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Contents of the Volumes

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Editors' Preface

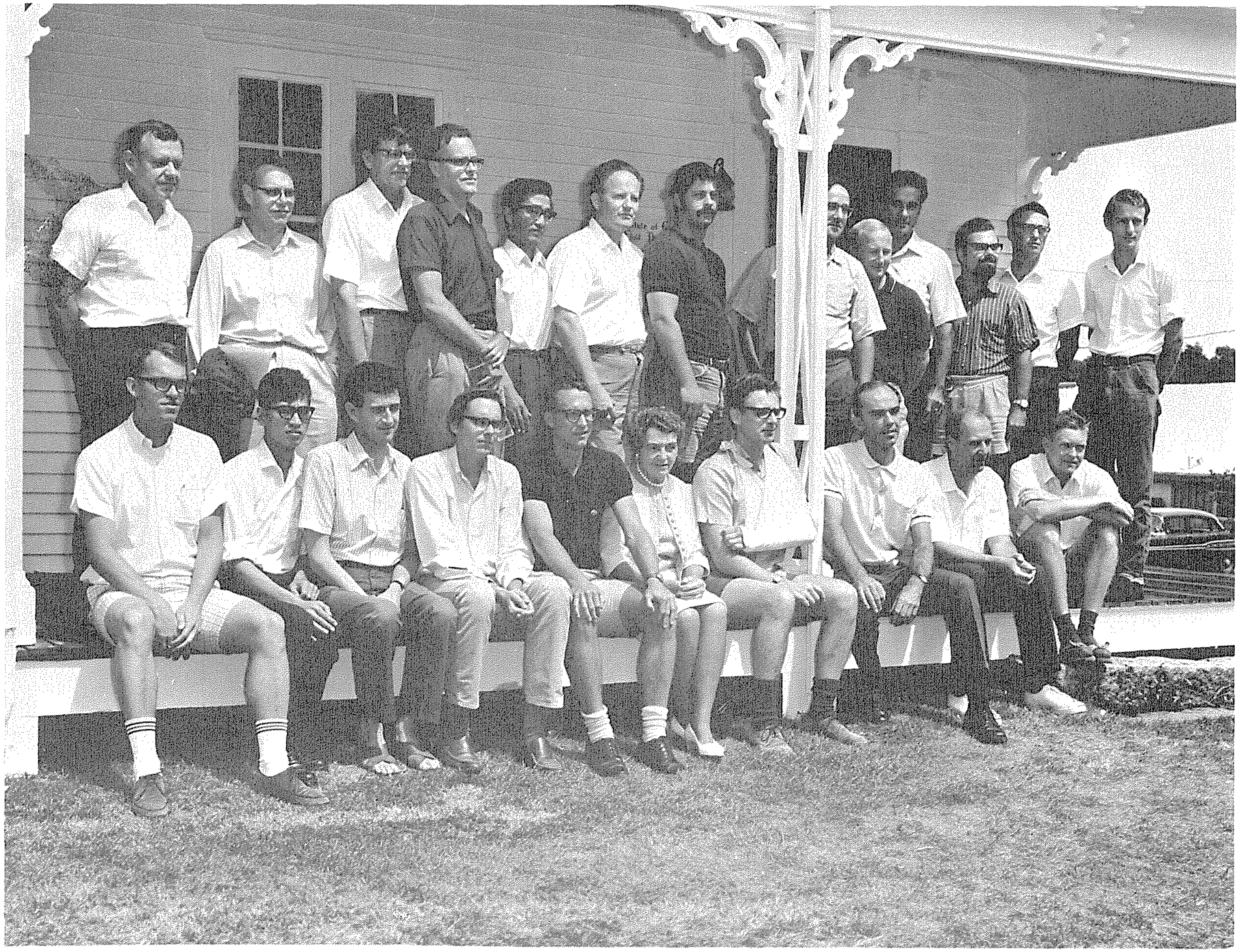
This volume contains the final reports by the student participants of their research activities of the summer.

As in previous years the efforts reported by the students reflect varying degrees of originality and completeness. A few of the students were guided carefully; with others there was little contact between student and staff. Some of the students either posed or were given precisely formulated problems which could be essentially completed during the summer. Others accepted problems which were not well-posed and which involved a period of groping and searching for a tractable and reasonably finite problem with a significant goal.

Time limitations did not allow the participants to rework the manuscripts and the present records must be interpreted as only interim reports.

We who took part in the program this summer are deeply indebted to the National Science Foundation for its continued support of the program and to the Woods Hole Oceanographic Institution for its support and encouragement and for the use of its facilities.

Mary C. Thayer
George Veronis



BACK ROW, LEFT to RIGHT: Stommel, Charnock, Cox, Turner, Yoshida, Rooth, Rosati, Stern, Kullenberg, Veronis, Gough, Malkus, Hazel. FRONT ROW: Foster, Kuo, Morton, Luyten, Allen, Thayer, Mooers, Gregg, Longuet-Higgins, Howard.

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Fellows' Reports

SOME ASPECTS OF THE INVISCID MOTION OF A CONTAINED ROTATING AND STRATIFIED FLUID

John S. Allen

Abstract

The linear inviscid motion of a contained rotating and stratified fluid is considered in the limit of weak stratification. The limiting flow is of interest because the method of determining the steady "geostrophic" component of the stratified flow does not necessarily give a flow which reduces to the steady homogeneous flow. By including a consideration of slow unsteady motions in a time scale dependent on the stratification parameter the relationship of the limiting stratified flow to the homogeneous steady flow is established. The containers are assumed to possess closed contours of constant height but are of general shape otherwise.

1. Introduction

If the inviscid linearized motion of a contained homogeneous rotating fluid is described by a superposition of oscillatory modes and a steady "geostrophic" mode, Greenspan[1] has shown that the steady mode can be specified, in terms of the initial conditions, with the aid of the mean circulation theorem. When the fluid is also stratified, with the gravity vector antiparallel to the rotation vector, Howard[2,3] and Siegmann[2,4] have shown that the steady mode can be determined by the solution to an equation resulting from the time independence of the potential vorticity. As pointed out by Howard in his lecture series in this course (ref. 3, lecture 6), in the limit of weak stratification, i.e. for vanishingly small values of the parameter $4S = \frac{N^2}{\Omega^2}$ where N is the Brunt-Väisälä frequency and Ω the rotational frequency, the method of determining the steady mode in the stratified case does not in general reduce to that used in the homogeneous problem and therefore the relationship of the limiting flow to the homogeneous flow is not completely straightforward. To relate the two cases it is necessary to include a consideration of the unsteady motions.

In this paper we consider the limit of weak stratification for the inviscid motion of a contained rotating and stratified fluid and primarily investigate the relation of the limiting flow to the homogeneous steady flow. It turns out that the characteristics of the limiting flow are different for containers with contours of constant height that lie in planes perpendicular to the gravity and rotation vectors (we will refer to these as "flat" contours) and for more generally shaped containers which, for the homogeneous geostrophic flow, require a component of velocity parallel to the rotation and gravity vectors. Therefore, we first consider bodies with "flat" contours. For more general containers, which are assumed, however, to possess closed contours of constant height, there is again a difference between the limiting flows when the body does not have a uniquely defined set of contours and when it does. These two cases are treated in order.

2. Equations

We consider an incompressible fluid, which satisfies the Boussinesq approximation, in a frame of reference rotating with a uniform angular velocity $\underline{\Omega} = \Omega \underline{k}$ and with a gravitational body force $\underline{g} = -g\underline{k}$, which is antiparallel to the rotation vector.

The governing equations are

$$\nabla \cdot \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + 2\Omega \underline{k} \times \underline{u} = -\frac{1}{\rho_0} \nabla p - \frac{\rho}{\rho_0} g \underline{k} + \nu \nabla^2 \underline{u} + \frac{\rho}{\rho_0} \frac{\Omega^2}{2} \nabla \left| \underline{k} \times \underline{r} \right|^2$$

$$\frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T = \kappa \nabla^2 T$$

$$\rho = \rho_0 \left[1 - \alpha (T - T_0) \right]$$

where the variables have their usual meaning.

We assume the Froude number $\frac{\Omega^2 L}{g}$ small and consider a linear equilibrium temperature and density distribution

$$T_s = T_o + \Delta T_o z/L$$
$$\rho_s = \rho_o - \rho_o \alpha \Delta T_o z/L$$

where $\Delta T_o \geq 0$ is the basic temperature difference imposed over the typical height L.

The variables are non-dimensionalized in the following manner

$$\underline{u} = \mathcal{U}\underline{u}^*, \qquad \underline{x} = L\underline{x}^*, \qquad t = \Omega^{-1}t^*$$
$$p = p_o - \rho_o g L z^* + \tfrac{1}{2}\rho_o g L \alpha \Delta T_o z^{*2} + \rho_o \mathcal{U}_o \Omega L p^*$$
$$T = T_s + \Delta T T^*$$

The resulting dimensionless equations are (dropping the stars)

$$\nabla \cdot \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + R_o \underline{u} \cdot \nabla \underline{u} + 2\underline{K} \times \underline{u} + \nabla p - \delta T \underline{K} = E \nabla^2 \underline{u}$$
$$\frac{\partial T}{\partial t} + R_o \underline{u} \cdot \nabla T + \frac{4S}{\delta} \underline{u} \cdot \underline{k} = E/P_r \, \nabla^2 T$$

where $\quad E = \frac{\nu}{\Omega L^2}, \quad R_o = \frac{\mathcal{U}}{\Omega L}, \quad P_r = \frac{\nu}{\kappa}$

$\frac{\alpha g \Delta T_o}{\Omega^2 L} = \frac{N^2}{\Omega^2}$ (the square of the ratio of the Brunt-Vaisala frequency to the rotation frequency)

and $\quad \delta = \frac{\alpha g \Delta T}{\Omega \mathcal{U}}$

If we assume $P_r \sim O(1)$ and $R_o, E <<<< 1$ we obtain the linearized inviscid equations

$$\nabla \cdot \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + 2\underline{k} \times \underline{u} + \nabla p - \delta T \underline{k} = 0$$

$$\frac{\partial T}{\partial t} + \frac{4S}{\delta} \underline{u} \cdot \underline{k} = 0$$

The parameter δ involves a relation between the value of the characteristic velocity $\mathcal{U}$ and the value of the characteristic perturbation temperature ΔT and is usually chosen to achieve certain balances in the equations when the flow is driven by either a given $\mathcal{U}$ or a given ΔT. For solutions of the above equations in the limit $4S << 1$ an appropriate scaling for δ is apparently

This choice implies

$$\Delta T = \frac{R_o}{4S} \Delta T_o = \mathcal{U} \sqrt{\frac{\Delta T_o}{L \alpha g}}$$

We will therefore consider problems for which the dimensional initial values vary in accordance with the above scaling in the limit of $\sqrt{4S} << 1$.

The set of equations to be considered are then

$$\nabla \cdot \underline{u} = 0 \tag{1a}$$

$$\frac{\partial \underline{u}}{\partial t} + 2\, \underline{k} \times \underline{u} + \nabla p - \sqrt{4S}\; T \underline{K} = 0 \tag{1b}$$

$$\frac{\partial T}{\partial t} + \sqrt{4S}\; \underline{u} \cdot \underline{K} = 0 \tag{1c}$$

The boundary and initial conditions for these equations for flows in closed containers are

$$\underline{u} \cdot \underline{n} = 0 \quad \text{in the boundary} \tag{2}$$

and

$$\underline{u}(\underline{x}, 0) = \underline{u}_{IC}(\underline{x}) \tag{3a}$$

$$T(\underline{x}, 0) = T_{IC}(\underline{x}) \tag{3b}$$

If a solution to the above equations is sought by a superposition of oscillatory modes and a steady geostrophic component

$$\underline{u} = \underline{u}_g(\underline{x}) + \sum_m \underline{u}_m(\underline{x}) e^{i\sigma_m t} \tag{4a}$$

$$T = T_g(\underline{x}) + \sum_m T_m(\underline{x}) e^{i\sigma_m t} \tag{4b}$$

Howard and Siegmann[2,3,4] have shown that the steady component, which satisfies

$$\nabla \cdot \underline{u}_g = 0 \tag{5a}$$

$$2\underline{k} \times \underline{u}_g + \nabla p_g - \sqrt{4S}\, T_g \underline{k} = 0 \tag{5b}$$

$$\sqrt{4S}\, \underline{u}_g \cdot \underline{k} = 0 \quad , \tag{5c}$$

can be uniquely determined from the equation

$$2\nabla \times \underline{u}_g \cdot \underline{k} + \frac{4}{\sqrt{4S}} \frac{\partial T}{\partial z} = 2\nabla \times \underline{u}_{IC} \cdot \underline{k} + \frac{4}{\sqrt{4S}} \frac{\partial T_{IC}}{\partial z} \tag{6}$$

which, written in terms of p_g , is

$$\nabla_H^2 p_g + \frac{1}{S} \frac{\partial^2 p_g}{\partial z^2} = 2\nabla \times \underline{u}_{IC} \cdot \underline{k} + \frac{4}{\sqrt{4S}} \frac{\partial T_{IC}}{\partial z} \tag{7}$$

where ∇_H^2 is the horizontal Laplacian, with the boundary conditions

$$(a) \qquad \sqrt{4S} \int_{A(z)} T_g \, da = \int_{A(z)} \frac{\partial p_g}{\partial z} \, da = \sqrt{4S} \int_{A(z)} T_{IC} \, da \qquad \text{for all } A(z) \tag{8a}$$

where is an integral over the area in a horizontal plane enclosed by the boundaries of the container,

$$(b) \qquad \sqrt{4S}\, T_g = \frac{\partial p_g}{\partial z} = \sqrt{4S}\, T_{IC} \quad \text{on horizontal "flat" boundaries,} \tag{8b}$$

and (c) $\quad p$ = constant for z = constant and x,y on the boundaries (8c)

For a homogeneous fluid the governing equations are

$$\nabla \cdot \underline{u} = 0 \tag{10a}$$

$$\frac{\partial \underline{u}}{\partial t} + 2\underline{k} \times \underline{u} + \nabla p = 0 \tag{10b}$$

If a solution to these equations is sought in the form (4) the steady geostrophic flow satisfies

$$\nabla \cdot \underline{u}_g = 0 \tag{11a}$$

$$2\underline{K} \times \underline{u}_g + \nabla p_g = 0 \tag{11b}$$

From equations (11) we find

$$\underline{K} \cdot \nabla \underline{u}_g = 0 \tag{12a}$$

$$\frac{\partial p_g}{\partial z} = 0 \tag{12b}$$

and

$$\underline{u}_g = (\underline{u}_g \cdot \underline{k})\,\underline{k} + \tfrac{1}{2}\,\underline{K} \times \nabla p_g \tag{13}$$

If we consider a container whose surface is an envelope of closed contours of constant height, H, the geostropic pressure is a function of H and the geostrophic velocity can be written

$$\underline{u}_g = -\frac{1}{2}\frac{dp_g}{dH}\,\underline{n}_T \times \underline{n}_B \tag{14}$$

where, if the top and bottom surfaces are, respectively

$$z = z_T\,(x,y)$$
$$z = z_B\,(x,y)$$
$$\underline{n}_T = \underline{K} - \nabla z_T = \left[1 + (\nabla z_T)^2\right]^{1/2} \hat{\underline{n}}_T$$
$$\underline{n}_B = -\underline{K} + \nabla z_B = \left[1 + (\nabla z_B)^2\right]^{1/2} \hat{\underline{n}}_B$$

As shown by Greenspan[1], the oscillatory modes possess no mean circulation about a geostrophic contour, i.e.

$$\oint_{C(H)} d\underline{s} \cdot \frac{1}{H} \int_{z_B}^{z_T} \underline{u}_m \, dz = 0 \tag{15}$$

and this allows the specification of the arbitrary function $\frac{dp_g}{dH}$ in equation (14) in terms of the mean circulation of the initial velocity distribution. That is

$$\frac{dp_g}{dH} = -2H\,\frac{MC(H)}{J(H)} \tag{16}$$

where

$$MC(H) = \oint_{C(H)} d\underline{s} \cdot \frac{1}{H} \int_{z_B}^{z_T} \underline{u}_{IC} \, dz \tag{17}$$

and

$$J(H) = H \oint_{C(H)} \underline{n}_T \times \underline{n}_B \cdot d\underline{s} \tag{18}$$

where

$$d\underline{s} = \frac{\hat{\underline{n}}_T \times \hat{\underline{n}}_B}{|\hat{\underline{n}}_T \times \hat{\underline{n}}_B|} \cdot ds$$

If we consider solutions to the equations (1) in the limit $\sqrt{4S} << 1$ we expect to regain the solutions to the homogeneous equations (10). However, we can see that the steady solution to (1), satisfying equations (5) will, for all values of $\sqrt{4S}$ not identically zero, have

$$\underline{u}_g \cdot \underline{k} = 0 \ .$$

Therefore, in the limit these solutions can not possibly approach the steady geostrophic flow of a homogeneous fluid in a container whose closed contours of constant height require a non-zero value of $\underline{u}_g \cdot \underline{k}$. The exact relationship of the limiting flow and the homogeneous flow is then not completely obvious. In the next sections we consider the solutions to equations (1) in the limit $\sqrt{4S} << 1$.

3. Containers with "Flat" Contours of Constant Height

For a container with "flat" contours of constant height, for example a sphere, the steady geostrophic flow of a homogeneous fluid does not require any vertical component of velocity. In this case we will show that the steady mode of the homogeneous fluid is included in the geostrophic flow of the stratified fluid and that it is the limiting solution to equations (5) determined by equation (7) and boundary conditions (8).

First it is easy to show from equation (6) (and also directly from (1)) that the mean circulation of the geostrophic flow is equal to that of the initial flow.

If we integrate equation (6) over an area in the horizontal plane bounded by a geostrophic contour C(H) and then over the vertical distance H between the bounding top and bottom contours we find

$$\int_{z_B}^{z_T} dz \int_{A(H)} 2\nabla\times\underline{u}_g\cdot\underline{k}\,da + \frac{4}{\sqrt{4S}}\left[\int_{A(H)} T_g\,dA\Bigg|_{z=z_B(H)}^{z=z_T(H)}\right] =$$

$$= \int_{z_B}^{z_T} dz \int_{A(H)} 2\nabla\times\underline{u}_{IC}\cdot\underline{k}\,da + \frac{4}{\sqrt{4S}}\left[\int_{A(H)} T_{IC}\,dA\Bigg|_{z=z_B(H)}^{z=z_T(H)}\right] \qquad (19)$$

Using boundary condition (8) and Stokes's theorem we have

$$\oint_{C(H)} d\underline{s}\cdot\frac{1}{H}\int_{z_B}^{z_T}\underline{u}_g\cdot dz = \oint_{C(H)} d\underline{s}\cdot\frac{1}{H}\int_{z_B}^{z_T}\underline{u}_{IC}\,dz$$

or

$$MC_g(H) = MC_{IC}(H) \qquad (20)$$

Next, for $\sqrt{4S} << 1$ we consider the solution to equations (5) determined by equation (7) and boundary conditions (8). Assuming that the solution has the following expansion in $\sqrt{4S}$

$$\underline{u}_g = \underline{u}_{g0} + \sqrt{4S}\,\underline{u}_{g1} + \dots$$

$$p_g = p_{g0} + \sqrt{4S}\,p_{g1} + 4S\,p_{g2} + \dots$$

$$T_g = T_{g0} + \sqrt{4S}\,T_{g1} + \dots$$

and substituting this in equations (5) and (7), we get the equations

<u>0^{th} Order</u>

Equation (7) is $$\frac{\partial^2 p_{g0}}{\partial z^2} = 0$$

Equation (5) gives

$$\underline{u}_{go} = \frac{1}{2}\underline{k} \times \nabla p_{go} \tag{21a}$$

$$\frac{\partial p_{go}}{\partial z} = 0 \tag{21b}$$

Therefore $p_{go} = p_{go}(x, y)$

and boundary condition (8c) implies

$$p_{go} = P_{go}(H) \tag{21c}$$

1st Order

Equation (7) gives

$$\frac{\partial^2 p_{g1}}{\partial z^2} = \frac{\partial T_{IC}}{\partial z}$$

which, with equation (5)

$$\frac{\partial p_{g1}}{\partial z} = T_{go}$$

shows $T_{go} = T_{IC} + C_2(x,y)$

2nd Order

Equation (7) gives

$$\frac{\partial^2 p_{g2}}{\partial z^2} = 2\nabla \times \underline{u}_{IC} \cdot \underline{k} - 2\nabla \times \underline{u}_{go} \cdot \underline{k} \tag{22}$$

Integrating this equation over the area in a horizontal plane bounded by a geostrophic contour and then over the vertical distance H between bounding contours and using boundary condition (8a) we find

$$M.C_{go}(H) = M.C(H)_{IC} \tag{23}$$

Equations (21) show that the lowest order flow is a possible geostrophic flow of a homogeneous fluid and equation (23) shows that it has the mean circulation of the initial velocity field. Therefore, for containers with "flat" contours of constant height the lowest order steady flow determined by equation (7) for $\sqrt{4S} << 1$ is the same as the geostrophic flow of a homogeneous fluid determined by the mean circulation theorem.

4. Containers of General Shape

For a container with a more general shape where the homogeneous steady flow requires a non-zero component of velocity in the vertical direction we have to include a consideration of the oscillatory modes in the stratified fluid to relate the limiting flow to the homogeneous steady flow.

If we let

$$\underline{u}(\underline{x},t) \rightarrow \underline{u}(\underline{x}) e^{i\sigma t}$$
$$T(\underline{x},t) \rightarrow T(\underline{x}) e^{i\sigma t}$$

equations (1) become

$$\nabla \cdot \underline{u} = 0 \tag{24a}$$

$$i\sigma \underline{u} + 2\underline{k} \times \underline{u} + \nabla p - \sqrt{4S}\, T\underline{k} = 0 \tag{24b}$$

$$i\sigma T + \sqrt{4S}\, \underline{u} \cdot \underline{k} = 0 \tag{24c}$$

Equation (24b) gives for the vertical component of velocity $\underline{u} \cdot \underline{k} = w$,

$$i\sigma w + \frac{\partial p}{\partial z} - \sqrt{4S}\, T\underline{k} = 0. \tag{24d}$$

If, for $\sqrt{4S} << 1$, we look for a balance between $\underline{u} \cdot \underline{k}$ and T in equations (24c) and (24d) and require $\underline{u} \cdot \underline{k} \sim O(1)$, we find

$$\sigma \sim O(\sqrt{4S}).$$

We therefore expand the variables as

$$\begin{aligned} \underline{u} &= \underline{u}_0 + \sqrt{4S}\, \underline{u}_1 + \dots \\ T &= T_0 + \sqrt{4S}\, T_1 + \dots \\ \sigma &= \sqrt{4S}\, \sigma_1 + 4S\sigma_2 + \dots \end{aligned} \tag{25}$$

This gives the following set of equations:

0th Order

$$\nabla \cdot \underline{u}_0 = 0 \tag{26a}$$

$$2\underline{K} \times \underline{u}_0 + \nabla p_0 = 0 \tag{26b}$$

$$\text{B.C.} \quad \underline{u}_0 \cdot \underline{n} = 0 \tag{26c}$$

1st Order

$$\nabla \cdot \underline{u}_1 = 0 \tag{27a}$$

$$i\sigma_1 \underline{u}_0 + 2\underline{K} \times \underline{u}_1 + \nabla p_1 - T_0 \underline{K} = 0 \tag{27b}$$

$$i\sigma_1 T_0 + \underline{u}_0 \cdot \underline{K} = 0 \tag{27c}$$

$$\text{B.C.} \quad \underline{u}_1 \cdot \underline{n} = 0 \tag{27d}$$

From equation (20b) we have

$$\underline{u}_0 = (\underline{u}_0 \cdot \underline{K})\underline{K} + \tfrac{1}{2}\underline{K} \times \nabla p_0 \tag{28a}$$

$$\text{or} \quad \underline{u}_{0H} = \tfrac{1}{2}\underline{K} \times \nabla p_0 \tag{28b}$$

where subscript H denotes the horizontal component,

$$\text{and} \quad \frac{\partial p_0}{\partial z} = 0 \tag{28c}$$

where $p_0 = p_0(x,y)$ is undetermined and where $(\underline{u}_0 \cdot \underline{k})$ can be expressed in terms of $\underline{u}_{0H}$ by applying the boundary condition (26c).

To derive an equation for p_0 we solve equation (27b) for $\underline{K} \cdot \nabla \underline{u}_1$, integrate, and apply the boundary condition (27d).

Taking the curl of equation (27b) we find

$$\underline{K} \cdot \nabla \underline{u}_1 = \frac{i\sigma_1}{2} \nabla \times \underline{u}_0 - \tfrac{1}{2}\nabla \times T_0 \underline{K} \tag{29}$$

If we take the curl of equation (28a) and substitute the resulting expression for $\nabla \times \underline{u}_0$ in (29) and also if we solve for T_0 from equation (27c) and substitute this in (29) we have

$$\underline{k} \cdot \nabla \underline{u}_1 = \frac{i\sigma_1}{4} \nabla^2 p_0 \underline{k} - \frac{1}{2}\left(\frac{\sigma_1^2 - 1}{i\sigma_1}\right) \nabla \times (\underline{u}_0 \cdot \underline{k}) \underline{k} \tag{30}$$

where the right-hand side is a function of x and y only. Integrating equation (30) with respect to z yields

$$\underline{u}_1 = \left[\frac{i\sigma_1}{4} \nabla^2 p_0 \underline{k} - \frac{1}{2}\left(\frac{\sigma_1^2 - 1}{i\sigma_1}\right) \nabla \times (\underline{u}_0 \cdot \underline{k}) \underline{k}\right] z + \underline{A}(x,y) \tag{31}$$

where $\underline{A}(x,y)$ is an arbitrary vector.

To satisfy boundary condition (27d) we require

$$\underline{u}_1 \cdot \underline{n}_T = 0 = z_T \frac{i\sigma_1}{4} \nabla^2 p_0 + z_T\left(\frac{\sigma_1^2 - 1}{i\sigma_1}\right) \nabla \times (\underline{u}_0 \cdot \underline{k}) \underline{k} \cdot \nabla z_T + \underline{A} \cdot (\underline{k} - \nabla z_T) \tag{32}$$

for $z = z_T$ and

$$-\underline{u}_1 \cdot \underline{n}_B = 0 = z_B \frac{i\sigma_1}{4} \nabla^2 p_0 + z_B\left(\frac{\sigma_1^2 - 1}{i\sigma_1}\right) \nabla \times (\underline{u}_0 \cdot \underline{k}) \underline{k} \cdot \nabla z_B + \underline{A}(\underline{k} - \nabla z_B) \tag{33}$$

for $z = z_B$

We can add and subtract (32) and (33) to give, respectively,

$$\frac{i\sigma_1}{4} \nabla^2 p_0 (z_T + z_B) + \frac{1}{2}\left(\frac{\sigma_1^2 - 1}{i\sigma_1}\right) \nabla \times (\underline{u}_0 \cdot \underline{k}) \underline{k} \cdot (z_T \nabla z_T + z_B \nabla z_B) + \underline{A} \cdot (2\underline{k} - (\nabla z_T + \nabla z_B)) = 0 \tag{34}$$

and

$$\frac{i\sigma_1}{4} \nabla^2 p_0 (z_T - z_B) + \frac{1}{2}\left(\frac{\sigma_1^2 - 1}{i\sigma_1}\right) \nabla \times (\underline{u}_0 \cdot \underline{k}) \underline{k} \cdot (z_T \nabla z_T - z_B \nabla z_B) - \underline{A} \cdot (\nabla z_T - \nabla z_B) = 0 \tag{35}$$

If we consider a container formed by a general cylinder, with generators parallel to the z-axis and with top and bottom surfaces z_T and z_B such that $\nabla z_T = \nabla z_B$, i.e. $\underline{n}_T = -\underline{n}_B$, equation (35) becomes

$$\nabla^2 p_0 - 4\left(\frac{\sigma_1^2 - 1}{\sigma_1^2}\right) \nabla \times (\underline{u}_0 \cdot \underline{k}) \underline{k} \cdot \nabla z_T = 0 \tag{36}$$

Equation (34) then requires

$$\underline{A} \cdot \underline{n}_T = -\underline{A} \cdot \underline{n}_B = 0 \tag{37}$$

The term $(\underline{u}_o \cdot \underline{k})$ in equation (36) can be expressed as a function of p_o by applying boundary condition (26c) to expression (28a)

$$\underline{u}_o \cdot \underline{n}_T = \underline{u}_o \cdot \underline{k} - \underline{u}_o \cdot \nabla z_T = 0$$

or

$$\underline{u}_o \cdot \underline{k} = \nabla z_T \cdot \underline{u}_{o_H} = \frac{1}{2} \nabla z_T \cdot \underline{k} \times \nabla p_o \tag{38}$$

Substituting (38) in equation (36) we have an equation for p_o.

$$\nabla^2 p_o - 2\left(\frac{\sigma_1^2 - 1}{\sigma_1^2}\right) \nabla \times \left(\left(\nabla z_T \cdot \underline{k} \times \nabla p_o \right) \underline{k} \right) \cdot \nabla z_T = 0 \tag{39}$$

The boundary condition is

$$p_o = \text{constant} = 0 \quad \text{in the boundary} \tag{40}$$

which follows from the condition that $\underline{u}_o \cdot \underline{n} = 0$ on the side walls of the cylinder and from the form of $\underline{u}_o$ in equation (28).

For an example, consider a container where the top and bottom surfaces are formed by parallel planes. That is, let

$$z_T = 1 - \frac{b}{a} x$$

and

$$z_B = -\frac{b}{a} x$$

Therefore

$$\nabla z_T = \nabla z_B = -\frac{b}{a} \underline{i} \tag{41}$$

and

$$\underline{n}_T = \underline{k} + \frac{b}{a} \underline{i} = -\underline{n}_B$$

where a and b are the direction cosines

and

$$a^2 + b^2 = 1 .$$

In this case

$$\underline{u}_o \cdot \underline{k} = -\frac{b}{2a} \underline{i} \cdot \underline{k} \times \nabla p_o$$

$$= \frac{b}{2a} \left(\underline{j} \cdot \nabla \right) p_o \tag{42}$$

and equation (39) becomes

$$\nabla^2 p_o + \frac{b^2}{a^2}\left(\frac{\sigma_1^2 - 1}{\sigma_1^2}\right)\frac{\partial^2 p_o}{\partial y^2} = 0 \qquad (43)$$

This can be written

$$\frac{\partial^2 p_o}{\partial x^2} - \mu^2 \frac{\partial^2 p_o}{\partial y^2} = 0 \qquad (44)$$

where

$$\mu^2 = \left(\frac{b^2 - \sigma_1^2}{a^2 \sigma_1^2}\right) \qquad (45)$$

The boundary condition is again

$$p_o = 0 \text{ on the boundary} \qquad (46)$$

Equation (44), with boundary condition (46) was given as a simple example of the Poincaré eigenvalue problem by Howard (ref. 3, Lecture 2). The solution for a rectangular region

$$0 \leq x \leq 1$$
$$0 \leq y \leq 1$$

is

$$p_o = \sum_m \sum_n A_{mn} \sin\left(\frac{n\pi x}{X_o}\right) \sin\left(\frac{m\pi y}{y_o}\right) \qquad (47)$$

with

$$\mu^2 = \frac{n^2}{m^2}\left(\frac{y_o^2}{X_o^2}\right) \qquad (48)$$

and

$$\sigma_1^2 = \frac{m^2 b^2 X_o^2}{n^2 a^2 Y_o^2 + m^2 X_o^2} \qquad (49)$$

We can notice from equation (43) that when the top and bottom surfaces are perpendicular to the z axis, i.e. $b = 0$ ("flat" contours), we have

$$p_o = 0$$

and no modes of this type exist.

