relaxation. However, this extra term is not needed to get the relaxation effects, provided the latter are described in the entropy source σ through convenient internal variables (terms $\beta / T \cdot d\tilde{\pi} / dt$ of the form (1.14), which play a part similar to chemical reaction terms. The usual technique of the treatment of these effects by irreversible thermodynamics (Meixner and Reik, 1959) allows us to avoid an extra term in (2.3). Applying the latter method to turbulent isotropic diffusion, one finds (Perdang, 1969) the Goldstein diffusion equation (Goldstein, 1951). It should be mentioned that Luikov's paper contains the first attempt at description of turbulent diffusion by methods of irreversible thermodynamics.

To apply Curie's symmetry law, we have to specify the symmetry group of the configuration we have to deal with. As far as the gas of material particles is concerned the relevant symmetry group is the full orthogonal group $\mathcal{K}_{\infty h}$. A discussion of the relevant geometry for the gas of quasi-particles needs some remarks on experimental and theoretical results on convection. Consider first the absence of a macroscopic velocity field. In that case gravity imposes one preferred direction in the problem of convection. Without entering into all the details which have been published in a large number of papers we may say, roughly, that the onset of convection is characterized by regular hexagons(*). As the Rayleigh number increases, the cellular structure becomes less regular, and goes over to a disordered appearance, as the Rayleigh number becomes sufficiently large ("turbulent convection"). At that stage, the configuration may be considered as essentially isotropic in the planes normal to the axis of gravity, at least if we are interested in mean values over time scales which are large as compared to the time scale of turbulent fluctuations.

To cover some aspects of the steady state, as well as the turbulent state, we adopt a rotational invariance around the direction of gravity of angles π/n, $n \geq 2$. (This covers the case of a square and a hexagonal plan-form.) As far as our problem is concerned, this will be shown to be equivalent with invariance under rotations of any angle θ. In addition we assume invariance under reflections on planes σ_v going through the rotation axis. The latter symmetry elements are suggested by the figures in Stuart (1964) and Chandrasekhar (1961) (see Figs.6, 7b and 8, pp. 50 and 51). Invariance under reflections on horizontal planes σ_h is not assumed, as the velocity field is not invariant under σ_h. The relevant

---

(*) For exceptions and more precise statements we refer to recent papers by Davis (1967), Koschmieder (1967), Palm (1960), Palm and Øiann (1964), Krishnamurti (1968), Schluter, Lortz and Busse (1965).

symmetry groups are thus $\mathcal{C}_{4v}$, $\mathcal{C}_{6v}$ and $\mathcal{C}_{\infty v}$.

We would like to point out that these symmetry groups under consideration are essentially different from that considered by G.K.Batchelor (1946) and S. Chandrasekhar (1950) in the study of turbulence, who assume, in addition, a σ_h plane ($\mathcal{D}_{\infty h}$ group).

> This approximation seems to be significant in the case of mechanical turbulence. One notices that our problem differs from Batchelor's and Chandrasekhar's in the sense that the latter are interested in the discussion of correlation tensors, whereas the aim of this part of our study is the determination of the general structure of the transport tensors.

In the presence of a macroscopic velocity field $\vec{v}$, we limit ourselves to the case of turbulent convection. The direction of $\vec{v}_1$ at any point P of the medium defines a second preferred direction. Both preferred directions reduce, in general, the relevant symmetry group to a single plane σ_v defined by ($\vec{g}, \vec{v}$) (or Pxz-plane). We are thus left with a symmetry $\mathcal{C}_h$. However if $\vec{v}$ is normal to $\vec{g}$, a second symmetry plane σ_v' appears (Pyz-plane), so that the relevant group becomes $\mathcal{C}_h \times \mathcal{C}_{h'}$. If $\vec{v}$ coincides with the direction $\vec{g}$, the previous $\mathcal{C}_{\infty v}$ symmetry remains unaltered.

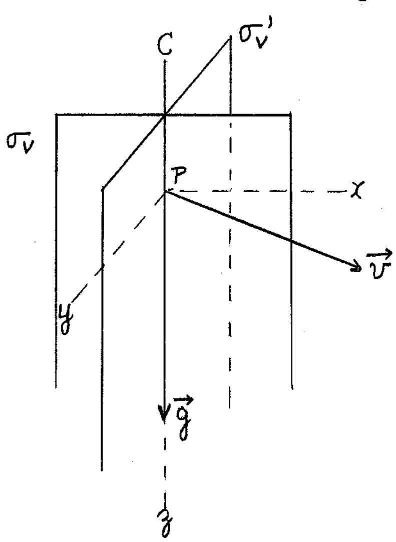

Fig.1 Symmetry elements in the presence of a velocity field.

The following table gives a summing up of the different symmetry configurations involved. $v_{\parallel}$ and $v_{\perp}$ respectively denote the components of the macroscopic velocity field $\vec{v}$ parallel and normal to the direction of gravity.

Table 1. - Symmetry Groups in the Presence of a Velocity Field

$v_\perp, v_\parallel \neq 0$	$v_\perp \neq 0, v_\parallel = 0$	$v_\perp = 0, v_\parallel \neq 0$	$v_\perp = 0, v_\parallel = 0$
$\mathcal{C}_h$	$\mathcal{C}_h \times \mathcal{C}_{h'}$	$\mathcal{C}_{\infty v}$	$\mathcal{C}_{\infty v}$

Though we do not discuss the case of the presence of an additional rotation field we mention that an arbitrary direction of the rotation vector $\vec{\Omega}$ leads to the destruction of all the symmetry elements. If $\vec{\Omega}$ coincides with $\vec{g}$, the relevant symmetry becomes $\mathcal{C}_\infty$. If $\vec{\Omega}$ is normal to $\vec{g}$, the symmetry group reduces to a single plane σ_h.

It seems physically reasonable to adopt the groups listed in the previous table as the characteristic symmetry groups of the gas of quasi-particles (assumption 2).

3. Explicit Requirements Imposed on the Phenomenological Coefficients by Onsager's Reciprocity Relations and Curie's Symmetry Principle

The gas G being isotropic, it is well-known that fluxes with subscripts G in (1.16)(*) can only couple with those forces which have special tensorial characters. The following couplings are allowed:

scalar-scalar, vector-vector, tensor-tensor, tensor-scalar (3.1)

We do not enter into the details of the forms of the couplings (3.1) for the isotropic gas G, as these forms are discussed in any book on irreversible thermodynamics.

As far as the gas of quasi-particles is concerned, we have, in addition to (3.1), the following couplings:

tensor-vector, vector-scalar (3.2)

(*) An exception is V_G which involves properties of the quasi-particles.

We now determine the specific forms of the coefficients of the Onsager matrix L in the case of the symmetries listed in Table 1 and for the different types of coupling (3.1) and (3.2).

To do so, we introduce the following matrices

$$
\left.
\begin{array}{l}
R^{(3)}_{yz} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}
\qquad
R^{(6)}_{yz} = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{array}\right)
\\[2ex]
R^{(3)}_{xz} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}
\qquad
R^{(6)}_{xz} = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \end{array}\right)
\\[2ex]
R^{(3)}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & \cdot \\ -\sin\theta & \cos\theta & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}
\\[2ex]
R^{(6)}(\theta) = \begin{pmatrix} \cos^2\theta & \sin^2\theta & \cdot & \cdot & \cdot & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & \cdot & \cdot & \cdot & -2\sin\theta\cos\theta \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cos\theta & -\sin\theta & \cdot \\ \cdot & \cdot & \cdot & \sin\theta & \cos\theta & \cdot \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cdot & \cdot & \cdot & \cos^2\theta - \sin^2\theta \end{pmatrix}
\end{array}
\right\} \quad (3.3)
$$

where $R^{(3)}_{xy}$ and $R^{(3)}_{zx}$ are 3-dimensional representations of the reflection operator on the O_{xy} and O_{zx} planes (the axes are defined on Fig.1), and $R^{(3)}(\theta)$ is a 3-dimensional representation of the rotation operator around the O_z axis. $R^{(6)}_{xy}$, $R^{(6)}_{zx}$ and $R^{(6)}(\theta)$ are 6-dimensional representations of the previous operators. The latter coorespond to the group elements of Table 1. The 6-dimensional representation is introduced for the following reason: the tensors of interest are symmetric, so they may be written as 6-dimensional vectors, according to the rule

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{pmatrix} \tag{3.4}$$

To fulfil the Curie symmetry, it is clear that if $\mathcal{G} = (G_1, G_2, \ldots)$ is the symmetry group of the quasi-particle gas, any coupling matrix L defining one of the couplings (3.1) and (3.2) must be invariant under the transformations of the group $\mathcal{G}$, or

$$G_i^{-1} L\, G_i = L, \qquad G_i \in \mathcal{G} \tag{3.5}$$

We now discuss the different possible couplings:

a) <u>scalar-scalar</u> ($S' = \Sigma S$)

always invariant under (3.5) for any scalar Σ

b) <u>vector-vector</u> ($\vec{V}' = \nu \vec{V}$)

ν is a 3-dimensional symmetric matrix, thus (3.5) should be fulfilled with G_i being 3-dimensional representations of the group elements of Table 1.

$\mathcal{C}_h$: $\quad R^{(3)}_{zx}\, \nu\, R^{(3)}_{zx} = \nu \qquad \nu = \begin{pmatrix} \nu_{11} & \cdot & \nu_{13} \\ \cdot & \nu_{22} & \cdot \\ \nu_{13} & \cdot & \nu_{33} \end{pmatrix}$

$\mathcal{C}_h \times \mathcal{C}_{h'}$: $\quad R^{(3)}_{xy}\, \nu\, R^{(3)}_{xy} = \nu \qquad \nu = \begin{pmatrix} \nu_{11} & \cdot & \cdot \\ \cdot & \nu_{22} & \cdot \\ \cdot & \cdot & \nu_{33} \end{pmatrix}$

$\mathcal{C}_{\infty v}$(*): $\quad R^{(3)}(-\theta)\, \nu\, R^{(3)}(\theta) = \nu \qquad \nu = \begin{pmatrix} q & \cdot & \cdot \\ \cdot & q & \cdot \\ \cdot & \cdot & s \end{pmatrix}$

(*) Instead of taking θ arbitrary (leading to $\mathcal{C}_{\infty v}$), we could have taken $\pi/2$ or $\pi/3$.

Taking account of the conditions of Table 1, we see that the most general vector-vector coupling is described by the matrix

$$\nu = \begin{pmatrix} q + V_{\parallel} q' + V_{\perp}\sigma_{11} & \cdot & V_{\parallel} V_{\perp}\sigma_{13} \\ \cdot & q + V_{\parallel} q' + V_{\perp}\sigma_{22} & \cdot \\ V_{\parallel} V_{\perp}\sigma_{13} & \cdot & s + V_{\parallel} s' + V_{\perp}\sigma_{33} \end{pmatrix} \tag{3.6}$$

In this expression q and s are independent of the velocity whereas the other parameters may depend on it(**).

c) <u>tensor-tensor</u> $(\tau' = T\,\tau)$

T is a 6x6 symmetric matrix. The G_i are thus 6-dimensional representations of the group elements of Table 1. Studying successively

$$\begin{array}{ll} R^{(6)}_{3x}\, T\, R^{(6)}_{3x} = T & v_{\parallel} \neq 0,\ v_{\perp} \neq 0 \\ R^{(6)}_{xy}\, T\, R^{(6)}_{xy} = T & v_{\perp} \neq 0,\ v_{\parallel} = 0 \\ R^{(6)}(-\theta)\, T\, R^{(6)}(\theta) = T & v_{\perp} = 0 \end{array}$$

one finally obtains

$$T = \begin{pmatrix} a + V_{\parallel} a' + V_{\perp} T^{11}_{11} & f + V_{\parallel} f' + V_{\perp} T^{11}_{22} & e + V_{\parallel} e' + V_{\perp} T^{11}_{33} & \cdot & V_{\parallel} V_{\perp} T^{11}_{31} & \cdot \\ f + V_{\parallel} f' + V_{\perp} T^{11}_{22} & a + V_{\parallel} a' + V_{\perp} T^{22}_{22} & e + V_{\parallel} e' + V_{\perp} T^{22}_{33} & \cdot & V_{\parallel} V_{\perp} T^{22}_{13} & \cdot \\ e + V_{\parallel} e' + V_{\perp} T^{11}_{33} & e + V_{\parallel} e' + V_{\perp} T^{22}_{33} & b + V_{\parallel} b' + V_{\perp} T^{33}_{33} & \cdot & V_{\parallel} V_{\perp} T^{33}_{13} & \cdot \\ \cdot & \cdot & \cdot & c + V_{\parallel} c' + V_{\perp} T^{23}_{23} & \cdot & V_{\parallel} V_{\perp} T^{23}_{12} \\ V_{\parallel} V_{\perp} T^{11}_{31} & V_{\parallel} V_{\perp} T^{22}_{13} & V_{\parallel} V_{\perp} T^{33}_{13} & \cdot & c + V_{\parallel} c' + V_{\perp} T^{13}_{13} & \cdot \\ \cdot & \cdot & \cdot & V_{\parallel} V_{\perp} T^{23}_{12} & \cdot & (a-f) + V_{\parallel}(a'-f') + V_{\perp} T^{12}_{12} \end{pmatrix} \tag{3.7}$$

a, f, e, b, c are independent of the velocity.

d) <u>scalar-vector</u> $(s' = \vec{w} \cdot \vec{V})$

The most general form of the coupling is given by

$$\vec{w} = (0,\ V_{\parallel} V_{\perp} w,\ W + V_{\parallel} w' + V_{\perp} w'') \tag{3.8}$$

W is independent of the velocity.

(**) $V_{\parallel}$ and $V_{\perp}$ are the moduli of the macroscopic velocity components in a dimensionless form (for example divided by the square root of the mean square velocity of fluctuations).

e) <u>scalar-tensor</u> ($S' = \Theta\,\tau$)

The coupling tensor Θ is represented by a 6-vector. Thus the symmetry requirements (3.5) become

$$\begin{array}{ll} R^{(6)}_{3x}\Theta = \Theta & v_\parallel \neq 0,\ v_\perp \neq 0 \\ R^{(6)}_{xy}\Theta = \Theta & v_\parallel = 0,\ v_\perp \neq 0 \\ R^{(6)}(\theta)\Theta = \Theta & v_\perp = 0 \end{array}$$

It follows that Θ must have the form

$$\Theta = (\varphi + V_\parallel \varphi' + V_\perp \Theta_{11},\ \varphi + V_\parallel \varphi' + V_\perp \Theta_{22},\ \Psi + V_\parallel \Psi' + V_\perp \Theta_{33},\ V_\parallel V_\perp \Theta_{32},\ 0,\ 0) \qquad (3.9)$$

φ and Ψ independent of the velocity.

f) <u>vector-tensor</u> ($\vec{V}' = K\,\tau$)

The coupling matrix K is represented by a 3x6 matrix, whose form is determined in a similar fashion as in the previous cases

$$K = \begin{pmatrix} V_\parallel V_\perp K_{111} & V_\parallel V_\perp K_{122} & V_\parallel V_\perp K_{133} & \cdot & \gamma+\gamma' V_\parallel + V_\perp K_{131} & \cdot \\ \cdot & \cdot & \cdot & \gamma+\gamma' V_\parallel + V_\perp K_{232} & \cdot & V_\parallel V_\perp K_{212} \\ \alpha+\alpha' V_\parallel + V_\perp K_{311} & \alpha+\alpha' V_\parallel + V_\perp K_{322} & \beta+\beta' V_\parallel + V_\perp K_{333} & \cdot & V_\parallel V_\perp K_{331} & \cdot \end{pmatrix} \qquad (3.10)$$

α, β and γ being independent of the velocity.