Also from equation (49) we see

$$\sigma_1^2 < 1 \qquad (50)$$

and therefore, for the rectangular cylinder, all these frequencies are smaller in absolute value than $\sqrt{4S}$.

For the containers just considered, where $\nabla z_T = \nabla z_B$, the contours of constant height are not uniquely defined. In this case we have found that there exist oscillatory modes with $\sigma \sim O(\sqrt{4S})$. The equation that determines the lowest order pressure is (39) with boundary condition (40). These modes are independent of z, to lowest order, and approach possible homogeneous steady flows in the limit $\sqrt{4S} \longrightarrow 0$.

If we try a similar method for containers which possess a uniquely defined set of closed contours of constant height, we find in contradiction, that the frequency of the modes depends on $H = H(x,y)$. We therefore use a different procedure and do not specify the time dependence to be a simple exponential but allow for a more general time variation on a scale of $O(\sqrt{4S})$. Also in this case we consider the initial value problem for equations (1) (but we only treat it in part) and look for a solution in the form of a superposition of modes.

With $\tau = \sqrt{4S}\, t$ we assume that the variables have the following expansion

$$\underline{u} = \underline{u}_o(\underline{x},\tau) + \sum_m \underline{u}_{om}(\underline{x})\, e^{i\sigma_m t} + \sqrt{4S}\left(\underline{u}_1(\underline{x},\tau) + \sum_m \underline{u}_{1m}(\underline{x},t)\right) + \ldots$$

$$T = T_o(\underline{x},\tau) + \sum_m p_{om}(\underline{x},t_1) + \ldots$$

$$p = p_o(\underline{x},\tau) + \sum_m p_{om}(\underline{x})e^{i\sigma_m t} + \sqrt{4S}\left(p_1(\underline{x},\tau) + \sum_m p_{1m}(\underline{x},t)\right) + \ldots \qquad (51)$$

This form of the expansion can be justified by the formalism of the method of multiple time scales but we do not include that here. The similarity of this procedure to that used by Greenspan[1] in the spin-up problem can be noted.

Substituting expressions (51) in equation (1) we get a sequence of equations:

0th Order

Geostrophic

$$\nabla \cdot \underline{u}_o \tag{52a}$$

$$2\underline{k} \times \underline{u}_o + \nabla p_o = 0 \tag{52b}$$

$$\text{B.C.} \quad \underline{u}_o \cdot \underline{n} = 0 \tag{52c}$$

Modes

$$\nabla \cdot \underline{u}_{om} = 0 \tag{53a}$$

$$i\sigma_m \underline{u}_{om} + 2\underline{k} \times \underline{u}_{om} + \nabla p_{om} = 0 \tag{53b}$$

$$\frac{\partial T_o}{\partial t} = 0 \tag{53c}$$

$$\text{B.C.} \quad \underline{u}_{om} \cdot \underline{n} = 0 \tag{53d}$$

1st Order

Geostrophic

$$\nabla \cdot \underline{u}_1 = 0 \tag{54a}$$

$$\frac{\partial \underline{u}_o}{\partial \tau} + 2\underline{k} \times \underline{u}_1 + \nabla p_1 - T_o \underline{k} = 0 \tag{54b}$$

$$\frac{\partial T_o}{\partial \tau} + \underline{u}_o \underline{k} = 0 \tag{54c}$$

$$\text{B.C.} \quad \underline{u}_1 \cdot \underline{n} = 0 \tag{54d}$$

Modes

$$\nabla \cdot \underline{u}_{1m} = 0 \tag{55a}$$

$$\frac{\partial \underline{u}_{1m}}{\partial t} + 2\underline{k} \times \underline{u}_{1m} + \nabla p_{1m} - T_{om}\underline{k} = 0 \tag{55b}$$

$$\frac{\partial T_{1m}}{\partial t} + \underline{u}_{1m} \cdot \underline{k} = 0 \tag{55c}$$

$$\text{B.C.} \quad \underline{u}_1 \cdot \underline{n} = 0 \tag{55d}$$

The zeroth order equations are the same as those for the homogeneous case. The initial geostrophic solution can be determined in terms of the initial conditions by the mean circulation theorem, as mentioned before. Therefore, is given by equation (14)

$$\underline{u}_o(\underline{x},\tau) = -\frac{1}{2}\frac{\partial p_o}{\partial H}(H,\tau)\,\underline{n}_T \times \underline{n}_B \tag{56}$$

The mean circulation of the oscillatory modes $\underline{u}_{om}$ is zero. From equation (55b) we can see that the mean circulation of $T_{om}\underline{k}$ also has to be zero. This means that the initial value of $T_o\underline{k}$ in equation (54b) is determined by the mean circulation of $T_{IC}\underline{k}$ and in particular is independent of z.

By taking the curl of equation (54b) we find

$$\underline{k}\cdot\nabla\underline{u}_1 = \frac{1}{2}\left[\nabla\times\frac{\partial\underline{u}_o}{\partial\tau} - \nabla\times T_o\underline{k}\right] \tag{57}$$

Since the right-hand side is independent of z we can integrate (57) to obtain

$$\underline{u}_1 = \frac{1}{2}\left[\nabla\times\frac{\partial\underline{u}_o}{\partial\tau} - \nabla\times T_o\underline{k}\right]z + \underline{A}(x,y) \tag{58}$$

where $\underline{A}(x,y)$ is an arbitrary vector.

Substituting (58) in equation (54a)

$$\nabla\cdot\underline{u}_1 = 0$$

we find that

$$\underline{A} = \frac{1}{2}\underline{k}\times\frac{\partial\underline{u}_o}{\partial\tau} + \nabla\times\underline{B}(x,y) \tag{59}$$

and therefore

$$\underline{u}_1 = \nabla\times\left(\frac{1}{2}z\frac{\partial\underline{u}_o}{\partial\tau}\right) - \nabla\times\left(\frac{1}{2}z\,T_o\underline{k}\right) + \nabla\times\underline{B}(x,y) \tag{60}$$

Now, integrating $\underline{u}_1\cdot\underline{n}$ over the top and bottom surfaces of the boundary enclosed by a geostrophic contour, C = C(H), using boundary condition (54d), and applying Stokes's theorem, we have

$$0 = \int_{S_T}\underline{n}_T\cdot\underline{u}_1\,da_T = \frac{1}{2}\oint_C z_T\frac{\partial\underline{u}_o}{\partial\tau}\cdot d\underline{s} - \frac{1}{2}\oint_C z_T T_o\underline{k}\cdot d\underline{s} + \oint_C \underline{B}\cdot d\underline{s} \tag{61}$$

and

$$0 = \int_{S_B}\underline{n}_B\cdot\underline{u}_1\,da_B = \frac{1}{2}\oint_C z_B\frac{\partial\underline{u}_o}{\partial\tau}\cdot d\underline{s} - \frac{1}{2}\oint_C z_B T_o\underline{k}\cdot d\underline{s} + \oint_C \underline{B}\cdot d\underline{s} \tag{62}$$

Subtracting equations (61) and (62) we find

$$0 = \oint_{C(H)} (z_T - z_B)\,\frac{\partial \underline{u}_o}{\partial \tau}\cdot d\underline{s} - \oint_{C(H)} (z_T - z_B)\, T_o\, \underline{k}\cdot d\underline{s} \tag{63}$$

Taking a derivative of equation (63) with respect to τ, substituting for $\frac{\partial T_o}{\partial \tau}$ from equation (54c) and noting that $H = z_T - z_B$ is independent of position along C, we have

$$\oint_{C(H)} \frac{\partial^2 \underline{u}_o}{\partial \tau^2}\cdot ds + \oint_{C(H)} (\underline{u}_o\cdot \underline{k})\,\underline{k}\cdot d\underline{s} = 0 \tag{64}$$

If we substitute from equation (56) for $\underline{u}_o(\underline{x},\tau)$, equation (64) becomes

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{dp_o}{dH}\right) + \frac{K}{J}\,\frac{dp_o}{dH} = 0 \tag{65}$$

where

$$J = J(H) = H\oint_{C(H)} \underline{n}_T \times \underline{n}_B \cdot d\underline{s} \tag{66}$$

or $$J = H\oint_{C(H)} \left[1 + (\nabla z_T)^2\right]^{1/2}\left[1 + (\nabla z_B)^2\right]^{1/2} \left|\underline{\hat{n}}_T \times \underline{\hat{n}}_B\right|^2 ds$$

and

$$K = K(H) = H\oint_{C(H)} (\underline{n}_T \times \underline{n}_B\cdot \underline{k})\,\underline{k}\cdot d\underline{s} \tag{67}$$

or $$K = H\oint_{C(H)} \left[1 + (\nabla z_T)^2\right]^{1/2}\left[1 + (\nabla z_B)^2\right]^{1/2} \left|\underline{\hat{n}}_T \times \underline{\hat{n}}_B\cdot \underline{k}\right|^2 ds$$

Since $\frac{K}{J} \geqslant 0$ the solution to equation (65) is

$$\frac{dp_o}{dH}(H,\tau) = \frac{dp_o}{dH}(H,0)\, e^{+i\left(\frac{K}{J}\right)^{1/2}\tau} \tag{68}$$

The lowest order solution, therefore, is similar in structure to the homogeneous geostrophic flow except that it has an oscillatory character on a long time scale $\tau = \sqrt{4S}\,t$. The frequency of this flow depends only on H and therefore can be viewed as a sum (or integral) of oscillatory modes (of very low frequency) which represent a flow only in an infinitesimal cylindrical shell about a geostrophic contour. In the limit $\sqrt{4S} \to 0$ this flow approaches the steady flow of the homogeneous fluid determined by the mean circulation theorem.

We note that for "flat" contours

$$\underline{n}_T \times \underline{n}_B \cdot \underline{k} = 0$$

and therefore $\quad K = 0$

and the behavior of the geostrophic flow is not present.

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PERIODIC MOTION IN A ROTATING STRATIFIED FLUID WITH SPECIAL REFERENCE TO THE ZERO-FREQUENCY LIMIT

Michael R. Foster

1. Introduction

Recently, the literature has contained several papers regarding steady motions in a rotating stratified fluid caused by the motion of boundaries or certain thermal boundary conditions or both. (Pedlosky and Barcilon, 1967, two papers; Veronis, 1967, II). These solutions are curious in the respect that the usual asymptotic theory for small Ekman number (or Rayleigh number) gives interior solutions that are controlled by the diffusion of momentum and heat in the fluid. The solutions for either stratification or rotation alone exhibit interiors controlled strongly by the singular layers on the boundaries and are thus in marked contrast to these rotating stratified solutions, which have no Ekman or buoyancy layers to leading order.

The purpose of this investigation is to consider periodic solutions for a rotating stratified fluid confined between two infinite vertical parallel plates and subjected to one-dimensional temperature distributions on those plates. Subsequently, it will be shown that when ω is $O(\Omega)$, the buoyancy layers on the plates control the interior motion which is of $O(E^{1/2})$ where E is the Ekman number based on the plate spacing, ℓ . One example of such a solution in Section 3 shows some similarities to the spin-up problem (Walin, 1968). In Section 4, the structure of the solution is given for $\omega \ll \Omega$ and it is shown that there are 3 distinct regimes in the frequency domain. If $\omega \gtrsim \sqrt{\nu\Omega/\ell^2}$, the buoyancy layers exert a strong effect on the interior flow which now splits into a part of $O(\nu\Omega/\omega^2\ell)$ or $O(E^{1/2}/f)$, $f = \omega/\Omega$, which consists of the temperature and transverse velocity (perpendicular to the plane of the temperature distribution), and the u-w velocity field in the x - z plane normal to this transverse direction and still of $O(E^{1/2})$. This range of frequencies is also characterized by the appearance of a Stokes layer (Stokes, 1851) on the plates of thickness $(\nu/\omega)^{1/2}$ which is thicker than the buoyancy layer and which takes the large transverse velocity to zero on the wall. This arises because the vorticity leaving the wall is trapped near the wall by the vorticity coming toward the wall one-half cycle later. One can see at this stage that diffusive effects have now extended into the fluid beyond the confines of the buoyancy layers. If ω is in the range $\sqrt{\Omega\nu/\ell^2} \gg \omega \gg \nu/\ell^2$, then the strong constraints of the rotation begin to be really felt; the buoyancy layers cannot exist in this range, and the thermal field calculated in the interior is uniformly valid to the wall. However, the transverse velocity again must be brought to zero through a Stokes layer. The buoyancy layers do exist to $O(f/E^{1/2})$ and feed an $x-z$ flow of $O(f)$ which is now much smaller than the primary transverse motion which is now $O(1)$. The Stokes layer is still thin here, but larger than previously, in fact, much thicker than $E^{1/4}$. Finally, taking $\omega \leq \nu/\ell^2$, the diffusion having penetrated into the whole body of the fluid through the mechanism of the Stokes layer, the primary flow is entirely controlled by diffusive processes. Hence, the buoyancy layer convergence or divergence is important all the way down to $O(E^{1/2})$ in f, at which point the flow begins to take a more diffusive character, though the transition is incomplete until f is $O(E)$. By elementary considerations, one can easily show that, in a purely stratified fluid, periodic solutions will be essentially like their steady equivalents if $\tilde{f} \leq R^{1/3}$ where $\tilde{f}$ is non-dimensionalized with the Brunt-Väisälä frequency and R is the Rayleigh number. So, the addition of rotation makes the requirement for a quasi-steady motion much stronger, i.e., $f \leq E$.

2. Mathematical Formulation

The equations of motion for a rotating stratified fluid under the usual Boussinesq approximations are

$$\frac{\partial \underline{u}^*}{\partial t} + (\underline{u}^* \cdot \tilde{\nabla})\underline{u}^* + 2\,\underline{\Omega} \times \underline{u}^* + \frac{1}{\rho_0}\nabla p^* = \nu \tilde{\nabla}^2 \underline{u}^* + g\alpha T^* \underline{k} \qquad (2.1)$$

$$\frac{\partial T^*}{\partial t} + (\underline{u}^* \cdot \tilde{\nabla}) T^* + \frac{4\Delta T}{L} w^* = \frac{\nu}{\sigma} \tilde{\nabla}^2 T^* \qquad (2.2)$$

$$\mathrm{div}\, \underline{u}^* = 0 \qquad (2.3)$$

where $(\ )^*$ denotes a dimensional quantity, T^* is a perturbation temperature, σ the Prandtl number, ν the kinematic viscosity, g the acceleration of gravity, α the thermal expansion coefficient, and $(4\Delta T/L)$ is the imposed temperature gradient. Here, we suppose that $\underline{\Omega} = \Omega \underline{k}$, $\underline{k}$ the vertical unit vector.

Writing $\underline{u}^* = \epsilon \frac{g\alpha\Delta T}{\Omega} \underline{u}$, $T^* = (\epsilon\Delta T)T$, $p^* = (\rho_0 g\alpha\epsilon\Delta T \ell) P$, $t = \tau/\Omega$ where $\epsilon = T_0/\Delta T$, T_0 being the scale of the non-linearity, gives

$$\frac{\partial \underline{u}}{\partial \tau} + \bar{\epsilon}(\underline{u}\cdot\nabla)\underline{u} + 2\underline{k}\times\underline{u} + \nabla P = E\nabla^2\underline{u} + \underline{k}\,T \tag{2.4}$$

$$\frac{\partial T}{\partial \tau} + \bar{\epsilon}(\underline{u}\cdot\nabla)T + 4\beta^2 W = \frac{E}{\sigma}\nabla^2 T \tag{2.5}$$

$$\nabla\cdot\underline{u} = 0 \tag{2.6}$$

In these equations, $E = \nu/\Omega\,\ell^2$, $\beta^2 = \frac{g\alpha\Delta T}{\Omega^2 L}$, $\bar{\epsilon} = \frac{g\alpha\Delta T}{\ell\Omega^2}\epsilon$, and the coordinates are scaled with ℓ. Hereafter, we will suppose that T_0 is sufficiently small to make the convection of heat and momentum unimportant, i.e., $\bar{\epsilon} = 0$. Supposing there is an $e^{if\tau}$ time dependence, $f = \omega/\Omega$, then the $\bar{\epsilon} = 0$ equations are

$$if\underline{u} + 2\underline{k}\times\underline{u} + \nabla P = E\nabla^2\underline{u} + \underline{k}\,T \tag{2.7}$$

$$ifT + 4\beta^2 W = \frac{E}{\sigma}\nabla^2 T \tag{2.8}$$

$$\nabla\cdot\underline{u} = 0 \tag{2.9}$$

One can write (2.7) - (2.9) as one-eighth order partial differential equation in P of obvious elliptic character. If there are operators

$L = E\nabla^2 - if$, $L_t = \frac{E}{\sigma}\nabla^2 - if$ then one can

write easily

$$L(LL_t + 4\beta^2)\nabla_1^2 P + L_t(L^2 + 4)\frac{\partial^2 P}{\partial z^2} = 0 \tag{2.10}$$

which is of little real use except to notice that the Ekman and buoyancy layer scaling are immediate by setting just the first group of terms or the second equal to zero.

3. Solution for Small Ekman Number; $f = O(1)$.

The procedure in this section is to look for solutions of (2.7) - (2.9) in the strip $-\infty < z < \infty, -\infty < y < \infty, |x| < 1$ subject to the boundary conditions that

$$T = \alpha_{\pm}(z) \text{ on } x = \pm 1 \qquad (3.1a)$$

$$U = V = W = 0 \text{ on } x = \pm 1 \qquad (3.1b)$$

A. The Interior Solution; $O(1)$

Away from regions of strong gradients near $|x| = 1$, we put

$$\underline{u} = \underline{u}^{(0)} + E^{1/2}\underline{u}^{(1)} + E\,\underline{u}^{(2)} + \ldots \quad \underline{et}\ \underline{al}$$

where $\underline{u}^{(k)} = \underline{u}^{(k)}(x,z)$. Insertion of this sequence into (2.7) - (2.9) and taking the limit $E \rightarrow 0$, with (x,z) fixed, gives

$$\begin{aligned} & if\,\underline{u}^{(0)} + 2\,\underline{k} \times \underline{u}^{(0)} + \nabla P^{(0)} = \underline{k}\,T^{(0)} \\ & if\,T^{(0)} + 4\beta^2 W^{(0)} = 0 \\ & \operatorname{div} \underline{u}^{(0)} = 0 \end{aligned} \qquad (3.2)$$

or, if $\frac{\partial}{\partial y}(\ ) = 0$,

$$\frac{4\beta^2 - f^2}{4 - f^2}\frac{\partial^2 p^{(0)}}{\partial x^2} + \frac{\partial^2 p^{(0)}}{\partial z^2} = 0 \qquad (3.3a)$$

and

$$u^{(0)} = -\frac{if}{4 - f^2}\frac{\partial^2 p^{(0)}}{\partial x} \qquad (3.3b)$$

$$w^{(0)} = -\frac{if}{4\beta^2 - f^2}\frac{\partial p^{(0)}}{\partial z} \qquad (3.3c)$$

$$v^{(0)} = \frac{2}{4 - f^2}\frac{\partial p^{(0)}}{\partial x} \qquad (3.3d)$$

$$T^{(0)} = \frac{4\beta^2}{4\beta^2 - f^2}\frac{\partial p^{(0)}}{\partial z} \qquad (3.4e)$$

where we will assume that

$$f^2 < \mathrm{MIN}(4\beta^2, 4) \qquad (3.5)$$

throughout the treatment to assure the ellipticity of (3.3a) uniformly as
Now, the boundary condition appropriate to (3.3a) is no flow through the vertical walls, ie

$$\frac{\partial p^{(0)}}{\partial x} = 0 \quad on \ x = \pm 1. \tag{3.6}$$

which also, as it happens, makes $v^{(0)}(\pm 1, z) = 0$. Applying (3.6) to (3.3a) means that $p^{(0)}$ = constant is the only solution, the constant being arbitrary. Hence,

$$u^{(0)} = v^{(0)} = w^{(0)} = T^{(0)} = 0 \tag{3.7}$$

B. <u>The Buoyancy Layers on x = ± 1</u>

Putting $x = \pm 1 + \sqrt{\frac{E}{\beta\sqrt{\sigma}}}\,\xi$ and the inner expansion

$$u = \sqrt{\frac{E}{\beta\sqrt{\sigma}}}\,\hat{u}^{(0)}(\xi, z) + \frac{E}{\beta\sqrt{\sigma}}\,\hat{u}^{(1)} + \ldots$$

$$(v, w) = \left(\hat{v}^{(0)}(\xi, z), \hat{w}^{(0)}(\xi, z)\right) + O\left(\sqrt{\frac{E}{\beta\sqrt{\sigma}}}\right)$$

$$P = p^{(0)}(\xi, z) + O(E^{1/2}) \ \underline{et}\ \underline{al}.$$

into (2.7) - (2.9) and letting $E \longrightarrow 0$ with ξ fixed will give the usual buoyancy layer equations with unsteady effects, <u>viz.</u>,

$$\frac{\partial^4 \hat{T}^{(0)}}{\partial \xi^4} - \frac{if}{\beta}\left(\frac{1}{\sqrt{\sigma}} + \sqrt{\sigma}\right)\frac{\partial^2 \hat{T}^{(0)}}{\partial \xi^2} + \left(4 - (f/\beta)^2\right)\hat{T}^{(0)} = 0 \tag{3.8a}$$

$$4\hat{w}^{(0)} = \frac{1}{\sqrt{\beta^3\sigma}}\,\frac{\partial^2 \hat{T}^{(0)}}{\partial \xi^2} - i\,\frac{f}{\beta^2}\,\hat{T}^{(0)} \tag{3.8b}$$

$$\frac{\partial \hat{u}^{(0)}}{\partial \xi} + \frac{\partial \hat{w}^{(0)}}{\partial z} = 0 \tag{3.8c}$$

which do <u>not</u> contain the vertical pressure gradient from the interior solution since 3.A. indicates that $\partial p^{(0)}/\partial z \equiv 0$, $\propto \frac{\partial P}{\partial z} = o(1)$. After some algebra, the solution is given as

$$T^{(0)}(\pm 1,z) = \alpha_{\pm}(z)\left[\left(\frac{2B^2\beta}{\sqrt{\sigma}}+f\right)e^{\pm(1+i)A\xi} + \left(\frac{2A^2\beta}{\sqrt{\sigma}}-f\right)e^{\pm(1-i)B\xi}\right] \times \frac{\sqrt{\sigma}}{2(A^2+B^2)} \tag{3.9a}$$

and

$$w^{(0)}(\pm 1,z) = \frac{i\sqrt{\sigma}\,\alpha_{\pm}(z)}{8\beta^3(A^2+B^2)}\left[\left(\frac{2A^2\beta}{\sqrt{\sigma}}-f\right)\left(\frac{2B^2\beta}{\sqrt{\sigma}}+f\right)\right]\left(e^{\pm(1+i)A\xi} - e^{\pm B(1-i)B\xi}\right) \tag{3.9b}$$

where
$$\binom{A}{B} = \sqrt{\sqrt{\left(\frac{f}{4\beta}\right)^2\left(\frac{1}{\sqrt{\sigma}}-\sqrt{\sigma}\right)^2+1} \pm \frac{f}{4\beta}\left(\frac{1}{\sqrt{\sigma}}+\sqrt{\sigma}\right)}$$

which are always positive provided (3.5) holds.

If $F_{\pm}(z) \equiv \int_0^{\mp\infty} \hat{w}_{\pm}^{(0)}(\xi,z)\,d\xi$ is the volume flux of fluid in the buoyancy layer, then (3.8c) will give

$$\hat{u}_{\pm}^{(0)}(\mp\infty,z) = -\frac{\partial F_{\pm}(z)}{\partial z}$$

Carrying out the algebra will give the following boundary condition for the $O(E^{1/2})$ interior fluid equations.

So $$u^{(1)}(\pm 1,z) = \pm\frac{i}{\sqrt{\beta\sigma^{-1/2}}}\,\frac{\alpha'_{\pm}(z)}{16\beta^3(A^2+B^2)}\left(\frac{2A^2\beta}{\sqrt{\sigma}}-f\right)\left(\frac{2B^2\beta}{\sqrt{\sigma}}+f\right)\left(\frac{1-i}{A}-\frac{1+i}{B}\right) \equiv \varphi_{\pm}(z) \tag{3.10}$$

which reduces to the familiar $\pm\frac{1}{4}\frac{\alpha'_{\pm}(z)}{(\beta\sqrt{\sigma})^{3/2}}$ when $f \to 0$. Notice that the phase angle is

$$\arctan\left(\frac{B-A}{B+A}\right) < 0 \text{ since } B < A \text{ from definition from A and B.}$$

C. <u>The Interior Solution, $O(E^{1/2})$</u>

Doing the same thing as described in A for the $O(E^{1/2})$ term gives identically the same equations for $P^{(1)}$, $u^{(1)}$, $w^{(1)}$ <u>et al.</u> as in A for the $O(1)$ terms, i.e. replacing superscript (0) by superscript (1) in (3.3) gives the $O(E^{1/2})$ equations. Therefore, the $O(E^{1/2})$ solution is reduced to the solution of the Neumann Problem,

$$\frac{\partial^2 p^{(1)}}{\partial z^2} + \frac{4\beta^2 - f^2}{4 - f^2}\frac{\partial^2 p^{(1)}}{\partial x^2} = 0 \tag{3.11a}$$

$$\frac{\partial p^{(1)}}{\partial x} = -\frac{4 - f^2}{if}\varphi_{\pm}(z) \text{ on } x = \pm 1. \tag{3.12b}$$

where $\varphi_{\pm}(z)$ is symbolic of the complex expression in (3.10)

If $\int_{-\infty}^{\infty} |\alpha_{\pm}(z)| dz < \infty$, a convenient representation of the solution is in terms of Fourier transforms in the z-direction.

If
$$p^{(1)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi(k,x) e^{ikz} dk \tag{3.13a}$$

then one can easily show that

$$\Pi(k,x) = -\frac{4 - f^2}{2if\lambda \sin\lambda \cosh\lambda}\left[(\Phi_+ - \Phi_-)\cosh\lambda \cosh\lambda x + (\Phi_+ + \Phi_-)\sin\lambda \sin\lambda x\right] \tag{3.13b}$$

where

$$\Phi_{\pm}(k,x) \equiv \int_{-\infty}^{\infty} e^{-ikz}\varphi_{\pm}(z) dz \tag{3.13c}$$

and $\lambda \equiv \sqrt{\frac{4 - f^2}{4\beta^2 - f^2}}\, k$. Notice exhibited the explicit dependence on the symmetric and antisymmetric parts of the boundary conditions.

D. <u>Two Examples</u>

(i) <u>Temperature pulses of arbitrary phase</u>

Suppose $\alpha_+(z) = H(z) - H(z - H)$

$\alpha_-(z) = e^{i\psi}[H(z) - H(z - H)]$

then $\varphi_+(z) = \gamma[\delta(z) - \delta(z - H)]$

$\varphi_-(z) = -\gamma e^{i\psi}[\delta(z) - \delta(z - H)]$

$= \gamma e^{i(\psi - \pi)}[\delta(z) - \delta(z - H)]$

where γ is the phase information, <u>viz</u>,

$$\gamma = \frac{\sigma^{1/4} i/16}{\sqrt{\beta\sigma}\,\beta^3(A^2+B^2)} \left[\frac{2A^2\beta}{\sqrt{\sigma}} - f\right]\left[\frac{2B^2\beta}{\sqrt{\sigma}} + f\right]\left[\frac{1-i}{A} - \frac{1+i}{B}\right]$$

Then, using (3.13c), we get

$$\Phi_{\pm}(k) = \gamma e^{i\left(\frac{1}{2}\mp\frac{1}{2}\right)(\psi-\pi)}\left(1-e^{-ikH}\right)$$

and putting this into (3.13b) and doing the inversion of Π gives, evaluating the inversion integral by means of contour integration in the complex k plane.

$$p^{(1)} = -\frac{4-f^2}{2if}\gamma\ \left(1+e^{i\psi}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\left(e^{-n\pi|z'|} - e^{-n\pi|z'-H'|}\right)\cos n\pi x +$$
$$+\left(1-e^{i\psi}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})\pi}\left(e^{-(n+\frac{1}{2})\pi|z'|} - e^{-(n+\frac{1}{2})\pi|z'-H'|}\right)\sin(n+\tfrac{1}{2})\pi x$$

where $z' = z\sqrt{\frac{4\beta^2-f^2}{4-f^2}}$, $H' = H\sqrt{\frac{4\beta^2-f^2}{4-f^2}}$. These solutions derivatives are not convergent on z = 0 or z = H in the classical sense. This is the direct result of using the Dirac distribution as if it were classically defined. This use of point sources and sinks to analyze the flow resulting from temperature pulses is worrisome at first since one can not expect (3.10) to be applicable over variations in the vertical of scale smaller than $O(E^{1/2})$. Now, in the case of pure stratification, there are, at least in the steady state, $R^{1/4}$ and $R^{1/3}$ layers to carry this flux of fluid. However, one can easily convince oneself that the only possible horizontal layers, viz, Ekman layers, do not have structure sufficiently complex to carry the mass. So, the flux must be handled by the interior flow. Certainly this outflow takes place, for f of $O(1)$, in small regions of dimension $E^{1/2} \times E^{1/2}$. Asymptotically this is of zero width for the interior flow, and hence the use of point-sources and sinks seems to be quite acceptable in this context. This source- and sink-like character of the interior flow is exactly what is found by Wälin in his treatment of the spin-up problem for (2.4) - (2.6) (Wälin, 1968).

In general, it is easy to show that

$$\left(\frac{\partial z}{\partial x}\right)_{p^{(1)}=\text{const.}} \left(\frac{\partial z}{\partial x}\right)_{\text{streamline}} = -\frac{4-f^2}{4\beta^2-f^2} \tag{3.15}$$

ie, the streamlines make a constant (and small, as $\beta \to \infty$) angle with the $p^{(1)}$= const. lines, the isobars. In the (z',x) plane, the curves are orthogonal families of curves.

This solution is understandable simply only when $\sqrt{\frac{4\beta^2-f^2}{4-f^2}}$ is very large. Then, convergence of the series, away from $z=0$, $H/2$ and H, is extremely rapid. In this case, the solution should approach the stratified solution. One can show, when $\psi = 0$, that the streamlines are horizontal except in regions near $|x| = 1$ and $x = 0$ of width $O(L^{3/2}\Omega/\sqrt{g\alpha\Delta T})$ where they curve into $|x| = 1$, $z = 0$, H or $x = 0$, $z = 0$, H. If $\psi = \pi$, the streamlines are straight across and curve into $|x| = 1$, $z = 0$, H in a region of the order stated. Hence, when the stratification is extremely large, the flow is quasi-steady almost everywhere; most of the dynamics occurring in the $1/\beta$ regions near the vertical boundaries. This observation is consistent with the associated phenomenon in the spin-up problem as pointed out by Wälin. A sketch of the flow field given by (3.14) when β is large is shown in Figure 1 for each case discussed above, viz, $\psi = 0$ and $\psi = \pi$. The double line indicates what must be $R^{1/3}$ layers to complete the circulation; the physics of such a singular region is clearly not a part of the solution (3.14).

(ii) Sinusoidal Temperature Distributions

Suppose that $\alpha_\pm(z) = -K_\pm \cos\alpha z$. Then, by (3.10) $\varphi_\pm(z) = \pm\,\alpha K_\pm \gamma' \sin\alpha z$ where γ' is exactly the phase information called γ in (i). Simple separation of variables technique in deference to an inconvenient (3.13) formulation will give just

$$p^{(1)} = -\frac{4-f^2}{2if\sigma}\alpha\gamma'\sin\alpha z\left[\frac{K_+-K_-}{\cosh\sigma}\sinh\sigma x + \frac{K_++K_-}{\sinh\sigma}\cosh\sigma x\right] \tag{3.16}$$

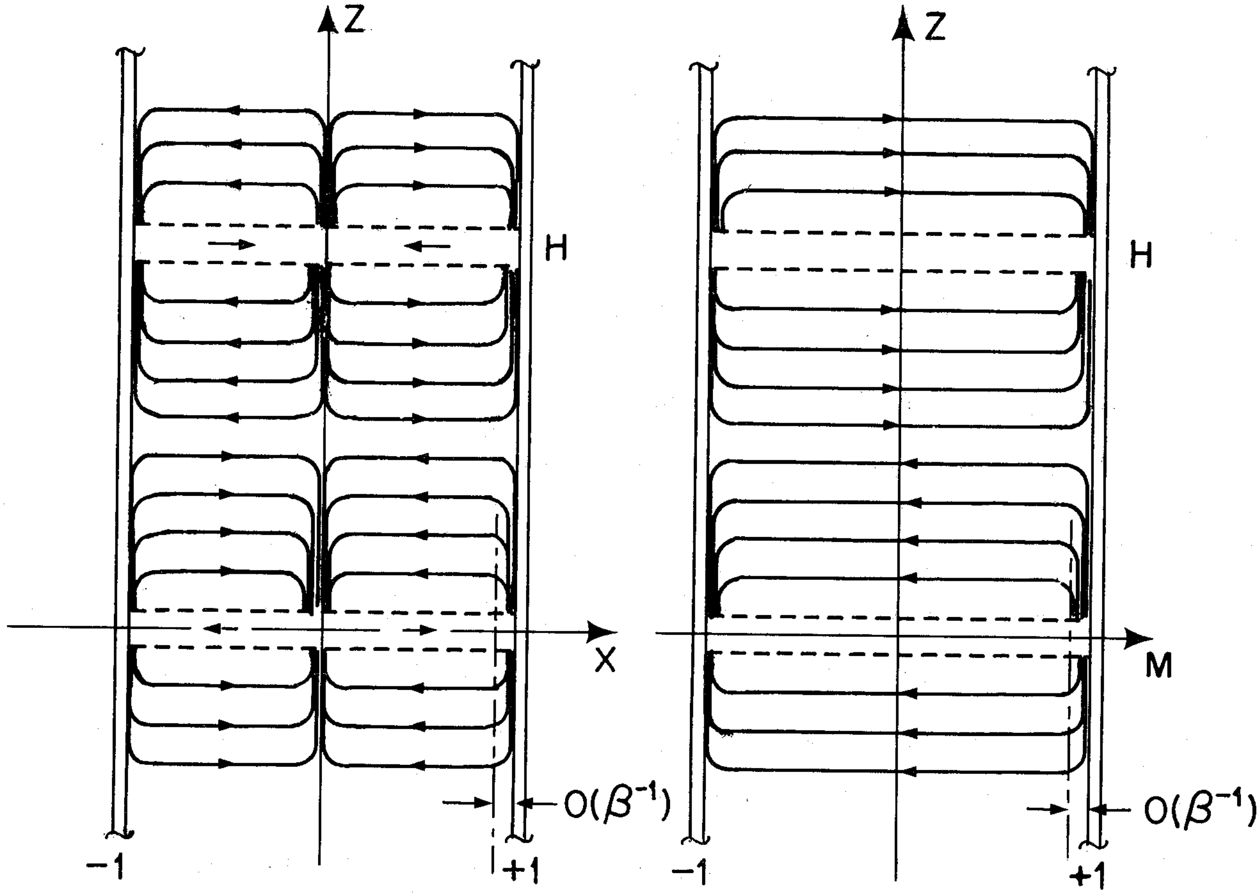

FIGURE 1.

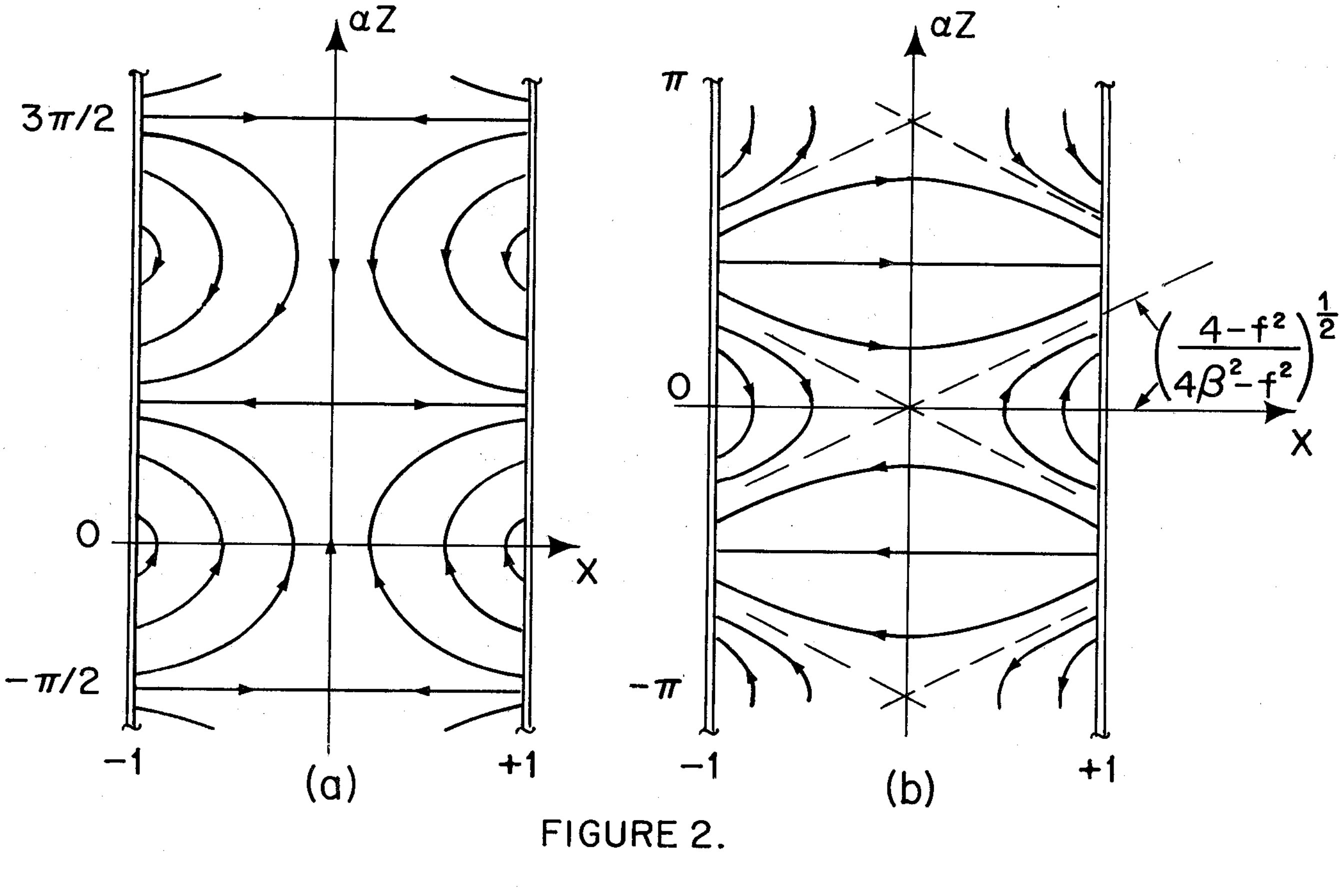

FIGURE 2.

where $$\sigma \equiv \alpha \sqrt{\frac{4-f^2}{4\beta^2-f^2}}$$

(a) Symmetric case, $K_+ = K_-$

Here, $$p^{(1)} = -\frac{4-f^2}{if} K_+ \Upsilon'\left(\frac{\alpha}{\sigma}\right) \sin \alpha z \, \frac{\cosh \sigma x}{\sinh \sigma}$$

and use of (3.15) will give the stream function

$$\psi_s = \cos \alpha z \sinh \sigma x \qquad (3.17)$$

The qualitative nature of (3.17) is given in Figure 2a. Notice that $\sigma \to 0$ does not alter the streamline pattern. However, since $|x| \le 1$, $\psi_s \propto \sigma$ as $\sigma \to 0$ and hence this flow carries no mass and the streamlines are irrelevant. One can see that as $\Omega \to 0$, $\omega \to 0$ to preserve f of $O(1)$ and hence the $\beta \to \infty$ limit must give an essentially steady solution. The steady solution has been given in the literature (Veronis, 1967,I) and it does indeed exhibit this zero interior flow property. The only effect is to restratify the fluid, steepening and shallowing the mean gradient in a distance π/α.

(b) Antisymmetric case; $K_- = -K_+$

Again $$p^{(1)} = -\frac{4-f^2}{if\sigma} \alpha K_+ \Upsilon' \sin \alpha z \, \frac{\sinh \sigma x}{\cosh \sigma}$$

and (3.15) produces the stream function

$$\psi_a = \cos \alpha z \cosh \sigma x \qquad (3.18)$$

Here, clearly $\psi_a \geq \cos \alpha z$ as $\sigma \to 0$, so there is a non-vanishing amount of mass carried by the fluid $\forall \sigma$. The streamline pattern is shown in Figure 2b. The peculiar thing is the stagnation points located on $x = 0$ at $z = m\pi$, and the part of the volume flux out of the buoyancy layers that reenters the same layer. One can show easily from (3.18) that the dotted asymptote that bounds this region in Figure 2b has a slope $\sqrt{(4-f^2)/(4\beta^2-f^2)}$.

4. Solution for Small Ekman Number and $f = O(1)$

In this section, we seek an asymptotic theory for the problem previously stated under the double limit $E \to 0$, $f \to 0$. This means that the interior solution has a representation

$$\underline{u} = \sum_{n=0}^{\infty} \epsilon_n \underline{u}^{(n)} \tag{4.1}$$

where $\epsilon_{n+1} = o(\epsilon_n)$ and the sequence $\{\epsilon_n\}$ as a functional of f and E must also be such that $\epsilon_n \to 0$ as $E, f \to 0$. Under this limit with gradients of $O(1)$, (2.7) - (2.9) give very simple equations for the $\{\underline{u}^{(0)}, p^{(0)}, T^{(0)}\}$ solution, viz,

$$\underline{u}^{(0)} = \frac{1}{2} \underline{k} \times \nabla p^{(0)} \tag{4.2a}$$

$$T^{(0)} = \frac{\partial p^{(0)}}{\partial z} \tag{4.2b}$$

These equations already have a different character from (3.3) when f was $O(1)$. There, $p^{(0)}$ was a sort of velocity potential and here $p^{(0)}$ is clearly identifiable as a stream function. Note that (4.2) already obeys the inviscid boundary condition $\underline{u} \cdot \underline{n} = 0$ on boundaries.

A. The Buoyancy Layers on $x = \pm 1$

As in section 3.B., we write

$$x = \pm 1 + \sqrt{\frac{E}{\beta \sqrt{\sigma}}}\, \xi \qquad \text{and}$$

use exactly the same inner expansion as in that place for the variables. Because $f = o(1)$ however, the structure is essentially that of a steady buoyancy layer as can be seen by letting $f \to 0$ in the equations of (3.B). Nevertheless, there is substantial difference here because $\partial p^{(0)}/\partial z|_{\pm 1}$ is not necessarily zero, so it must be retained in the equations. The essential result of the solution is the boundary condition it produces on the external flow which, in this case, takes the form

$$u(\pm 1, z) = \pm \frac{E^{1/2}}{4\sqrt{(\beta\sqrt{\sigma})^3}} \frac{d}{dz}\left[\alpha_\pm(z) - \frac{\partial P}{\partial z}\bigg|_{\pm 1}\right] \tag{4.3}$$

This compatibility condition plays an important role in what follows.

B. <u>The Interior Solution; $\Omega >> \omega \gtrsim \sqrt{\nu\Omega/\ell^2}$</u>

In this range of f, which may be stated mathematically as follows:

$$\lim_{E\to 0}\left(\frac{E^{1/2}}{f}\right) < \infty \tag{4.4}$$

one can show that the leading term in the outer expansion of the solution is $O(E^{1/2}/f)$. This is easily seen by examination of the boundary conditions (4.3) which apply to the solution. The only terms that could possibly be larger are $O(f^k)$ for some integer values of k, and the solutions to the associated equations can easily be seen to be zero. Hence, write

$$\underline{u} = \frac{E^{1/2}}{f}\left[\underline{u}_*^{(0)} + f\,\underline{u}_*^{(1)} + o(f)\right] \tag{4.5}$$

where the * is used to distinguish this $\underline{u}^{(0)}$ as to order from the previous $\underline{u}^{(0)}$ in section 3. Substitution of this and similar expansions for T and P into (2.7) - (2.9) and successive application of the limit will yield the two sets of equations,

$$\underline{u}_*^{(0)} = \frac{1}{2}\,\underline{k} \times \nabla P_*^{(0)} \tag{4.6a}$$

$$T_*^{(0)} = \frac{\partial P_*^{(0)}}{\partial z} \tag{4.6b}$$

and

$$2\,\underline{k} \times \underline{u}_*^{(1)} + i\,\underline{u}_*^{(0)} + \nabla P_*^{(1)} = \underline{k}\,T_*^{(1)} \tag{4.7a}$$

$$i\,T_*^{(0)} + 4\beta^2 W_*^{(1)} = 0 \tag{4.7b}$$

$$\frac{\partial u_*^{(1)}}{\partial x} \quad \frac{\partial w_*^{(1)}}{\partial z} = 0 \tag{4.7c}$$

Substitution into (4.3) also gives just, on $x = \pm 1$,

$$u_*^{(0)} = 0$$

$$u_*^{(1)} = \pm \frac{1}{4\sqrt{(\beta\sqrt{\sigma})^3}} \frac{d}{dz}\left[\alpha_\pm(z) - \left(\frac{E^{1/2}}{f}\right)\frac{\partial P_*^{(0)}}{\partial z}\right] \tag{4.8}$$

The y-component of (4.7a) gives just

$$V_*^{(0)} = 2i\,u_*^{(1)} \tag{4.9}$$

and elimination of the $O(f)$ variables in (4.7) leads, with (4.8) and (4.9) to the following boundary value problem,

$$\beta^2 \frac{\partial^2 P_*^{(0)}}{\partial x^2} + \frac{\partial^2 P_*^{(0)}}{\partial z^2} = 0 \tag{4.10}$$

and on $x = \pm 1$, $\dfrac{\partial P_*^{(0)}}{\partial x} = \dfrac{\pm i}{\sqrt{(\beta\sqrt{\sigma})^3}} \dfrac{d}{dz}\left[\alpha_\pm(z) - \left(\dfrac{E^{1/2}}{f}\right)\dfrac{\partial P_*^{(0)}}{\partial z}\right]$

So, in general, one has to solve this problem for the interior. The difficulty with this solution is, of course, that it does not satisfy no-slip on $x = \pm 1$ since $V_*^{(0)} \not\equiv 0$ there. In fact,

$$V_*^{(0)}(\pm 1, z) = \frac{\pm i}{2\sqrt{(\beta\sqrt{\sigma})^3}} \frac{d}{dz}\left[\alpha_\pm(z) - \frac{E^{1/2}}{f}\frac{\partial P_*^{(0)}}{\partial z}\right] \tag{4.11}$$

One can easily convince oneself that the buoyancy layers of section A cannot remove this non-zero velocity on the walls.

<u>The Stokes Layer</u>

It is clear that the peculiar details of (4.10) are related to the periodicity of the solution directly, so one might expect that any new singular layers would also be related directly to the unsteadiness of the flow. Without giving the details, a substitution of the form

$$V = \frac{E^{1/2}}{f}\tilde{V}_*^{(0)} + \ldots$$

$u = E^{1/2}\tilde{u}_*^{(1)} + \ldots$ <u>etc</u>. into

(2.7) - (2.9) with the independent variable η,

$x = \pm 1 + \sqrt{\dfrac{2E}{f}}\,\eta$ and doing the limit process will yield (4.6b) and the x and z components of (4.6a) plus

$$\frac{1}{2}\frac{\partial^2 \tilde{V}_*^{(o)}}{\partial \eta^2} - i\,\tilde{V}_*^{(o)} = -i\,V_*^{(o)}(x,z)_{x=\pm 1}$$
$$\frac{\partial \tilde{u}_*^{(1)}}{\partial \eta} = 0 \tag{4.12}$$

which is an equation used by Stokes in 1851 to describe the flow induced by small harmonic oscillations of a plate in a viscous fluid. The solution is

$$\tilde{V}_*^{(o)} = V_*^{(o)}(\pm 1, z)\left[1 - e^{\pm(1+i)\eta}\right] \tag{4.13}$$

Now, the thickness of this layer is $\sqrt{2\nu/\omega}$ and under (4.4) it is certainly thin and yet thick compared to the buoyancy layer, so the technique of using, for $V_*^{(o)}(\pm 1, z)$ the value at the edge of the buoyancy layer is justified.

So, the solution in this range is complete, the solution of (4.10) being used to evaluate $V_*^{(o)}(\pm 1, z)$ from (4.9) and (4.8) and $\partial P_*^{(o)}/\partial z$ being calculated for insertion into the buoyancy layer solution to complete it. For f is this range, the buoyancy layers still play a dominant role in the dynamics, but one begins to see hints that viscosity is going to be important in the interior of the fluid because of the existence of the Stokes layer outside the buoyancy layer. In addition, lowering f has the effect of raising the order of the interior flow which is now $O(E^{1/2}/f)$ and as $f \rightarrow E^{1/2}$ from above, the velocities in the interior $\rightarrow O(1)$.

C. <u>The Interior Solution</u>; $\underline{\sqrt{\Omega\,\nu/\ell^2} >> \omega >> \nu/\ell^2}$

In this parameter range we have the double constraint,

$$\lim_{E\to 0} \frac{E}{f} = 0 \qquad \text{and} \quad \lim_{E\to 0} \frac{f}{E^{1/2}} = 0 \tag{4.14}$$

that is $1/\omega$ is small compared to the diffusion time and large compared to the "spin-up" time. (Greenspan and Howard, 1963). Now, in the expansion scheme for the interior, $\epsilon_0 = 1$, $\epsilon_1 = E^{1/2}$, $\epsilon_2 = f$, . . . It is quite clear that both the $O(1)$

and $O(E^{1/2})$ terms in the expansion obey (4.6) which obviously, then, makes buoyancy layers impossible since the strong constraint in the interior associated with the stratification and rotation, in this geometry, imply that there can exist no horizontal (i.e. x-direction) or vertical velocity to $O(E^{1/2})$. So, (4.3) with boundedness at ∞ implies

$$\frac{\partial p^{(0)}}{\partial z} = \alpha_{\pm}(z) \text{ on } x = \pm 1 \tag{4.15}$$

Now, the order f equations may be used in the usual way to give the $O(1)$ equation as

$$\beta^2 \frac{\partial^2 p^{(0)}}{\partial x^2} + \frac{\partial^2 p^{(0)}}{\partial z^2} = 0 \tag{4.16a}$$

as it must be under (4.14), and is the same equation as (4.10), and

$$\underline{u}^{(0)} = \frac{1}{2} \underline{k} \times \nabla p^{(0)} \tag{4.16b}$$

As before, there is a Stokes Layer on the walls which has the solution

$$\tilde{V}^{(0)}_{\pm 1} = \frac{1}{2} \frac{\partial p^{(0)}}{\partial x} (\pm 1, z) \left[1 - e^{\pm (1+i)\eta} \right] \tag{4.17}$$

with

$$\eta \equiv \frac{x \mp 1}{\sqrt{2E/f}}$$

which is even a thicker layer than before.

So, here the temperature field contains no regions of large gradients; however, diffusive character in the flow is still isolated to the boundary region but, as $f \rightarrow E$, this layer will fill the interior and then clearly diffusive action will dominate the interior flow.

D. <u>The Interior Solution; $\omega \lesssim \nu/\ell^2$</u>

In this final range of frequencies, the constraint on the mathematics is

$$\lim_{E \to 0} (f/E) < \infty \tag{4.18}$$

In this outer expansion $\epsilon_0 = 1$, $\epsilon_1 = E^{1/2}$, $\epsilon_2 = E$, $\epsilon_3 = o(E)$, etc. Buoyancy layer suction is again prevented by the $O(E^{1/2})$ equation, but the $O(E)$ equation is now

$$
\begin{aligned}
&i(f/E)\underline{u}^{(0)} + 2\underline{k} \times \underline{u}^{(2)} + \nabla p^{(2)} = \underline{k}\,T^{(2)} + \nabla^2 \underline{u}^{(0)} \\
&i(f/E)T^{(0)} + 4\beta^2 w^{(2)} = \frac{1}{\sigma}\nabla^2 T^{(0)} \\
&\operatorname{div} \underline{u}^{(2)} = 0
\end{aligned}
\tag{4.19}
$$

Conservation of mass to this order then yields an equation for $P^{(0)}$, viz.,

$$\beta^2\left[\nabla^2 - if/E\right]\frac{\partial^2 p^{(0)}}{\partial x^2} + \left[\frac{1}{\sigma}\nabla^2 - if/E\right]\frac{\partial^2 p^{(0)}}{\partial z^2} = 0 \tag{4.20a}$$

$$\underline{u}^{(0)} = \frac{1}{2}\underline{k} \times \nabla p^{(0)} \tag{4.20b}$$

$$T^{(0)} = \frac{\partial p^{(0)}}{\partial z} \tag{4.20c}$$

Now the no-slip boundary condition and the thermal condition give

$$\left.\begin{aligned} \frac{\partial p^{(0)}}{\partial x} &= 0 \\ \frac{\partial p^{(0)}}{\partial z} &= \alpha_{\pm}(z) \end{aligned}\right\} \text{on } x = \pm 1 \tag{4.21}$$

The solution is now entirely controlled by diffusion and there are no singular layers anywhere in the fluid. It is inconsistent to retain the (f/E) term in (4.20a) if f is sufficiently small. If, in a particular case, $P^{(1)} \not\equiv 0$, then $P^{(0)}$ will contain terms of uniformly larger order than the $P^{(1)}$ terms only if f is in the range

$$E \gg f \gg E^{3/2}$$

For all f's not obeying this restriction, (4.20a) is to be solved without the f/E terms. Only then is the motion precisely a "steady" motion with the interior precisely in phase with the boundary fluctuations.

E. An Example; Sinusoidal Temperature Distributions

We now do a particular example of the things set forth in B, C, and D frequency domains. Take

$$\alpha_{\pm}(z) = -K_{\pm}\cos\alpha z \tag{4.22}$$

which is just the problem done in Section 3.D, (ii).

(i) $\Omega >> \omega \geq \sqrt{\nu\Omega/\ell^2}$; B

The problem here is to solve (4.10) under the above condition (4.22). It is quite easy to obtain the solution as

$$P_*^{(0)} = \frac{i\sin\alpha z}{(\beta\sqrt{\sigma})^{3/2}}\left[\frac{(K_+-K_-)\cosh\lambda x}{\lambda\sinh\lambda - b\cosh\lambda} + \frac{(K_+-K_-)\sinh\lambda x}{\lambda\cosh\lambda - b\sinh\lambda}\right] \tag{4.23}$$

where $\lambda \equiv \alpha/\beta$ and $b = i\alpha^2\sqrt{\frac{E}{f^2\beta^3\sigma^{-3/2}}}$

and $$\frac{1}{2}\left.\frac{\partial P_*^{(0)}}{\partial x}\right|_{\pm 1} = \frac{i\lambda\sin\alpha z}{2(\beta\sqrt{\sigma})^{3/2}}\left[\frac{\pm(K_++K_-)\sinh\lambda}{\lambda\sinh\lambda - b\cosh\lambda} + \frac{(K_+-K_-)\cosh\lambda}{\lambda\cosh\lambda - b\sinh\lambda}\right] \tag{4.24}$$

For the symmetric case $K_+ = K_-$, the stream function is

$$\psi_s = \cos\alpha z \sinh\lambda x$$

and

$$V_*^{(0)}(\pm 1, z) = \pm\frac{iK_+}{(\beta\sqrt{\sigma})^{3/2}}\sin\alpha z\,\frac{1}{1-\frac{b}{\lambda}\coth\lambda} \tag{4.25a}$$

so the streamlines are just as in 3.D.(ii)(b).

For the antisymmetric case, the stream function is

$$\psi_a = \cos\alpha z \cosh\lambda x$$ which is

identical with (3.18) with $f \equiv 0$. Then v-velocity at the plate is

$$V_*^{(0)}(\pm 1, z) = \frac{K_+ i\sin\alpha z}{(\beta\sqrt{\sigma})^{3/2}\left[1-\frac{b}{\lambda}\tanh\lambda\right]} \tag{4.25b}$$

Hence, the streamlines are just those given in section 3.D(ii) with $f \equiv 0$, and they represent a periodic flow in the x-z plane with velocities of $O(E^{1/2})$. However, those velocity components do <u>not</u> represent the primary motion in the interior since there is y-velocity of $O(E^{1/2}/f)$. In addition, there are shear layers to bring this velocity component to zero on the plates, and these layers are thick compared to the buoyancy layers which also still play an important role in the dynamics of the interior. The structure of the shear layers has been given in (4.13).

(ii) $\sqrt{\Omega\nu/\ell^2} >> \omega >> \nu/\ell^2$; C

Equations (4.15) and (4.16) give, if $\alpha(z) = -K_\pm \cos\alpha z$,

$$p^{(0)}(x,z) = -\frac{\sin\alpha z}{\alpha}\left[(K_+ + K_-)\frac{\cosh\lambda x}{\cosh\lambda} + (K_+ - K_-)\frac{\sinh\lambda x}{\sinh\lambda}\right] \quad (4.26)$$

and $V^{(0)}$ is the x-derivative and is now $O(1)$ in the interior. The shear layers given by (4.17) reduce the velocity to zero. The buoyancy layers do not enter the solution to this order due to the strong constraints on the interior motion. The u and w velocities are $O(f)$ and the streamline pattern has the same structure as in (i) for the purely symmetric and purely antisymmetric cases. However, this u-w flow is becoming less and less significant as $f \longrightarrow 0$.

(iii) $\omega \leqslant \nu/\ell^2$, D

In this section, the u and w velocities are $O(E)$ and the solution is now without singular layers (4.20) and (4.21) giving the solution, in symbolic form,

$$p^{(0)} = -\frac{\sin\alpha z}{\alpha}\sum_1^4 A_n e^{\lambda n x} \quad (4.27)$$

where the A_n are determined from the equations

$$\sum_1^4 A_n e^{\pm\lambda n} = K_\pm$$

$$\sum_1^4 A_n \lambda_n e^{\pm\lambda n} = 0$$

and the λ's are roots of the quadratic

$$\lambda^2 = \frac{1}{2}\left[\alpha^2\left(1+\frac{1}{\beta^2\sigma}\right)+if/E\right] \pm \frac{1}{2}\sqrt{\left(\alpha^2\left(1-\frac{1}{\beta^2\sigma}\right)+if/E\right)^2 + \frac{4\alpha^2}{\beta^2}\left(\frac{1}{\sigma}-1\right)\left(\alpha^2+if/E\right)} \quad (4.28)$$

A lot of algebra will eventually give the explicit form of (4.27) as just

$$-p^{(0)}\frac{\alpha}{\sin\alpha z} = \frac{K_+ + K_-}{2}\,\frac{\lambda_2 \sinh\lambda_2 \cosh\lambda_1 x - \lambda_1 \sinh\lambda_1 \cosh\lambda_2 x}{\lambda_2 \sinh\lambda_2 \cosh\lambda_1 - \lambda_1 \sinh\lambda_1 \cosh\lambda_2} +$$
$$+\frac{K_+ - K_-}{2}\,\frac{\lambda_2 \cosh\lambda_2 \sinh\lambda_1 x - \lambda_1 \cosh\lambda_1 \sinh\lambda_2 x}{\lambda_2 \cosh\lambda_2 \sinh\lambda_1 - \lambda_1 \cosh\lambda_1 \sinh\lambda_2}$$

where λ_1 and λ_2 are absolute values of the roots (4.28), $\underline{u}$, the roots are in pairs, $\pm\lambda_1$, $\pm\lambda_2$. These have simple forms in two cases. If $f/E \rightarrow 0$

$$\left.\begin{aligned} \lambda_1 &\sim \alpha\left[1+\frac{if}{2\alpha^2 E}\ \frac{\sigma(\beta^2-1)}{\sigma\beta^2-1}\right] \\ \lambda_2 &\sim \frac{\alpha}{\beta\sqrt{\sigma}}\left[1+\frac{if\sigma\beta^2}{2\alpha^2 E}\ \frac{\sigma(\sigma-1)}{\beta^2\sigma-1}\right] \end{aligned}\right\} \tag{4.29}$$

If $\sigma \equiv 1$, then (4.28) readily will yield just

$$\left.\begin{aligned} \lambda_1 &= (\alpha^4+f^2/E^2)^{1/4} e^{i\tan^{-1}(f/\alpha^2 E)} \\ \lambda_2 &= \alpha/\beta \end{aligned}\right\} \tag{4.30}$$

5. Conclusion

So, we have seen that when a rotating stratified fluid is subjected to periodic temperature boundary conditions, the fluid motion is characterized by buoyancy layer suction when ω is $O(\Omega)$. As ω gets smaller, it passes through two distinct regimes before becoming so small that the fluid exhibits the typical diffusion-controlled steady structure. These two intermediate regimes both exhibit a thin Stokes layer as an important part of the structure of the leading order interior solution. The buoyancy layers become less and less significant in the dynamics of the interior region, the importance of diffusion in the interior becoming manifest in the growth of the Stokes layer from a size just larger than the buoyancy layer, through a size $O(E^{1/4})$ at the transition between the two regimes, to $O(1)$ when $f \rightarrow E$. Thus, the unsteady motion becomes less and less successful in confining the z-vorticity to the walls and eventually it fills the entire fluid. For elucidation of the details, consider the table on the following page. This is shown schematically in Figure 3. Some future work might include inertia effects say first in the Stokes layer in region III where the shears are very large. In all of this treatment, we have taken $\bar{\epsilon} = 0$, which is hopefully a good lowest order solution. In view of the analogy, (Veronis, 1967, II), what is done here will be essentially the same as moving horizontally infinite boundaries

	ω REGIME	INTERIOR FLOW		BUOYANCY LAYER	STOKES LAYER
		$O(V,T)$	$O(u,w)$	$O(W,T)$	$O(V)$
I	$\omega = O(\Omega)$	$E^{1/2}$	$E^{1/2}$	1	$E^{1/2}$
II	$\sqrt{\frac{\nu\Omega}{\ell^2}} \leq \omega << \Omega$	$E^{1/2}/f$	$E^{1/2}$	1	$E^{1/2}/f$
III	$\nu/\ell^2 << \omega << \sqrt{\frac{\nu\Omega}{\ell^2}}$	1	f	$f/E^{1/2}$	1
IV	$\omega \leq \nu/\ell^2$	1	E	$E^{1/2}$	NONE

with some specified speed in a stratified rotating fluid. Then all of the phenomena are tilted through 90°.