The following table indicates the number of free parameters involved in the coupling matrices for the different coupling types, and the different symmetries:

Table 2. - Number of independent parameters in the coupling matrices.

type of coupling \ symmetry	$\mathcal{C}_h$	$\mathcal{C}_h \times \mathcal{C}_{h'}$	$\mathcal{C}_{\infty v}$ (or $\mathcal{C}_{4v}$, $\mathcal{C}_{6v}$)
a) scalar-scalar	1	1	1
b) vector-vector	4	3	2
c) tensor-tensor	13	9	5
d) scalar-vector	2	1	1
e) scalar-tensor	4	3	2
f) vector-tensor	10	5	3

Condition (2.5) finally implies the following inequalities:

$$
\left.
\begin{array}{ll}
\text{a)} & \Sigma \geqslant 0 \\
\text{b)} & q, s \geqslant 0 \\
 & \nu_{11}, \nu_{22}, \nu_{33} \geqslant 0 \\
 & \nu_{11}\,\nu_{33} \geqslant \nu_{13}^{2} \\
\text{c)} & \text{For tensor-tensor coupling we only mention the most} \\
 & \text{important cases} \\
 & a, b, c \geqslant 0 \\
 & a \geqslant f \\
 & e \leqslant \sqrt{\frac{b(a+f)}{2}} \quad \text{if } f \neq a \\
 & e \quad \text{arbitrary if } f = a
\end{array}
\right\} \qquad (3.11)
$$

From the previous results we conclude that the explicit form of (2.3) is given by

$$
\begin{bmatrix} \vec{F}_G \\ \vec{F}_Q \\ \underset{\sim}{\vec{J}} \\ \tau_G \\ \tau_Q \\ \underset{\sim}{\Sigma} \\ \rho\dot{\xi} \\ V_G \\ V_Q \end{bmatrix}
=
\left[\begin{array}{ccc|cc|cccc}
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & \nu_{22} & \nu_{23} & 0 & K_1 & \tilde{\vec{W}}_{11} & \tilde{\vec{W}}_{21} & \tilde{\vec{W}}_{31} & \tilde{\vec{W}}_{41} \\
* & \tilde{\nu}_{23} & \nu_{33} & 0 & K_2 & \tilde{\vec{W}}_{12} & \tilde{\vec{W}}_{22} & \tilde{\vec{W}}_{32} & \tilde{\vec{W}}_{42} \\
\hline
0 & 0 & 0 & * & * & * & * & * & * \\
0 & -\tilde{K}_1 & -\tilde{K}_2 & * & T & \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\
\hline
0 & \vec{W}_{11} & \vec{W}_{12} & * & -\tilde{\Theta}_1 & \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\
0 & \vec{W}_{21} & \vec{W}_{22} & * & -\tilde{\Theta}_2 & \Sigma_{12} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\
0 & \vec{W}_{31} & \vec{W}_{32} & * & -\tilde{\Theta}_3 & \Sigma_{13} & \Sigma_{23} & \Sigma_{33} & \Sigma_{34} \\
0 & \vec{W}_{41} & \vec{W}_{42} & * & -\tilde{\Theta}_4 & \Sigma_{14} & \Sigma_{24} & \Sigma_{34} & \Sigma_{44}
\end{array}\right]
\left[\begin{array}{c}
\partial/\partial\vec{r}\,(1/T_G) \\
\partial/\partial\vec{r}\,(1/T_Q) \\
\partial/\partial\vec{r}\left(\frac{\mu_G}{T_G} + \frac{\mu_Q}{T_Q}\right) \\
\hline
\frac{1}{2T_G}\left(\frac{\partial}{\partial\vec{r}}\vec{v} + \widetilde{\frac{\partial}{\partial\vec{r}}\vec{v}}\right) \\
\frac{1}{2T_Q}\left(\frac{\partial}{\partial\vec{r}}\vec{v} + \widetilde{\frac{\partial}{\partial\vec{r}}\vec{v}}\right) \\
\hline
-\left(\frac{\mu_G}{T_G} + \frac{\mu_Q}{T_Q}\right) \\
-A_Q/T_Q \\
-\eta_G/T_G \\
-\eta_Q/T_Q
\end{array}\right]
\qquad (3.12)
$$

In this expression the symbol * means usual isotropic coupling. The other symbols are defined by (3.6), (3.7), (3.8), (3.9), and (3.10). Subscripts are used to distinguish the different coupling matrices of same nature. The stress-tensors τ_Q and τ_G as well as the rate of deformation tensor

$\frac{1}{2}\left(\partial/\partial\vec{\pi}\ \vec{v} + \widetilde{\partial/\partial\vec{\pi}\ \vec{v}}\right)$ are assumed to be written in 6-dimensional vector form according to (3.4).

Equations (3.12) give the most general constitutive relations comparable with the principles and assumptions formulated in Sections 1 and 2.

In most of the usual astrophysical situations one is concerned with turbulent convection processes, which are by far more efficient than the corresponding "molecular" (i.e. radiative) effects. In addition one is mainly interested in the heat transfer, and sometimes, in stability problems, in the turbulent stress tensor. If the star is chemically homogeneous, we find

$$\vec{f}_Q = -\frac{1}{T_Q^2}\begin{pmatrix} (\underline{q} + V_\parallel q' + V_\perp \nu_{11})\frac{\partial}{\partial x}T_Q + V_\parallel V_\perp \nu_{13}\frac{\partial}{\partial z}T_Q \\ (\underline{q} + V_\parallel q' + V_\perp \nu_{22})\frac{\partial}{\partial y}T_Q \\ (\underline{s} + V_\parallel s' + V_\perp \nu_{33})\frac{\partial}{\partial z}T_Q + V_\parallel V_\perp \nu_{13}\frac{\partial}{\partial x}T_Q \end{pmatrix} -$$

$$-\begin{pmatrix} 0 \\ V_\parallel V_\perp w_1 \\ \underline{W_1} + V_\parallel w_1' + V_\perp w'' \end{pmatrix}\frac{A_Q}{T_Q} - \begin{pmatrix} 0 \\ V_\parallel V_\perp w_2 \\ \underline{W_2} + V_\parallel w_2' + V_\perp w_2'' \end{pmatrix}\frac{\eta_G}{T_G} - \begin{pmatrix} 0 \\ V_\parallel V_\perp w_3 \\ \underline{W_3} + V_\parallel w_3' + V_\perp w_3'' \end{pmatrix}\frac{\eta_Q}{T_Q} +$$

$$+\begin{pmatrix} V_\parallel V_\perp\left(K_{111}\frac{\partial v_x}{\partial x} + K_{122}\frac{\partial v_y}{\partial y} + K_{133}\frac{\partial v_z}{\partial z}\right) + (\underline{\gamma} + \gamma' V_\parallel + V_\perp K_{131})\frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ (\underline{\gamma} + V_\parallel \gamma' + V_\perp K_{232})\frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) + V_\parallel V_\perp K_{212}\frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \\ (\underline{\alpha} + \alpha' V_\parallel + V_\perp K_{311})\frac{\partial v_x}{\partial x} + (\underline{\alpha} + \alpha' V_\parallel + V_\perp K_{322})\frac{\partial v_y}{\partial x} + (\underline{\beta} + \beta' V_\parallel + V_\perp K_{333})\frac{\partial v_z}{\partial z} + V_\parallel V_\perp K_{331}\frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \end{pmatrix} \quad (3.13)$$

similarly the stress-tensor is a sum of five components

$$\tau_Q = \tau_{Q\vec{v}} + \tau_{QT_Q} + \tau_{QA_Q} + \tau_{Q\eta_G} + \tau_{Q\eta_Q}$$

The different components are given by

$$
\tau_{\theta\vec{v}} = \frac{1}{T_\theta}
\left[\begin{array}{l|l|l}
\begin{array}{l}
a\frac{\partial v_x}{\partial x}+f\frac{\partial v_y}{\partial y}+e\frac{\partial v_z}{\partial z} \\
+V_\parallel\left(a'\frac{\partial v_x}{\partial x}+f'\frac{\partial v_y}{\partial y}+e'\frac{\partial v_z}{\partial z}\right) \\
+V_\perp\Big[\left(T^{11}_{11}\frac{\partial v_x}{\partial x}+T^{11}_{22}\frac{\partial v_y}{\partial y}+T^{11}_{33}\frac{\partial v_z}{\partial z}\right) \\
+V_\parallel T^{11}_{31}\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)
\end{array}
&
\begin{array}{l}
(a-f)\frac{1}{2}\left(\frac{\partial v_y}{\partial x}\ \ \frac{\partial v_x}{\partial y}\right) \\
+V_\parallel(a'-f')\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right) \\
+V_\perp\Big[T^{12}_{12}\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)+ \\
V_\parallel T^{23}_{12}\frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\Big]
\end{array}
&
\begin{array}{l}
c\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right) \\
+V_\parallel c'\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right) \\
+V_\perp\Big[T^{13}_{13}\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right) \\
+V_\parallel\left(T^{11}_{31}\frac{\partial v_x}{\partial x}+T^{22}_{13}\frac{\partial v_y}{\partial y}+T^{33}_{13}\frac{\partial v_z}{\partial z}\right)\Big]
\end{array}
\\ \hdashline
\begin{array}{l}
(a-f)\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right) \\
+V_\parallel(a'-f')\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right) \\
+V_\perp\Big[T^{12}_{12}\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right) \\
+V_\parallel T^{23}_{12}\frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\Big]
\end{array}
&
\begin{array}{l}
f\frac{\partial v_x}{\partial x}+a\frac{\partial v_y}{\partial y}+e\frac{\partial v_z}{\partial z} \\
+V_\parallel\left(f\frac{\partial v_x}{\partial x}+a'\frac{\partial v_y}{\partial y}+e'\frac{\partial v_z}{\partial z}\right) \\
+V_\perp\Big[\left(T^{11}_{22}\frac{\partial v_x}{\partial x}+T^{22}_{22}\frac{\partial v_y}{\partial y}+T^{22}_{33}\frac{\partial v_z}{\partial z}\right) \\
+V_\parallel T^{22}_{13}\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)\Big]
\end{array}
&
\begin{array}{l}
c\frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right) \\
+V_\parallel c'\frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right) \\
+V_\perp\Big[T^{23}_{23}\frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right) \\
+V_\parallel T^{23}_{12}\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\Big]
\end{array}
\\ \hdashline
\begin{array}{l}
c\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right) \\
+V_\parallel c'\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right) \\
+V_\perp\Big[T^{13}_{13}\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right) \\
+V_\parallel\left(T^{11}_{31}\frac{\partial v_x}{\partial x}+T^{22}_{13}\frac{\partial v_y}{\partial y}+T^{33}_{13}\frac{\partial v_z}{\partial z}\right)\Big]
\end{array}
&
\begin{array}{l}
c\frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right) \\
+V_\parallel c'\frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right) \\
+V_\perp\Big[T^{23}_{23}\frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right) \\
+V_\parallel T^{23}_{12}\frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial v_y}\right)\Big]
\end{array}
&
\begin{array}{l}
e\frac{\partial v_x}{\partial x}+e\frac{\partial v_y}{\partial y}+b\frac{\partial v_z}{\partial z} \\
+V_\parallel\left(e'\frac{\partial v_x}{\partial x}+e'\frac{\partial v_y}{\partial y}+b'\frac{\partial v_z}{\partial z}\right) \\
+V_\perp\Big[T^{11}_{33}\frac{\partial v_x}{\partial x}+T^{22}_{33}\frac{\partial v_y}{\partial y}+T^{33}_{33}\frac{\partial v_z}{\partial z} \\
+V_\parallel T^{33}_{13}\frac{1}{2}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)\Big]
\end{array}
\end{array}\right]
\tag{3.14}
$$

$$\tau_{QT_Q} = \frac{1}{T_Q^2}\left(\begin{array}{c|c|c} V_\parallel V_\perp K_{111}\frac{\partial}{\partial x}T_Q + & V_\parallel V_\perp K_{212}\frac{\partial}{\partial z}T_Q & (\underline{\gamma}+\gamma' V_\parallel + V_\perp K_{131})\frac{\partial}{\partial x}T_Q + \\ +(\underline{\alpha}+\alpha' V_\parallel + V_\perp K_{311})\frac{\partial}{\partial z}T_Q & & +V_\parallel V_\perp K_{331}\frac{\partial}{\partial z}T_Q \\ \hline V_\parallel V_\perp K_{212}\frac{\partial}{\partial z}T_Q & V_\parallel V_\perp K_{122}\frac{\partial}{\partial x}T_Q + & (\underline{\gamma}+\gamma' V_\parallel + V_\perp K_{232})\frac{\partial}{\partial y}T_Q \\ \cdot & +(\underline{\alpha}+\alpha' V_\parallel + V_\perp K_{322})\frac{\partial}{\partial z}T_Q & \\ \hline (\underline{\gamma}+\gamma' V_\parallel + V_\perp K_{131})\frac{\partial}{\partial x}T_Q + & (\underline{\gamma}+\gamma' V_\parallel + V_\perp K_{232})\frac{\partial}{\partial y}T_Q & V_\parallel V_\perp K_{133}\frac{\partial}{\partial x}T_Q + \\ + V_\parallel V_\perp K_{331}\frac{\partial}{\partial z}T_Q & & +(\underline{\beta}+\beta' V_\parallel + V_\perp K_{333})\frac{\partial}{\partial z}T_Q \end{array}\right) \tag{3.15}$$

$$\tau_{Qi} = \begin{pmatrix} [\underline{\varphi_i} + V_\parallel \varphi_i' + V_\perp \Theta_{11}^{(i)}]s_i & \cdot & \cdot \\ \cdot & [\underline{\varphi_i} + V_\parallel \varphi_i' + V_\perp \Theta_{22}^{(i)}]s_i & [V_\parallel V_\perp \Theta_{32}^{(i)}]s_i \\ & [V_\parallel V_\perp \Theta_{32}^{(i)}]s_i & [\underline{\psi_i} + V_\parallel \psi_i' + V_\perp \Theta_{33}^{(i)}]s_i \end{pmatrix} \tag{3.16}$$

with $s_i = -A_Q/T_Q\,,\ -\eta_G/T_G\,,\ -\eta_Q/T_Q$

In particular, in a medium with a small velocity field (for example in the linear theory of stellar oscillations or stability), only the underlined coefficients should be taken into account.

4. Comparison with Previous Approaches

A relation of the type (3.6) (with $V_\perp = V_\parallel = 0$) between vectorial fluxes and forces was first obtained in the context of turbulence problem by H. Ertel (1937), starting from a mixing length approach. In the linear relation between the stress tensor and the rate of deformation tensor, Ertel's method leads to only two independent parameters. Similarly, Wasintýnski's (1946) treatment, and more recently Elsässer's (1966) discussion do not introduce the largest number of "eddy viscosity coefficients" compatible with the symmetry of the problem.

In addition to the larger number of phenomenological coefficients, our treatment shows the possibility of scalar-vector and vector-tensor coupling which are entirely absent from previous approaches. Thus the present formalism allows, for example, for a heat flux even in the absence of gradients.

We also notice that the dependence of the phenomenological coefficients on the macroscopic velocity field is in accordance with experiment (A.P.Ingersoll).

Finally, we may point out some analogies with situations occurring in crystallography: If $V_{\parallel} = V_{\perp} = 0$ the tensor-tensor coupling (3.7) has exactly the same form as the matrix of the elastic coefficients in a hexagonal crystal (Sneddon and Berry, 1958) (the difference of a factor ½ in (a-4) results from the fact that crystallographers use a different rule of writing a symmetric tensor in a vector form than (3.4). Similarly, the vector-tensor coupling (3.10) coincides with the matrix relating the electric field to the deformation field in a pyroelectric hexagonal class (Forsbergh, 1956). These analogies are of course simple symmetry consequences.

We would like to point out that the absence of a symmetry plane is essential for the vector-tensor coupling. In the presence of σ_h (3.10) becomes

$$K' = \begin{pmatrix} V_{\parallel} V_{\perp} K_{111} & V_{\parallel} V_{\perp} K_{122} & V_{\parallel} V_{\perp} K_{133} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & V_{\parallel} V_{\perp} K_{212} \\ \cdot & \cdot & \cdot & V_{\parallel} V_{\perp} K_{331} & \cdot \end{pmatrix}$$

as one easily shows, applying (3.5). This matrix vanishes if $V_{\parallel}$ or $V_{\perp} = 0$.

The thermodynamic method has allowed us to find the most general structure of the relevant transport mechanisms. It does not, however, lead to any information on the phenomenological coefficients (except for inequalities deduced from (2.4).

Part II. - Statistical Formulation

The aim of the second part of this paper is to provide a technique to determine the coefficients involved in (3.12), starting from a statistical point of view. For the sake of simplicity we do not consider the general case described by Eqs.(1.1), but we limit ourselves to a Boussinesq-type system of equations. The only "macroscopic" convective effect we are going to investigate with the aid of this model is convective heat flux. It is thought, however, that the statistical formulation displayed in the following pages, is sufficiently flexible to be easily extended to situations which are more realistic in the astrophysical context(*).

1. Relevant Equations and Definition of the Phase Space

The equations governing the Boussinesq approximation may be found in Chandrasekhar's book (1961). We write these equations in a form similar to that given by Platzman (1965).

$$\left.\begin{array}{c} \frac{\partial}{\partial t} S = \mathcal{L} S - \vec{v} \cdot \frac{\partial}{\partial \vec{r}} S - G \\ \partial/\partial\vec{r} \cdot \vec{v} = 0 \end{array}\right\} \qquad (1.1)$$

with $$S = (\vec{v}, T), \quad \mathcal{L} = \begin{pmatrix} \nu\, \partial^2/\partial\vec{r}^{\,2} & \vec{k} \\ R^* \vec{k} & K\, \partial^2/\partial\vec{r}^{\,2} \end{pmatrix}, \quad G = \begin{pmatrix} \partial/\partial\vec{r}\ \varpi \\ 0 \end{pmatrix} \qquad (1.2)$$

S being the vector of the fluctuation field ($\vec{v}$ and T are the fluctuations of the velocity and temperature field due to convective motions). The system of units adopted is the following:

- lengths are expressed in terms of the depth d of the fluid layer

(*) A discussion of the validity of the Boussinesq approximation is given by Spiegel and Veronis (1960).

- times are given in terms of $d^2/\sqrt{K\nu}$ (K, molecular conductivity;

  ν molecular viscosity

- the unit of temperature is $K\nu/g\alpha d^3$, α expansion coefficient;

R is the Rayleigh number, and P the Prandtl number. $R = g\alpha\beta d^4/K\nu$; $P = \nu/K$, β constant temperature gradient. R* is the Rayleigh number in the presence of the actual temperature gradient imposed by convection. (The appearance of R* instead of R is the only difference between our formulation and Platzmann's.) ϖ is the pressure fluctuation (for its precise definition see Chandrasekhar, 1961). In addition to (1.1) describing the fluctuations, we have to add the equations of mean motion, which correspond to the complete equations from which the fluctuations are subtracted out. In the latter appears the convective heat flux, according to the results of Part I.

We briefly recall that the usual technique of obtaining the fluctuation equations consists in separating the different physical variables according to (I.1.18), where the mean values are understood as averages over the horizontal coordinates. At this stage we prefer to understand these averages as time averages over a period τ large as compared to the fluctuation time scale (but short in comparison to the time scale of "macroscopic" variations; in the present problem, the latter time scale is considered infinite.

In Eq.(1.1) two kinds of non-linear terms appear (see for instance Spiegel, 1966): the so-called fluctuation interactions $\vec{v}\cdot\vec{\nabla}S$, and the effect of terms of the type R* $\vec{k}\cdot\vec{v}$, in which quadratic fluctuations occur in R*. In our description however R* is given by the mean temperature field, and is not considered as directly non-linear: the non-linearity plays a role only through the mean equations. The effect of this second kind of non-linear terms is that the fluctuations appear as functions or possibly as functionals of the mean field. As, on the other hand, one expects that the phenomenological coefficients are related to the fluctuations, one concludes that the phenomenological coefficients are functions or functionals of the mean field(**).

The set of vectors S satisfying convenient boundary conditions defines a Hilbert space $\mathcal{H}$, in which we adopt the following definition of the inner produce

$$(S', S) = \int_{v} dv\left(\vec{v}' \cdot \vec{v} + T'T/R\right), \quad R > 0 \tag{1.3}$$

where the volume integration extends over the whole configuration of the fluid, and where the inverse of the Rayleigh number has been introduced for physical reasons ($-\frac{1}{2}T^2/R$ plays the role of an available potential energy density; Lorenz, 1955).

Let $\underline{\xi}'$ be a complete set of orthonormal functions subtending $\mathcal{H}$

$$\underline{\xi}' = (\xi_1, \xi_2, \ldots) \tag{1.4}$$

Then

$$\forall S \in \mathcal{H} : S = \underline{x}' \cdot \underline{\xi}' \equiv \sum_{i=1}^{\infty} x_i' \, \xi_i \tag{1.5}$$

(**) The latter remark obviously applies to more complicated systems than the mere Boussinesq approximation. In particular, it allows us to understand the presence of velocity components in the phenomenological coefficients (I. 3.6, 3.7, 3.8, 3.9, 3.10).

where the coefficients x_i are vectors in 4-space. We introduce an N-dimensional enclidean space (N finite), in the following way: Let $\underline{\xi}$ be the set of the N first functions in (1.4)

$$\underline{\xi} = (\xi_1, \xi_2, \ldots \xi_N) \tag{1.6}$$

According to (1.5) we may write

$$\forall S \in \mathcal{H} : S = \underline{x} \cdot \underline{\xi} + \varepsilon(N) \equiv \sum_{i=1}^{N} x_i \, \xi_i + \sum_{i=N+1}^{\infty} x_i \, \xi_i \tag{1.7}$$

The N dimensional space of the coefficients

$$\underline{x} = (x_1, x_2, \ldots x_N) \tag{1.8}$$

will be called the <u>phase-space</u> Γ of the convection problem. As any numerical attempt to solve the fluctuation Eqs.(II.1.1) has made use of a truncated expansion of the form (1.7) (with $\varepsilon(N)$ taken equal to zero) we have introduced a <u>finite</u> dimensional phase-space. The most important expansions used up to now have been the Fourier expansion (introduced by Malkus and Veronis, 1958, in the time-independent case, and by Saltzman, 1962 in the time-dependent case), an expansion in terms of the eigenfunctions of the linearized equations associated with (I.1.1) (i.e. with $\vec{v} \cdot \vec{\nabla} S$ neglected) (Platzman, 1965), or a finite difference scheme.

From the previous definitions, it follows that to any linear operator $\mathcal{O}$ in $\mathcal{H}$ we associate a matrix Ω in Γ which is univocally determined by

$$\mathcal{O} \in \mathcal{H} : \mathcal{O} \longrightarrow (\underline{\xi}, \mathcal{O}\, \underline{\xi}) = \Omega \in \Gamma \tag{1.9}$$

conversely to any matrix Ω in Γ we associate an operator in $\mathcal{H}$

$$\Omega \in \Gamma : \Omega \longrightarrow \underline{\xi} \cdot \Omega \cdot \underline{\xi}^* = \mathcal{O} \in \mathcal{H} \tag{1.10}$$

with $\mathcal{O}\, S$ defined by $\underline{\xi} \cdot \Omega \cdot (\underline{\xi}, S)$ (1.11)

(*) $(\underline{\xi}, \vec{\nabla}\varpi)_i = (\vec{v}_i, \vec{\nabla}\varpi) = \int_v dv\, \vec{v}_i \cdot \vec{\nabla}\varpi = \int_v dv\, \vec{\nabla} \cdot (\vec{v}_i \varpi) - \int_v dv\, \varpi \underbrace{\vec{\nabla} \cdot \vec{v}_i}_{0} = \int_f df\, \underbrace{\vec{n} \cdot \vec{v}_i}_{0}\, \varpi$

Introducing (1.7) into (1.1), we see that these equations take the form

$$\frac{\partial}{\partial t}\underline{x} = \mathsf{L}\cdot\underline{x} + \mathcal{N} : \underline{x}\,\underline{x} = \underline{f}\,(\underline{x}) \tag{1.12}$$

where the matrix L is obtained from the operator $\mathcal{L}$ by (1.9). The contribution of pressure fluctuation vanishes, as a consequence of incompressibility provided we adopt as boundary conditions $\vec{v}\cdot\vec{n}=0$ (*). The quadratic term $\mathcal{N}:\underline{x}\,\underline{x}$ corresponds to the fluctuation interactions:

$$\begin{aligned}(\underline{\xi}, -\vec{v}\cdot\vec{\nabla} S) &= -(\underline{\xi}, \xi\cdot\vec{\nabla}\underline{\xi}) : \underline{x}\,\underline{x} \\ &\equiv \mathcal{N} : \underline{x}\,\underline{x}\end{aligned} \tag{1.13}$$

where N is a matrix with three indices, N_{ijk} . In a Fourier representation, or in an expansion in terms of the linear eigenfunctions we have the following property

$$\mathcal{N} : \underline{x}^*\underline{x}\,\underline{x} \equiv \sum_{i,j,k}^{N} \mathcal{N}_{ijk}\, x_i^* x_j\, x_k = 0 \tag{1.14}$$

which again is a consequence of incompressibility and the boundary condition $\vec{v}\cdot\vec{n}=0$ (**). Equation (1.12) is the generalization of the spectral-dynamical equation to an arbitrary expansion set (1.6). (1.14) corresponds to the well-known selection rule on the wave numbers in the Fourier expansion, or the analogous orthogonality condition in the eigenvector expansion.

Without loss of generality, the matrix N may be replaced by a symmetric matrix in its last two subscripts:

$$\mathcal{S}_{i,jk} = \frac{1}{2}\left(\mathcal{N}_{ijk} + \mathcal{N}_{ikj}\right) \tag{1.15}$$

Successive operations on (1.14) with $\partial/\partial\underline{x}$ lead to the conclusion

$$\left.\begin{aligned} &\mathcal{S}_{i,ii} = 0 \\ &\mathcal{S}_{i,jj} + 2\,\mathcal{S}_{j,ij} = 0 \\ &\mathcal{S}_{i,jk} + \mathcal{S}_{j,ki} + \mathcal{S}_{k,ij} = 0 \end{aligned}\right\} \tag{1.16}$$

If the set of coefficients is complex, we define

(**) $\mathcal{N} : \underline{x}^*\underline{x}\,\underline{x} \equiv (\underline{\xi}\cdot\underline{x}, \underline{x}\cdot\underline{\vec{v}}\;\vec{\nabla}_{\underline{x}}\cdot\vec{v}) = \int_V dV \left[\vec{v}^*(\vec{v}\cdot\vec{\nabla})\vec{v} + T^*\vec{v}\cdot\vec{\nabla}T/R\right] = 0$ as previously.

$$x_i^* \equiv x_{-i} \tag{1.17}$$

Successive derivations yield

$$f_{-i,ii} = 0; \; f_{-i,jj} + 2f_{-j,ij} = 0; \; f_{-i,jk} + f_{-j,ik} + f_{-k,ij} = 0 \tag{1.18}$$

As the linear operator $\mathcal{L} \in \mathcal{H}$ is hermitian (Platzman, 1965), $\mathsf{L} \in \Gamma$ is always diagonalizable. Without loss of generality, we thus may assume L diagonal if needed.

2. General Properties of the Phase Space

Equation (1.12) defines a non-linear, autonomous ordinary differential equation with a second member analytic in $\underline{x}$. A well-known theorem on autonomous differential equations states that the equilibrium points given by

$$\underline{f}(\underline{x}_\ell) = \mathsf{L} \cdot \underline{x}_\ell + f : \underline{x}_\ell \, \underline{x}_\ell = 0 \tag{2.1}$$

can only be reached asymptotically, i.e.

$$\lim_{t \to \infty} \underline{x}(t) = \underline{x}_\ell \tag{2.2}$$

On the other hand, the application of the Lie method (Gröbner and Lesky, 1966) shows that this system has a unique solution

$$\left. \begin{array}{c} \underline{x}(t) = \exp(t\mathcal{L})\underline{x}_0 \quad \text{in } t \in [t_0, t_1] \\ \text{with } \mathcal{L} = \underline{f}(\underline{x}_0) \cdot \dfrac{\partial}{\partial \underline{x}_0} = (\mathsf{L} \cdot \underline{x}_0 + f : \underline{x}_0 \underline{x}_0) \cdot \dfrac{\partial}{\partial \underline{x}_0} \\ \underline{x}_0 \equiv \underline{x}(t_0) \end{array} \right\} \tag{2.3}$$

where (t_0, t_1) is a finite interval. As a condition of physical consistency of the model Eq.(1.12), we require that all the solutions $\underline{x}(t)$, for any finite initial condition $\underline{x}_0$, are bounded, i.e. a positive constant B exists such that

$$\forall \underline{x}_0 \text{ and } \forall t : |\underline{x}(t)| \leq B \tag{2.4}$$

Successive applications of (2.3) allow us to conclude that a given initial condition $\underline{x}_0$ defines, for any finite time interval, a path $\mathscr{C} \in \Gamma$, which has no double points. Similarly, a set of initial conditions

($\underline{x}_{01}, \underline{x}_{02} \ldots$) leads, for any finite time interval, to a system of non-intersecting paths $\mathcal{C}_1, \mathcal{C}_2 \cdots \in \Gamma$. In fact, if two paths $\mathcal{C}_i, \mathcal{C}_j$ intersected, and if the point $\underline{x}_I$ at time t_I is the first point of intersection, we may choose $\underline{x}_I$ as an initial condition and change t in -t. The unicity condition implies that the paths $\mathcal{C}_i, \mathcal{C}_j$ coincide at previous times $[t_I, t_I - t_1]$, which leads to a contradiction(*). These properties are mere consequences of the autonomous character of (1.12), and do not rely on the particular structure of the right-hand side of this equation, which is related to the particular physics of the problem.