Acknowledgments

The author is grateful to Dr. G. Veronis for his suggestion of this problem and the importance of the zero frequency limit, and for several hours of discussion and helpful comments. Thanks also to Drs. Malkus and Stern and especially Dr. L. N. Howard for their interest in the problem.

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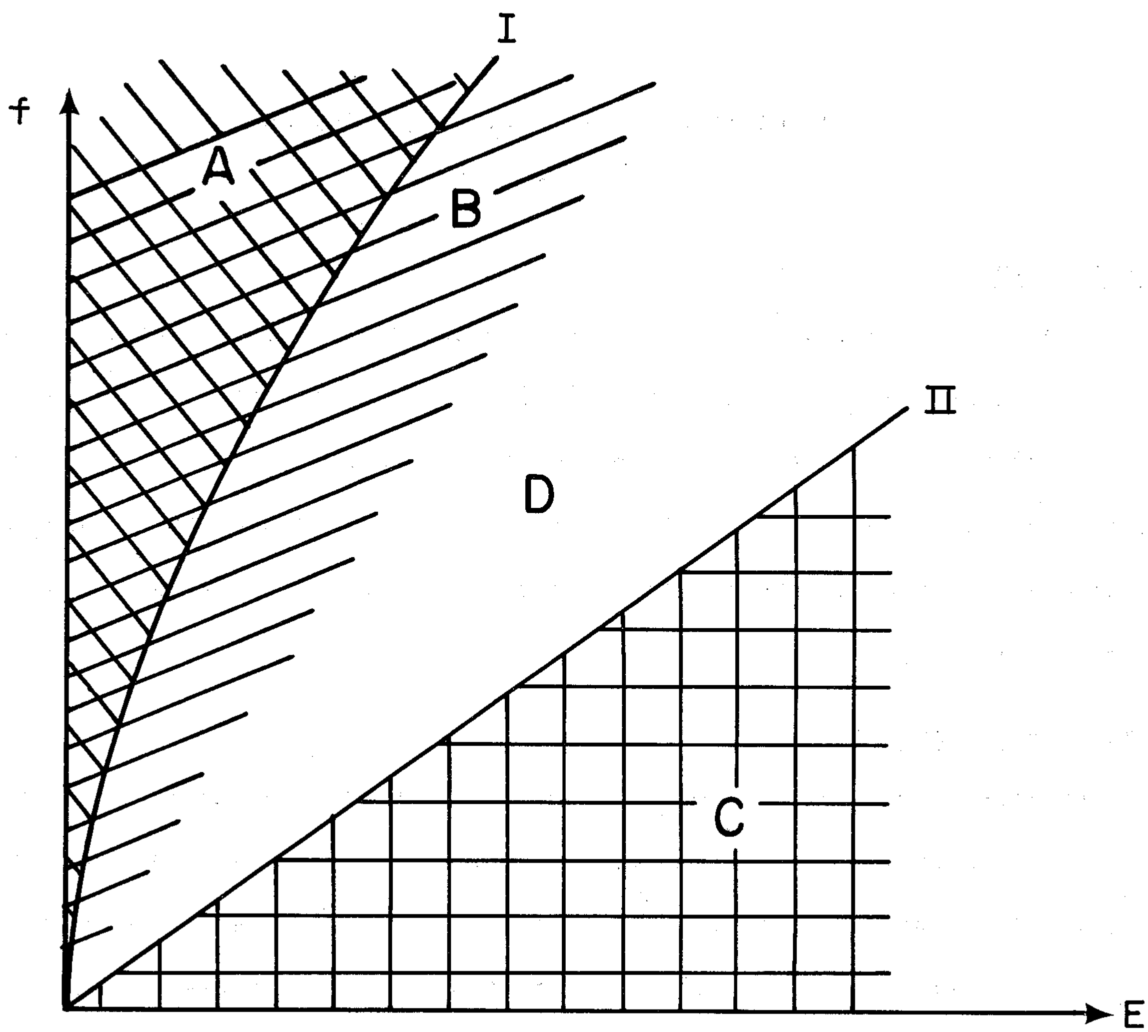

A – REGION WHERE BUOYANCY LAYER CONTROLS THE STRUCTURE OF THE SOLUTION

B – REGION WHERE STOKES LAYER PLAYS A SIGNIFICANT ROLE IN THE SOLUTION

C – REGION OF DIFFUSIVE CONTROL IN THE INTERIOR

D – SMALL TRANSITIONAL REGION NOT ANALYZED IN DETAIL HERE

I – THE LINE $f = E^{\frac{1}{2}}$; A DISTINGUISHED LIMIT

II – THE LINE $f = E$; A DISTINGUISHED LIMIT

Figure 3

MECHANICAL STIRRING AND SALT FINGERS

Michael C. Gregg

1. Introduction

Much interest has developed recently in the role of salt fingers in the ocean. Laboratory experiments, Turner (1967), have shown that stirring inhibits the formation of salt fingers. This has cast some doubt on whether salt fingers can occur in the ocean. As a first step in approaching this problem the present measurements were done to observe the effect of various rates of mechanical stirring on salt fingering.

2. Experimental Method

The experiments were performed in a 24.5x25x45 cm plexiglass tank shown in Figure 1. Experiments were done both with the stirring grids as shown and with only the top grid. Data runs were made with hot salty water above cold fresh water and with salt water above sugar water. In both cases the density difference between layers was approximately 5 parts per thousand.

For the salt fingering runs the tank was first filled with tap water and then the heating coil was used to warm the upper layer to produce a temperature contrast of approximately 25^{o}C between the layers. Intermittent stirring was used to produce a homogeneous layer above the interface. In the single grid runs dye was used in the upper layer so that the interface could be visually brought to the 24 cm level. When the upper layer was at the proper temperature the heater was turned off and 30 grams of salt dissolved in hot water were carefully added to the upper layer. The density of the added hot salty water was nearly the same as that of the hot water in the upper layer. Addition of the salt water was done slowly over a period of several minutes. Both the stirrer and the timer were started as soon as the salt water began to be added.

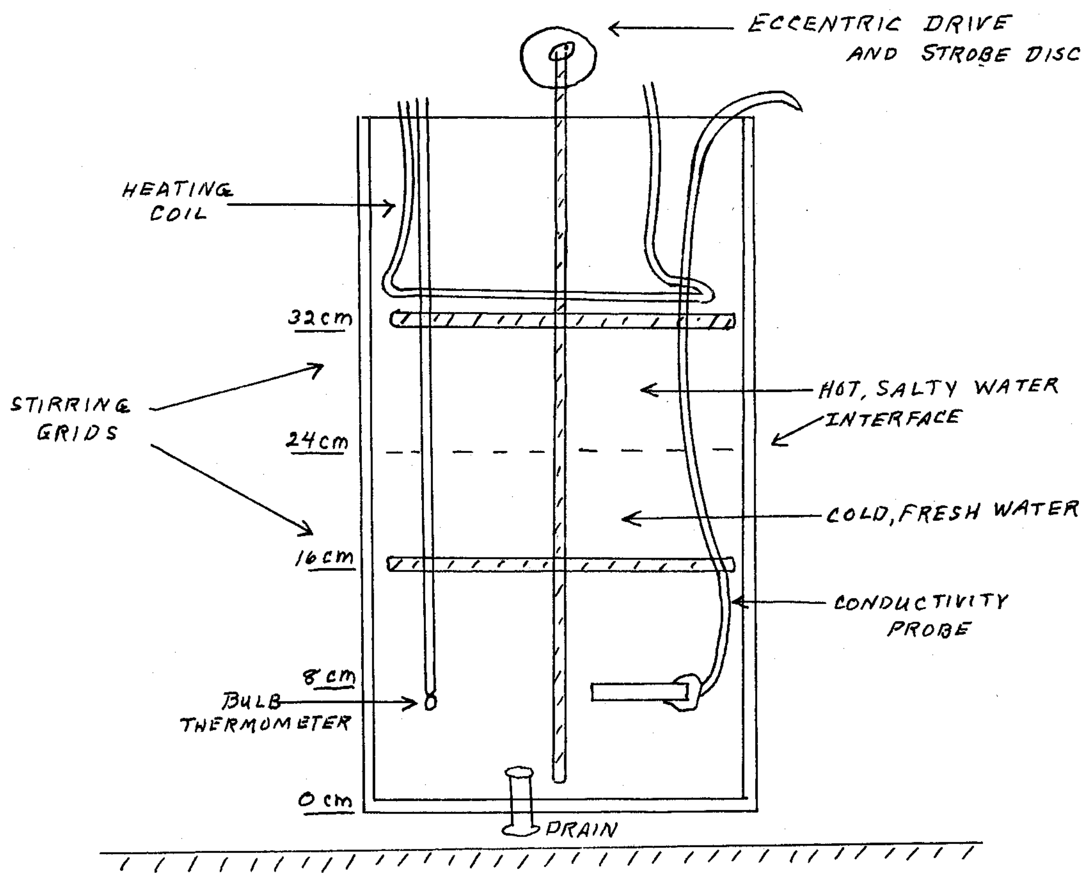

Fig. 1. Experimental tank with insulation removed.

For each of the various stirring rates measurements were made of the temperature and conductivity in the bottom layer as a function of time. For most measurements the bulb of the thermometer and the conductivity probe were located at 6-8 cm above the bottom of the tank. The thermometer was graduated in units of 0.1°C and was read to the nearest tenth of a degree. The conductivity instrument was a commercially available dip type probe that was calibrated directly against salt concentration and temperature. Unfortunately the threshold of the probe (.18 ‰ salt at 22°C) was too high to observe the initial stages of salt fingering.

During some runs temperature profiles were taken through the interface. At the conclusion of some runs using both grids the tank was drained slowly to permit a final profile of temperature and salinity.

3. Data and Discussion

Salt concentration and temperature observed at 6 cm above the bottom of the tank for the single grid case are shown in Figures 2 and 3. Similar results for the double grid case are shown in Figures 4 and 5. The curves shown are those of individual data runs. In most cases at least two runs were made for each stirring frequency. In some cases the curves duplicated, in others there were variations, but in all cases the relative positions of the curves for different stirring rates were the same as shown in the figures.

Both Figures 2 and 4 show that the salt flux is strongly inhibited by stirring. For both cases the salt flux at the probe has a minimum and then begins to increase at still higher stirring rates. The temperature increase in the lower layer has a minimum as a function of stirring rate in the one grid case and increases continuously in the double grid case.

The lower set of curves in Figure 4 were an attempt to determine the transport of salt independently of the salt fingering. It is interesting to note that at a moderate stirring rate the salt was transported downward much more effectively in the diffusively stable system.

In the one grid case there are two processes acting. On one hand there is the salt fingering situation, in which a gravitationally stable system is diffusively unstable, which acts to decrease the potential energy of the system by transporting density downward. The other process is entrainment, i.e. the advance of a turbulent layer into one at rest, which acts to increase the potential energy of a stratified system.

From Figure 2 it seems that salt fingering is completely inhibited at the higher stirring rates while the interface is descending. Once the interface is reasonably stable salt fingering seems to begin. Temperature profiles for various

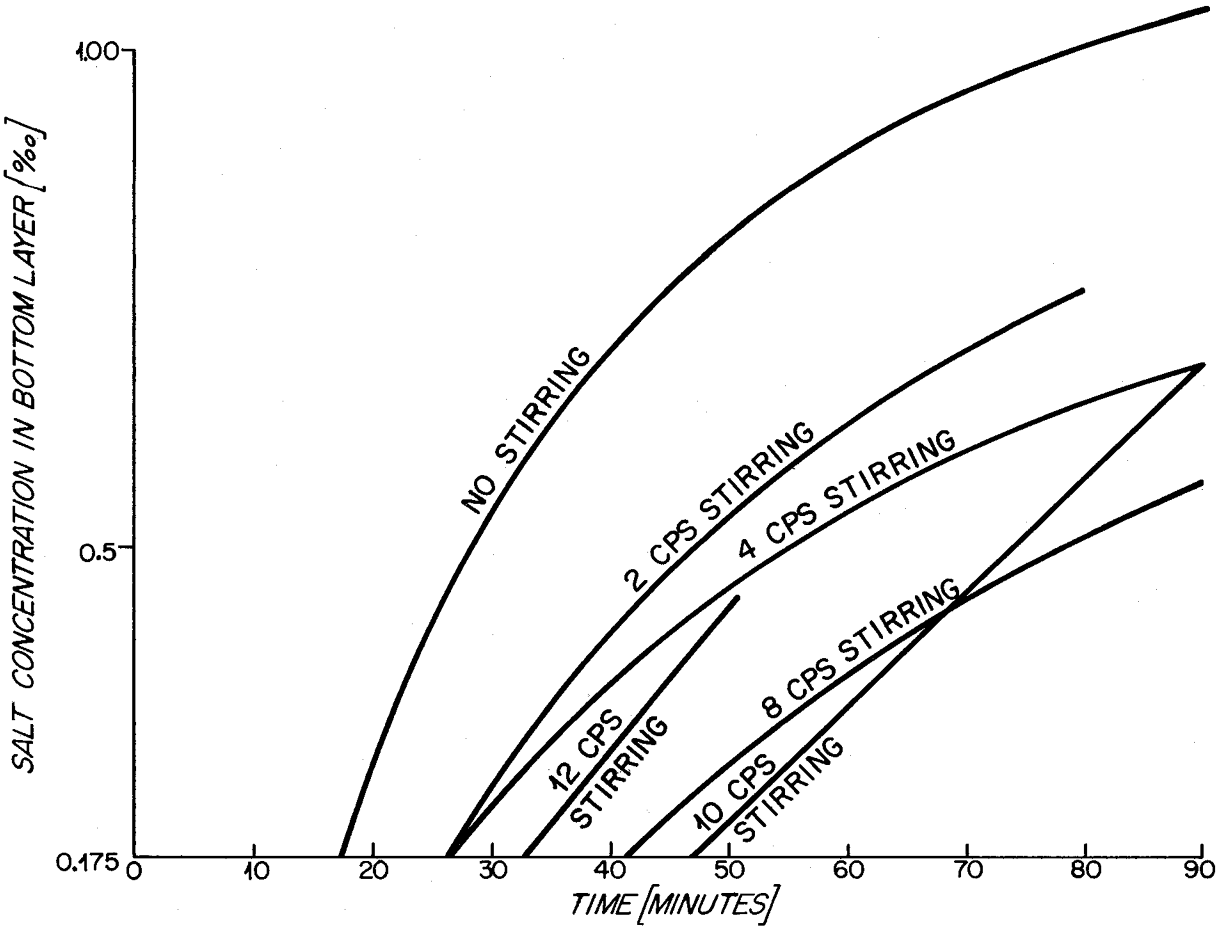

Figure 2

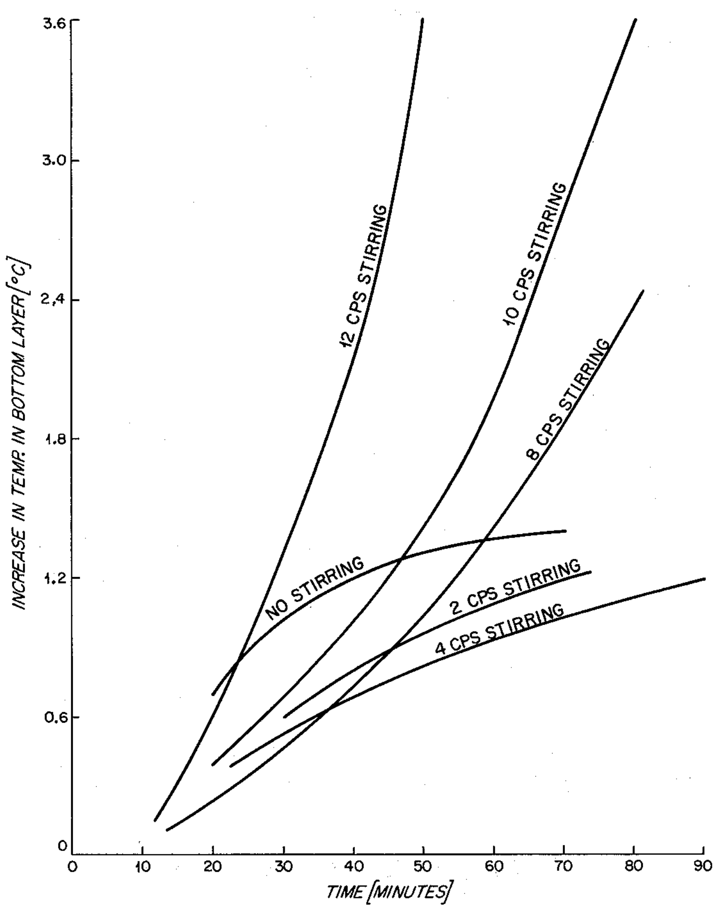

Figure 3

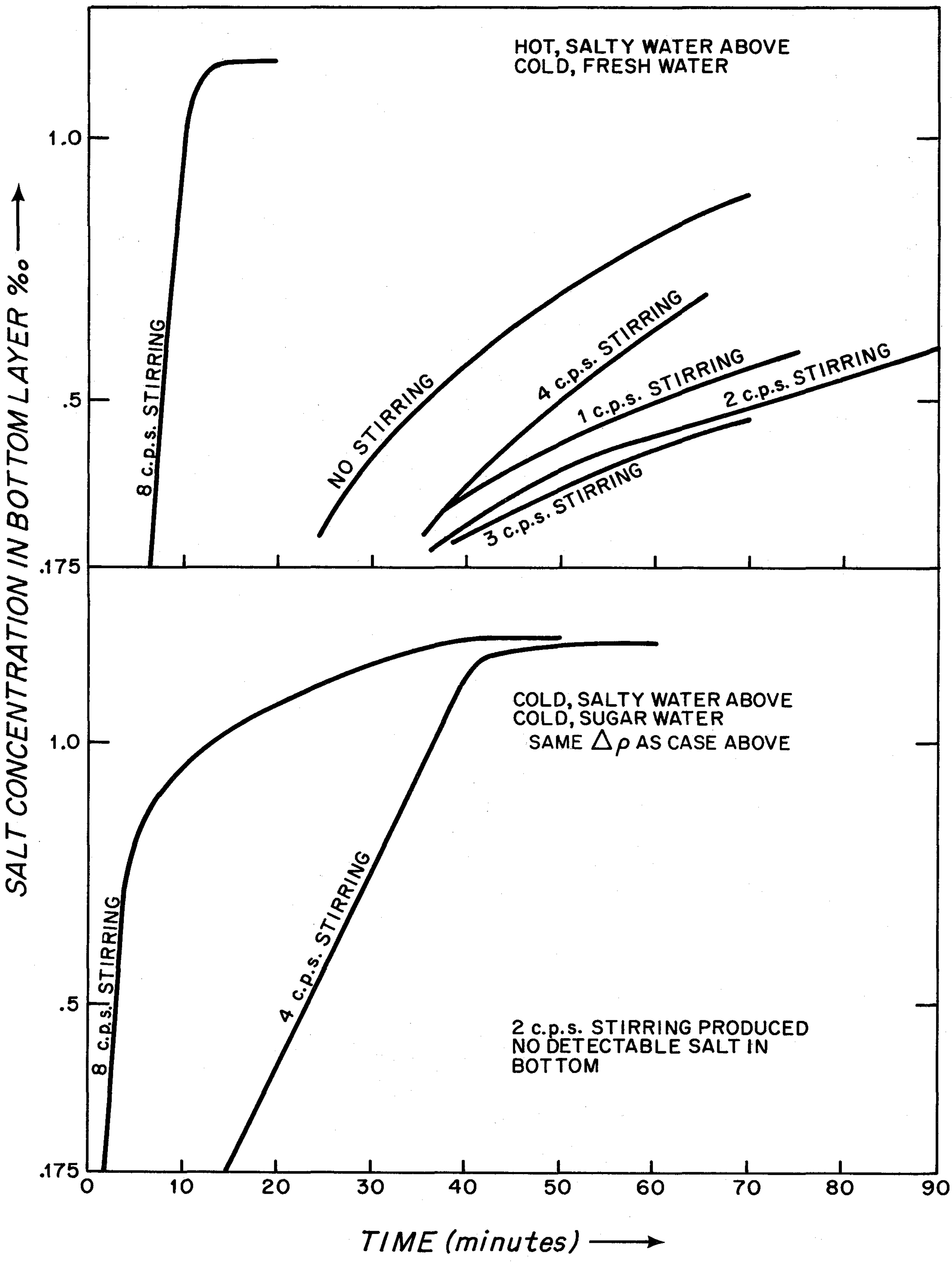
HOT, SALTY WATER ABOVE
COLD, FRESH WATER
8 c.p.s. STIRRING
NO STIRRING
4 c.p.s. STIRRING
1 c.p.s. STIRRING
2 c.p.s. STIRRING
3 c.p.s. STIRRING
1.0
.5
.175
SALT CONCENTRATION IN BOTTOM LAYER ‰
COLD, SALTY WATER ABOVE
COLD, SUGAR WATER
SAME $\Delta\rho$ AS CASE ABOVE
8 c.p.s. STIRRING
4 c.p.s. STIRRING
2 c.p.s. STIRRING PRODUCED
NO DETECTABLE SALT IN
BOTTOM
0
10
20
30
40
50
60
70
80
90
TIME (minutes)

Figure 4

shaking rates are shown in Figure 6. Differences of temperature of the bottom layer are partly due to variations of tap water temperature.

If the total heat flux across the interface is represented by Ht and the salt flux is represented by F_S, then the density flux due to heat is given by αHt and that due to salt by βF_S. α is the volume coefficient of thermal expansion and β is the similar coefficient for salt. Turner (1967) has shown that $\alpha H/\beta F_S$ due to salt fingering is approximately 0.5 for $\alpha\Delta T/\beta\Delta S$ across the interface in the range of 2 to 10. For the present measurements $\alpha\Delta T/\beta\Delta S = 3.8$.

Several persons have performed tank experiments with a stirring grid to observe entrainment of stably stratified fluids. These experiments have shown that the interface between turbulent and non-turbulent fluids is sharp and that entrainment occurs by the detachment of wisps or streamers from cusps of internal waves at the interface. Consequently, for diffusively stable systems the layer below the entraining interface is not mixed with the upper layer. Hence, for the single grid experiments one expects that the increase of temperature and salt in the lower layer is due only to salt fingers and molecular diffusion.

The $\frac{\alpha Ht}{\beta F_S}$ ratios obtained for both the single and double grid cases are shown in Figure 7. The fluxes Ht and F_S were computed from the increase in temperature and salt at the probe location. In most cases the probes were in regions of negligible gradients of temperature and salt and hence the flux ratios do not reflect the conductive flux of heat across the interface.

For the single grid case the $\alpha HT/\beta F_S$ values remain constant for stirring rates less than 4 cps and increase for greater stirring rates. Part of this increase may be due to the interfaces descending so that the thermometer was in the temperature gradient region near the interface.

For the double grid case the ratio $\frac{\alpha H t}{\beta F_s}$ increases from 1/2 at no stirring to about 3.8 at high stirring rates. Since the ratio $\frac{\alpha \Delta T}{\beta \Delta s}$ across the interface is 3.8 this indicates complete mixing for stirring rates greater than 4 cps.

In order to relate these measurements to the world outside the tank it is necessary to determine the stirring energies involved. This was not possible but an argument of Turner and Kraus (1967) was used to obtain a rough idea of the energy developed by the grid. The rate of change of potential energy due to the descent of the interface is

$$\frac{dE}{dt} = \frac{A}{2}(g h \Delta \rho)\frac{dh}{dt}$$

where h is the height of the interface and A is the cross-sectional area of the tank. Assuming that the kinetic energy due to stirring is put in at a constant rate and is all being used to increase the potential energy of the system, then the interface should descend at a constant rate. To determine the rate of energy input by the grid using this argument a layer of cold salt water was placed above a layer of cold sugar water with the same $\Delta \rho$ as for the salt fingering case. It is interesting to note from Figure 8 that for the same density difference the salt/sugar interface descends faster than the hot salty/cold fresh one.

The initial velocity of the salt/sugar interface corresponds to a rate of energy input to the fluid by the grid of approximately 120 ergs/sec. A measurement at 4 cps gave value of about 60 ergs/sec. For comparison the rate of change of potential energy due to straight salt fingering was about 30 ergs/sec. Thus for the two-grid case salt fingering seemed almost completely inhibited at 4 cps, corresponding to a rate of energy input to the upper layer of twice that of the straight salt fingering case.

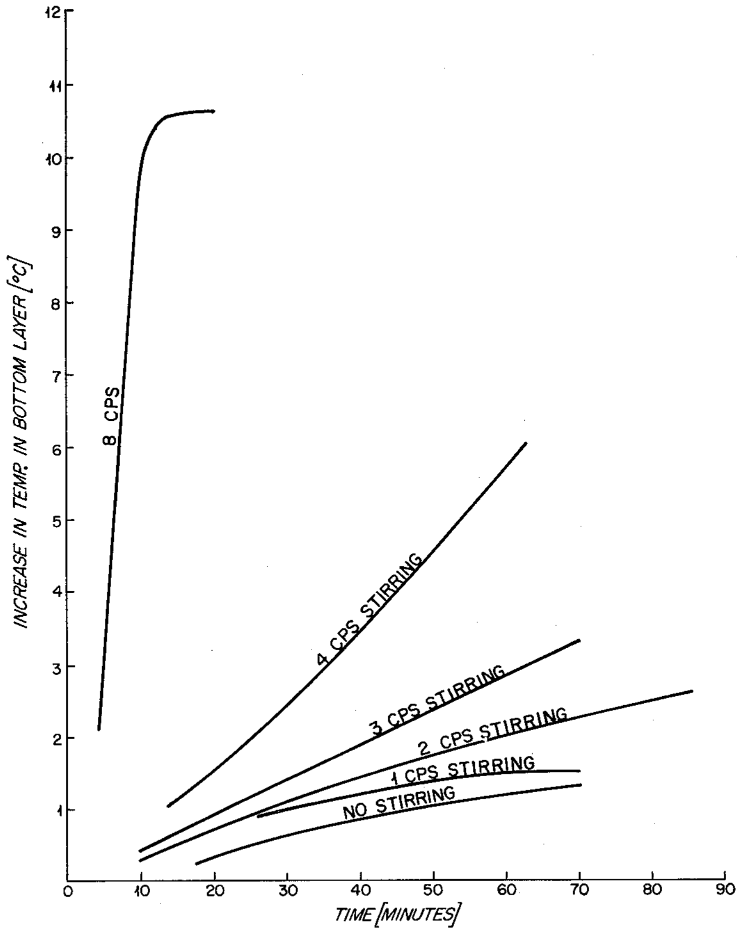

Figure 5

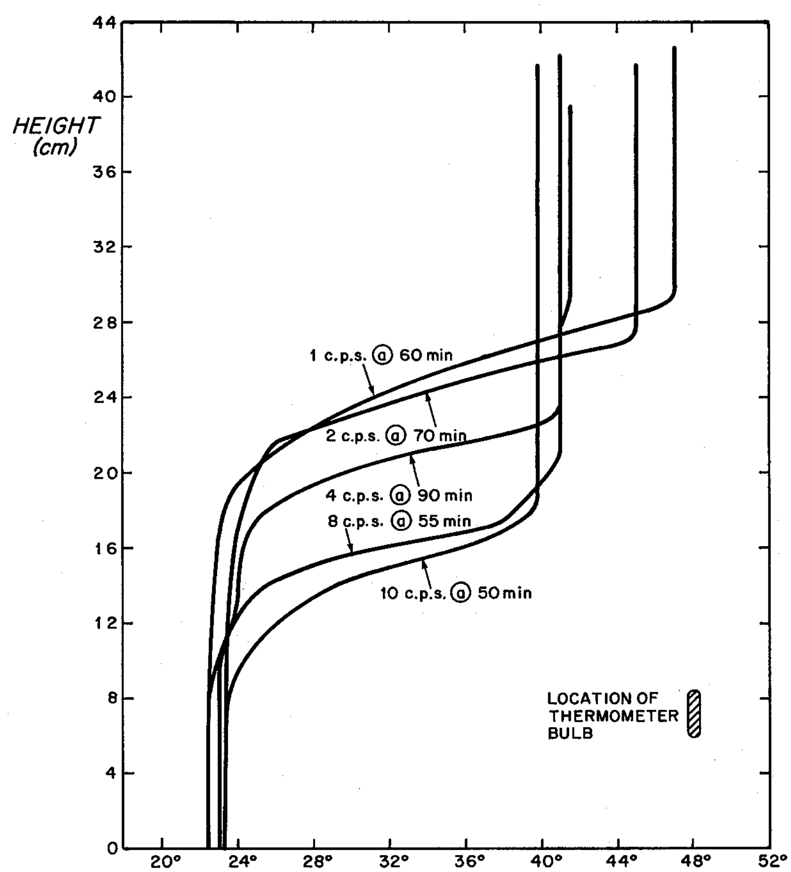

Figure 6

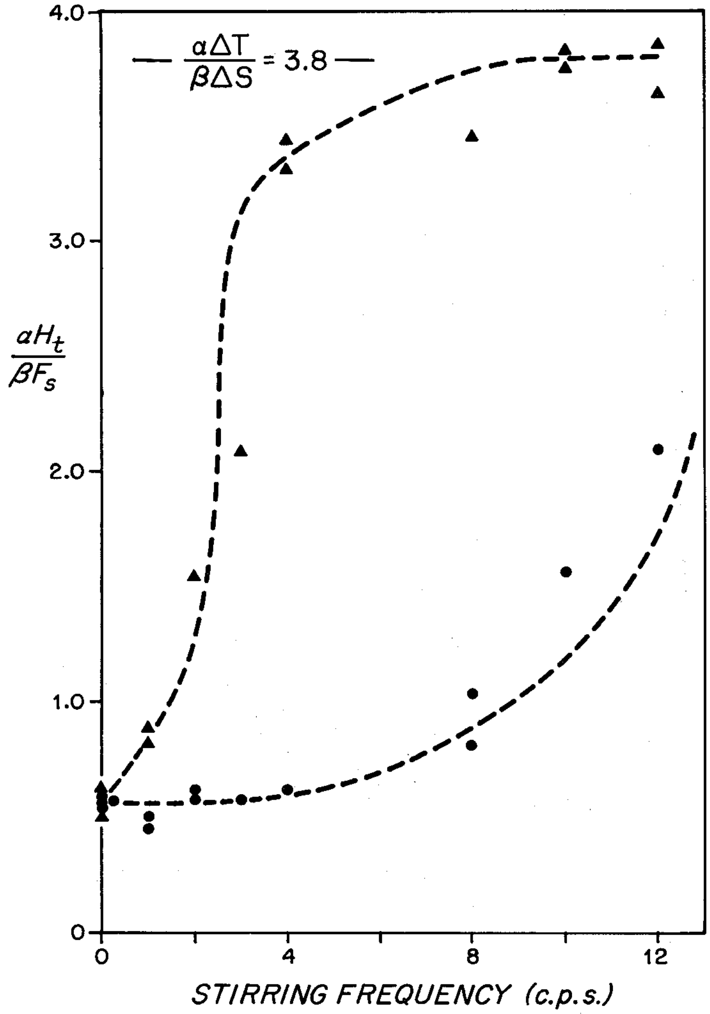

Figure 7

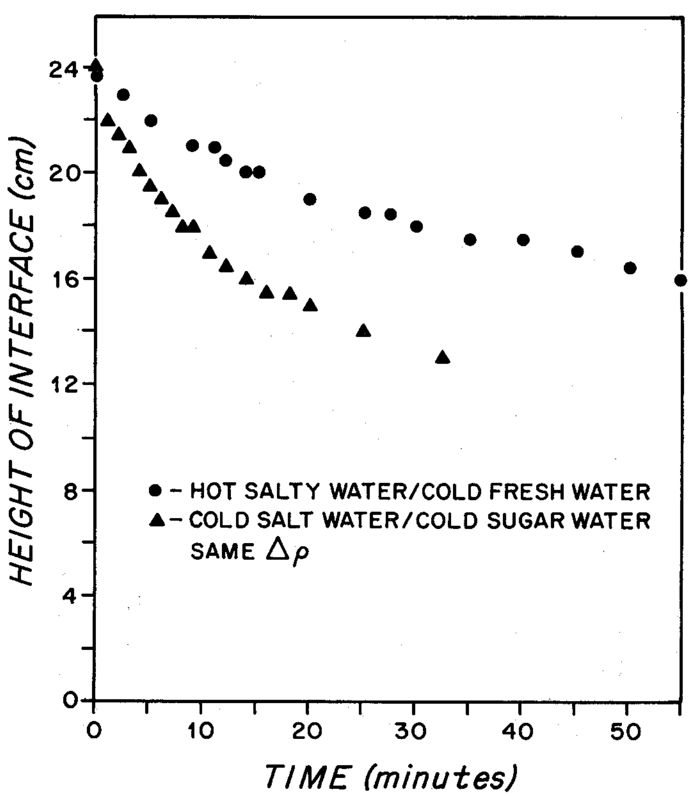

Figure 8

4. Future Work

These measurements are a first attempt to study the effect of an imposed fluid motion on the salt fingering process. It would be of interest to repeat them using a thermistor, a small moveable salinity probe with a low threshold, and a hot film anemometer.