The following properly involves the special nature of the problem under consideration: If $\mathcal{V}_0$ is any measurable N-dimentional set of points in Γ , the natural evolution (2.3) continuously deforms $\mathcal{V}_0$ into a measurable set $\mathcal{V}_t$, as it follows from the previous remarks. We prove that the measure

$$M(\mathcal{V}_t) = \int_{\mathcal{V}_t} d\underline{x} \tag{2.5}$$

may be rendered arbitrarily small for $t - t_0 > \tau$, being a positive number, provided the dimension N of the phase space is sufficiently large. We may write

$$\frac{1}{M(\mathcal{V}_t)} \frac{d}{dt} M(\mathcal{V}_t) = \frac{1}{M(\mathcal{V}_t)} \int_{\mathcal{S}} \underline{n} \cdot \underline{v}\, d\mathcal{S} = \frac{1}{M(\mathcal{V}_t)} \int_{\mathcal{V}} \frac{\partial}{\partial \underline{x}} \cdot \underline{v}\, d\mathcal{V}$$

where $\underline{v} = d\underline{x}/dt$ is the velocity in phase space, $\underline{n}$ the normal to the boundary $\mathcal{S}$ of the set $\mathcal{V}_t$. From (1.12) it follows that

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{v} \equiv tr\, L + 2\, tr\,(\mathcal{S} \cdot \underline{x})$$

$tr\, L$ is equal to the sum of the N eigenvalues of the linearized system taken into account in the N-dimensional representation. The properties of

(*) A simpler argument is that the tangent to a path $\mathcal{C}$ at the point $\underline{x}(\neq \underline{x}_{ei}) \in \Gamma$ is given by $(d\underline{x}/dt)/|d\underline{x}/dt|$ and thus is unique. It is undefined at the points $\underline{x}_{ei}$, which thus may be reached (as asymptotically) by distinct paths.

the spectrum of eigenvalues of the linearized system (1.1) are well-known. One distinguishes (Platzman, 1965)

- thermal modes, with eigenvalues $\delta_n \div -n^2$
- kinetic modes $\delta_n \div -n^2$
- convective modes, (two sets of eigenvalues if $R > 0$):

forced modes $\delta_n^- < 0$

free modes $\delta_n^+ > 0, 0$ or < 0

The number of unstable free modes δ_n^+ is limited, and in addition these values are bounded (see also Spiegel, 1966). We thus conclude that, provided N is sufficiently large, n will be sufficiently large, such that

$$\mathrm{tr}\, L < -A_N$$

A_N being a positive number depending on N, and increasing as N^2 according to the results on the eigenvalues.

On the other hand, the matrix $f \cdot \underline{x}$ increases roughly as N, as it follows from a Fourier expansion introduced in (1.13). Hence, provided N is chosen to be sufficiently large, the contribution of $2\,\mathrm{tr}\, f \cdot \underline{x}$ is of order 1/N as compared to $\mathrm{tr}\, L$, so that we have, by taking into account the condition of boundedness (2.4)

$$\frac{1}{M(\mathcal{V}_t)} \frac{d}{dt} M(\mathcal{V}_t) < -A_N \text{ for } N > N_0$$

The proof follows on integrating the previous inequality, and shows that for $t > 1/A_N$ the measure $M(\mathcal{V}_t)$ is rendered arbitrarily small. In other words any N-dimensional variety $\mathcal{V}_0$ evolves toward a variety of a dimension smaller than N, in a sufficiently short time interval τ . From a practical point of view the variety $\lim_{t\to\infty} \mathcal{V}_t = \gamma$ plays a part analogous to the energy surface in usual statistical mechanics. If the variety γ is defined by

$$\varphi_i(\underline{x}(t)) = C_i \quad , \quad i = 1, 2, \cdots m \tag{2.6}$$

where N-m is the dimension of γ , we obtain, after operating with d/dt on (2.6)

$$\frac{\partial \varphi_i}{\partial \underline{x}} \cdot \underline{f}(\underline{x}) = 0 \tag{2.7}$$

showing that the normal to the "surface" γ is orthogonal to the direction of the tangents to the path at any point $\underline{x} \in \gamma$.

The variety γ displays the following properties:

- a path $\mathcal{C}$ belongs to γ with a precision ε after a time τ_ε.
- once a path $\mathcal{C}$ "enters" γ with a precision ε , it remains in γ with a precision ε . These properties are consequences of the definition of γ .

It follows that γ necessarily contains all the equilibrium points defined by (2.1), as the latter are invariant under the Lie-transformation (2.3). In particular $\underline{x}_e = 0 \in \gamma$, the origin being always an equilibrium point (physically it corresponds to the state of absence of convective motions). In addition, if periodic solutions exist, they belong to γ , as follows again from (2.3)

The previous remarks suggest a new definition of the turbulent state which, according to the generally accepted intuitive picture corresponds to an irregular variation of the fluctuating quantities:

<u>Given a finite dimension N of the phase space, we call a solution turbulent if it is bounded and neither an equilibrium point, nor a periodic solution of (1.12).</u>

One may expect that these non-steady and non-periodic solutions will cover γ in the sense that, given a point $P \in \gamma$, a path $\mathcal{C}$ will approach this point arbitrarily close and arbitrarily often in a sufficiently long time interval.

3. Possibility of Non-steady and Non-periodic Solutions, and a Remark on Quasi

From (2.1) we conclude that the steady state convection solutions are given by

$$(L + \mathcal{S} \cdot \underline{x}_\ell) \cdot \underline{x}_\ell = 0 \tag{3.1}$$

Except for the trivial solution $\underline{x}_\ell = 0$, which is assumed to be unstable, as otherwise we do not have a convective state, (3.1) defines an N-1 dimensional variety γ_ℓ containing all the equilibrium points $\underline{x}_\ell$:

$$\gamma_\ell : \operatorname{dtm} [L + \mathcal{S} \cdot \underline{x}_\ell] = 0 \tag{3.2}$$ *

We show that for a sufficiently large Rayleigh number R, all the bounded steady state solutions are unstable. To do so, it is sufficient to show that among the set of eigenvalues $\lambda_i, i = 1, 2 \ldots N$ of the equation (1.12) linearized in the neighborhood of any point $\underline{x}_\ell$, at least one value exists with $R_e \lambda_j > 0$ (**) for R sufficiently large. The set of λ_i is given by

$$\operatorname{dtm} [(\underline{L + 2 \mathcal{S} \cdot \underline{x}_\ell}) - \lambda E] = 0 \tag{3.3}$$

From (1.1) and (1.2) it is obvious that the matrix L is of the form

$$L = L_1 + RL_2 \tag{3.4}$$

where L_1 describes the molecular dissipation processes and L_2 the destabilizing buoyancy effects. As $R > R_0$ the underlined part in (3.3) may be written

$$L(1 + \varepsilon)$$

($\underline{x}_e$ being bounded). As at least one eigenvalue of L satisfies $R_e \lambda_k > 0$, (this corresponds to the instability of the $\underline{x}_e = 0$ state), we conclude that for R sufficiently large $R_e \lambda_k > 0$, and any steady state solution becomes unstable.

It is interesting to point out that the numerical analyses considered

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(*) γ_e does not coincide with γ , as $\underline{x}_\ell = 0 \notin \gamma_\ell$, whereas $\underline{x}_\ell = 0 \in \gamma$.

(**) In the usual Boussinesq approximation, the eigenvalues λ_i are all real.

up to now all show a transition of steady state convection to non-steady state convection, provided the Rayleigh number is sufficiently high. This is for instance the case in the model discussed by Lorenz (1963), who limits the dimension of the phase space to N=3, or the model by Welander (1967) (the latter model has been discussed in detail by Keller (1966), who proves the appearance of a periodic solution, once the (unique, in this model) steady state solution becomes unstable), or the extensive numerical results of J.Toomre (1965).

The previous result has been accepted up to now, implicitly or explicitly, by many writers on convection. For instance Howard (1963) states "We must expect that all steady state convective solutions become unstable for sufficiently high Rayleigh numbers". Several authors have assumed, however, that convection may remain stationary at very high Rayleigh numbers. In particular, this is the attitude of Wesseling (1969). As far as the set of periodic solutions is concerned, let us first mention that the latter may be found from a process of successive approximations. The condition of periodicity is

$$\underline{x}(t+P) = \underline{x}(t)$$

$$\text{or} \quad \exp(P\mathcal{L})\,\underline{x}_0 = \underline{x}_0 \tag{3.4}$$

Expanding the exponential operator, the latter requirement in general leads to conditions on the class of initial conditions $\underline{x}_0$, on P, and/or on $\mathcal{L}$. Under the assumption of boundedness (2.4) of the solutions $\underline{x}(t)$, one notices that periodic solutions are excluded for $R > R_p$, R_p being a sufficiently large critical Rayleigh number.

To show this, we simply note that

$$\underline{x}(t) = \exp\left[tRL_1\left(1+\frac{n}{R}\right)\cdot\underline{x}\,\frac{\partial}{\partial\underline{x}_0}\right]\underline{x}_0 \rightarrow \exp\left[tR\left(L_1\cdot\underline{x}_0\right)\cdot\frac{\partial}{\partial\underline{x}_0}\right]\underline{x}_0$$

if R sufficiently large.

As the linear operator has real eigenvalues, this excludes a periodic behavior.

Non-periodic (and non-steady) convection states have been found by Lorenz (1963), Welander (1967), and J.Toomre (1969). This type of motion is also well established by experimental investigations.

Assuming now the existence of bounded non-periodic and non-steady solutions of (1.12), the following requirement of quasi-ergodicity in a bounded variety $\mathcal{W}$, which may be identified with γ) in a "loose sense", may always be fulfilled: For a sufficiently long time interval τ , any trajectory P(t) of the non-periodic and non-stationary solutions $\underline{x}(t)$ approaches any point $G \in \gamma$ a sufficiently large number of times $\mathcal{N}(N)$, in such a way that

$$|G - P(t)| \leq \varepsilon(N) \tag{3.5}$$

($\mathcal{N}$ and ε depend on the dimension of the phase space.) The numerical results of Lorenz (1963) show a quasi-ergodicity in this sense. Quasi-ergodicity stricto sensu is achieved provided N arbitrary large, ε and $\mathcal{N}$ may be rendered arbitrarily small, respectively large. A vigorous proof of this point, however, does not seem to be feasible for the system we are concerned with, as the usual proof of quasi-ergodicity in statistical mechanics crucially involves the volume integral invariant, which does not exist for (1.12) (see Birkhoff, 1931).

These preliminaries show that it is not unreasonable to seek a statistical description of turbulent convection, in the sense defined in this section. It may be noticed that the previous discussion is not limited to the restrictive conditions of the Boussinesq approximation.

4. The "Liouville Equation" and the Fokker-Planck Approximation

The statistical approach used in this section is essentially modelled on the usual approach of classical statistical mechanics.

Consider an ensemble $\mathcal{E}$ of $\mathcal{N}$ systems $\mathcal{S}_i$, i.e. $\mathcal{N}$ solutions of our equations with different initial conditions. For statistical purposes we specify $\mathcal{E}$ by a density ρ in Γ , describing the distribution of phase points. As a measure of the probability of any system $\mathcal{S}_i$, selected at random from $\mathcal{E}$, to be found in a region $\Omega_t \in \mathcal{V}$ at time t we adopt the definition

$$P(\Omega_t) = \int_{\Omega_t} d\underline{x}\, \rho \tag{4.1}$$

in accordance with standard conventions in statistical mechanics of particles (R.C.Tolman, 1938). The density ρ is normed by

$$\int_{\mathcal{V}_\infty} d\underline{x}\, \rho = 1 \tag{4.2}$$

$\mathcal{V}_\infty$ being the phase space. The density function ρ is assumed to have the following properties

$$\rho \begin{cases} \text{real and positive,} \\ \text{continuous function of } \underline{x}, \\ \text{integral (4.2) exists} \end{cases} \tag{4.3}$$

An ensemble mean value of any function $F(\underline{x})$ is given by

$$\langle F \rangle = \int_{\mathcal{V}_\infty} d\underline{x}\, F \rho \tag{4.4}$$

and is to be interpreted as the expectation value of F in any system $\mathcal{S}_i \in \mathcal{E}$ selected at random. The property of quasi-ergodicity implies that the ensemble mean is equivalent with a time mean.

Consider the representative points P_i of an ensemble $\mathcal{E}$, and let $\mathcal{V}_0$ be the region in phase space which they occupy at $t = t_0$. Under the natural evolution of the system, $\mathcal{V}_0$ is continuously deformed into $\mathcal{V}_t$ at t , the deformation being described by the equations of motion (1.12).

The number of representative phase points being conserved, the density satisfies a continuity equation in Γ

$$\frac{\partial}{\partial t}\rho + div(\rho \underline{v}) = 0 \qquad (4.5)$$

with $div \equiv \partial/\partial \underline{x}$, $\underline{v} \equiv \frac{\partial}{\partial t}\underline{x}$

As the velocity $\underline{v}$ in Γ is a known function of the coordinates, we may write

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial \underline{x}}\cdot \underline{f}(\underline{x})\rho \quad (*) \\ \text{or} \quad &\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial \underline{x}}\cdot\left[(L\cdot\underline{x} + \mathcal{P}:\underline{x}\,\underline{x})\rho\right] \end{aligned}\right\} \qquad (4.6)$$

The latter equation plays a part analogous to the Liouville equation in mechanics.

As a particular case, Eq.(4.6) includes Hopf's equation, provided we add a fluctuating force term in the brackets and consider $\underline{x}$ as the Fourier coefficients of the velocity field only (see Edwards, 1964, Eq. 2.15). Equation (4.6), just as the Liouville equation in mechanics, provides a complete description of our system (1.12), as it is rigorously equivalent to the latter. For statistical purposes, we have to introduce some smoothing approximation. To do so, we follow a line of reasoning similar to that of Edwards (1964): The components of the operator of the right-hand side of (4.6) are of three types:

- a term corresponding to the buoyancy effect, describing the input of "variance" (Platzman, 1964), $\mathcal{L}_B$
- a term corresponding to dissipative effects of molecular friction, $\mathcal{L}_M$
- a term allowing for a flow of variance from unstable convective modes to stable convective modes, and to the kinetic and thermal modes, $\mathcal{L}_F$ (fluctuation interaction).

The variance sharing activity is pictured in a paper by Spiegel

(*) We may mention that this equation provides a statistical description for any system described by equations of motion of the form $\partial/\partial t\, \underline{x} = \underline{f}(\underline{x})$.

(1966). As a consequence of incompressibility, the latter term neither creates nor destroys variance.

Explicitly
$$\mathcal{L}_B + \mathcal{L}_M = - \frac{\partial}{\partial \underline{x}} \cdot L \cdot \underline{x}$$
$$\mathcal{L}_F = - \frac{\partial}{\partial \underline{x}} \cdot (f : \underline{x}\,\underline{x}) \tag{4.7}$$

We now replace the correct fluctuation interaction operator, $\mathcal{L}_F$, by an approximate operator $\mathcal{F}$, essentially displaying the same gross properties as $\mathcal{L}_F$. The following physical remarks allow us to introduce a significant approximation operator $\mathcal{F}$: Any component x_i is influenced by all the other components x_j, j=1, 2 . . . N. In the state of turbulent convection, this effect may be described, at least to a first approximation, by an operator with random properties. Let us then select a time scale τ sufficiently small that $\underline{x}$ is subjected to small variations, during times of order τ, but sufficiently large, that during the same interval $d\underline{x}/dt$ fluctuates and changes its sign sufficiently often. For systems which fulfill the preceding requirements, we may apply the well-known Fokker-Planck equation
$$\tau \frac{\partial}{\partial t} W = - \frac{\partial}{\partial \underline{x}} \cdot \langle R \rangle W + \frac{1}{2} \frac{\partial^2}{\partial \underline{x}\, \partial \underline{x}} : \langle \underline{R}\,\underline{R} \rangle W \tag{4.8}$$
W being the distribution function, $\langle R \rangle$ the vector of mean displacements and $\langle R R \rangle$ the tensor of square mean displacements. The domain of validity of this equation is restricted to time scales larger than τ. (For a clear exposition of the Fokker-Planck equation, we refer to Planck, 1917.)

With Chandrasekhar (1943) the quantities $\langle R \rangle$ and $\langle R R \rangle$ may be estimated as
$$\frac{1}{\tau} \langle R \rangle = -\beta \underline{x} \; ; \quad \frac{1}{\tau} \langle R R \rangle = 2 E q \tag{4.9}$$
to order 1 in τ. β measures the deterministic loss, and q the fluc-

tuating part (*).

Equation (4.8), with coefficients defined by (4.9), is the usual Fokker-Planck equation in the case of real variables and coefficients. To take account of the fact that x_i and $x_{-i}\,(=x_i^*)$ are not independent in the case of complex variables, we replace the fluctuating part of (4.9) by

$$\frac{1}{2\tau}\langle R_i R_j\rangle = Q_{ij}\left(\delta_{ii}+\delta_{i-i}\right)$$

In addition, to ensure the reality of ρ , we have to symmetrize the Fokker-Planck operator in (4.8), in the following way

$$\mathcal{F}=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\beta_i x_i+Q_{i-i}\frac{\partial}{\partial x_{-i}}\right)+\sum_{i=1}^{N}\frac{\partial}{\partial x_{-i}}\left(\beta_{-i}+Q_{-ii}\frac{\partial}{\partial x_i}\right) \qquad (4.10)$$

The coefficients β_i and Q_{i-i} (4x4 matrices) are unknown up to this stage. They will be determined later. When adopting a representation in which the linear matrix is diagonal (in Γ), Eq.(4.6) may be rewritten in the following form

$$\frac{\partial}{\partial t}\rho=\left\{\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[(\beta_i-L_i)x_i+Q_{i-i}\frac{\partial}{\partial x_{-i}}\right]+\sum_{i=1}^{N}\frac{\partial}{\partial x_{-i}}\left[(\beta_{-i}-L_{-i})x_{-i}+Q_{-ii}\frac{\partial}{\partial x_i}\right]+\mathcal{C}\right\}\rho\equiv(\mathcal{F}'+\mathcal{C})\rho \qquad (4.11)$$

where $\mathcal{C}=\mathcal{L}_F-\mathcal{F}$ is a correction operator taking into account the non-Fokker-Planckian contributions.

In a first approximation, we neglect the contribution of $\mathcal{C}$, and we notice that Eq.(4.11) is equivalent to the statistical equation describing mechanical turbulence (Edwards, 1964) (provided we interpret the coefficients as Fourier coefficients).

To discuss the validity of the Fokker-Planck type approximation

$$\frac{\partial}{\partial t}\rho=\mathcal{F}'\rho$$

we consider a model equation of the form

(*) Whereas in the usual case β and q are scalars, they correspond in our case to 4x4 matrices, as any component x_i of the phase space describes 4 coefficients (velocity and temperature expansion coefficients).

$$\frac{\partial}{\partial t}\rho = \left[\sum_{i=1}^{M} \beta_i \frac{\partial}{\partial x_i} x_i + q \sum_{i=1}^{M} \frac{\partial^2}{\partial x_i^2}\right]\rho \tag{4.12}$$

where q is a constant. The latter may be expected to reflect the correct time behavior of (4.11), as the only essential modification as compared to (4.11) is a simplification in the fluctuation interaction. For an initial δ-distribution the solution of (4.12) is given by

$$\rho(x_i, t) = \exp(\mathrm{tr}\,\beta)t \prod_{i=1}^{M} \left\{ \left(\frac{\beta_i}{2\pi q}\right)^{1/2} \frac{\exp\left[-x_i^2 \frac{\beta_i}{2q}\left(\frac{1}{1-e^{-2\beta_i t}}\right)\right]}{\left(e^{2\beta_i t}-1\right)^{1/2}} \right\} \tag{4.13}$$

(see Appendix I)

This model equation shows that the system evolves toward a statistically steady gaussian distribution for

$$t > \sup_{i=1,\dots M} \left\{\frac{1}{\beta_i}\right\} = \tau_R$$

The smallest time constant β_i thus determines the relaxation time. On the other hand, the characteristic time scale of fluctuations is to be expected as a mean value of the time scales $1/\beta_i$

$$\tau_F \sim \frac{1}{M} \sum_{i=1}^{M} \frac{1}{\beta_i} \ll \tau_R \tag{4.14}$$

The latter condition shows, a posteriori, that the Fokker-Planck approximation to the "Liouville equation" (4.6) is not unreasonable.

A statistically steady state solution of (4.11) with $\mathcal{G} = 0$ is easily obtained: Noticing that $\mathcal{F}'$ has the structure $\partial/\partial \underline{x} \cdot \underline{\varphi}$, we conclude that steady state solutions correspond to $\underline{\varphi}\rho = 0$ (the arguments of M.Planck (1917) allow us to reject other solutions). Thus

$$\left.\begin{aligned} \varphi_i \rho &\equiv \left[(\beta_i - L_i)x_i + Q_{i-i}\frac{\partial}{\partial x_{-i}} + Q_{ii}\frac{\partial}{\partial x_{-i}}\right]\rho = 0 \\ \varphi_{-i}\rho &\equiv \left[(\beta_{-i} - L_i)x_{-i} + Q_{-ii}\frac{\partial}{\partial x_i} + Q_{i-i}\frac{\partial}{\partial x_i}\right]\rho = 0 \end{aligned}\right\} \tag{4.15}$$

showing that the components in phase space are decoupled.

One finds that (4.15) admits of the solution

$$\left.\begin{aligned} \rho &= \prod_{i=1}^{N} \rho_i(x_i, x_{-i}) \\ \rho_i(x_i, x_{-i}) &= C_i \exp\left[-x_{-i} Q_{-ii}^{-1} (\beta_i - L_i) x_i\right] \end{aligned}\right\} \tag{4.16}$$

provided

$$\begin{gathered} \beta_i = \beta_{-i} \quad , \quad Q_{i-i} = Q_{-ii} \\ \left[\beta_i \,, Q_{i-i}\right] = 0 \end{gathered} \tag{4.17}$$

As up to this stage the (4x4) matrices β_i and Q_{i-i} were arbitrary, the previous requirements (4.17) are always possible to be fulfilled. The commutation condition, on the other hand, implies that the symmetric matrices β_i and Q_{i-i} have the same principal directions. The quadratic form in (4.16) is thus hermitian. To fulfill the normalization condition (4.3) the matrices $Q_{-ii}^{-1}(\beta_i - L_i)$ have to be positive definite.

As the matrices β_i and Q_{i-i} appear in the Fokker-Planck equation as coupling matrices between vector-vector (velocity components), scalar-scalar (temperature components), and vector-scalar fields (velocity components and temperature components), the discussion of Part I applies. With the choice of axes as defined in Fig.1, the coupling expressions imply that the matrices β_i , Q_{i-i} as well as $Q_{-ii}^{-1}(\beta_i - L_i)$ are of the form

$$\left(\begin{array}{ccc|c} a & \cdot & \cdot & \cdot \\ \cdot & a & \cdot & \cdot \\ \cdot & \cdot & b & e \\ \hline \cdot & \cdot & e & c \end{array}\right) \tag{4.18}$$

In the absence of temperature effects the previous expression describes the statistical behavior of non-isotropic turbulence. In the isotropic case, and provided we interpret the expansion coefficients as Fourier components, we recover S.F.Edwards' expression (5.14).

To obtain the normalization constants C_i , we have to take account of the constraint of incompressibility. Writing the latter

$$x_{i3} = x_{i1} \Delta_{i1} + x_{i2} \Delta_{i2}$$

The zero-order moment in Appendix II leads to the following result

$$\rho(\underline{x}) = \sqrt{\prod_{i=1}^{N} \frac{\det\left[\mathcal{P}_i \cdot Q_{i-i}^{-1} \cdot (\beta_i - L_i) \cdot \tilde{\mathcal{P}}_i\right]}{\pi^3}}\; \exp - \sum_{i=1}^{N} x_{-i}\, Q_{-ii}^{-1} \cdot (\beta_i - L_i) \cdot x_i \qquad (4.19)$$

with

$$\mathcal{P}_i = \begin{pmatrix} 1 & 0 & \Delta_{i1} & 0 \\ 0 & 1 & \Delta_{i2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We now determine the 4x4 matrices β_i and Q_{i-i}, in such a way that the Fokker-Planck approximation describes (4.6) "at best". Denoting by $\mathcal{L}$ the Liouville operator in (4.6), and by $\underline{\mathcal{F}}'(P)$ the associated Fokker-Planck operator, depending on the set of parameters P, both operators being integrated once, with constants of integration put equal to zero as previously, we determine the parameters P (matrices β_i and Q_{i-i}) in such a way as to have

$$\int_{V_\infty} d\underline{x} \left(\underline{\mathcal{L}} - \underline{\mathcal{F}}'(P)\right)^2 \rho \longrightarrow \text{Minimum} \qquad (4.20)$$

where ρ is the steady state distribution function.

Writing out the minimum condition (4.20) explicitly, one notices that it leads to equations of the form

$$\left.\begin{aligned} &\left(\int_{V_\infty} d\underline{x}\; 2\,\underline{x}\,\underline{x}\,\rho\right)\beta + \left(\int_{V_\infty} d\underline{x}\left[\underline{x}\left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right) + \left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right)\underline{x}\right]\rho\right) Q = \left(2\int_{V_\infty} d\underline{x}\; \underline{x}\cdot L\cdot \underline{x} - 2\int_{V_\infty} d\underline{x}\; \underline{x}\cdot \underline{f}\right) \\ &\left(\int_{V_\infty} d\underline{x}\left[x\left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right) + \left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right)\underline{x}\right]\rho\right)\beta + \left(\int_{V_\infty} d\underline{x}\; 2\left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right)\left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right)\rho\right) Q = \\ &= \left(\int_{V_\infty} d\underline{x}\left[(L\cdot \underline{x})\left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right) + \left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right) L\,\underline{x}\right] - \int_{V_\infty} d\underline{x}\left[\left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right)\underline{f} + \underline{f}\left(\frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}^*}\right)\right]\right) \end{aligned}\right\} (4.21)$$

These equations are linear in the β_i and Q_{i-i} matrices, and may thus easily be solved. However, to compute the relevant integrals, one has to know ρ. The latter quantity is approximated by (4.16). In this form (4.21) gives β_i and Q_{i-i} as functions of $Q_{i-i}^{-1}(\beta_i - L_i)$. This formal solution is similar to that indicated by S.F.Edwards in the case of mechanical turbulence.

For a practical evaluation, one may first determine $Q^{-1}_{i-i}(\beta_i - L_i)$ in an approximate manner. To do so we may start from

$$\oint(x)\rho = 0, \text{hence} \int_{V_\infty} d\underline{x}\, \oint(\underline{x})\cdot f(\underline{x})\rho = 0 \tag{4.22}$$

The latter equation implies second and fourth moments, which are explicitly given in Appendix II.

This procedure completely determines the distribution function (4.19).

5. Remarks on the Determination of the Heat Flux

Although similar to the corresponding expression in statistical mechanics, the distribution law (4.16) shows two essential new features:

- the coefficient of x_{-i}, x_i in the exponential is a 4x4 matrix
- this matrix differs, in general, for the different subscripts i.

In fact, these differences from the usual statistical mechanics of particles introduce "irreversible effects" already at this stage of the approximation.

The temperature of the quasi-particles, T_Q, may be defined from the following expression of the argument of the exponential

$$x_{-i}H_i x_i \equiv (\vec{v}_{-i}, T_{-i})\begin{pmatrix}\mathcal{H}_i & h_i \\ h_i & \eta_i\end{pmatrix}\begin{pmatrix}\vec{v}_i \\ T_i\end{pmatrix} = \left(\tfrac{1}{3}\operatorname{tr}\mathcal{H}_i\right)\vec{v}_{-i}\cdot\vec{v}_i + \vec{v}_{-i}\cdot\mathcal{H}'_i\cdot\vec{v}_i - T_{-i}\eta_i T_i + T_{-i}h_i\vec{v}_i + \vec{v}_{-i}h_i T_i \tag{5.1}$$

where $\mathcal{H}'_i$ is a traceless 3x3 matrix. From (1.10) we conclude that

$$T_Q(\vec{r}) = \frac{6k}{\mu}\sum_{i=1}^{N}\xi_i(\vec{r})(\operatorname{tr}\mathcal{H}_i)^{-1}\xi_i^*(\vec{r}) \tag{5.2}$$

We have written a factor $6k/\mu$ to have an expression analogous to that occurring in the distribution function of an ordinary gas. μ may be interpreted as a characteristic mass (for example the mean mass of the molecules). Equation (5.2) describes a non-uniform temperature field $T_Q(\vec{r})$. In a similar way we may introduce a pressure p_Q and an "internal energy" u_Q, which are obtained from the mean motion equation through the fluctuation term $\langle \vec{v}_f \cdot \vec{v}_f \rangle$, and may thus be related to T_Q.

Irreversible effects are related to the non-Maxwellian character of (4.16). In particular, the convective heat flux, as deduced from the mean energy equation, is given by

$$\vec{\mathcal{F}} = \rho c_v \langle \vec{v}_f T_f \rangle \tag{5.3}$$

From (1.7), we conclude that

$$\langle \vec{v}_f T_f \rangle = \sum_{i,j} \xi_i \xi_j^* \langle \vec{v}_{fi} T_{fj}^* \rangle = \sum_i \xi_i \langle \vec{v}_{fi} T_{fi}^* \rangle \xi_i^* \tag{5.4}$$

the latter step being a consequence of the orthogonality property of the set $\underline{\xi}$. The ensemble mean $\langle \vec{v}_{fi} T_{fi}^* \rangle$ obviously is contained in the 4x4 matrix $\langle x_i x_i \rangle$, whose expression is given in Appendix II.

If we write the z-component of the heat transport by convection in the form

$$\mathcal{F}_z = -K \frac{\partial}{\partial z} T_Q \tag{5.5}$$

The transport coefficient K is given by

$$K = -\rho c_v \frac{\sum_{i=1}^{N} \xi_i \langle v_{z fi} T_{fi}^* \rangle \xi_i^*}{\sum_{i=1}^{N} \xi_i \left(6k/\mu \, \mathrm{tr}\, \mathcal{H}_i\right)(\alpha_i + \alpha_i') \xi_i^*} \tag{5.6}$$

where, we have $\frac{\partial}{\partial z} \xi_i = \alpha_i \xi_i$, $\frac{\partial}{\partial z} \xi_i^* = \alpha_i' \xi_i$

This form is particularly interesting if we approximate the expansion (1.8) by a single term: in that case K appears to be a constant.

Our theory allows us to give a rough evaluation of the dependence of the Nusselt number on the Rayleigh number: Limiting (5.4) to a single mode, denoted by M, we have

$$\mathcal{F} \div \langle v_{fM} , T_{fM} \rangle \div \langle x_M x_M \rangle \tag{5.7}$$

From (4.22) we derive that $H_M \div L_M^2$, assuming the lateral wavelength a is small as compared to the vertical wavelength (equal to π). The equation of the convective modes (see Platzman,G.W., Eq.3.13) leads, through a maximization of the eigenvalue with respect to a, to $L_M \div R^{1/6}$. As $H_M \div \langle x_M x_M \rangle$ (see Appendix II), we conclude that $\mathcal{F} \div R^{1/3}$, which corresponds to the experimentally well-established power law for the Nusselt number.

I wish to express my gratitude to Professors E.A.Spiegel and J.B.Keller for stimulating discussions.