It would also be of more oceanographic interest to study the effect of a shear flow on the salt fingering. This could be done in a laboratory using an overflow tank. A first approach to the shear case was done by placing a beaker with salt fingers on a rotating table. The differential spin-up between the two stratified layers provided shear at the interface.

5. Acknowledgements

I wish to express my gratitude to the G.F.D. program for a very stimulating summer. Special thanks to Prof. Claes Rooth for help in obtaining the apparatus and for many helpful discussions. Beneficial discussions were also enjoyed with Drs. Stern and Turner.

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SIDE-WALL EFFECTS ON SPIN-UP ABOVE A POROUS MEDIUM

Philip Hazel

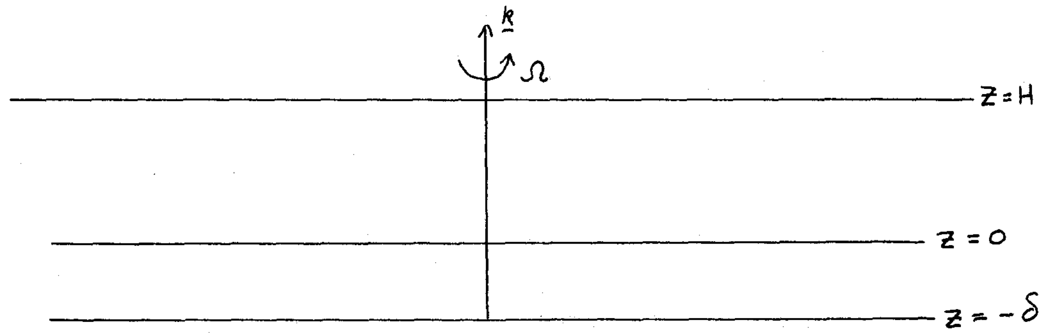
1. Introduction

The spin-up problem for a fluid in a container having a layer of porous medium along part of its boundary has recently come into prominence. The case of a spherical porous shell surrounding the fluid has been treated by Bretherton and Spiegel (1968), while the flow between two infinite porous sheets has been solved independently by a number of people. Veronis' analysis is reviewed briefly here in Section 2. It is found that, for suitable values of the porosity, the spin-up time can be greatly reduced compared to the equivalent problem with rigid boundaries.

Laboratory experiments must of necessity be performed in vessels of finite size; it is the purpose of this discussion to try to determine what effect rigid, vertical sidewalls have on the spin-up process. We shall consider the case of an inviscid fluid in a right circular cylinder with a layer of porous medium on the bottom; the neglect of viscosity leads to some singularities in the flow, but nevertheless, several important trends are indicated. The central part of the fluid is found to spin up on a time scale virtually the same as that for the fluid of infinite horizontal extent, but near the side walls, the time scale becomes longer, and viscosity evidently becomes important.

2. Review of Veronis' Analysis

Consider a fluid of infinite horizontal extent, above a porous medium layer of depth δ . The upper boundary condition at $z = H$ is that the vertical velocity should be zero. This can be interpreted as either



(a) a free surface with negligible curvature, or

(b) the mid-point of a symmetric system with a porous lid.

The lengths are dimensionless, and usually we would take $H = 1$, but it is kept here for comparison with later sections. The governing equations for the fluid in $0 \leq z \leq H$ are the usual spin-up equations, viz:

$$\begin{cases} \underline{u}_t + \epsilon\, \underline{u} \cdot \nabla \underline{u} + 2\, \underline{k} \wedge \underline{u} + \nabla P = E \nabla^2 \underline{u} \\ \nabla \cdot \underline{u} = 0 \end{cases} \tag{2.1}$$

(cf. Prof. Howard's lectures in Vol. 1)

These have been non-dimensionalized in the usual way, with the time scaled on Ω^{-1}. (We are interested only in axisymmetric motions, and so will always take $\frac{\partial}{\partial \theta} = 0$.) When considering the flow <u>inside</u> the porous layer, the viscous dissipation term is replaced by a D'Arcy law term, where resistance is proportional to the velocity. If the proportionality constant is $\lambda^*_{secs^{-1}}$ we have

$$\begin{cases} \underline{u}_t + \epsilon\, \underline{u} \cdot \nabla \underline{u} + 2\, \underline{k} \wedge \underline{u} + \nabla P = -\frac{\lambda^*}{\Omega} \underline{u} \\ \nabla \cdot \underline{u} = 0 \end{cases} \tag{2.2}$$

with the same scaling as (2.1). λ^*/Ω, the dimensionless D'Arcy constant, will be called λ.

Veronis considered the case of small Rossby number $(\epsilon = 0)$ and large λ. Equation (2.1) is then as for "usual" spin up, while for the porous medium the coriolis terms are negligible, giving

$$\nabla^2 P = 0 \tag{2.3}$$

The boundary conditions are

$$w = 0 \quad \text{at} \quad z = -\delta, H$$
$$\underline{u} \text{ and } P \text{ to match at } z = 0 \tag{2.4}$$

while initially we have $v = r$ except near the bottom. By applying the usual Ekman layer technique to the upper flow the following solution to the problem was found:

$0 \leq z \leq H$:

$$u = r\left(-\frac{2}{\lambda}\cos\xi - \sin\xi\right)e^{-\xi}e^{-\sigma t} + \frac{1}{2}r\sigma e^{-\sigma t} + o(E)$$
$$v = re^{-\sigma t} + r\left(-\cos\xi + \frac{2}{\lambda}\sin\xi\right)e^{-\xi}e^{-\sigma t} + o(E^{1/2})$$
$$w = \sigma(H-z)e^{-\sigma t} - E^{1/2}\left[\left(1+\frac{2}{\lambda}\right)\cos\xi + \left(1-\frac{2}{\lambda}\right)\sin\xi\right]e^{-\xi}e^{-\sigma t} + o(E) \tag{2.5}$$
$$p = r^2 e^{-\sigma t} + o(E^{1/2})$$

$-\delta \leq z \leq 0$:

$$u = -\left(\frac{2}{\lambda}\right)r e^{-\sigma t}$$
$$v = 0$$
$$w = \left(\frac{4}{\lambda}\right)(z+\delta)e^{-\sigma t} \tag{2.6}$$
$$p = \left(r^2 - 2z(z+2\delta)\right)e^{-\sigma t}$$

where ξ is the stretched Ekman layer coordinate $(\xi = E^{-1/2}z)$ and

$$\sigma = \frac{1}{H}\left(1+\frac{2}{\lambda}\right)E^{1/2} + \frac{4\delta}{\lambda H} \tag{2.7}$$

The spin-up time is σ^{-1}, and for suitable values of λ and δ, the last term can be made very much greater than the first, thus considerable shortening the spin-up time; e.g. in a typical laboratory experiment using water we may have

$$\lambda = 25, \quad \delta/H = 0.1, \quad (E^{1/2}/H) = 5\times10^{-3}$$

$$\Longrightarrow \begin{cases} \sigma = 21.4 \times 10^{-3} \\ \sigma_{viscous} = 5.4 \times 10^{-3} \end{cases}$$

The spin-up time is shortened by 75% by introduction of the porous medium.

3. Correction Term when λ is Small

When it is not valid to assume $\lambda >> 1$, but still $\lambda >> E^{1/2}$, we have to keep the Coriolis term in Equation (2.2). Instead of $\nabla^2 P = 0$, we now have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r P_r\right) + \left(1 + \frac{4}{\lambda^2}\right) P_{zz} = 0 \tag{3.1}$$

Repeating Veronis' analysis using this equation gives a small value of v inside the porous medium and a small correction to σ , viz:

$\underline{-\delta \leq z \leq 0}$:

$$\begin{aligned} u &= \frac{-2r\lambda}{\lambda^2+4} e^{-\sigma^* t} \\ v &= \frac{4r}{\lambda^2+4} e^{-\sigma^* t} \\ w &= \frac{4\lambda(z+\delta)}{\lambda^2+4} e^{-\sigma^* t} \end{aligned} \tag{3.2}$$

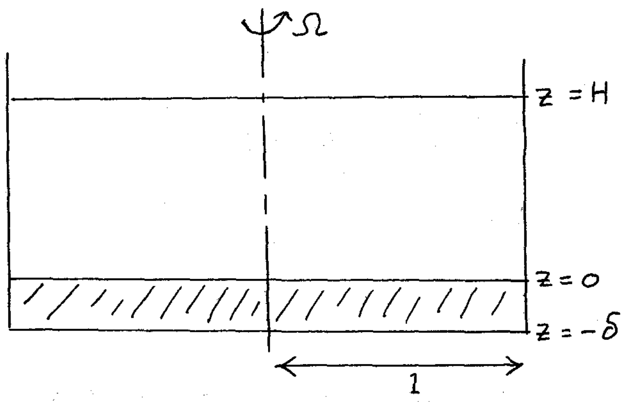
where

$$\sigma^* = \frac{1}{H}\left(1 + \frac{2\lambda-4}{\lambda^2+4}\right) + \frac{4\delta\lambda}{(\lambda^2+4)H} \tag{3.3}$$

For most laboratory applications the correction is negligible.

4. Discussion of the Effect of Vertical Sidewalls

We now consider the spin-up problem in a right circular cylinder, with a layer of porous medium on the bottom. We take the (dimensionless) radius of the cylinder to be unity.



In any real fluid flow in this system there is going to be an Ekman layer of thickness $E^{1/2}$ above the porous bottom, and in addition, some viscous sidewall layers, probably similar to the $E^{1/4}$ and $E^{1/3}$ layers found in the "usual" spin-up problem. Indeed, the solution of this problem should tend to the solution of Greenspan and Howard (1963) for the "usual" problem as we let $\lambda \to \infty$. The analysis for the full problem is somewhat complicated, so for an initial investigation an <u>inviscid</u> fluid model was used. In doing this, we must think a bit about what we are neglecting. By neglecting the bottom Ekman layer we introduce a discontinuity in u and w at $z = 0$, and also a possible smallish error into the spin-up time. Neglect of the sidewall layers leads to much greater effects. The process of spin-up with sidewalls consists essentially of the diffusion of vorticity from the walls into the fluid, a process in which viscosity plays a vital part near the walls. In the central flow, the vortex lines are being stretched, while at the walls they are being squished, adding to the unbalance of vorticity there. However in this region, the viscous sidewall effects are able to remove excess vorticity, and this is how it leaves the system.

In considering an inviscid fluid with a porous bottom, we are forcing a pseudo "Ekman suction" which transports vorticity as above, but we are not providing any means for the vorticity to leave the system. Hence we must expect a singularity in the vorticity field at the wall as all the vorticity piles up there.

5. An Approximate Solution in Terms of a Fourier-Bessel Series

We treat only the case $\lambda >> 1$, which is satisfied under most laboratory conditions. In $0 \leq z \leq H$ we again have Equation (2.1) with $\epsilon = 0$, and we drop the term $\frac{\partial u}{\partial t}$ in the first equation, because it is of order$\left(\delta/\lambda\right)$ which is $<< 1$. Thus:

$$\left.\begin{aligned} -2v + \frac{\partial P}{\partial r} &= 0 \\ \frac{\partial v}{\partial t} + 2u &= 0 \\ \frac{\partial w}{\partial t} + \frac{\partial P}{\partial z} &= 0 \\ \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \tag{5.1}$$

In $-\delta \leq z \leq 0$ we have

$$\left.\begin{aligned} \frac{\partial P}{\partial r} &= -\lambda u \\ \frac{\partial P}{\partial z} &= -\lambda w \\ \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \quad \text{i.e. } \nabla^2 P = 0 \tag{5.2}$$

The boundary conditions are

$$\left.\begin{aligned} &w = 0 \quad \text{at} \quad z = -\delta,\ H \\ &u = 0 \quad \text{at} \quad r = 1 \\ &\text{Continuous } P \text{ and } w \text{ at } z = 0 \end{aligned}\right\} \tag{5.3}$$

Consider first the flow in the porous medium. The problem becomes

$$\left\{\begin{aligned} &\nabla^2 P = 0 \\ &\frac{\partial P}{\partial r} = 0 \qquad r = 1 \\ &\frac{\partial P}{\partial z} = 0 \qquad z = -\delta \end{aligned}\right. \tag{5.4}$$

This is an elliptic problem, so conditions must be given on all boundaries. We pretend that we know P on $z = 0$, solve the problem in $-\delta \leq z \leq 0$, presuming that P will eventually be given by the upper solution and the matching conditions.

Suppose $$P(r,0,t) = \sum_{1}^{\infty} A_n(t) J_0(k_n r) \tag{5.5}$$

where k_n are the roots of $J_1(x)=0$. Consider one mode in the porous medium; a solution satisfying (5.4) is

$$P_n = A_n J_0(k_n r) \frac{\cosh k_n(z+\delta)}{\cosh k_n \delta} \tag{5.6}$$

$$(5.2) \Rightarrow \qquad w_n = -\frac{1}{\lambda} k_n A_n J_0(k_n r) \frac{\sinh k_n(z+\delta)}{\cosh k_n \delta} \tag{5.7}$$

$$\left. w_n \right)_{z=0} = -\frac{k_n A_n}{\lambda} J_0(k_n r) \tanh(k_n \delta) \tag{5.8}$$

In $0 \le z \le H$ we suppose, by analogy with Greenspan and Howard's solution for a rigid container, that w is linear in z to a first approximation. Hence, by matching to (5.8)

$$w_n = \frac{k_n A_n}{\lambda} J_0(k_n r) \tanh(k\delta)\left(\frac{z-H}{H}\right) \qquad [0 \le z \le H] \tag{5.9}$$

Integrating the third equation of (5.1) gives

$$u_n = -\frac{A_n}{\lambda H} J_1(k_n r) \tanh(k_n \delta) \qquad [0 \le z \le H] \tag{5.10}$$

$$\frac{\partial v_n}{\partial t} = \frac{2A_n}{\lambda H} J_1(k_n r) \tanh(k_n \delta) \qquad [0 \le z \le H] \tag{5.11}$$

Now u_n and v_n are independent of z by our assumption of linearity of w, so by (5.5) and the first equation of (5.1) we have

$$v_n = \frac{1}{2}\frac{\partial P_n}{\partial r} = \left.\frac{1}{2}\frac{\partial P_n}{\partial r}\right)_{z=0} = -\frac{1}{2} A_n k_n J_1(k_n r) \tag{5.12}$$

$$\frac{\partial v_n}{\partial t} = -\frac{1}{2}\frac{\partial A_n}{\partial t} k_n J_1(k_n r)$$

And so from (5.11) we get an equation for

$$\frac{\partial A_n}{\partial t} = \frac{-4}{\lambda H}\frac{\tanh(k_n\delta)}{k_n}\cdot A_n \tag{5.13}$$

$$A_n = A_n^{\circ} e^{-\sigma_n t}$$

where $$\sigma_n = \frac{4\tanh k_n\delta}{\lambda H k_n} \tag{5.14}$$

Thus the low order modes of large horizontal scale (k_n small) have $\sigma_n \cong \frac{4\delta}{\lambda H}$, essentially the same as σ for the infinite fluid, but for the high order modes $\sigma_n \cong \frac{4}{\lambda H k_n}$ and the spin-up time tends to infinity. This indicates that near the wall, viscosity must become important.

Validity of this Solution:

By assuming w to be a linear function of z in $0 \le z \le H$ we get a P that is independent of z. This does not satisfy the third equation of (5.1) exactly, because $P_z = 0$. However, if we look at the sizes of the various terms, we find that the first term in negligible.

$$\overline{W}_n = \int_0^H w_{nt}\, dz = -\frac{1}{\lambda} k_n \frac{\partial A_n}{\partial t} J_0(k_n r)\tanh(k_n\delta)\frac{1}{2}H$$

$$= \frac{2}{\lambda^2} A_n J_0(k_n r)\tanh^2(k_n\delta)$$

$$P_n = A_n J_0(k_n r)$$

Thus $\left|\frac{\overline{W}_n}{P_n}\right| = \frac{2\tanh^2(k_n\delta)}{\lambda^2} << 1$ because $\lambda >> 1$

Initial Conditions

Initially we have $v = r$, which means that $\left.\frac{\partial P}{\partial r}\right)_{t=0} = 2r = \sum_1^\infty \frac{-4}{J_0(k_n)} J_1(k_n r)$

$$A_n^{\circ} = \frac{4}{k_n^2 J_0(k_n)}$$

Thus the solution is (for $\lambda >> 1$)

$$\boxed{\begin{aligned} P &= \sum \frac{4e^{-\sigma_n t}}{k_n^2 J_0(k_n)} J_0(k_n r) \\ \text{where} \quad \sigma_n &= \frac{4 \tanh(k_n \delta)}{\lambda H k_n} \end{aligned}} \tag{5.15}$$

$$w_{interface} = \sum \frac{-4 \tanh(k_n \delta) e^{-\sigma_n t}}{\lambda k_n J_0(k_n)} J_0(k_n t) \tag{5.16}$$

This representation of the solution is very slowly convergent, in addition to having troubles at the wall. The expansion of r in terms of J_1 cannot of course be uniformly convergent when $r \longrightarrow 1$, and this singularity is exhibited throughout the solution. The value of w at the interface for one set of parameters was computed on a desk calculator from Equation (5.16), and some of the results are shown in Fig. 1 (drawn for the spin-up case). Even when 20 terms of the series are taken, there are still substantial wiggles in the curve. The dashed line indicates the "interior" value of w as obtained from Veronis' analysis.

As expected, the vertical velocity blows up at the wall. To obtain a more accurate description of the flow field, one would have to model the viscous effect at the wall, either by putting in some sort of $E^{1/4}$-like layer, or possibly by a source-sink mechanism.

6. Solution Retaining z Dependence of w

A more accurate form of the Bessel series solution can be obtained by using a Hankel Transform. Equations (5.1) can be reduced to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial^2 P}{\partial t^2}\right) + 4\frac{\partial^2 P}{\partial z^2} = 0 \qquad [0 \le z \le H] \tag{6.1}$$

Define $\overline{P}_n = \int_0^1 r P(r,z,t) J_0(k_n r) dr$ where k_n are the roots of $J_1(x) = 0$ as before. Since we have $\frac{\partial P}{\partial r} = 0$ when $r = 1$, the transformed Equation (6.1) is

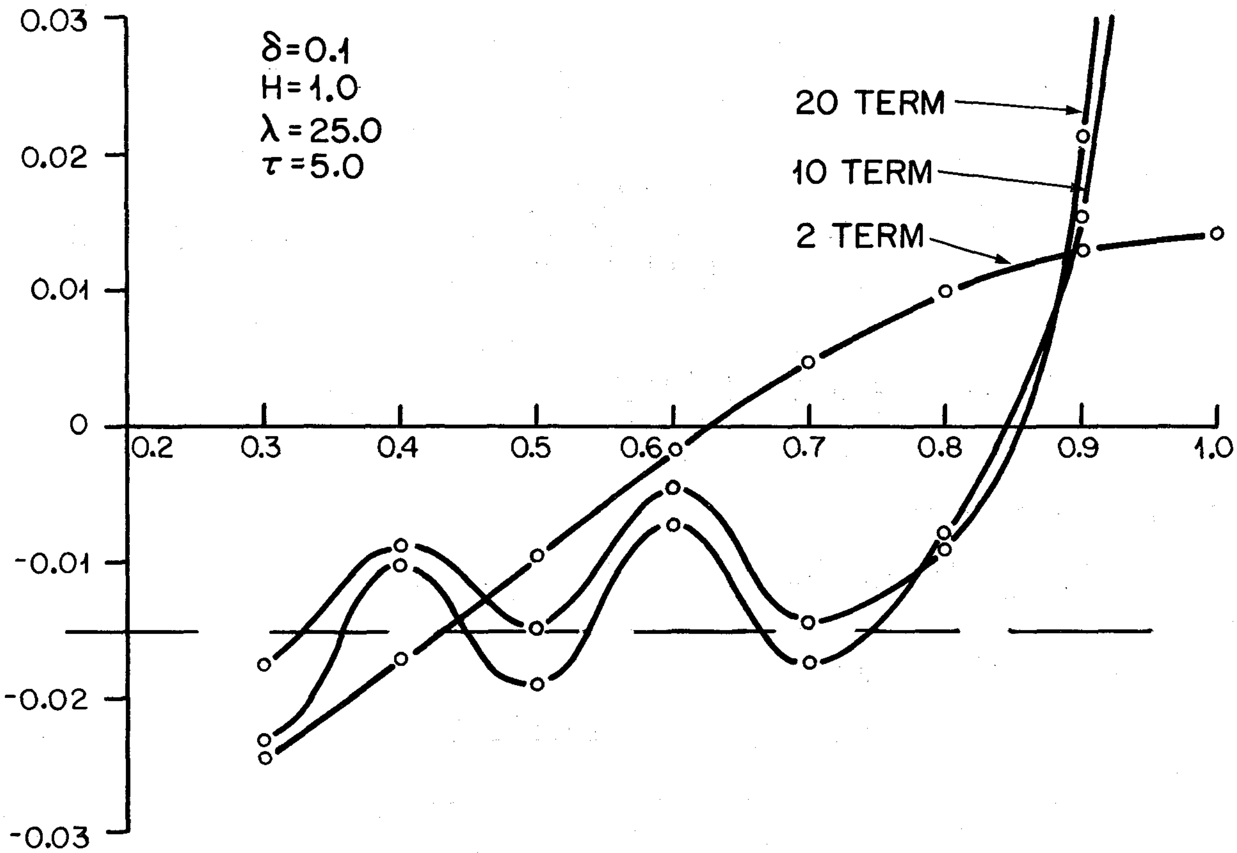


Figure 1

$$-k^2 \frac{\partial^2 \bar{P}_n}{\partial t^2} + 4 \frac{\partial^2 \bar{P}_n}{\partial z^2} = 0$$

We assume exponential decay of each mode $\propto e^{-\sigma_n t}$.

Thus
$$\bar{P}_n = \alpha_n e^{-\sigma_n t} \left\{ \frac{\cosh \frac{k_n \sigma_n}{2}(z-H)}{\cosh \frac{k_n \sigma_n}{2} H} \right\} \tag{6.2}$$

in $0 \le z \le H$.

Now in $-\delta \le z \le 0$ we have

$$\bar{P}_n = A_n \frac{\cosh k_n (z+\delta)}{\cosh k_n \delta} \tag{6.3}$$

from (5.6) since $\bar{P}_n$ on the boundary $= A_n$. Equating the two forms of $\bar{P}_n$ when $z = 0$ gives:

$$\alpha_n e^{-\sigma_n t} = A_n \tag{6.4}$$

Now also

$$\overline{w} = \frac{k_n}{2} \alpha_n e^{-\sigma_n t} \left\{ \frac{\sinh \frac{k_n \sigma_n}{2}(z-H)}{\cosh \frac{k_n \sigma_n}{2} H} \right\} \qquad [0 \leq z \leq H] \tag{6.5}$$

and

$$\overline{w} = -\frac{1}{\lambda} k_n A_n \frac{\sinh k_n(z+\delta)}{\cosh k_n \delta} \qquad [-\delta \leq z \leq 0] \tag{6.6}$$

Matching $\overline{w}$ at $z = 0$ gives

$$-\frac{k_n}{2} \alpha_n e^{-\sigma_n t} \tanh \frac{k_n \sigma_n}{2} H = -\frac{1}{\lambda} k_n A_n \tanh k_n \delta$$

together with (6.4) this gives

$$\tanh \frac{k_n \sigma_n}{2} H = \frac{2 \tanh k_n \delta}{\lambda} \tag{6.7}$$

$$\Longrightarrow \quad \boxed{\sigma_n = \frac{1}{k_n H} \log \left\{ \frac{1 + \frac{2}{\lambda} \tanh k_n \delta}{1 - \frac{2}{\lambda} \tanh k_n \delta} \right\}} \tag{6.8}$$

In the case $\lambda \gg 1$ this reduces to $\sigma_n = \frac{4}{\lambda k_n H} \tanh k_n \delta$ as found before. Since λ is usually large, the correction to σ_n given by (6.7) is a small one. Putting in the initial condition for $\frac{\partial P}{\partial r}$ at $z = 0$ as before gives

$$\alpha_n = \frac{+4}{k_n^2 J_0(k_n)}$$

and so we have

$$P = \sum_{n=1}^{\infty} \frac{4 e^{-\sigma_n t} \cosh \frac{k_n \sigma_n}{2}(z-H)}{k_n^2 J_0(k_n) \cosh \frac{k_n \sigma_n}{2} H} J_0(k_n r) \tag{6.9}$$

$$w = \sum \frac{2 e^{-\sigma_n t} \sinh \frac{k_n \sigma_n}{2}(z-H)}{k_n J_0(k_n) \cosh \frac{k_n \sigma_n}{2} H} \tag{6.10}$$

Thus, using (6.7), we find

$$w\Big)_{z=0} = \sum \frac{-4 e^{-\sigma_n t} \tanh k_n \delta}{k_n \lambda J_0(k_n)} J_0(k_n r) \tag{6.11}$$

which is as before except for the correction to σ_n.

7. Conclusion

The Bessel series solution to the inviscid problem is useful for obtaining some insight into the nature of the spin-up process, although fine details of the flow are not available, due to the singularities at the walls. In particular, we may deduce that the spin-up time of the interior is close to the value given by the "infinite" analysis, but that near the edge, the spin-up time is longer. The highest order modes have effectively infinite spin-up time on this model, but in a real fluid, viscous effects from the side and bottom viscous layers would alter the nature of these modes.

The solution to this problem clearly points out the two very different roles played by viscosity in the "usual" spin-up problem; through the bottom Ekman layer it creates Ekman suction which stretches vortex lines, and so spins up the interior, at the same time moving vorticity to the edges. In the side-wall layers viscosity plays a different, yet equally important role by removing the vorticity anomaly there, which is continuously fed by the Ekman layer. Replacement of the Ekman layer by a porous medium takes over the pumping mechanism, but cannot account for the loss of vorticity at the side walls.

Acknowledgements

I should like to thank Prof. Howard for suggesting the problem and for many fruitful discussions, Prof. Veronis for the loan of his original analysis, Prof. Rooth for his physical insight, and Dr. Spiegel for discovering an ideal porous medium for experimental purposes.

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A WAVE PROBLEM

Gunnar E. B. Kullenberg

Introduction

A study of a long surface wave in a basin with a uniform source in the closed end producing a uniform current through the basin with the water flowing over a sill at the open end. The oceanographical background to the problem is the observed internal overflows over sills from one basin to an outside basin where the overflowing water sinks to a proper level. These overflows sometimes occur in bursts, as in the Denmark Strait. Dr. P. Welander, who suggested the problem to me, had the idea that it might be interesting to investigate if these bursting overflows could be generated by some wave mechanism in the basin, like a long wave affected by the Coriolis force (Kelvin wave type). Due to the uniform steady current through the basin the internal surface fills across the basin, and a wave motion on top of this increases and decreases the tilting, so that during the outward part of the wave motion the surface reaches over the sill, but during the inward part the tilt is so decreased that the surface does not reach over the sill. To investigate this mechanism I have here considered only the linear part of the problem, with a free surface and the water flowing free over a sill. The sill depth (depth from surface to level of the sill) is large compared to the wave amplitude, but small compared to the total depth of the basin. Thus we have a picture of the flow as in Figure 1.

Into the basin of length L and depth H flows $q\ \mathrm{cm^3/cm,s}$, which causes a uniform current U and which flows out freely over the sill. The overflow is related to the sill depth ζ and the current by the expression:

$$q = \frac{2}{3} C_c \sqrt{2g}\left[\left(\frac{U^2}{2g} + \bar{\zeta}\right)^{3/2} - \left(\frac{U^2}{2g}\right)^{3/2}\right] \qquad (1)$$

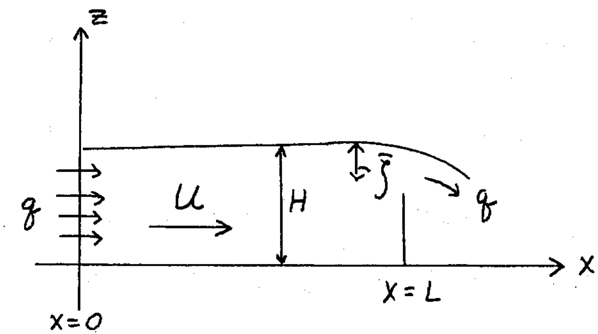


Fig. 1

which is essentially an integrated Bernoulli equation. C_c is a contraction coefficient.

We now suppose that $\frac{U^2}{2g\bar{\zeta}} << 1$, and get $q = \frac{2}{3} C_c\sqrt{2g} \cdot \bar{\zeta}^{3/2}$

In the basin we also have $q = H \cdot U$. Now suppose we introduce a small wave disturbance on the surface with the amplitude ζ. The rate of overflow is then given by:

$$q = \frac{2}{3} C_c\sqrt{2g}\left[\bar{\zeta}^{3/2} + \frac{3}{2}\bar{\zeta}^{1/2} \cdot \zeta\right] = H \cdot (U + u), \quad \text{or}$$

$$H \cdot u = C_c\sqrt{2} \cdot \sqrt{g\bar{\zeta}} \cdot \zeta = C_1 \cdot \zeta \quad , \quad \zeta << \bar{\zeta}$$

C_1 is treated as a constant. From the expression $H \cdot U = q$ we find

$$(\bar{\zeta})^{1/2} = \sqrt{1.3} \cdot \frac{(H \cdot U)^{1/3}}{g^{1/6}}$$

which gives $$C_1 = 1.1\left(\frac{U}{\sqrt{gH}}\right)^{1/3} \cdot \left(\frac{g}{H}\right)^{1/2} \cong \varepsilon^{1/3} \cdot \left(\frac{g}{H}\right)^{1/2} \quad \text{where } \varepsilon = \frac{U}{\sqrt{gH}} .$$

We have then assumed that $C_c \simeq 0.7$. In this way we have a relation between the wave current and the amplitude at the sill, which will be used as a boundary condition in the problem. If we wanted to consider directly the bursting overflow this boundary condition would have been non-linear. The problem can further be divided into a non-rotating and a rotating case.