Appendix I Solution of the model equation

$$\frac{\partial}{\partial t}\rho = \left(\frac{\partial}{\partial \underset{\sim}{x}}\cdot\beta\cdot\underset{\sim}{x} + Q\frac{\partial^2}{\partial x^2}\right)\rho, \qquad \begin{array}{l}\beta \text{ diagonal matrix}\\ Q \text{ scalar}\end{array}$$

Following S. Chandrasekhar (1943) we make the follosing substitution

$$\underset{\sim}{x} = \exp(-\beta t)\underset{\sim}{q}$$

which yields

$$\frac{\partial}{\partial t}W = (\mathrm{tr}\,\beta)W + Q\,e^{2\beta t} : \frac{\partial^2}{\partial \underset{\sim}{q}\,\partial \underset{\sim}{q}}W$$

with $\rho(\underset{\sim}{x}(t), t) = W(\underset{\sim}{q}(t), t)$

Introducing $\chi = e^{-(\mathrm{tr}\,\beta)t}W$

leads to the elimination of the first term of the right-hand side

$$\frac{\partial}{\partial t}\chi = Qe^{2\beta t} : \frac{\partial^2}{\partial \underset{\sim}{q}\,\partial \underset{\sim}{q}}\chi$$

Separation of variables

$$\chi(\underset{\sim}{q}) = \prod_{i=1}^{M}\chi_i(q_i)$$

yields standard diffusion equations

$$\frac{\partial}{\partial t}\chi_i = Qe^{2\beta_i t}\frac{\partial^2}{\partial q_i^2}\chi_i$$

For an initial point distribution, the latter equation admits the solution

$$\chi_i(q_i, t) = \frac{1}{\sqrt{2\pi\varphi_i}}\exp - q_i^2/4\varphi_i \quad ; \quad \varphi_i = \frac{Q}{2\beta_i}(\exp 2\beta_i t - 1)$$

Thus

$$\rho(\underset{\sim}{x}, t) = e^{(\mathrm{tr}\,\beta)t}\prod_{i=1}^{M}\frac{\exp\left[-\frac{x_i^2\beta_i}{2Q}\left(\frac{e^{2\beta_i t}}{e^{2\beta_i t}-1}\right)\right]}{\sqrt{\frac{2\pi Q}{\beta_i}\left(e^{2\beta_i t}-1\right)}}$$

Appendix II Computation of integrals

$$I_p(H) = \int_{]-\infty,\infty[^\nu} d\underset{\sim}{x}\ \underbrace{\underset{\sim}{x}\,\underset{\sim}{x}\cdots\underset{\sim}{x}}_{p}\ \exp -\underset{\sim}{x}\cdot H\cdot\underset{\sim}{x} = \langle\underset{\sim}{x}\,\underset{\sim}{x}\cdots\underset{\sim}{x}\rangle$$

with H hermitian positive definite, under the constraint $\underset{\sim}{\beta}\cdot\underset{\sim}{x} = 0$, $\underset{\sim}{x}$ and $\underset{\sim}{\beta}$ being ν dimensional vectors.

We define the following $\nu-1$ dimensional vectors:

$$\underset{\wedge}{\chi} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots x_\nu)$$

$$\underset{\wedge}{\Delta} = (-\beta_1/\beta_i, -\beta_2/\beta_i, \dots, -\beta_{i-1}/\beta_i, -\beta_{i+1}/\beta_i, \dots, -\beta_\nu/\beta_\nu) = (\Delta_1, \Delta_2, \dots \Delta_\nu)$$

The constraint is taken into account through a δ-distribution

$$I_p(H) = \int_{]-\infty,\infty[^\nu} d\underset{\wedge}{\chi}\, dx_i \; \underbrace{\underset{\sim}{x}\,\underset{\sim}{x} \dots \underset{\sim}{x}}_{p} \; exp(-\underset{\sim}{x}\cdot H \cdot \underset{\sim}{x}) \; \delta(x_i - \underset{\wedge}{\Delta}\cdot\underset{\wedge}{\chi})$$

Integration over x_i may be performed immediately

$$I_p(H) = \int_{]-\infty,\infty[^{\nu-1}} d\underset{\wedge}{\chi}\; exp(-\underset{\wedge}{\chi}\cdot\mathcal{P}\cdot H \cdot \widetilde{\mathcal{P}}\cdot\underset{\wedge}{\chi}) \quad (\mathcal{P}\cdot\underset{\wedge}{\chi})(\mathcal{P}\cdot\underset{\wedge}{\chi})\dots(\mathcal{P}\cdot\underset{\wedge}{\chi})$$

where $\mathcal{P}$ is a $\nu\times\nu-1$ matrix given by

$$\mathcal{P} \quad \overset{i}{\updownarrow} \begin{pmatrix} 1 & 0 & \dots & 0 & \Delta_1 & & & \\ 0 & 1 & & 0 & \Delta_2 & & 0 & \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \Delta_{i-1} & & & \\ & & & & \Delta_{i+1} & 1 & 0 & \dots & 0 \\ & & 0 & & \vdots & 0 & 1 & & 0 \\ & & & & \Delta_\nu & 0 & 0 & \dots & 1 \end{pmatrix} \Bigg\updownarrow \nu-1$$

which plays the rôle of a projection operator, in the sense that it transforms ν-dimensional vectors into ν-1-dimensional vectors. To proof the appearance of $\mathcal{P}$ in the integrand, we assume, without loss of generality $i = \nu$. Then $\underset{\sim}{x} = (\underset{\wedge}{\chi}, x_\nu)$

and integration over x_ν replaces $\underset{\sim}{x}$ by $(\underset{\wedge}{\chi}, \underset{\wedge}{\Delta}\cdot\underset{\wedge}{\chi}) = (\mathcal{E}, \underset{\wedge}{\Delta})\cdot\underset{\wedge}{\chi} \equiv \mathcal{P}\cdot\underset{\wedge}{\chi}$ $\mathcal{E}$ being the identity matrix in $\nu-1$ dimensions. Similarly

$$\underset{\sim}{x}\cdot H\cdot\underset{\sim}{x} = (\underset{\wedge}{\chi}, x_\nu)\cdot H\cdot\binom{\underset{\wedge}{\chi}}{x_\nu} = (\underset{\wedge}{\chi}, \underset{\wedge}{\Delta}\cdot\underset{\wedge}{\chi})\cdot H\cdot\binom{\underset{\wedge}{\chi}}{\underset{\wedge}{\Delta}\cdot\underset{\wedge}{\chi}} = \underset{\wedge}{\chi}\cdot(\mathcal{E}, \underset{\wedge}{\Delta})\cdot H\binom{\mathcal{E}}{\underset{\wedge}{\Delta}}\cdot\underset{\wedge}{\chi}$$
$$= \underset{\wedge}{\chi}\cdot\mathcal{P}\cdot H\cdot\widetilde{\mathcal{P}}\cdot\underset{\wedge}{\chi}$$

If U is the unitary matrix diagonalizing $\mathcal{P}\cdot H\cdot\widetilde{\mathcal{P}}$, and $\underset{\wedge}{y} = U\cdot\underset{\wedge}{\chi}$, the integral may be rewritten, with self-explanatory notations

$$I_p(H) = (\mathcal{P}\cdot\widetilde{U})^{p(p)} \vdots \int d\underset{\wedge}{y}\; exp\left(-\sum_{i=1}^{\nu-1} D_i y_i^2\right) \underbrace{\underset{\wedge}{y}\,\underset{\wedge}{y}\dots\underset{\wedge}{y}}_{p}$$

Writing $\mathcal{J}_{ij\dots k} = \int dy_1 dy_2 \dots dy_{\nu-1}\; exp\left(-\sum_{i=1}^{\nu-1} D_i y_i^2\right) \underline{y_i y_j \cdots y_k}$

one notices that this integral vanishes except if the underlined quantity is of the form

$$y_m^{2\alpha_m}\, y_n^{2\alpha_n} \dots, \; \alpha_m, \alpha_n \dots \text{integers}$$

One finds

$$\mathcal{J}_{2\alpha_i\, 2\alpha_j \dots} = \sqrt{\frac{\pi^{\nu-1}}{dtm(\mathcal{P}\cdot H\cdot\widetilde{\mathcal{P}})}}\; \mathcal{D}^{2\alpha_i}\mathcal{D}^{2\alpha_j}\dots$$

with $\mathcal{D}^{2\alpha_i} \equiv \frac{(2\alpha_i-1)!!}{2^{\alpha_i}} \frac{1}{D^{\alpha_i}}$

The general result is then given by

$$\langle x_i x_j \ldots x_k \rangle = \sqrt{\frac{\pi^{\nu-1}}{dtm(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})}} \left\{ \sum_{\substack{i'i''\\j'j''\\k'k''}} \mathcal{P}_{ii'} \tilde{U}_{i'i''} \mathcal{D}_{i''} \mathcal{P}_{jj'} \tilde{U}_{j'j''} \mathcal{D}_{j''} \ldots \mathcal{P}_{kk'} \tilde{U}_{k'k''} \mathcal{D}_{k''} \delta_{i''j''\ldots k''} \right\}$$

where the symbol $\delta_{i''j''\ldots k''}$ is 1 provided the indices are pairwise equal, and is zero otherwise.

We indicate the most important moments

1) $\langle 1 \rangle = \int d\underset{\sim}{x}\ \exp - \underset{\sim}{x} \cdot H \cdot \underset{\sim}{x} = \sqrt{\frac{\pi^{\nu-1}}{dtm(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})}} = C^{-1}$

2) $\langle \underset{\sim}{x} \rangle = 0$

3) $\langle \underset{\sim}{x}\underset{\sim}{x} \rangle = \int d\underset{\sim}{x}\ \exp(-\underset{\sim}{x}\cdot H\cdot\underset{\sim}{x})\, \underset{\sim}{x}\underset{\sim}{x} = \frac{1}{2} C^{-1} \mathcal{P}\cdot(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1}\cdot\tilde{\mathcal{P}}$

(note that $(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1} \neq \mathcal{P}\cdot H^{-1}\cdot\tilde{\mathcal{P}}$ due to the fact that $\mathcal{P}$ is not a quadratic matrix)

4) $\langle \underset{\sim}{x}\underset{\sim}{x}\underset{\sim}{x} \rangle = 0$

5) $$\langle \underset{\sim}{x}\underset{\sim}{x}\underset{\sim}{x}\underset{\sim}{x} \rangle_{ijkl} = C^{-1} \frac{1}{4} \Big[(\mathcal{P}\cdot(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1}\cdot\tilde{\mathcal{P}})_{ij} (\mathcal{P}\cdot(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1}\cdot\tilde{\mathcal{P}})_{kl} + (\mathcal{P}\cdot(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1}\cdot\tilde{\mathcal{P}})_{ik} (\mathcal{P}\cdot(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1}\cdot\tilde{\mathcal{P}})_{jl} + (\mathcal{P}\cdot(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1}\cdot\tilde{\mathcal{P}})_{il} (\mathcal{P}\cdot(\mathcal{P}\cdot H\cdot\tilde{\mathcal{P}})^{-1}\cdot\tilde{\mathcal{P}})_{jk} \Big]$$

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GAS FLOW AROUND A STAR

John D. Trasco

Introduction

Previous work on the problem of gas flow in the vicinity of a star has, in general, been concerned with the early stages during which the ionized hydrogen (H II) region is being formed. We will be concerned here with the case of a steady state H II region i.e. one which has come essentially into pressure equilibrium with its surroundings; but, it will be useful for the orientation of the problem to consider the results of investigations on expanding H II regions.

Interstellar Medium and H II Regions

The interstellar medium consists mainly of neutral hydrogen gas with a density distribution which shows "lumps" over a wide range of length scales. The classical picture has been of "clouds" of gas with a mean size of ~ 10 pc and a mean density of ~ 10 atoms/cm^3 covering about 10% of the volume of interstellar space. We will neglect these density fluctuations and deal with an infinite medium with an initially smooth density distribution. An H II region will be created either if a hot star is formed in a strong local condensation of the gas or if a hot star encounters a previously neutral gas cloud. Some concept of the type of star needed to create an H II region as well as of the typical parameters involved can be obtained from the data in Table I (taken from Mathews, 1965). The first column gives the mass of the star in solar masses; the second, the temperature of the star which excites the H II region; the third, the equilibrium temperature of the ionized gas, and the fourth the main sequence lifetime of the star.

Table I

$m/m_\odot$	T_*	T_{eq}	t_{eff} (10^6 yrs)
30	42,000	7700	4.2
20	35,100	7400	6.2
11	27,400	7000	12.6
6	20,200	6500	35.5

Table II gives the results of Mathews (1965) for a star which is not in motion with respect to the gas. Here R_i is the radius at which the ionizing radiation is first balanced by recombinations, $\tau_i = R_i/c(T_{eq})$ is the time for a hydrodynamical disturbance to traverse the nebula, R_f is the radius of the final H II region which is in static equilibrium, and τ_f is the time needed to reach this configuration. All of these results are given for three values of the density.

As was noted above, a typical density for an interstellar cloud is 10 atom cm^{-3}; however, the higher densities given in Table II might be characteristic of regions where star formation is occurring. It can be seen

Table II

$n = 10\ cm^{-3}$

$\mathcal{M}/\mathcal{M}_\odot$	R_i (pc)	τ_i (10^4yrs)	R_f (pc)	τ_f (10^6yrs)
30	13.0	85	370	25.0
20	8.5	56	238	16.0
11	4.0	27	110	7.7
6	1.7	24	44	3.2

$n = 100\ cm^{-3}$

$\mathcal{M}/\mathcal{M}_\odot$	R_i (pc)	τ_i (10^4yrs)	R_f (pc)	τ_f (10^6yrs)
30	2.8	18.0	80.0	5.3
20	1.8	12.0	50.0	3.4
11	0.86	5.9	23.0	1.6
6	0.36	2.5	9.2	0.67

$n = 1000\ cm^{-3}$

$\mathcal{M}/\mathcal{M}_\odot$	R_i (pc)	τ_i (10^4yrs)	R_f (pc)	τ_f (10^6yrs)
30	0.60	3.9	17.0	1.1
20	0.39	2.6	11.0	0.75
11	0.18	1.2	4.8	0.34
6	0.08	0.57	2.0	0.15

from these two tables that the most massive stars evolve off the main sequence before the H II region comes into equilibrium with its surroundings. However these stars ($\mathcal{M} > 20\mathcal{M}_\odot$) are fairly rare and for stars in the mass range of 5 to 10 $\mathcal{M}_\odot$, equilibrium is reached well before the main sequence stage ends.

The results discussed above refer to a star which is not moving with respect to the interstellar gas. Recently, Rasiwala (1969) has studied the case where the star is moving at supersonic velocities with respect to the gas. Two main problems were considered (1) the deformation of the H II region which results because the shock does not detach from the ionization front at the same time in all directions and (2) the flow of the gas along the axis defined by the motion of the star. The results of this work showed that the wind velocity is reduced in the region upstream of the star with the ionized gas then being continuously accelerated throughout the

H II region until reaching the shock front downstream of the star. The density distribution showed an increase upstream of the star with a leveling off to the original density throughout the central part of the H II region and then increasing sharply as the downstream shock is reached. At later times, these effects are magnified with a drop in the density occurring both near the center of the H II region and just in front of the downstream shock.

Isothermal Problem

We will be concerned with a star moving slowly relative to the gas i.e. the velocity (W) is small with respect to the velocity of sound in the medium. Further, we are considering a steady state, equilibrium model, so there is no expansion of the H II region. As a result of these assumptions, there will be no shock fronts formed. (The results of Rasiwala discussed above are associated with the occurrence of shocks, so our results are not directly comparable with his.) In addition, since H II regions are typically 10 pc or greater, gravitational effects are small over the bulk of the region and will be neglected here. These assumptions can be used to specify the dynamical problem and it is now necessary to consider the thermal problem.

It is known from the calculations for a stationary star in a gas cloud that there is a fairly sharp spatial boundary between the ionized and neutral gas. If the star is in motion, the values of W which would be appropriate for our problems are of the order of a few km/sec. Since the time for ionization is typically $\sim 10^4$ yrs, this gives a length scale of $\sim .1$ pc as the distance the star would move during the time needed to ionize the gas. As this distance is small with respect to the nebula, it will be possible to simplify the problem by considering it as an ionized region of fixed dimension through which gas may flow being ionized when it enters the region and recombining when it leaves the region. The temperature within the H II region is not likely to vary over a large range both the heating and the cooling mechanisms are strongly temperature sensitive. The temperature of the neutral gas should, in addition, not be greatly altered except very near the H II region.

These assumptions lead to the following definition of the problem. A star moves at a velocity $-W$ with respect to an infinite compressible medium. We consider the two-dimensional steady flow of the gas in a coordinate system centered on the star and therefore one in which the gas velocity at large distances from the star reduces to $W\hat{k}$ where $-\hat{k}$ is a unit vector in the direction of the star's motion. (Fig.1).

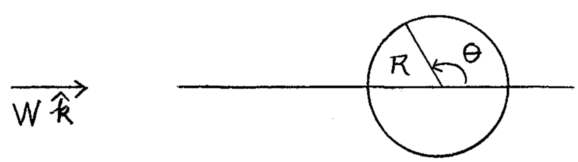

Fig.1

Within a sphere of radius R centered on the star, the gas is completely ionized and at a uniform temperature T_1; outside this sphere, the gas is neutral and at a uniform temperature T_2 (subscripts 1 and 2 will be used to refer to the regions $r < R$ and $r > R$ respectively.)

The relevant equations governing the behavior of the gas can be written as $\frac{\partial}{\partial x^i}$

conservation of mass $\frac{\partial}{\partial x^i}(\rho u_i) = 0$

conservation of momentum $\frac{\partial}{\partial x^i}(P_{ij}) = 0$

where ρ is the density, u_i the velocity and P_{ij}., the stress tensor is given by $P_{ij} = \rho u_i u_j + p\delta_{ij}$. The pressure p of an isothermal gas is given by $p = a^2\rho$ where a is the constant isothermal sound speed. The boundary conditions at the ionization front are given by

$$[\rho u_i n_i] = 0$$
$$[P_{ij} n_i] = 0$$

where n_i is a unit vector normal to the ionization front and the square brackets denote the difference in the quantity within the brackets on either side of the front.

It was assumed that the variables could be expanded about a zero-order solution

$$\begin{pmatrix} \rho \\ p \\ u_i \end{pmatrix} = \begin{pmatrix} \rho^{(0)} \\ p^{(0)} \\ u_i^{(0)} \end{pmatrix} + \varepsilon \begin{pmatrix} \rho^{(1)} \\ p^{(1)} \\ u_i^{(1)} \end{pmatrix} + \cdots$$

and that for the zero order $u_i^{(0)} \equiv 0$. The equations to zero order reduce to

$$\frac{\partial}{\partial x^i} P_{ij}^{(0)} = \frac{\partial}{\partial x^j} p^{(0)} = 0$$
$$[p^{(0)} \delta_{ij} n_i] = 0$$

The solution for this is

Exterior $(r > R)$
$$p_2^{(0)} = const = p_0$$
$$u_{i2}^{(0)} = 0$$
$$\rho_2^{(0)} = \frac{p_2^{(0)}}{a_2^2} = const = \rho_0$$

Interior $(r < R)$
$$p_1^{(0)} = p_0$$
$$u_{i1}^{(0)} = 0$$
$$\rho_1^{(0)} = \frac{p_1^{(0)}}{a_1^2} = \eta \rho_0$$

where $\eta = \left(\frac{a_2}{a_1}\right)^2$. Since $a^2 \propto T$ with $T_2 \approx 100°K$ and $T_1 \approx 10^4 °K$; $\eta \simeq 10^{-2}$.

The first order equations are

$$(\rho u_i)^{(1)} = \rho^{(0)} u_i^{(1)} + \rho^{(1)} u_i^{(0)} = \rho^{(0)} u_i^{(1)}$$

$$P_{ij}^{(1)} = \rho^{(1)} u_i^{(0)} u_j^{(0)} + \rho^{(0)} u_i^{(1)} u_j^{(0)} + \rho^{(0)} u_i^{(0)} u_j^{(1)} + p^{(1)} \delta_{ij} = p^{(1)} \delta_{ij}$$

$$\frac{\partial}{\partial x^i}(\rho u_i)^{(1)} = \rho^{(0)} \frac{\partial}{\partial x^i} u_i^{(1)} = 0 \qquad (1)$$

$$\frac{\partial}{\partial x^i}(P_{ij}^{(1)}) = \frac{\partial}{\partial x^j} p^{(1)} = 0$$

with the boundary conditions

$$[P_{ij} n_i] = [p] = 0 \qquad (2)$$

$$[\rho^{(0)} u_i^{(1)} n_i] = 0$$

Therefore $p^{(1)}$ is a constant and continuous at $r = R$. Since this is the same result as was obtained for the zero order pressure, we can assume that this is incorporated in the zero order result and set $p^{(1)} = 0$ and therefore $\rho^{(1)} = \frac{p^{(1)}}{a^2} = 0$.

Assuming potential flow $\left(u_i^{(1)} = \frac{\partial \psi^{(1)}}{\partial x^i}\right)$, the equation for the velocity becomes $\nabla^2 \psi^{(1)} = 0$ with the additional condition that the velocity goes to $W\hat{k}$ at large distances from the star. Therefore we have

$$\psi_2^{(1)} = W r P_1(\mu) + \sum_{n=0}^{\infty} \frac{\psi_{2n} P_n(\mu)}{r^{n+1}} \qquad (3)$$

$$\psi_1^{(1)} = \sum_{n=0}^{\infty} \psi_{1n} r^n P_n(\mu)$$

where $P_n(\mu)$ is the Legendre polynomial of n^{th} order and $\mu = \cos\theta$. The boundary condition gives

$$\rho_2^{(0)} u_{r2}^{(1)} = \rho_1^{(0)} u_{r1}^{(1)} \quad at \; r = R$$

or

$$\rho_2^{(0)} \left[W P_1(\mu) - \sum_{n=0} (n+1) \frac{\psi_{2n} P_n(\mu)}{R^{n+2}} \right] = \rho_1^{(0)} \left[\sum_{n=0}^{\infty} n \psi_{1n} P_n(\mu) R^{n-1} \right]$$

Equating the coefficients of each of the Legendre polynomials we obtained

$$\left.\begin{array}{ll} n=0 & \psi_{20} = 0 \\ n=1 & W - \frac{2\psi_{21}}{R^3} = \eta \psi_{11} \\ n>1 & \psi_{1n} = -\frac{1}{\eta} \frac{(n+1)}{n} \frac{\psi_{2n}}{R^{2n+1}} \end{array}\right\} \qquad (4)$$

This does not completely specify the velocity of the first order flow and it is necessary to go to the second order equations to obtain the condition which determines $u_i^{(1)}$. Since $u_i^{(0)} = 0$ and $\rho^{(1)} = 0$, the second order equations reduce to

$$\frac{\partial}{\partial x^i}\left(\rho^{(0)} u_i^{(2)}\right) = 0 \tag{5}$$

$$\frac{\partial}{\partial x^i}\left(\rho^{(0)} u_i^{(1)} u_j^{(1)} + p^{(2)}\delta_{ij}\right) = 0 \tag{6}$$

$$\left.\begin{array}{l} \left[\rho^{(0)} u_i^{(2)} n_i\right] = 0 \\ \left[\left\{\rho^{(0)} u_i^{(1)} u_j^{(1)} + p^{(2)}\delta_{ij}\right\} n_i\right] = 0 \end{array}\right\} \text{at } r = R \tag{7}$$

The second boundary condition can be rewritten as

$$\left[\rho^{(0)} u_r^{(1)} u_r^{(1)} + p^{(2)}\right] = 0$$

and

$$\left[\rho^{(0)} u_n^{(1)} u_\theta^{(1)}\right] = 0 \tag{8}$$

Combining Eqs.(2) and (8) gives

$$\left[u_\theta^{(1)}\right] = 0 \quad \text{at } r = R$$

Substituting for $u_\theta^{(1)}$ in terms of the velocity potential (3) and equating coefficients of each of the Legendre polynomials yields

$$n = 1 \qquad \psi_{11} = W + \frac{\psi_{21}}{R^3}$$

$$n > 1 \qquad \psi_{1n} = -\frac{1}{\eta}\,\frac{(n+1)}{n}\,\frac{1}{R^{2n+1}}\,\psi_{2n}$$

Combining this with (4) gives

$$\psi_{1n} = \psi_{2n} = 0 \qquad n \neq 1$$

$$\psi_{11} = \frac{3W}{\eta+2} \quad ; \quad \psi_{21} = \frac{(1-\eta)}{(2+\eta)} R^3 W$$

The velocity potential is then

$$\psi_1^{(1)} = \frac{3W}{(\eta+2)} r\mu \qquad \psi_2^{(1)} = W r\mu + \frac{(1-\eta)}{(\eta+2)} W \frac{R^3\mu}{r^2}$$

and the velocity is given by

$$\underset{\sim}{u}_1^{(1)} = \frac{3W}{\eta+2}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$\underset{\sim}{u}_2^{(1)} = W\left[\cos\theta\left\{1 - \frac{2R^3}{r^3}\frac{(1-\eta)}{(\eta+2)}\right\}\hat{r} - \sin\theta\left\{1 + \frac{R^3}{r^3}\frac{(1-\eta)}{(\eta+2)}\right\}\hat{\theta}\right] \tag{9}$$

This flow is shown in Fig.2.

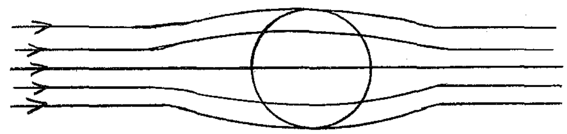

Fig.2

From Eq.(5)

This is the same equation and boundary conditions as applied for $u_i^{(1)}$. Therefore, we can assume that these results are incorporated in the results for $u_i^{(1)}$ and set $u_i^{(2)}=0$. Equation (6) admits of a first integral

$$\frac{1}{2}u_i^{(1)}u_i^{(1)}+\frac{p^{(2)}}{\rho^{(0)}} = \text{constant on streamlines} \tag{10}$$

For the external flow $p^{(2)}\to 0$ and $|u^{(1)}|^2\to W^2$ and the constant is $\frac{1}{2}W^2$ which is the same for all streamlines. Since $p^{(2)}=a^2\rho^{(2)}$ we can solve this for $\rho_2^{(2)}$

$$\rho_2^{(2)}=-\frac{\rho_2^{(0)}}{a_2^2}\frac{(1-\eta)}{(\eta+2)}R^3W^2\left\{\frac{(1-\eta)R^3(1+3\cos^2\theta)}{2(\eta+2)r^6}+\frac{(1+3\cos^2\theta)}{r^3}\right\} \tag{11}$$

Within the H II region $\frac{1}{2}\left(u_{i1}^{(1)}\right)^2+\frac{a_1^2\,\rho_1^{(2)}}{\rho_1^{(0)}}$ is constant on streamlines. These streamlines are in the $\hat{k}$ direction and $u_{i1}^{(1)}$ is constant along them. Therefore $\rho_1^{(2)}$ is also constant on these streamlines. The boundary condition (8) specifies $\rho_1^{(2)}$ at $r=R$; this then makes it possible to solve for $\rho_1^{(2)}$

$$\rho_1^{(2)}=\rho_1^{(0)}\left[9\left(\frac{W}{a_2}\right)^2\frac{\eta(\eta-1)}{(\eta+2)^2}\left(1-\frac{r^2\sin^2\theta}{R^2}\right)-\frac{(1-\eta)}{(\eta+2)}\left(\frac{W^2}{a^2}\right)\left\{\frac{1}{2}\frac{(1-\eta)}{(\eta+2)}\left(4-\frac{3r^2\sin^2\theta}{R^2}\right)-\left(2-\frac{3r^2\sin^2\theta}{R^2}\right)\right\}\right] \tag{12}$$

The second order density distribution is shown in Fig.3. The results along a line in the direction of motion of the star are shown in Fig.3a while those along the line perpendicular to the star's motion are given in Fig.3b where $m_2=\dfrac{W}{a_2}$.

The third order equations are

$$\begin{aligned}&\frac{\partial}{\partial x^j}p^{(3)}=0\\&\frac{\partial}{\partial x^i}\left(\rho^{(2)}u_i^{(1)}+\rho^{(0)}u_i^{(3)}\right)=0\end{aligned} \tag{13}$$

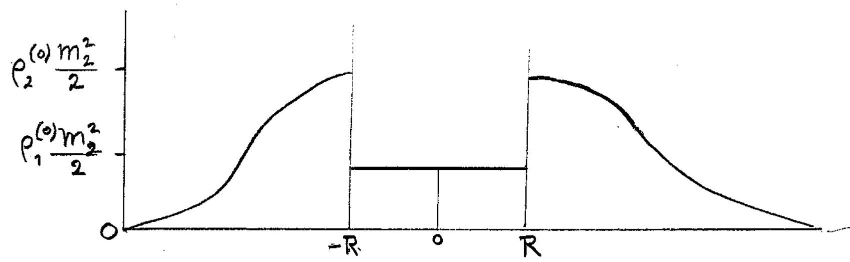

Fig.3a

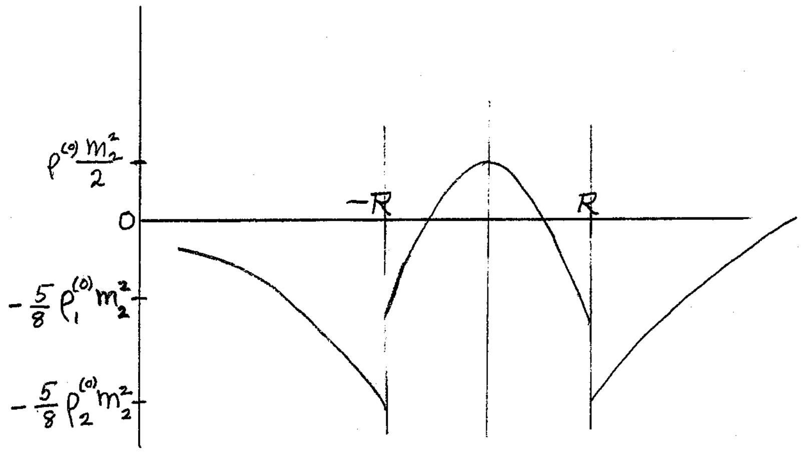

Fig.3b

$$[p^{(3)} n_j] = 0$$
$$[(\rho^{(2)}u_i^{(1)} + \rho^{(0)}u_i^{(3)})n_i] = 0 \tag{14}$$

Following the same arguments used for the first order equation, we can set $p^{(3)} = 0$. Combining Eqs.(6) and (13) yields

$$\rho^{(0)}\frac{\partial u_i^{(3)}}{\partial x^i} = \frac{1}{a^2}\frac{\partial}{\partial x^j}\left(u_i^{(1)}u_i^{(1)}u_j^{(1)}\right) \tag{15}$$