Non-rotating case

With a coordinate system as in Figure 1, the linearized equations are:

$$u_t + U \cdot u_x + g \cdot \zeta_x = 0 \qquad (1)$$

$$\zeta_t + U \cdot \zeta_x + H \cdot u_x = 0 \qquad (2)$$

The continuity equation (2) has been vertically integrated assuming constant depth H and using the free surface condition $\frac{d}{dt}(z-\zeta)=0$. From (1) and (2) one obtains the wave equation

$$\zeta_{tt} + 2U \cdot \zeta_{xt} + \zeta_{xx}\left(U^2 - gH\right) = 0 \qquad (3)$$

The wave current u satisfies the same equation. The boundary conditions, $u=0$ for $x=0$ and $u=C_1 \cdot \zeta$ for $x=L$, are also formulated in amplitude ζ only, which is done by eliminating u_x from (1) and (2). Then we get:

$$(HC_1 - U) \cdot \zeta_t + (gH - U^2) \cdot \zeta_x = 0, \qquad x = L$$

$$-U \cdot \zeta_t + (gH - U^2) \cdot \zeta_x = 0, \qquad x = 0$$

In addition to these something must be said about the incoming fluid at $x=0$. We there impose the condition that the cross current $v=0$. This condition however does not enter this part of the problem. The equation is solved by setting $\zeta \propto e^{-\sigma t} \cdot e^{-\lambda x}$ which gives $\sigma^2 + 2U\sigma\lambda + U^2\lambda^2 - gH\lambda^2 = 0$ or

$$\lambda = \sigma\left(\frac{U \pm \sqrt{gH}}{gH - U^2}\right) = \sigma(\alpha \pm \beta);$$

$$\alpha = \frac{U}{gH - U^2}, \qquad \beta = \frac{\sqrt{gH}}{gH - U^2}$$

By putting

$$\zeta = e^{-\sigma t}\left(A_1 e^{-\sigma(\beta+\alpha)x} + A_2 e^{\sigma(\beta-\alpha)x}\right)$$

in the boundary conditions we get finally

$$e^{-2\beta\sigma L} = \frac{1 - \varepsilon^{1/3}}{1 + \varepsilon^{1/3}},$$

having used that $\frac{HC_1}{\sqrt{gH}} = \varepsilon^{1/3}$. Now $\sigma = \sigma_1 + i\sigma_2$, and we get $\sigma_2 = \mu\pi\sigma_0(1-\varepsilon^2), \mu = 1,3,5\ldots$ where $\sigma_0 = \frac{\sqrt{gH}}{2L}$. The real part is expanded and using that $\varepsilon \ll 1$ we get $\sigma_1 = 2\sigma_0 \cdot \varepsilon^{1/3}(1-\varepsilon^2)$. Thus $\sigma = \sigma_0(1-\varepsilon^2)(i\mu\pi + 2\varepsilon^{1/3})$.

The effect of the uniform current is thus a slight change of the frequency and a damping of the wave. Consider a real basin, with $H \sim 100$ m, $u \sim 10$ cm/s, which gives $\varepsilon \sim 5 \cdot 10^{-3}$. The frequency change is very small, and with the approximation $u^2 \ll 2g\bar{\zeta}$ already done, we should not include the ε^2 in the solution. The damping, given by the factor $2\varepsilon^{1/3} \sim 0.3$, however, is a more noticeable effect.

The phase velocity of the wave is $c' = \sqrt{gH}(1+\varepsilon)$ for the wave going against the current and $c'' = \sqrt{gH}(1-\varepsilon)$ for the other part. The change relative to the ordinary long wave is thus small for ordinary ranges of ε. The wave numbers are given by

$$\lambda' = -\frac{(1-\varepsilon)(in\pi + 2\varepsilon^{1/3})}{2L} \quad \text{and} \quad \lambda'' = +\frac{(1+\varepsilon)(in\pi + 2\varepsilon^{1/3})}{2L}$$

and the amplitude is determined but for a constant. To determine this some initial value has to be given. However, to understand the physics this is not necessary. The damping of the wave due to the current might be seen as a washing out of the wave, the reflection at the open end being very poor.

In the model no friction has been included. If friction through the wave motion is included we get frictional damping also. For a real basin with a depth of 50-100 m the linear friction coefficient is of the order 10^{-5} s^{-1}, and with a period of 10^3 - 10^4 s, the frictional damping factor is 0.01 - 0.1, while the other is $2\varepsilon^{1/3} \sim 0.5$. Thus this can in some cases be more important than the friction, according to these assumptions. In order to look at the plane-effect on waves a laboratory experiment was done.

Experiment

For this a 320 cm long, 8.1 cm wide, 20 cm deep wave tank with a sill height of 14 cm was used. The flow was started at the closed end by injecting water through a diffuser (tube with holes) connected to the tap by a hose. By this different flow rates could be used. Having established a flow, a wave was started by slightly tilting the horizontal tank, i.e. raising it at the closed end about 1 cm and putting it back again slowly. In this way a long wave was started. The following was measured: flowrate q, water depth H, current velocity U, wave period T, wave amplitude ζ . The current was measured using a balasted float. The flow was quite uniform. The wave amplitude was measured by observing the wave, directly or through a tube, relative to a mm scale. The amplitude varied between 1 and 5 mm. The depth $\bar{\zeta}$ over the sill varied between about 5 to 15 mm for the different flow rates used.

The observed values were used to calculate $\varepsilon = \frac{U}{\sqrt{gH}}$ and the total damping factor δ . This is a sum of frictional damping kT and flow damping $2 \cdot \varepsilon^{1/3}$, so that $\delta = kT + 2\varepsilon^{1/3}$. The frictional damping was also measured for zero flow. The values are given in Table 1.

TABLE 1. Observed values of q , U , δ , ε , kT, ε_*

q (cm³/s)	U (cm/s)	δ	$\varepsilon \cdot 10^3$	$2\varepsilon^{1/3}$	kT	$\varepsilon_* \cdot 10^3$
0	0	-	-	-	.128	-
49	.28	.408	2.3	.27	.138	2.75
64	.45	.458	3.7	.31	.148	4.5
77	.59	.497	4.8	.335	.162	6.3
96	.94	.565	7.7	.395	.17	10.2
126	1.24	.610	10	.432	.178	14.0
155	1.44	.724	11.5	.45	.174	15
218	1.85	.69	14.7	.49	.20	22=
282	2.2	.87			.35	55
370	2.96	1.31			.74	215

In Figure 2 the calculated frictional damping factors $kT = \delta \cdot 2\varepsilon^{1/3}$ are given as a function of the current U . When the calculated line is extrapolated for $U = 0$ we get a value of $(kT)_0$ which is very close to the observed*. On the other hand the

*The frictional damping increases with the velocity.

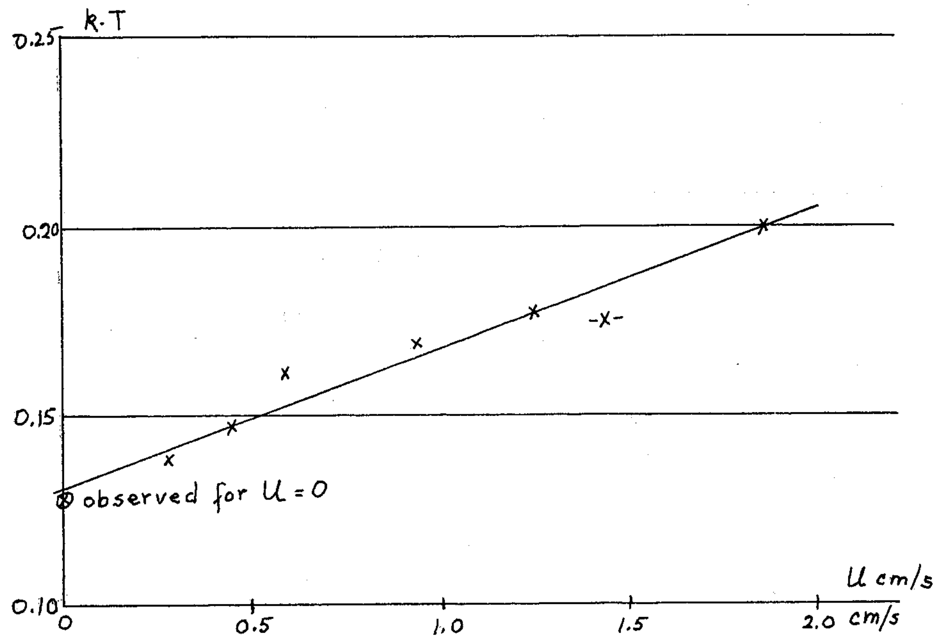


Fig. 2. Frictional damping factor kT as a function of current velocity U.

observed $(kT)_o$ can be used to derive $2\mathcal{E}_*^{1/3} = \delta - (kT)_o$ (Table 1). In Figure 3 $\mathcal{E}_*$ is given as a function of the current U. The calculated line has a slope close to 1, and passes very close to the origin. The error is 11% of the lowest observed $\mathcal{E}$-value. The experimental results are interpreted as confirming the theoretical damping effect of the flow in the channel. For current velocities larger than 2 cm/s in the tank, the frictional damping rises sharply. This occurs at a Reynolds number $\frac{UB}{\nu} \sim 1600\text{-}2000$, i.e. near the critical Re for pipe flow.

The rotating case

The non-rotating problem was done in order to understand what influence the flow in the basin would have on the wave. The damping effect in this case is what one should expect. In the rotating case, however, something different might turn up. For the bursting overflow the rotational effect might be essential, and this could be so also in the linear case. However, the rotating problem turned out to be rather intricate, and no solution of the problem has yet been

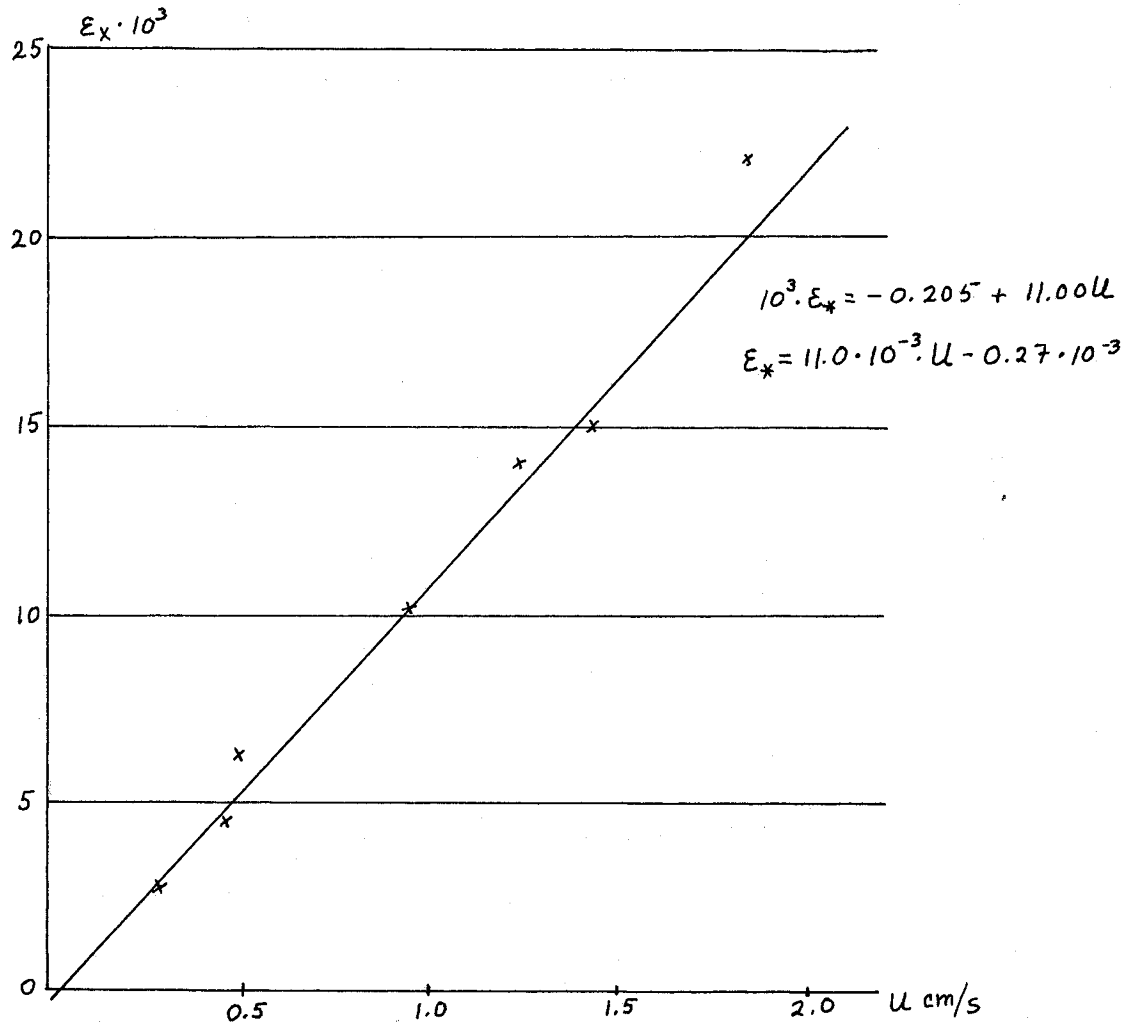


Fig. 3. Theoretical damping factor ε_* as a function of the current velocity U.

found. With the same coordinate system as before we have the linearized equations:

$$u_t + U \cdot u_x + g \cdot \zeta_x - f \cdot v = 0 \tag{4}$$

$$v_t + U \cdot v_x + g \cdot \zeta_y + f \cdot u = 0 \tag{5}$$

$$\zeta_t + U \cdot \zeta_x + H \cdot v_x + H \cdot v_y = 0 \tag{6}$$

The continuity equation (6) has been vertically integrated and the free surface condition is used as before. The boundary conditions are:

$$x = 0 \quad : \quad u = 0, \quad v = 0$$

$$x = L \quad : \quad u = C_i \zeta$$

$$y = 0, B : \quad v = 0$$

We use the same condition on u at the sill as before. To the same approximation, $u^2 << 2g\bar{\zeta}$, the integrated Bernoulli type equation for q, should be valid also in this case. Also in this case the boundary condition on v at $x=0$ is essential. We note that we cannot go from the rotating case directly to the non-rotating because of the crosscurrent here.

From the equations one can form a wave equation in the amplitude ζ. However, the boundary conditions are difficult to express in ζ only, and furthermore they cannot be separated. The equation thus cannot be solved by the method of separation. There is no appropriate scaling possibility for simplifying the system so that this method can be used. Another way could be to assume $fv=0$, i.e. geostrophy only in the y-direction, but this is not successful. I have tried to introduce the rotational effect as a perturbation, but that does not work. The method of solution given below was suggested to me by Professor Howard. The technique is to express everything in v and derive an equation only in v which is solved by inserting a sine-series for v in y with coefficients in X .

We now assume all functions to be proportional to $e^{i\sigma t}$ and write equations (4) and (6) in the form:

$$i\sigma u + U\cdot u_x + g\cdot\zeta_x = f\cdot v$$

$$i\sigma\zeta + U\cdot\zeta_x + H\cdot u_x = -H\cdot v_y$$

From these ζ_x and u_x are eliminated giving

$$u_x + \frac{i\sigma}{gH-U^2}\left(-U\cdot u + g\cdot\zeta\right) = -\frac{1}{(gH-U^2)}\left(fUv + gHv_y\right) \tag{6}$$

$$\zeta_x + \frac{i\sigma}{gH-U^2}\left(Hu - U\zeta\right) = \frac{1}{gH-U^2}\left(Hfu + UHv_y\right) \tag{7}$$

Multiplying these equations by A_1 and A_2 respectively, and adding we get

$$\left(A_1 \cdot u + A_2 \cdot \zeta\right)_x + \frac{i\sigma}{gH - U^2}\left(A_1 A_2\right)\begin{pmatrix} -U & g \\ H & -U \end{pmatrix}\begin{pmatrix} u \\ \zeta \end{pmatrix} = F(v) \tag{8}$$

The eigenvalues λ_1 and λ_2, and the coefficients A_1, A_2 are determined for

$$\lambda_1 = -U + \sqrt{gH} \quad , \quad A_1 = \frac{1}{\sqrt{gH}} \quad , \quad A_2 = \frac{1}{H}$$

$$\lambda_2 = -U - \sqrt{gH} \quad , \quad A_1 = \frac{1}{g} \quad , \quad A_2 = \frac{1}{\sqrt{gH}}$$

Then the equation (8) becomes in the two cases:

$$\frac{\partial}{\partial x}\left[\left(\frac{u}{\sqrt{gH}} + \frac{\zeta}{H}\right) e^{\frac{i\sigma(\sqrt{gH} - U)}{gH - U^2} x}\right] = \frac{e^{\frac{i\sigma(\sqrt{gH} - U)}{gH - U^2}}}{\sqrt{gH}\,(1 + \varepsilon)}\left[\frac{f}{\sqrt{gH}} \cdot v - v_y\right] \tag{9}$$

$$\frac{\partial}{\partial x}\left[\left(-\frac{u}{g} + \frac{\zeta}{\sqrt{gH}}\right) e^{-\frac{i\sigma(\sqrt{gH} - U)}{gH - U^2} x}\right] = \frac{e^{-\frac{i\sigma(\sqrt{gH} + U)}{gH - U^2} x}}{g(1 - \varepsilon)}\left[\frac{f}{\sqrt{gH}} \cdot v + v_y\right] \tag{10}$$

These two equations are used to determine u, ζ by integration from $0 - X$, and using the boundary condition that $u = 0$ at $X = 0$, we get

$$2\zeta(x,y) = e^{ax}\int_0^x \frac{e^{-ax'}}{1-\varepsilon}\sqrt{\frac{H}{g}}\left(\frac{f v}{\sqrt{gH}} + v_y\right)dx' + e^{-bx'}\int_0^x \frac{e^{bx'}}{1+\varepsilon}\sqrt{\frac{H}{g}}\left(\frac{f v}{\sqrt{gH}} - v_y\right)dx' + \zeta(0,y)\left(e^{ax} + e^{-bx}\right) \tag{11}$$

$$2u = -e^{ax}\int_0^x \frac{e^{-ax}}{1-\varepsilon}\left(\frac{f v}{\sqrt{gH}} + v_y\right)dx' + e^{-bx}\int_0^x \frac{e^{bx'}}{1+\varepsilon}\left(\frac{f v}{\sqrt{gH}} - v_y\right)dx' + \sqrt{\frac{g}{H}} \cdot \zeta(0,y)\left(e^{-bx} - e^{ax}\right) \tag{12}$$

where $a = \frac{i\sigma}{\sqrt{gH}} \cdot \frac{1}{1-\varepsilon}$, $\quad b = \frac{i\sigma}{\sqrt{gH}} \cdot \frac{1}{1+\varepsilon}$.

The unknown function $\zeta(0,y)$ is to be determined by using boundary conditions at $X = L$. The values of ζ and u are put into Equation (5) written in the form $2(f u + g \cdot \zeta_y) = -2(i\sigma v + U \cdot v_x)$. However, at this stage we first make use of $\varepsilon \ll 1$ and put $a = b = \frac{i\sigma}{\sqrt{gH}}$. Putting u and ζ_y in we get:

$$e^{\frac{i\sigma x}{\sqrt{gH}}}\int_0^x e^{-\frac{i\sigma x}{\sqrt{gH}}}\cdot\sqrt{gH}\left(v_{yy}-\frac{f^2}{gH}v\right)dx'+e^{-\frac{i\sigma x}{\sqrt{gH}}}\int_0^x e^{\frac{i\sigma x}{\sqrt{gH}}}\cdot\sqrt{gH}\left(\frac{f^2 v}{\sqrt{gH}}-v_{yy}\right)dx'+$$

$$+g\cdot\zeta_y(0,y)\left(e^{\frac{i\sigma x}{\sqrt{gH}}}+e^{-\frac{i\sigma x}{\sqrt{gH}}}\right)+f\sqrt{\frac{g}{H}}\left(e^{-\frac{i\sigma x}{\sqrt{gH}}}-e^{\frac{i\sigma x}{\sqrt{gH}}}\right)+2\left(i\sigma v+Uv_x\right)=0 \tag{13}$$

which is differentiated twice with respect to x, and we end up with:

$$\frac{\sigma^2}{gH}\left(i\sigma v+U\cdot v_x\right)+i\sigma v_{yy}-i\sigma\frac{f^2}{gH}v+\left(i\sigma v_{xx}+U\cdot v_{xxx}\right)=0$$

This equation is non-dimensionalized using

$\sigma=\sigma_*\cdot\frac{\sqrt{gH}}{L}$, $x=x_*\cdot L$, $y=y_*\cdot B$, $v=v_*\cdot\sqrt{gH}$ and we proceed with the case when the ε-terms are much less than the others, and get:

$$\frac{\partial^2 v_*}{\partial x_*^2}+\delta^2\cdot\frac{\partial^2 v_*}{\partial y_*^2}+v_*\left(\sigma_*^2-\gamma^2\cdot\delta^2\right)=0 \tag{14}$$

where $\delta=\frac{L}{B}$ and $\gamma=\frac{fB}{\sqrt{gH}}$

In this we set $v_*=\sum_1^\infty a_k(x_*)\cdot\sin k\pi y_*$ which satisfies the boundary conditions $v_*=0$ at $y_*=0,1$. We get

$$a_k=A_k\cdot\sinh\left(\sqrt{\left((k\pi)^2+\gamma^2\right)\cdot\delta^2-\sigma_*^2}\right)\cdot x_* \tag{15}$$

The coefficients A_k are to be determined by using the boundary condition $v_*=0$ at $x_*=0$. First the functions $\zeta(0,y)$, which enter in u and ζ, must be determined by integrating (9) and (10) from x to L and using the boundary condition $u=c_i\zeta$ at $x=L$. By doing this, eliminating ζ and setting $x=0$ using $u(0)=0$ we get the scaled equation $(\zeta=\zeta_*\cdot H)$:

$$\zeta_*(1,y_*)\left[(1-\varepsilon^{1/3})e^{-i\sigma_*}-(1+\varepsilon^{1/3})e^{i\sigma_*}\right]=-\int_0^1 e^{i\sigma_* x_*}\left(\gamma v_*-\frac{\partial v_*}{\partial y_*}\right)\cdot\delta\,dx+$$

$$+\int_0^1 e^{-i\sigma_* x_*}\left(\gamma v_*+\frac{\partial v_*}{\partial y_*}\right)\delta\,dx_* \tag{16}$$

On the other hand we have Equation (11) and setting $x_*=1$ then we have another expression for $\zeta_*(1,y_*)$. Putting the two together we find an expression for $\zeta_*(0,y_*)$ in v_* only. Now we go back to Equation (13) scale it and put the expressions for v_*, $\zeta_*(0,y_*)$, $\frac{\partial\zeta_*(0,y_*)}{\partial y_*}$ into it and use this equation putting $x_*=1$ to determine the A_k. There is a lot of algebra involved. The integrals are done by integrating by parts twice. The expression is multiplied with $\sin m\pi y_*$ and integrated from $y_* = 1$ to 0 setting $k = m$ in the sine-expressions. We finally end up with Equation (17). The solution is only sketched here in this way. All the steps have been mentioned.

Equation 17 has to be used to determine A_k, but no definite results have come out of this yet. Putting both $\gamma = 0$ and $\varepsilon = 0$ in Equation (17) one finds a frequency $\sigma_* = \pm k\pi\delta$ or with $\delta = 1$, $\sigma = \frac{k\pi\sqrt{gH}}{B}$, i.e. the ordinary long wave frequency.

$$
\begin{aligned}
&\frac{A_m}{(m\pi)^2+\gamma^2}\left[(m\pi)^2 \cdot i\sigma_* \sinh F - \gamma^2\varepsilon^{1/3} \cdot F(\cosh F - 1)\right] \cdot \left(e^{2i\sigma_*} - e^{-2i\sigma_*}\right) - \\
&- \frac{A_m}{(m\pi)^2+\gamma^2}\left[(m\pi)^2 F(\cosh F - 1) + \gamma^2\varepsilon^{1/3} \cdot i\sigma_* \sinh F\right]\left(e^{-2i\sigma_*} + e^{2i\sigma_*}\right) - \\
&- \frac{2A_m}{(m\pi)^2+\gamma^2}\left[(m\pi)^2 F(\cosh F - 1) - \gamma^2\varepsilon^{1/3} \cdot i\sigma_* \sinh F\right] + \\
&+ \frac{2A_m}{(m\pi)^2+\gamma^2}\left[(m\pi)^2 F(\cosh F - 1) + \gamma^2\varepsilon^{1/3} \cdot i\sigma_* \sinh F\right]\left(e^{-i\sigma_*} + e^{+i\sigma_*}\right) - \\
&- A_m \cdot i\sigma_* \sinh F\left(e^{2i\sigma_*} - e^{-2i\sigma_*}\right) - A_m i\sigma_* \cdot \sinh F \cdot \varepsilon^{1/3}\left(e^{2i\sigma_*} + e^{-2i\sigma_*} + 2\right) = \\
&= \sum_{(m+k)\,odd} \frac{A_k}{(k\pi)^2+\gamma^2}\, 4\gamma \frac{mk}{m^2-k^2}\Big[2\left(i\sigma_* \sinh F + \varepsilon^{1/3} F(\cosh F - 1)\right)\left(1 - \left(e^{-i\sigma_*} + e^{i\sigma_*}\right)\right) + \\
&\quad + \left(e^{i\sigma_*} - e^{-i\sigma_*}\right)\left(F(\cosh F - 1) + \varepsilon^{1/3} \cdot i\sigma_* \sinh F\right)\Big] \cdot \left(e^{-i\sigma_*} + e^{i\sigma_*}\right). \qquad (17)
\end{aligned}
$$

Since we have made no difference between y and x in the scaling process, thus assuming the amplitudes to be of the same order in both x and y, it is most appropriate to put $\delta = \frac{L}{B} = 1$ here. Putting only $\varepsilon = 0$ one finds $\sigma_* = \pm 1 \cdot \sqrt{(k\pi)^2 + \gamma^2}$. This is the frequency of a wave caused by a travelling pressure disturbance in the channel. Thus these expected things come out of the solution. To put $\gamma = 0$ and expect the non-rotating solution given before to result, is not correct, though, since before we had no cross current at all. This alters the picture a great deal. To make this case energe one has at least to assume different amplitudes in the x and y-direction when scaling.

The next step is to put $\sigma_* = \sigma_1 + i\sigma_2$ and investigate what happens when σ_2 is small, so the hyperbolic functions can be expanded. Nothing has resulted yet, however, and it is not evident what will result.

One might say that the solution suffers a great deal because of the approximations made, but on the other hand in real cases $\varepsilon << 1$. However, the order of equation in v is lowered and that might be serious.

Conclusion

The flow has in the non-rotating case a considerable influence on the long wave. The wave is washed out of the channel.

The rotating case has not been completely solved, and the effects of the flow on the wave are not evident. Possibly more will result when different amplitudes are assumed in x and y-directions, and the scaling accordingly changed.

Acknowledgements

The problem was suggested to me by Dr. P. Welander, whom I thank very much for help during the work. I also thank Dr. L. N. Howard and Dr. G. Veronis for their stimulating help during the latter part of the work.

DISTRIBUTION OF CHEMICAL TRACERS IN DEEP PACIFIC AND ATLANTIC

Han-Hsiung Kuo

I. Introduction

With the abyssal circulation proposed by Stommel and Arons (1960, 1967), we are going to study the horizontal distribution of tracers in deep ocean basins. Two model basins are constructed to examine if the observed data of dissolved oxygen and Carbon 14 in the Pacific and the Atlantic are consistent with the proposed flow pattern and what are the relative roles of large-scale lateral mixing and advective process in the distribution of tracers. To this purpose, the following assumptions are made:

(1) The region of the ocean in which we are interested is beneath the main thermocline to the bottom (about 2 km to 5 km depth of the ocean).

(2) The flow in the deep basin is steady, geostrophic and hydrostatic except in the western boundary (and northern boundary in the Pacific Ocean) where there exists an intense flow.

(3) The interior flow is induced by the upwelling just beneath the thermocline. This is based on the results of thermohaline circulation theory.

(4) The density in the deep basin is uniform.

(5) Bottom friction is negligible. Thus the upwelling velocity is zero at the bottom.

(6) Tracers are homogeneous in vertical direction. In other words, we are interested in vertically averaged properties.

(7) The effects of the concentration of surface water and sediments to that of deep water are negligible.

II. <u>Model Basins and Their Flow Patterns.</u> (Figure 1)

The model basins used to simulate the abyssal circulation of Pacific and Atlantic Ocean basins are:

(1) Model basin for the Pacific Ocean is bounded by meridians 180^{o}W and 90^{o}W and latitude circles 60^{o}S and 50^{o}N.

(2) Model basin for the Atlantic Ocean is bounded by meridians 60^{o}W and 0^{o}W and latitude circle 60^{o}S.

The flow patterns based on the above assumptions in both basins written in geophysical coordinates (i.e. $\lambda \sim$ longitudinal distance measured from western boundary eastward; $\phi \sim$ latitude, positive northward; $z \sim$ vertical coordinate, positive upward) will be:

⟨1⟩ Interior flow pattern (same for both oceans)

$$U = 2 W_o \frac{a}{H} \cos\phi \, (\lambda_B - \lambda)$$
$$V = W_o \frac{a}{H} \tan\phi \, (\lambda_B - \lambda)$$
$$W = W_o \left(1 + z/H\right)$$

where $W_o = 0.5\times10^{-5} \sim 3.5\times10^{-5}$ cm/sec $\sim$ upwelling velocity just beneath the main thermocline.