Within the H II region $u_i^{(1)}$ is constant, so $\frac{\partial u_{i1}^{(3)}}{\partial x^i} = 0$. Although this is the same equation as was obtained for $u_i^{(2)}$, the boundary conditions will be different since $\frac{\partial}{\partial x^i}u_{i2}^{(3)} \neq 0$ and therefore it is not possible to set $u_{i,1}^{(3)} = 0$. However we still assume potential flow, so

$$(u_i^{(3)})_1 = \frac{\partial}{\partial x^i}\Psi_{11}$$
$$\Psi_{11} = \sum_{n=0}^{\infty}\Psi_n r^n P_n(\mu) \tag{16}$$

For the external flow, we know the right-hand side of Eq.(15). Assuming potential flow and using (9) for $\underset{\sim}{u}_1$, (15) reduces to

$$\nabla^2\psi = P_1(\mu)\mathcal{F}_1(r) + P_3(\mu)\mathcal{F}_3(r)$$

where $\mathcal{F}_1(r)$ and $\mathcal{F}_3(r)$ are known functions of r alone. We looked for solutions of the form

$$\psi = \sum \frac{A_n(r)\,P_n(\mu)}{r^{n+1}}$$

The result is

$$\psi = \frac{1}{a_2^2}\left\{\frac{2}{3}\frac{W^3\alpha^3}{r^8} - \frac{6}{5}\frac{W^3\alpha^2}{r^5}\right\}P_1(\mu) + \frac{1}{a_2^2}\left\{-\frac{6}{5}\frac{W^3\alpha^3}{r^2} + \frac{3}{11}\frac{W^3\alpha^3}{r^8} - \frac{12}{5}\frac{W^3\alpha^2}{r^5}\right\}P_3(\mu)$$

and the velocity is given by

$$\left(u_r^{(3)}\right)_2 = \frac{W^3}{a_2^2}\mu^3\left\{\frac{6\alpha}{r^3} + \frac{30\alpha^2}{r^6} - \frac{60}{11}\frac{\alpha^3}{r^9}\right\} + \frac{W^3}{a_2^2}\mu\left\{-\frac{18}{5}\frac{\alpha}{r^3} - \frac{12\alpha^2}{r^6} - \frac{68}{33}\frac{\alpha^3}{r^9}\right\}$$

$$\left(u_\theta^{(3)}\right)_2 = \frac{W^3}{a_2^2}\sqrt{1-\mu^2}\left\{-\frac{9}{5}\frac{\alpha}{r^3} - \frac{84}{5}\frac{\alpha^2}{r^6} - \frac{17}{66}\frac{\alpha^3}{r^9}\right\} + \frac{W^3}{a_2^2}\sqrt{1-\mu^2}\,\mu^2\left\{+\frac{9\alpha}{r^3} + \frac{30\alpha^2}{r^6} - \frac{45}{22}\frac{\alpha^3}{r^9}\right\}$$

where $$\alpha = \frac{(1-\eta)R^3}{(\eta+2)}$$

Using the boundary condition (14), we can now determine the coefficients in (16) for the internal flow. All the terms vanish except for $n = 1$ and 3

$$\psi_{11} = \psi_1 r P_1(\mu) + \psi_3 r^3 P_3(\mu)$$

where

$$\psi_1 = \frac{1}{a_1^2}\left(\frac{3W}{\eta+2}\right)^3 + \frac{W^3}{a_2^2}\left\{-1 + \frac{23}{5}\frac{\alpha}{R^3} + \frac{\alpha^2}{R^6} - \frac{12}{5}\frac{\alpha^3}{R^9}\right\}$$

$$\psi_3 = \frac{1}{3R^2a^2}\left(\frac{3W}{\eta+2}\right)^3 + \frac{W^3}{3R^2a_2^2}\left\{+\frac{12}{5}\frac{\alpha}{R^3} - \frac{44}{15}\frac{\alpha^3}{R^9}\right\}$$

The velocity within the H II region is then given by

$$\left(u_r^{(3)}\right)_1 = \psi_1\cos\theta + \frac{3}{2}\psi_3 r^2\cos\theta\,(5\cos^2\theta - 3)$$

$$\left(u_\theta^{(3)}\right)_1 = -\psi_1\sin\theta - 3\,\frac{\psi_3 r^2}{2}\sin\theta\,(5\cos^2\theta - 1)$$

In this order, the flow within the H II region is a quadruple flow superimposed on a constant flow in the $\hat{k}$ direction. For the external flow, the leading terms are a quadruple flow. In third order, the flow pattern is shown in Fig.4.

The result of adding this to the first order velocity flow (Fig.3) would be to deform it slightly, increasing the "bump" at the sphere which appears in Fig.3. However, these new effects are of importance only very near the H II region. Higher order effects will influence the flow only

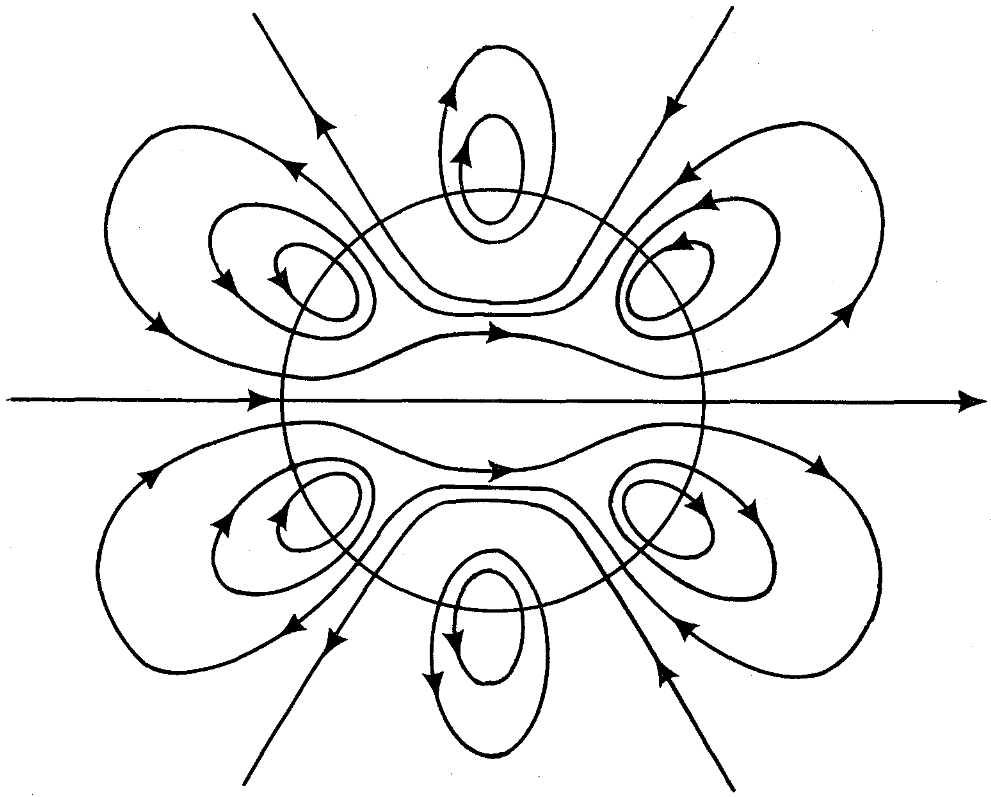

Fig.4

in the immediate vicinity of the H II region. These terms were therefore not calculated.

Thermal Problem

The previous approach to the problem used certain assumptions to eliminate the thermal problem. We next attempted to include this aspect in considering the case of a heat source moving through an infinite compressible medium at a constant velocity. In addition to the equations for the conservation of mass and momentum used above, we also must specify conservation of the thermal energy.

$$\frac{\partial}{\partial x^i}\left(\rho u_i\left(\frac{1}{2}u_i^2+w\right)-\varkappa\frac{\partial}{\partial x^i}T\right)=0$$

where T is the temperature, χ is the thermal conductivity and w, the enthalpy, $C_p T$ for an ideal gas where C_p is the specific heat at constant pressure.

Unsing the same technique as for the previous problem, we can expand all the variables in a series, the zero order equations are then

$$\frac{\partial}{\partial x^i}\left(\rho^{(0)} u_i^{(0)} u_j^{(0)} + p^{(0)}\delta_{ij}\right) = 0$$
$$\frac{\partial}{\partial x^i}\left(\rho^{(0)} u_i^{(0)}\right) = 0$$
$$\frac{\partial}{\partial x^i}\left(\rho^{(0)} u_i^{(0)}\left\{\frac{1}{2}u_i^{(0)}u_i^{(0)} + c_p T^{(0)}\right\} - \chi\frac{\partial T^{(0)}}{\partial x^i}\right) = 0$$

As before, we assume $u_i^{(0)} = 0$ giving

$$\frac{\partial p^{(0)}}{\partial x^j} = 0 \qquad \frac{\partial}{\partial x^i}\left(\chi\frac{\partial T^{(0)}}{\partial x^i}\right) = 0$$

If we take χ = const, the solution is

$$p^{(0)} = \text{constant} = p_0$$
$$\nabla^2 T^{(0)} = 0 \implies T^{(0)} = \sum_n \frac{A_n P_n(\mu)}{r^{n+1}} + \sum_n B_n P_n(\mu) r^n$$

We want $T \to$ constant $= T_0$ as $r \to \infty$ which implies $B_n = 0$ for $n \geq 1$. So, we have

$$T^{(0)} = T_0 + \sum_n \frac{A_n P_n(\mu)}{r^{n+1}}$$

The only other boundary condition that must be satisfied is that the flux is constant and equal to the flux put out by the star. This condition will specify A_0. As there is no need for any other terms to satisfy the boundary conditions, it will be convenient to set $A_n = 0$ for $n > 0$ and assume that these terms will be determined at the higher orders. So we have

$$T^{(0)} = T_0 + \frac{A}{r}$$

where $A = L/4\pi\chi$ with L being the luminosity of the star. Assuming an ideal gas equation of state

$$p = \frac{k}{m_H \mu}\rho T$$
$$\rho^{(0)} = \frac{m_H \mu}{k} p_0 \frac{r}{T_0 r + A}$$

This completely specifies the zero order solution. The first order equations are

$$\frac{\partial}{\partial x^i} p^{(1)} = 0 \tag{17}$$

$$\frac{\partial}{\partial x^i} \rho^{(0)} u_i^{(1)} = 0 \tag{18}$$

$$\frac{\partial}{\partial x^i}\left(\rho^{(0)} u_i^{(1)} c_p T^{(0)} - \chi \frac{\partial T^{(0)}}{\partial x^i}\right) = 0 \tag{19}$$

$$p^{(1)} = \frac{k}{m_H \mu} \left(\rho^{(0)} T^{(1)} + \rho^{(1)} T^{(0)}\right)$$

Equation (17) implies $p^{(1)}$ = constant which can be set equal to zero following the same reasoning as used in the previous problem.

The second equation gives

$$\rho^{(0)} u_i^{(1)} = \mathrm{curl}\, \psi \hat{\Phi} \tag{20}$$

where ψ is an arbitrary function.

Equation (19) gives

$$\chi \nabla^2 T^{(1)} = c_p \frac{\mu m_H}{k} p_0 \frac{\partial}{\partial x^i} u_i^{(1)}$$

So the temperature distribution in first order is determined by the first order velocity.

As in the previous problem, the first order velocity is not entirely specified by the first order equations. The second order equations are

$$\frac{\partial}{\partial x^i}\left[\rho^{(1)} u_i^{(1)} + \rho^{(0)} u_i^{(2)}\right] = 0 \tag{21}$$

$$\frac{\partial}{\partial x^i}\left[\rho^{(0)} u_i^{(1)} u_j^{(1)} + p^{(2)} \delta_{ij}\right] = 0 \tag{22}$$

$$\frac{\partial}{\partial x^i}\left[\rho^{(0)} u_i^{(1)} c_p T^{(1)} + \rho^{(1)} u_i^{(1)} c_p T^{(0)} + \rho^{(0)} u_i^{(2)} T^{(0)} c_p - \chi \frac{\partial}{\partial x^i} T^{(1)}\right] = 0 \tag{23}$$

Equation (22) can be rewritten as

$$\rho^{(0)} u_i^{(1)} \frac{\partial}{\partial x^i} u_j^{(1)} = -\frac{\partial}{\partial x^j} p^{(2)}$$

So $\rho^{(0)} u_i^{(1)} \frac{\partial}{\partial x^i} u_j^{(1)}$ is the gradient of a scalar and

$$\mathrm{curl}\left[\rho^{(0)} u_i^{(1)} \frac{\partial}{\partial x^i} u_j^{(1)}\right] = 0 \tag{24}$$

Since $\rho^{(0)}$ is a known function, we can substitute (20) into (24) to obtain an equation for ψ which is in general a third order non-linear differential equation. Using Cartesian coordinates, we can reduce this equation to the form

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial x}, \psi\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial y}, \psi\right) = 0 \tag{25}$$

where
$$(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

Equation (25) admits of a first integral

$$\left(\frac{1}{\rho}\frac{\partial \psi}{\partial x}, \psi\right) = \frac{\partial \chi}{\partial y}$$
$$\left(\frac{1}{\rho}\frac{\partial \psi}{\partial y}, \psi\right) = -\frac{\partial \chi}{\partial x}$$

where χ is an arbitrary function of x and y. It did not prove possible to proceed further along these lines, so we considered an alternate approach.

For an actual H II region, the temperature drops sharply at the boundary and in general the temperature in the H II region is considerably higher than in the exterior. Therefore, we considered the flow in the region near the star where $T^{(0)} = T_0 + \frac{A}{r} \approx \frac{A}{r}$ and so where $\rho \propto r$. Further, we set $\psi = r \sin\theta\, \phi(r)$. This gives a velocity

$$\underset{\sim}{u} = \frac{2\mu\phi}{r}\hat{r} - \sqrt{1-\mu^2}\left(\frac{d\phi}{dr} + \frac{2\phi}{r}\right)\hat{\theta} \tag{26}$$

So this choice of ψ is equivalent to keeping only the first term in a multipole expansion of the velocity. Substituting (26) in (24) gives a third order non-linear differential equation for ϕ

$$-2r\phi\phi''' - 4\phi\phi'' + \phi'^2 - \frac{12\phi^2}{r^2} + \frac{20\phi\phi'}{r} = 0 \tag{27}$$

where primes denote differentiation with respect to r. This equation admits of particular solutions $\phi = Ar^n$ where n = -2.8, +0.6, +3.74. This can be reduced to a first order non-linear differential equation

$$p\frac{dp}{dg} + p = H(g) \tag{28}$$

where

$$p = \frac{d}{dr}\left\{ r \frac{d\ln(\phi r^{-n})}{dr} \right\}$$

$$dg = (3n - 1 + 3f)\,df$$

$$f = r \frac{d}{de} \ln(\phi r^{-n})$$

$$H(g) = \frac{\left[\left(\frac{6n - 6n^2 + 20}{2}\right) f + \frac{(6n-3)f^2}{2} - f^3 \right]}{3n - 1 + 3f}$$

and n is any of the values for which $\phi = Ar^n$ solves Eq.(27). We have not been able to obtain any further results on this problem. Equation (28) is in a form which is convenient for numerical solution. However, we have not attempted such a numerical approach to the problem.

Once the solution to this problem is obtained, the next step would be to allow for ionization to occur. This would re-introduce the surface of the problem. A further generalization of the problem which in many respects would be more interesting would be to consider the case where W is comparable to or greater than the speed of sound in the medium. If this aspect of the problem could be solved, it should be possible to go back and consider the initial stages of the problem in which the ionization front itself moves into the interstellar gas with supersonic velocities.

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