$a = 6.4\times10^{8}$ cm $\sim$ radius of the earth

$H = 3.0\times10^{5}$ cm $\sim$ depth from the main thermocline to the bottom

$\lambda_B \sim$ width of the basin (in radians)

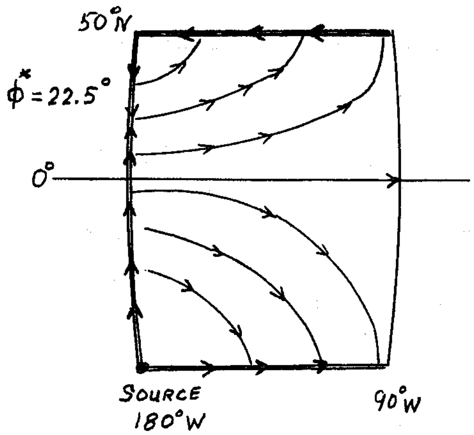
⟨2⟩ Mass transport in western boundary (T_W)

(a) Pacific

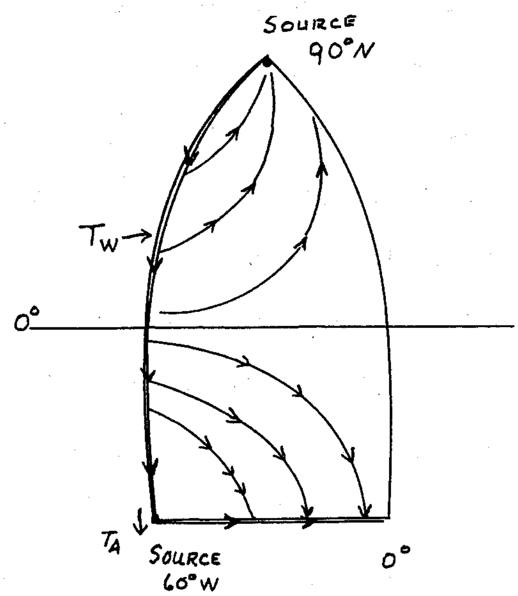
$$T_W = 2 W_o a^2 \lambda_B \left(\sin\phi^* - \sin\phi\right)$$

where ϕ^* is the singular point in the North Pacific Ocean due to the fact that there is a return northern boundary current and ϕ^* equals to 22.5^{o}.

<A> Model Basin for Pacific Ocean



<B> Model Basin for Atlantic Ocean



Velocity in the interior

$$U = 2\, W_0 \frac{a}{H} \cos\phi \,(\lambda_B - \lambda)$$

$$V = W_0 \frac{a}{H} \tan\phi$$

$$W = W_0 \left(1 + z/H\right)$$

Western Boundary Mass Transport

<A>

$$T_W = 2\, W_0\, a^2 \lambda_B \left(\sin\phi^* - \sin\phi\right)$$

<B>

$$T_W = -\,2\, W_0\, a^2 \lambda_B \left(\sin 60° + \sin\phi\right) - T_A$$

Figure 1: Model basins and their flow patterns

(b) Atlantic

$$T_W = -2 W_0 Q^2 \lambda_B (\sin 60^\circ + \sin \phi) - T_A$$

where T_A is the net transport across 60°S, and is assumed about 1/3 of the total strength of the western boundary current.

It should be noted that the source for the Pacific Ocean is at the west corner at 60°S and for the Atlantic Ocean, at the North Pole.

III <u>Governing Equations for Horizontal Distribution of Tracers and Boundary Conditions</u>

The equations governing the steady horizontal distribution of tracers in the ocean basins are:

for oxygen

$$K_H \nabla_H^2 C - \underset{\sim}{V}_H \cdot \underset{\sim}{\nabla} C - \nu = 0$$

for Carbon 14

$$K_H \nabla_H^2 C - \underset{\sim}{V}_H \cdot \underset{\sim}{\nabla} C - \mu C = 0$$

where $C \sim$ concentration of tracers

$K_H \sim$ horizontal eddy diffusivity

$$\nabla_H^2 = \frac{1}{a^2 \cos\phi} \frac{\partial}{\partial \phi} \cos\phi \frac{\partial}{\partial \phi} + \frac{1}{a^2 \cos^2\phi} \frac{\partial^2}{\partial \lambda^2} \sim \text{horizontal Laplacian operator}$$

$$\underset{\sim}{V}_H = u \underset{\sim}{i} + v \underset{\sim}{j}$$

$\nu \sim$ constant decay rate of oxygen. It has been estimated about 6×10^{-3} (ml/l) year^{-1} in the deep Atlantic basin by Riley (1961). Munk (1966) obtained the value of $2.7 \sim 5.3 \times 10^{-3}$ ml/l year^{-1} in a vertical diffusion model of abyssal waters in the North Pacific. We will use 2×10^{-3} ml/l year^{-1} in the present investigation.

$\mu = 1.24 \times 10^{-4}$ year^{-1} $\sim$ exponential decay rate of Carbon 14.

Boundary conditions will be read as:

(1) At the eastern boundary: We will assume no flux of tracers across the boundary.

(2) At the northern boundary: For the Pacific Ocean, the concentration of the tracer in the northern boundary current is assumed as the average concentration of the inflow. For the Atlantic Ocean, it is a source at the North Pole. The concentration is regarded as a fixed value.

(3) At the southern boundary: The basin is adjoined with the circumpolar circulation. We will take the concentration of the tracer as the observed value.

(4) At the western boundary: We assume that the concentration is uniform across the longitudinal section in the boundary current. The meridional variation of concentration in the current is governed by

$$K_H \frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\cos\phi\frac{\partial C}{\partial\phi} - V^b\frac{\partial C}{a\partial\phi} - \nu(\sigma\mu C) + \frac{1}{\lambda^*}K_H \cdot \frac{1}{a^2\cos^2\phi}\left.\frac{\partial C}{\partial\lambda}\right|_{\lambda=0} = 0$$

where $V^b \sim$ the velocity of the boundary current. It can be obtained easily from the known transport.

$\lambda^* \sim$ the width of the boundary current. We will take it as $1/100\ \lambda_B$.

The last term in the above equation represents the gradient of the concentration at the western boundary of the interior.

IV Numerical Results and Discussion

The governing equations with boundary conditions described above will be solved numerically in non-dimensional form for fixed decay rate and various combinations of horizontal eddy diffusion and upwelling velocity. The concentration of oxygen is calculated as observed value. For the Pacific Ocean, we use the observed data 4.5 ml/l for the fixed concentration in the southern boundary. For the Atlantic Ocean, the value of 6.5 ml/l in the North Pole and 5.15 ml/l in the southern boundary are used in the calculation. The concentration of Carbon 14 is normalized with respect to that of the source. Thus we take 1.0 as the value for the concentration at the southern boundary in both oceans and North Pole in the Atlantic. The results will be reported as relative concentration and age (in years). The detailed calculations with comparisons to observed data are discussed separately for the two oceans below.

(1) Pacific

With $K_H = 10^7$ cm^2/sec and $W_o = 1.5 \times 10^{-5}$ cm/sec, the calculated distributions of oxygen (Figure 2) and Carbon 14 (Figure 3) agree qualitatively well with the observed data. The 3.5 ml/l and 4 ml/l iso-concentration lines start about 20°N and fall apart widely in the middle of the basin as indicated by Stommel and Arons (1960). The dissolved oxygen at the northern part of the basin is about 3 ml/l (Reid, 1965). Bien, Rakestraw and Suess (1960, 1965) reported that the relative age of Carbon 14 is about 300-400 years between 40°S and 15°N and the age of water in the area north of 30°N is the oldest.

(2) Atlantic

With $K_H = 10^7$ cm^2/sec and $W_o = 0.75 \times 10^{-5}$ cm/sec, we have a better fit for the results of oxygen (Figure 4) and Carbon 14 (Figure 5). There exists an oxygen minimum 5.0 ml/l about 10°S near the eastern boundary (Stommel and Arons, 1960). The relative age of Carbon 14 between the west basin and the east basin in the North Atlantic is about 250 years. This is given by Brocker et al., (1960).

V. Conclusion

Based on a decrease in the apparent age , about 300 - 400 years. between 40°S and 15°N in the deep Pacific Ocean (Bien et al, 1960), it is concluded that there is correspondence to an average northward component of the velocity of the water of 0.06 $\pm$ 0.02 cm/sec. This seems misleading according to present

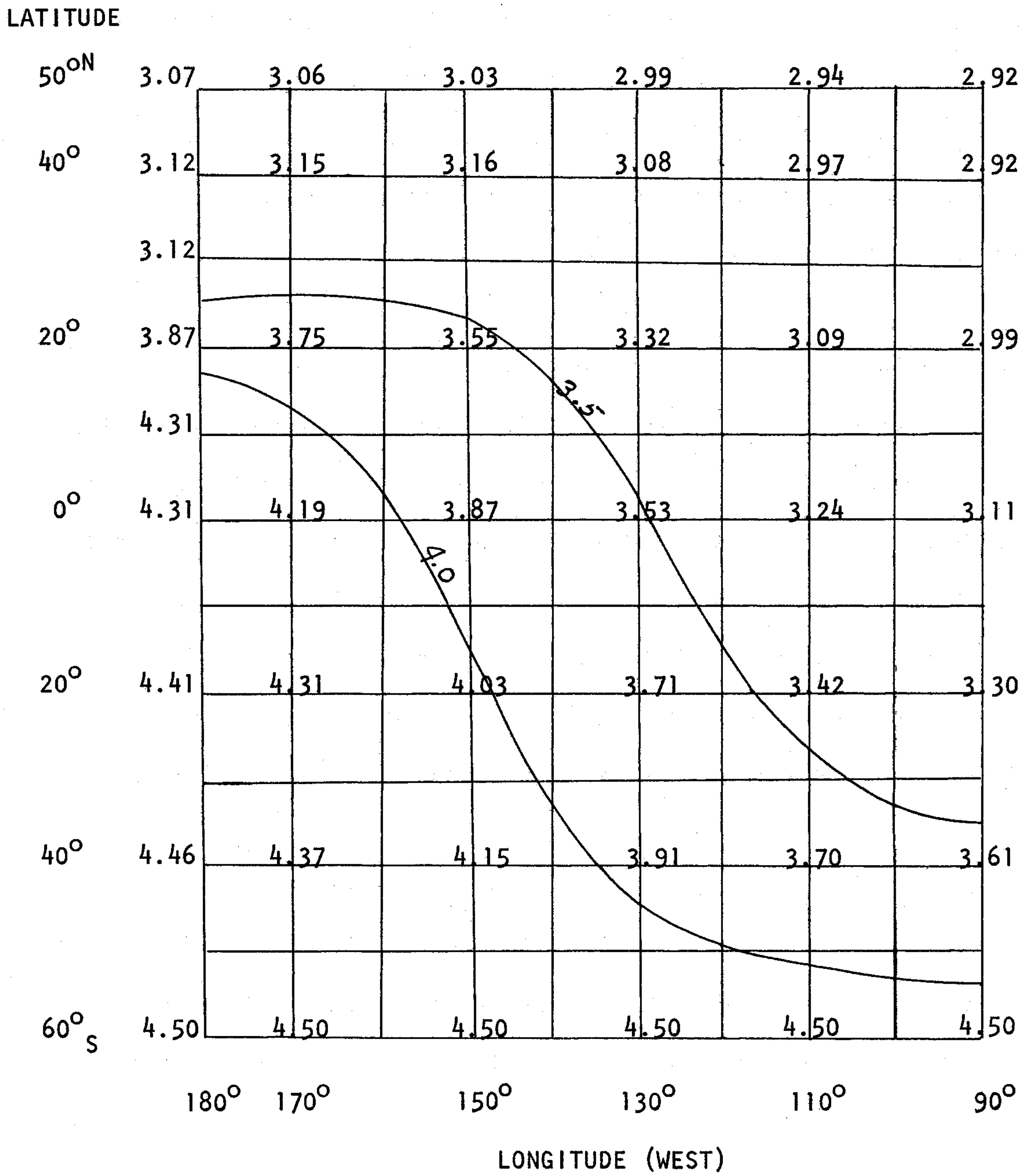


Figure 2: The calculated distribution of dissolved oxygen (ml/l) in the model basin for the Pacific Ocean

$K_H = 10^7\ cm^2/sec$

$W_o = 1.50 \times 10^{-5}\ cm/sec$

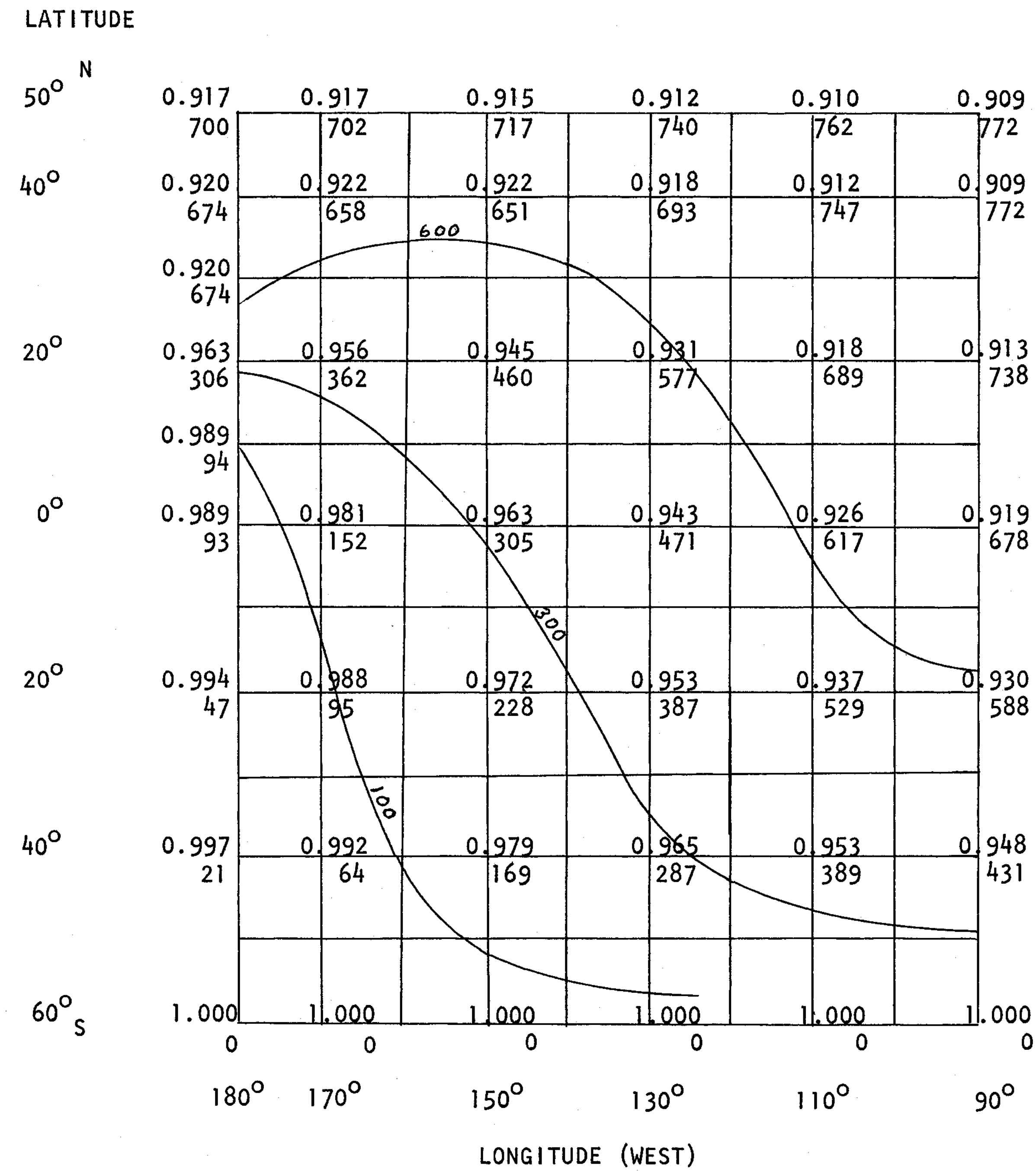
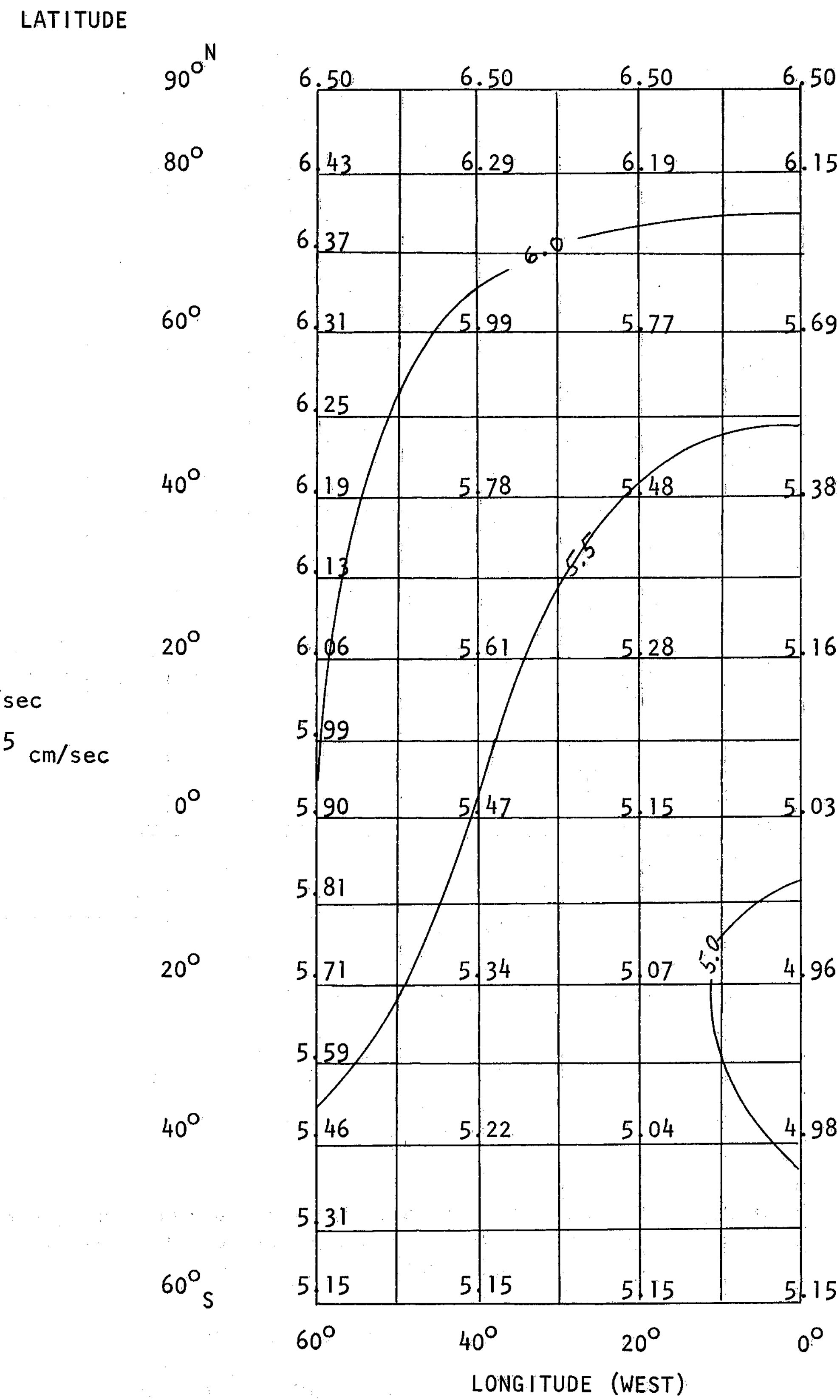


Figure 3: The calculated relative concentration (upper number) and age (year, lower number) of Carbon 14 in the model basin for the Pacific Ocean

$$K_H = 10^7 \text{ cm}^2/\text{sec}$$

$$W_o = 1.50\times10^{-5} \text{ cm/sec}$$



$K_H = 10^7\ cm^2/sec$

$W_o = 0.75 \times 10^{-5}$ cm/sec

Figure 4: The calculated distribution of dissolved oxygen (ml/l) in the model basin for the Atlantic Ocean.

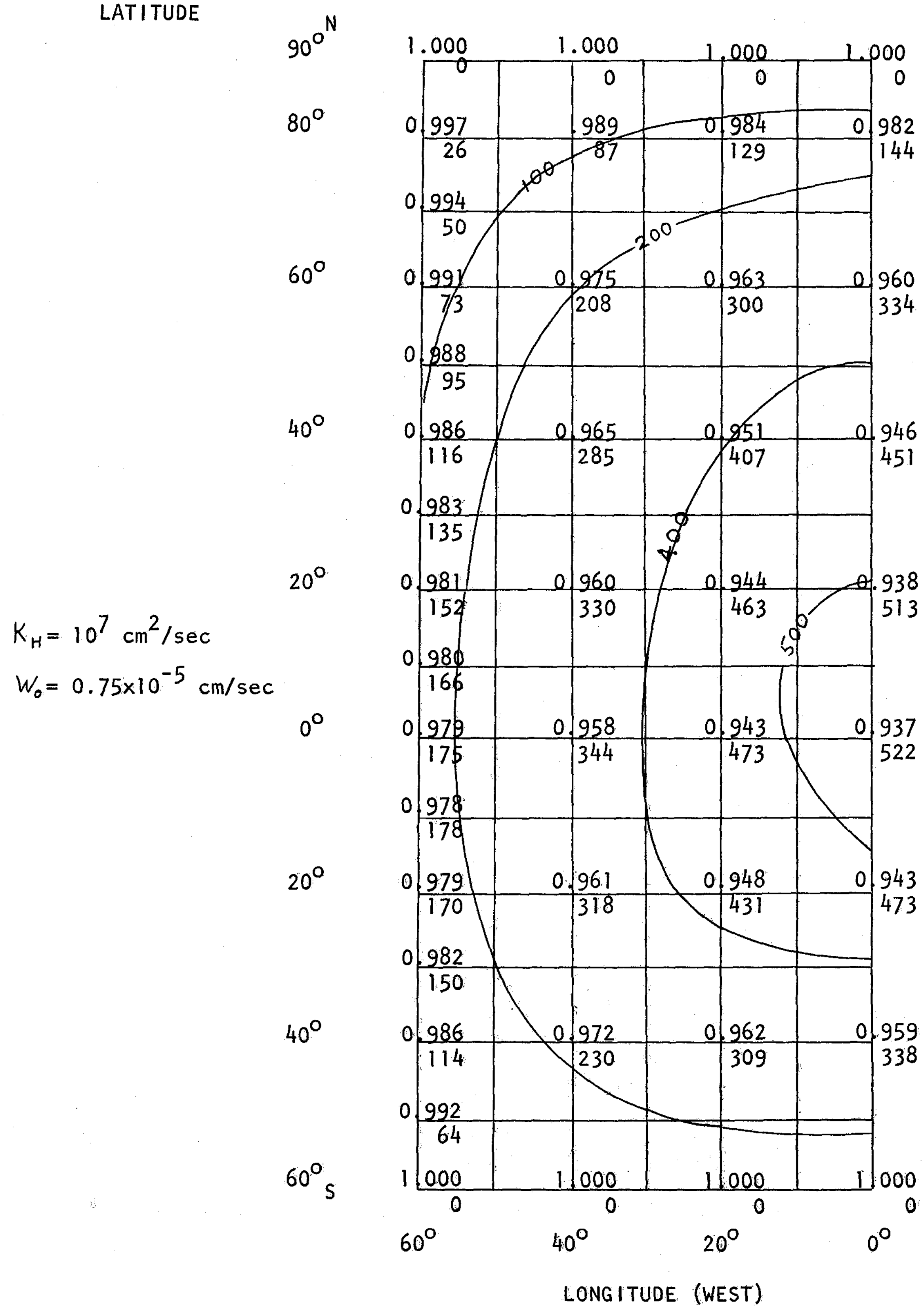


Figure 5: The calculated relative concentration (upper number) and age (year, lower number) of Carbon 14 in the model basin for the Atlantic Ocean.

study. The distributions of oxygen and Carbon 14 in both oceans are consistent with the abyssal circulation proposed by Stommel and Arons (1960) with suitable eddy diffusion. The horizontal eddy diffusivity and upwelling velocity in the ocean are about 1×10^7 cm^2/sec and 1.5 0.75×10^{-5} cm/sec respectively derived from our results.

Acknowledgements

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LAYERS OF HOMOGENEOUS AND STRATIFIED ROTATING FLUIDS

James R. Luyten

1. Introduction

Many oceanic problems are thought to involve the continuous stratification of the sea in an essential way. The main thermocline is a strongly stratified region of the ocean and has long resisted a complete analysis. It is tempting to approximate the continuous stratification by a system of many layers of homogeneous fluid of different density. The layers are pieced together by assuming that the velocity and stress are continuous across the interfaces. A continuously stratified rotating fluid can tolerate a vertical shear in its interior through the thermal wind relations whereas a homogeneous rotating fluid obeys the Taylor-Proudman theorem in its bulk and allows vertical shear only in Ekman layers. The dynamics is very different in these two cases so that the strong Ekman layers at the interfaces between layers are a poor model of the shear in a continuously stratified layer.

Two-layer models have been used in oceanic circulation theory but they have the drawback that the velocity at the bottom of the lower layer is a severe constraint on the motion in the upper layer. All of the shear must be transmitted through the strong Ekman layers at the interface. If this were a realistic model one would expect the bottom topography to be a strong influence on the circulation in the upper ocean. The general circulation in the upper ocean does not reflect the bottom topography strongly so that the main thermocline region must isolate the upper ocean from the deep ocean.

As a step in the direction of understanding how the continuous stratification produces such an isolation we have considered a three-layer system in a rigid rotating cylinder. Two of the layers are of different but homogeneous density while the third layer is linearly stratified. These three layers are fitted together to produce a nearly continuous density profile. We examine the steady flows in this system when the top and bottom of the cylinder are rotated differentially (as seen from the rotating frame). We shall assume that the three fluids are immiscible so that the vertical velocity vanishes at the interfaces. We match the velocity and stress across each interface to determine the relations between the boundary velocities and the interior flows.

If our thoughts about the "insulating" nature of a strongly stratified layer are correct, we expect that the interior velocity field in the upper (lower) homogeneous layer should be determined primarily by the prescribed velocity of the top (bottom). We shall show that this is true.

One must, of course, be very wary of trying to apply these calculations to the oceanic problem - it is not obvious that this model can be realized experimentally let alone be a model of the effects of the thermocline. In the real ocean the layers are not immiscible and there probably is a vertical velocity through the transition regions. It is possible that further calculations can make this model more realistic.

We begin by presenting the well-known results for the steady flows in a rotating cylinder of homogeneous fluid of density ρ. Let us assume the cylinder to be of depth L and radius aL and rotated uniformly at a rate Ω about the axis. The basic equations for steady flows as seen from the rotating frame are

$$(\underline{u}\cdot\nabla)\underline{u} + 2\Omega\,\underline{k}\times\underline{u} + \frac{1}{\rho}\nabla p + g\underline{k} - \frac{1}{2}\Omega^2(\underline{k}\times\underline{r})^2 = \nu\nabla^2\underline{u}\ ;\quad \nabla\cdot\underline{u} = 0$$

where ν is the viscosity.

Let us assume that the boundary conditions at $z = 0_g$, and L are

$$\underline{u}(r\,L) = U_T(r)\,\underline{r}_1 + V_T(r)\,\underline{\theta}_1$$

$$\underline{u}(r\,0) = U_B(r)\,\underline{r}_1 + V_B(r)\,\underline{\theta}_1$$

where $\underline{r}_1, \underline{\theta}_1$ and $\underline{k}$ are the usual radial, azimuthal and vertical unit vectors. We non-dimensionalize the equations by writing

$$\underline{r} = L\underline{r} \;,\; \underline{u}(r) = U\underline{u}(r), \quad p/\rho + gz - \tfrac{1}{2}\Omega^2(\underline{k}\times\underline{r})^2 = \rho U \Omega L p.$$

Then
$$\frac{U}{L\Omega}(\underline{u}\cdot\nabla)\underline{u} + 2\underline{k}\times\underline{u} + \nabla p = \frac{\nu}{L^2\Omega}\nabla^2\underline{u}\;,\quad \nabla\cdot\underline{u} = 0$$

or defining the Rossby number $\epsilon = U/L\Omega$ and the Ekman number $\nu/L^2\Omega = E$, we have

$$\epsilon(\underline{u}\cdot\nabla)\underline{u} + 2\underline{u}\times\underline{u} + \nabla p = E\nabla^2\underline{u}\;,\quad \nabla\cdot\underline{u} = 0$$

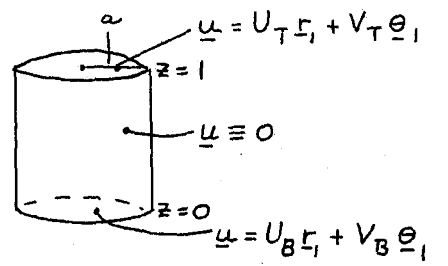
If we assume that $\epsilon \ll E \ll 1 \left(R_e = \frac{UL}{\nu} = \epsilon/E \ll 1\right)$ we are left with the linear problem:

$$2\underline{k}\times\underline{u} + \nabla p \pm E\nabla^2\underline{u}\;,\quad \nabla\cdot\underline{u} = 0\;,\quad E \ll 1.$$

We also have to look at the rotational Froude number $F = \dfrac{\Omega^2 L}{g}$ which measures the centripetal accelerations relative to the gravitational accelerations:

$$gz - \tfrac{1}{2}\Omega^2(\underline{k}\times\underline{r})^2 = gL\left(z - \tfrac{1}{2}F(\underline{k}\times\underline{r})^2\right)$$

We assume that this is very small throughout so that the interfaces will not be significantly warped. Our problem then becomes



$$2\underline{k}\times\underline{u} + \nabla p = E\nabla^2\underline{u}\;,\quad \nabla\cdot\underline{u} = 0$$

where the boundary conditions are written in non-dimensional form. We assume that the problem is axisymmetric so that away from the walls $r = a$, we have an interior solution (independent of z):

$$v(r) = \tfrac{1}{2}p_r(r)$$

$$u(r) = \tfrac{1}{4}E\,\partial_r\nabla^2 p(r)$$

$$w(r) = W_0(r)$$

In the Ekman layer solution $(z = 1, 0 \mp E^{+1/2}\zeta)$ the boundary layer corrections are

$$\tilde{u}(r\zeta) = \mathrm{Re}\, C(r) e^{-(i+i)\zeta}$$

$$\tilde{v}(r\zeta) = \mathrm{Im}\, C(r) e^{-(i+i)\zeta}$$

$$\tilde{w}(r\zeta) = \mp E^{1/2}/\sqrt{2}\, \mathrm{Re}\left\{\frac{1}{r}\left(rC(r)\right)_r e^{-(i+i)\zeta} e^{-i\pi/4}\right\}$$

where the upper and lower signs refer to the top (T) and bottom (B) respectively.

The vertical boundary conditions give:

$$u(rz=1,0) = \frac{1}{4} E\, \partial_r \nabla^2 p(r) + \mathrm{Re}\, C(r) = U_{T,B}(r)$$

$$v(rz=1,0) = \frac{1}{2} p_r(r) + \mathrm{Im}\, C(r) \quad = V_{T,B}(r)$$

$$w(rz=1,0) = W_0(r) \mp E^{1/2} \frac{1}{2}\, \mathrm{Re}\left(\frac{1}{r}\left(rC(r)\right)_r (1-i)\right) = 0$$

so that $\quad C(r) = U_{T,B}(r) + i\left(V_{T,B}(r) - v(r)\right)$,

and

$$W_0(r) = E^{1/2}/2\ \frac{1}{r}\left(r\left[U_T(r) + V_T(r) - v(r)\right]\right)_r$$

$$= -E^{1/2}/2\ \frac{1}{r}\left(r\left[U_B(r) + V_B(r) - v(r)\right]\right)_r$$

which gives

$$v(r) = \frac{1}{2}\left[V_T(r) + U_T(r) + V_B(r) + U_B(r)\right] \tag{11.1}$$

Then $\quad C(r) = U_{T,B}(r) + i\left(V_{T,B}(r) - \frac{1}{2} V_T(r) - \frac{1}{2} V_B(r) - \frac{1}{2}\left(V_T(r) + U_B(r)\right)\right)$

or $\quad C_T(r) = U_T(r) + i\frac{1}{2}\left(V_T(r) - V_B(r) - U_T(r) - U_B(r)\right) = U_T(r) + i\Delta V_T(r)$

$$C_B(r) = U_B(r) + i\frac{1}{2}\left(V_B(r) - V_T(r) - U_T(r) - U_B(r)\right) = U_B(r) - i\Delta V_B(r).$$

giving Ekman layer fields:

$$\tilde{u}(r\zeta) = U_{T,B}(r)\cos\zeta\, e^{-\zeta} \pm \Delta V_{T,B}(r)\sin\zeta\, e^{-\zeta}$$

$$\tilde{v}(r\zeta) = -U_{T,B}(r)\sin\zeta\, e^{-\zeta} \pm \Delta V_{T,B}(r)\cos\zeta\, e^{-\zeta}$$

$$\tilde{w}(r\zeta) = \mp E^{1/2}/\sqrt{2}\left\{\frac{1}{r}\left(rU_{T,B}(r)\right)_r \cos\left(\zeta + \pi/4\right) \pm \frac{1}{r}\left(r\Delta V_{T,B}(r)\right)_r \sin\left(\zeta + \pi/4\right)\right\} e^{-\zeta}.$$

The stress at the top and bottom is given by (using dimensional fields)

$$\underline{\sigma}(r_{T,B}) = (\sigma_{ij} n_j) = \mu u_z \underline{r}_1 + \mu v_z \underline{\theta}_1 + (2\mu w_z - p)\underline{k}$$
$$= \left(\frac{\nu\rho U}{L}\right)\left[u_z \underline{r}_1 + v_z \underline{\theta}_1 + 2 w_z \underline{k}\right] - p_* \underline{k}$$

using the non-dimensional fields (p_* is dimensional p). Now the only vertical shear is in the Ekman layers so that

$$u_z\Big|_{T,B} = \mp E^{-1/2}\partial_\zeta \tilde{u}(r\zeta)\Big|_{\zeta=0} = \mp E^{-1/2}\left\{\pm \Delta V_{T,B}(r) - U_{T,B}(r)\right\} = E^{-1/2}\left\{-\Delta V_{T,B}(r) \pm U_{T,B}(r)\right\}$$
$$v_z\Big|_{T,B} = \mp E^{-1/2}\delta_\zeta \tilde{v}(r\zeta)\Big|_{\zeta=0} = \mp E^{-1/2}\left\{-U_{T,B}(r) \mp \Delta V_{T,B}(r)\right\} = -E^{-1/2}\left\{-\Delta V_{T,B}(r) \mp U_{T,B}(r)\right\}$$
$$w_z\Big|_{T,B} = -\frac{1}{r}(ru)_r\Big|_{T,B} = -\frac{1}{r}\left(rU_{T,B}(r)\right)_r .$$

and

$$\underline{\sigma}(r_{T,B}) = \left(\frac{\nu\rho U}{L}\right)E^{-1/2}\left\{-\underline{r}_1\left(\Delta V_{T,B}u\right) \mp U_{T,B}(r) - \underline{\theta}_1\left(-\Delta V_{T,B}(r) \mp U_{T,B}(r)\right\} -$$
$$- \underline{k}\left\{p_*(r) + 2\,\frac{\nu\rho U}{L}\,\frac{1}{r}\left(rU_{T,B}(r)\right)_r\right\}$$

In order to satisfy the boundary conditions of $\underline{u} = 0$ at the side walls, we must involve the side wall layers. In our case, an $E^{1/4}$ layer is sufficient since the side wall condition is independent of depth. The side walls are not crucial to our problem and will not be discussed.

It is clear from this expression for the stress at the top and bottom in the fluid that if ΔV and U are $O(1)$, the stress is $O(E^{-1/2})$ which is very large. A stress of $O(1)$ requires that $\Delta V(r)$ and $U(r)$ are $O(E^{1/2})$ which in turn implies that the Ekman suction $W_o(r)$ is $O(E)$.

III. We turn now to the problem of the steady flows of a linearily stratified fluid in a rotating cylinder. We assume that the fluid is incompressible and Boussinesq so that

$$\rho(\underline{r}) = \rho\left(1 - \alpha T_o(z) - \alpha T(\underline{r})\right)$$

where $T_o(z)$ is the basic linear stratification and T the perturbation temperature.

The conservation equations become

$$(\underline{u}\cdot\nabla)\underline{u} + 2\Omega\,\underline{k}\times\underline{u} + \frac{1}{\rho}\nabla p + g\underline{k} - \nabla\frac{1}{2}\Omega^2\nabla(\underline{k}\times\underline{r})^2 - g\alpha T_0(z)\,\underline{k} - g\alpha T\underline{k} = \nu\nabla^2\underline{u}$$

$$(\underline{u}\cdot\nabla)(T_0+T) = \kappa\nabla^2 T$$

$$\nabla\cdot\underline{u} = 0$$

where κ is the diffusion constant. We non-dimensionalize these equations by writing

$$\underline{r}\to L\underline{r},\quad \underline{u}\to \mathcal{U}\underline{u},\quad \frac{1}{\rho}\nabla p + g\underline{k} - \nabla\frac{1}{2}\Omega^2(\underline{k}\times\underline{r})^2 - g\alpha T_0(z) \to \Omega\mathcal{U}\nabla p$$

$$T\to \Theta T \qquad \kappa = \nu/\sigma$$

Then

$$\frac{\mathcal{U}}{L\Omega}(\underline{u}\cdot\nabla)\underline{u} + 2\underline{k}\times\underline{u} + \nabla p - \frac{g\alpha\Theta}{\Omega\mathcal{U}}T\underline{k} = \frac{\nu}{L^2\Omega}\nabla^2\underline{u}$$

$$\frac{\mathcal{U}T_0'(z)}{\Theta\Omega}w + \frac{\mathcal{U}}{L\Omega}(u\cdot\nabla)T = \frac{\nu}{L^2\Omega}\cdot\frac{1}{\sigma}\nabla^2 T$$

$$\nabla\cdot\underline{u} = 0$$

Again $\epsilon = \mathcal{U}/L\Omega$, $E = \nu/L^2\Omega$. We set $\frac{\alpha g\Theta}{\Omega\mathcal{U}} = 1$ thus setting the scale for the temperature perturbations $\Theta = \mathcal{U}\frac{\Omega}{\alpha g}$. Then

$$\frac{\mathcal{U}T_0'(z)}{\Theta\,\Omega} = \frac{\mathcal{U}T_0'(z)}{[\mathcal{U}\Omega/g\alpha]\Omega} = \frac{g\alpha T_0'(z)}{\Omega^2} = \frac{N^2}{\Omega^2} = 4S$$

where N is the Brunt-Väisälä frequency $\left(g\frac{1}{\rho}\left|\frac{\partial\rho}{\partial z}\right|\right)^{1/2}$. $4S$ is the parameter measuring the strength of the basic stratification. Again assuming $\epsilon \ll E$, we have

$$2\underline{k}\times\underline{u} + \nabla p = E\nabla^2\underline{u} + T\underline{k}$$

$$4\sigma S w = E\nabla^2 T, \quad \nabla\cdot\underline{u} = 0$$

One of the most important consequences of the stratification is easily seen from the second equation, the heat equation representing the balance between vertical advection of the stable density field and diffusion. This equation constrains the vertical velocity. If the temperature field is $O(1)$, then w is constrained to be $O(E/\sigma S)$. This will prevent Ekman pumping with w of $O(E^{1/2})$ whenever $E/\sigma S < E^{1/2}$, i.e. whenever $\sigma S > E^{1/2}$. Note that this is independent of the vertical length scaling since it is the ratio of two dimensional depths.

$$\frac{\sigma S}{E^{1/2}} = \frac{\sigma \alpha g T_o'(z)}{\Omega^2} \left(\frac{1}{L}\sqrt{\frac{\nu}{\Omega}}\right)^{-1/2} = \frac{\sigma g}{\Omega^2}\left(\frac{\Delta \rho}{\rho}\right)\bigg/\sqrt{\frac{\nu}{\Omega}}$$

It is essentially the ratio of the Lineykin depth to the Ekman depth. It is the density jump $(\Delta\rho/\rho)$ that determines whether or not there is significant Ekman pumping.

The second effect that is of importance here is the ratio of the vertical dimensions to the horizontal dimensions - a measure of how thin the stratified layer is. Suppose we introduce a vertical scale $H = \delta L$ where δ is small. In order to retain the same balance in continuity equation, we scale z by H and w by $HU/L = \delta U$. We write a new Ekman number $\bar{E} = \frac{\nu}{H^2\Omega} = \delta^{-2} E$. The equations then become

$$-2v + \epsilon(\underline{u}\cdot\nabla)u + p_r = \bar{E}\left(\partial_z^2 + \delta^2(\nabla_1^2 - 1/r^2)\right)u$$

$$2u + \epsilon(\underline{u}\cdot\nabla)v = \bar{E}\left(\partial_z^2 + \delta^2(\nabla_1^2 - 1/r^2)\right)v$$

$$\epsilon(\underline{u}\cdot\nabla)(\delta^2 w) + p_z = T + \bar{E}\left(\partial_z^2 + \delta^2\nabla_1^2\right)(\delta^2 w)$$

$$4\sigma S(\delta^2 w) = \bar{E}\left(\partial_z^2 + \delta^2\nabla_1^2\right)T$$

$$\frac{1}{r}(ru)_r + w_z = 0$$

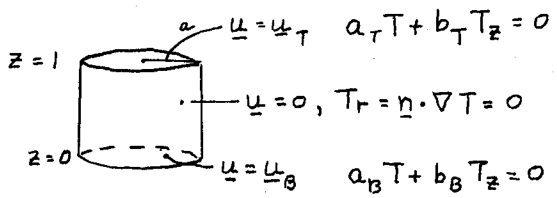
where $\nabla_1^2 = \nabla^2 - \partial_z^2 = \frac{1}{r}\partial_r(r\partial_r)$

For the fluid to remain essentially hydrostatic the temperature scale Θ becomes $\frac{U\Omega}{g\alpha\delta}$. For a given value of $\bar{E}^{1/2}/\sigma S$ which is independent of δ, we have that

$$4w = \left(\frac{\bar{E}^{1/2}}{\sigma S}\right)\left(\frac{\bar{E}^{1/2}}{\delta^2}\right)\nabla^2 T$$

so that an important constraint on w comes through $\bar{E}^{1/2}\delta^{-2}$. It is clear that whatever the stratification, a sufficiently small δ will knock out the constraint. In particular if σS were $O(1)$, then $\delta^2 = \bar{E}^{1/2} = \delta^{-1}E^{1/2}$ or $\delta = E^{1/6}$ would allow $O(1)$ Ekman layers to form. Since we are interested in the case of weak Ekman layers, we should keep δ away from these very small values.

To analyze this problem more thoroughly we consider a strongly stratified fluid in a thin rotating cylinder driven above and below by a prescribed velocity field. We assume that the side wall is insulated and at rest in the rotating frame. We imagine some homogeneous boundary conditions on the temperature field at the top and bottom of the cylinder. As before, we assume



$$-2v + p_r = \bar{E} u_{zz} + \delta^2 \bar{E}\left(\nabla_1^2 - 1/r^2\right) u$$
$$2u = \bar{E} v_{zz} + \delta^2 \bar{E}\left(\nabla_1^2 - 1/r^2\right) v$$
$$p_z = T + \bar{E}(\delta^2 w)_{zz} + \delta^2 \bar{E} \nabla_1^2 (\delta^2 w)$$
$$4\sigma S(\delta^2 w) = \bar{E} T_{zz} + \delta^2 \bar{E} \nabla_1^2 T$$
$$\frac{1}{r}(ru)_r + w_z = 0,$$

We resort to boundary layer methods and write, for the interior,

$$v(rz) = \frac{1}{2} p_r(rz)$$
$$u(rz) = \bar{E}/4\, \partial_r \left[\partial_z^2 + \delta^2 \nabla_1^2\right] p_1(z)$$
$$w(rz) = \frac{\bar{E}}{4\sigma S \delta^2} \partial_z \left[\partial_z^2 + \delta^2 \nabla_1^2\right] p_1(z)$$
$$T(rz) = p_z(rz).$$

Applying the continuity equation gives

$$\nabla_1^2\left(\partial_z^2 + \delta^2 \nabla_1^2\right) p_1(z) + \frac{1}{\sigma S \delta^2}\left(\partial_z^2 + \delta^2 \nabla_1^2\right) p_1(z) = 0$$

or

$$\left(\delta^2 \nabla_1^2 + \frac{1}{\sigma S}\partial_z^2\right)\left(\delta^2 \nabla_1^2 + \partial_z^2\right) p_1(z) = 0$$

Thus we see that the interior motion is governed by the small diffusion inherent in it.

We will need boundary layers to satisfy all of the boundary conditions because the original pressure equation is eighth order:

For $\delta = 1$, $$E \nabla^2\left(E^2 \nabla^6 + 4\partial_z^2 + 4\sigma S \nabla_1^2\right) p_1(z) = 0$$

and in the interior we have only a fourth order equation. Near the top and bottom, let $z = 1, 0 \mp \bar{E}^{1/2}\zeta$. The boundary layer corrections are

$$\tilde{u}(r\zeta) = \mathcal{R}e\, C(r)\, e^{-(1+i)\zeta}$$

$$\tilde{v}(r\zeta) = \mathcal{I}m\, C(r)\, e^{-(1+i)\zeta}$$

$$\tilde{w}(r\zeta) = \mp \frac{\bar{E}^{1/2}}{\sqrt{2}} \mathcal{R}e \left\{ \frac{1}{r}(rC(r))_r e^{-i\pi/4} e^{-(1+i)\zeta} \right\}$$

$$\tilde{T}(r\zeta) = \mp \frac{\bar{E}^{1/2}}{\sqrt{2}} \left[2\sigma S \delta^2\right] \mathcal{I}m \left\{ \frac{1}{r}(rC(r))_r e^{-i\pi/4} e^{-(1+i)\zeta} \right\}$$

Near the side walls, let $r = a - (\delta^2 \bar{E}/\sqrt{\sigma S})^{1/2} \rho = a - \frac{\delta}{\mu} \bar{E}^{1/2} \rho$ with $\mu^4 = \sigma S$.

The boundary layer corrections are

$$\bar{w}(\rho z) = \mathcal{R}e\, D(z)\, e^{-(1+i)\rho}$$

$$\bar{T}(\rho z) = 2\mu^2 \delta^2\, \mathcal{I}m\, D(z)\, e^{-(1+i)\rho}$$

$$\bar{u}(\rho z) = -\left(\frac{\delta \bar{E}^{1/2}}{\mu}\right) \frac{1}{\sqrt{2}} \mathcal{R}e \left[D'(z) e^{-i\pi/4} e^{-(1+i)\rho} \right]$$

$$\bar{v}(\rho z) = -\left(\frac{\delta \bar{E}^{1/2}}{r}\right) \frac{1}{\mu^2\sqrt{2}} \mathcal{I}m \left[D'(z) e^{-i\pi/4} e^{-(1+i)\rho} \right].$$

Applying the boundary conditions gives:

$$w(r\,z=1,0) = \frac{\bar{E}}{4\sigma S \delta^2} \partial_z (\partial_z^2 + \delta^2 \nabla_1^2) p(r\,1,0) \mp \frac{\bar{E}^{1/2}}{\sqrt{2}} \mathcal{R}e \left[\frac{1}{r}(rC(r))_r e^{-i\pi/4} \right] = 0$$

$$\Rightarrow \mathcal{R}e \left[\frac{1}{r}(rC(r))_r (1-i) \right] = \pm \frac{\bar{E}^{1/2}}{2\sigma S \delta^2} \partial_z (\partial_z^2 + \delta^2 \nabla_1^2) p(r\,1,0)$$

$$u(r\,z=1,0) = \frac{\bar{E}}{4} \partial_r (\partial_z^2 + \delta^2 \nabla_1^2) p(r\,1,0) + \mathcal{R}e\, C(r) = U_{T,B}(r)$$

$$\Rightarrow \mathcal{R}e\, C(r) = U_{T,B}(r) + O(\bar{E})$$

$$v(r\,z=1,0) = \frac{1}{2} p_r(r\,1,0) + \mathcal{I}m\, C(r) = V_{T,B}(r)$$

To get a consistent solution $C(r)$ must be $O(\bar{E}^{1/2})$ and thus $U_{T,B}(r)$ is $O(\bar{E}^{1/2})$. If it isn't then non-linear boundary layers must appear. Similarly we write

$$V_{T,B}(r) = V^{(0)}_{T,B}(r) + \bar{E}^{1/2} V^{(1)}_{T,B}(r)$$

This is, of course, not unique unless $V^{(1)}_{T,B}(r)$ is related, by some constraint, to $V^{(0)}_{T,B}(r)$.

Then we write

$$C(r) = \bar{E}^{1/2}\left\{ U^{(1)}_{T,B}(r) + i\, V^{(1)}_{T,B}(r)\right\} \;,\; p_r(r\,1,0) = 2\, V^{(0)}_{T,B}(r)\,.$$

One constraint appears immediately through w,

$$\bar{E}^{-1/2} \mathcal{Re}\, \frac{1}{r}\left(r\,C(r)\right)_r (1-i) = \pm\left(r\left[U^{(1)}_{T,B}(r) + V^{(1)}_{T,B}(r)\right]\right)r$$

$$= \pm\frac{1}{2\sigma S \delta^2}\, \partial_z\left(\partial_z^2 + \delta^2 \nabla_1^2\right) p(r\,1,0)$$

which gives an implicit relation between $V^{(1)}_{T,B}(r)$ and $V^{(0)}_{T,B}(r)$ through $p(rz)$.

The thermal boundary condition is

$$a_{T,B}\, T(rz=1,0) + b_{T,B}\, T_z(r\,1,0) = a_{T,B}\, p_z(r\,1,0) + b_{T,B}\, p_{zz}(r\,1,0) = 0.$$

At the side wall $r = a$

$$w(az) = \mathcal{Re}\, D(z) + \bar{E}/_{4\sigma S\delta^2}\, \partial_z\left(\partial_z^2 + \delta^2\nabla_1^2\right) p(az) = 0$$

$$\Rightarrow \mathcal{Re}\, D(z) = O(\bar{E})$$

$$v(az) = p_r(az) - \frac{\delta \bar{E}^{1/2}}{\mu}\, \frac{1}{2\mu^3}\, \mathrm{Im}\, D'(z)(1-i) = 0$$

$$\Rightarrow p_r(az) = 0$$

$$T_r(az) = p_{rz}(az) - \left(\frac{\delta\bar{E}^{1/2}}{\mu}\right)^{-1} \partial_\rho\left\{2\mu^2\delta^2\, \mathrm{Im}\, D(z)\, e^{-(1+i)\rho}\right\}_{\rho=0}$$

$$= p_{rz}(az) + 2\,\frac{3}{\mu}\,\delta\,\bar{E}^{-1/2}\, \mathrm{Im}\,(1+i)D(z) = 0$$

But if $p_r(az) = 0$ then $\partial_z p_r(az) = 0$ and

$$\mathrm{Im}\,(1+i)D(z) = O(\bar{E}) = \mathrm{Im}\, D(z) + \mathcal{Re}\, D(z) = \mathrm{Im}\, D(z).$$

For the tangential velocity

$$u(az) = \frac{\bar{E}}{4}\, \partial_r\left(\partial_z^2 + \delta^2\nabla_1^2\right) p(az) - \left(\frac{\delta\bar{E}^{1/2}}{\mu}\right)\frac{1}{2}\, \mathcal{Re}\, D'(z)(1-i) = 0$$

Since $D(z)$ is $O(\bar{E})$ we have

$$\partial r\left(\partial_z^2 + \delta^2 \nabla_1^2\right) p(a z) = 0$$

Using $p_r(az)=0$, we get

$$\partial r \nabla_1^2 p(a z) = 0$$

Thus we have the boundary conditions for the interior flow:

$$\left(\partial_z^2 + \delta^2 \nabla_1^2\right)\left(\delta^2 \nabla_1^2 + \frac{1}{\sigma S}\partial_z^2\right) p(r z) = 0$$

$$p_r(r\,1,0) = 2V_{T,B}^{(0)}(r) \quad , \quad a_{T,B}\, p_z(r\,1,0) + b_{T,B}\, p_{zz}(r\,1,0) = 0$$

$$p_r(a z) = 0 \quad , \quad \partial_r \nabla_1^2 p(a z) = 0$$

The simplest method for solving this equation is to use the eigen functions of $\nabla_1^2 : J_0(k_n r/a)$. However, there is no guarantee that an expansion of $p(rz)$ in $J_0(k_n r/a)$ will be four times differentiable, term by term. We must assume that $p(rz)$ can be written as

$$p(r z) = \sum_1^\infty A_n(z)\, J_0\left(k_n r/a\right) + F(r z)$$

where the series presumably can be differentiated term by term. Thus

$$\nabla_1^4 p(r z) = \sum_1^\infty \left(\frac{k_n}{a}\right)^4 A_n(z)\, J_0\left(k_n r/a\right) + \nabla_1^4 F(r z) = 0$$

To find the possible forms of $F(rz)$ we integrate $\nabla_1^4 F = 0$ and obtain

$$F(r z) = A(z)\, r^2 \log r + B(z)\, r^2 + C(z) \log r + D(z)$$

If these functions are to satisfy the differential equation, $A(z)$ and $B(z)$ must be linear in z, $C(z)$ and $D(z)$ must be cubic in z. The functions $A_n(z)$ must satisfy

$$\left(\partial_z^2 - \delta^2 (k_n/a)^2\right)\left(\frac{1}{\sigma S}\partial_z^2 - \delta^2 (k_n/a)^2\right) A_n(z) = 0$$

The boundary conditions become:

At $z = 1, 0$.

$$v(r\,1,0) = V_{T,B}^{(0)}(r) = \sum_1^\infty V_{T,B}^{(0)}(n)\, J_0'\left(k_n r/a\right)$$

so that $\frac{k_n}{2a} A_n(1,0) = V^{(0)}_{T,B}(n) \quad , F_r(r\,1,0) = 0.$

and at $r = a$, set $J_0'(k_n) = 0$, $F_r(a\,z) = 0$.

For definiteness, we choose $a_{T,B} = 1$, $b_{T,B} = 0$ so that

$$A_n'(1,0) = 0 \quad , \quad F_z(r\,1,0) = 0$$

Then we can show that $F(rz) =$ constant and that

$$A_n(z) = \alpha_n \phi_n(z) + \beta_n \phi_n(r\,z)$$

where $$\phi_n(z) = \mu^2 \sinh\left(\frac{\mu^2 \delta k_n}{a}\right) \sinh\left(\frac{\delta k_n z}{a}\right) - \sinh\left(\frac{\delta k_n}{a}\right) \sinh\left(\frac{\mu^2 \delta k_n z}{a}\right)$$

and $$\alpha_n = \frac{A_n(0)\phi_n(0) - A_n(1)\phi_n(1)}{\phi_n^2(0) - \phi_n^2(1)} \quad , \quad \beta_n = \frac{A_n(1)\phi_n(0) - A_n(0)\phi_n(1)}{\phi_n^2(0) - \phi_n^2(1)}$$

Using this solution, we can evaluate $V^{(1)}_{T,B}(r)$ by writing

$$\left(r\left[U^{(1)}_{T,B}(r) + V^{(1)}_{T,B}(r)\right]\right)_r = \pm \frac{1}{2\sigma S \delta^2} \sum_{n=1}^{\infty} \left[\partial_z^2 - \delta^2\left(\frac{k_n}{a}\right)^2\right] A_n'(1,0)\, J_0(k_n\, r/a)$$

$$= \pm \frac{1}{2\sigma S \delta^2} \sum_1^{\infty} A_n^{III}(1,0)\, J_0(k_n\, r/a).$$

If we expand both $U^{(1)}_{T,B}(r)$ and $V^{(1)}_{T,B}(r)$ in $J_0'(k_n\, r/a)$, then

$$\left[U^{(1)}_{T,B}(r) + V^{(1)}_{T,B}(r)\right] = \sum_1^{\infty} \left[U^{(1)}_{T,B}(n) + V^{(1)}_{T,B}(n)\right] J_0'(k_n\, r/a)$$

$$\frac{1}{r}\left(r\left[\quad\right]\right)_r = \sum_1^{\infty} \left[-\left(\frac{k_n}{a}\right)^2\right] \left[U^{(1)}_{T,B}(n) + V^{(1)}_{T,B}(n)\right] J_0(k_n\, r/a)$$

which gives

$$-\left(\frac{k_n}{a}\right)^2 \left[U^{(1)}_{T,B}(n) + V^{(1)}_{T,B}(n)\right] = \pm \frac{1}{2\sigma S \delta^2} A_n'''(1,0)$$

Now $A_n'''(z) = \alpha_n \phi_n^{(1)}(z) - \beta_n \phi_n'''(1-z)$

and $\phi_n^{(1)}(z) = \mu^2 \left(\frac{k_n \delta}{a}\right)^3 \left[\sinh \frac{\mu^2 \delta k_n}{a} \sinh \delta k_n z/a - \mu^4 \sinh \frac{\delta k_n}{a} \sinh \frac{\delta \mu^2 k_n z}{a}\right]$

$$\Rightarrow \phi_n'''(0) = 0$$

$$\phi_n'''(1) = \mu^2 \left(\frac{k_n \delta}{a}\right)^3 (1 - \mu^4) \sinh \frac{\mu^2 \delta k_n}{a} \sinh \frac{\delta k_n}{a}$$

so that

$$A_n'''(0) = -\beta_n \mu^2 \left(\frac{k_n\delta}{a}\right)^3 (1-\mu^4) \sinh \frac{\delta\mu^2 k_n}{a} \sinh \frac{\delta k_n}{a}$$

$$A_n'''(1) = \alpha_n \mu^2 \left(\frac{k_n\delta}{\alpha}\right)^3 (1-\mu^4) \sinh \frac{\delta\mu^2 k_n}{\alpha} \sinh \frac{\delta k_n}{a}$$

and

$$-\left(\frac{k_n}{a}\right)^2 \left[U_{T,B}^{(1)}(n) + V_{T,B}^{(1)}(n)\right] = (\alpha_n, \beta_n) \frac{\mu^2 (k_n\delta/a)^3 (1-\mu^4)}{2\sigma S \delta^2} \sinh \frac{\delta\mu^2 k_n}{a} \sinh \frac{\delta k_n}{a}$$

Thus we obtain 2 linear equations for $V_{T_1}^{(1)}$ and $V_B^{(1)}$ in terms of $U_{T,B}^{(1)}$ and $V_{T_1}^{(0)}$ $V_B^{(0)}$ both assumed to be known. But $V_{T,B}^{(0)}$ are known in terms of $V_{T,B}$ and $V_{T,B}^{(1)}$ so that they can now be determined.

We now calculate the stresses at the top and bottom of the cylinder. As before the stress is given by

$$\underline{\sigma}(rz) = \sigma_{ij}(rz) n_j = \mu u_z \underline{r}_1 + \mu v_z \underline{\theta}_1 + (2\mu w_z - p)\underline{k}$$
$$= \left(\frac{\nu\rho U}{H}\right)\left[u_z \underline{r}_1 + v_z \underline{\theta}_1\right] + \left(\frac{\nu\rho U}{L} 2 w_z - p_*\right)\underline{k}$$

in terms of the non-dimensional variables (p_* is dimensional p).

$$u_z(r\,T,B) = \mp\left(V_{T,B}^{(1)}(r) - U_{T,B}^{(1)}(r)\right) = \mp\left(V_{T,B}^{(1)}(r) - U_{T,B}^{(1)}(r)\right),$$

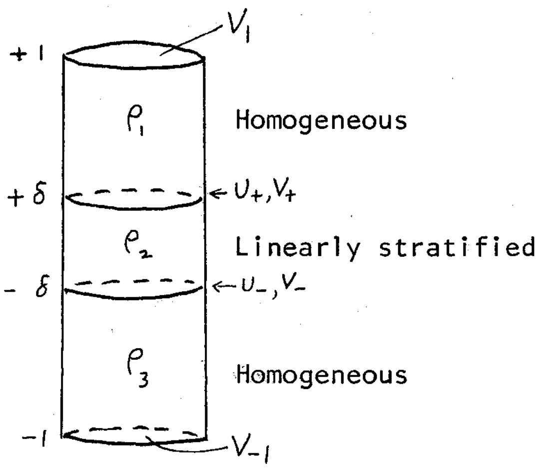
$$v_z(r\,T,B) = \tfrac{1}{2} p_{rz}(r\,T,B) \pm \left(V_{T,B}^{(1)}(r) + U_{T,B}^{(1)}(r)\right),$$

$$w_z(r\,T,B) = -\bar{E}^{1/2}\frac{1}{r}\left(r U_{T,B}^{(1)}(r)\right)_r .$$

so that

$$\underline{\sigma}(r\,T,B) = \left(\frac{\nu\rho U}{H}\right)\left[\mp \underline{r}_1\left(V_{T,B}^{(1)}(r) - U_{T,B}^{(1)}(r)\right) \pm \underline{\theta}_1\left(V_{T,B}^{(1)}(r) + U_{T,B}^{(1)}(r)\right) + \underline{\theta}_1 \tfrac{1}{2} p_{rz}(r\,T,B)\right] -$$
$$- \underline{k}\left[p_*(r\,T,B) + \frac{2\nu\rho U}{L} \bar{E}^{1/2} \frac{1}{r}\left(r U_{T,B}^{(1)}(r)\right)_r\right].$$

We are now in a position to look at the three-layers system described above. Let us consider a rotating cylinder filled with three fluids of different densities. The top and the bottom layers are homogeneous while the middle layer is linearly stratified. The top and bottom of the cylinder are rotated differentially. We match both velocity and stress across each interface. The interfaces are assumed to be flat. This is certainly an approximation, which we will discuss below. We neglect the differences between the three densities except in the hydrostatic term.



At $z = 1$ $\quad \underline{u}(r,1) = V_1(r)\,\underline{\theta}_1$

$z = \delta \quad \underline{u}(r,\delta) = U_+(r)\,\underline{r}_1 + V_+\underline{\theta}_1 \quad (w=0)$

$z = -\delta \quad \underline{u}(r,-\delta) = U_-(r)\,\underline{r}_1 + V_-(r)\,\underline{\theta}_1 \, (w=0)$

$z = -1 \quad \underline{u}(r,-1) = V_{-1}(r)\,\underline{\theta}_1$

The stress at $z = +\delta$ in the homogeneous layer is given by

$$\sigma(r\delta) = \frac{\nu\rho U}{L} E^{-1/2}\left\{-\underline{r}_1\left(\Delta V_+(r) + U_+(r)\right) - \underline{\theta}_1\left(-\Delta V_+(r) + U_+(r)\right)\right\} -$$
$$- \underline{k}\left\{p_*(r\delta) + \frac{2\nu\rho U}{L}\frac{1}{r}\left(rU_+(r)\right)_r\right\}$$

where $\Delta V_+(r) = \frac{1}{2}\left(V_1(r) - V_+(r)\right) + \frac{1}{2}U_+(r)$

$$\underline{\sigma}(r\delta) = \frac{\nu\rho U}{L} E^{-1/2}\left\{-\underline{r}_1\left(\tfrac{1}{2}(V_1 - V_+) + \tfrac{3}{2}U_+\right) - \underline{\theta}_1\left(-\tfrac{1}{2}(V_1 - V_+) + \tfrac{1}{2}U_+\right)\right\}$$
$$- \underline{k}\left\{p_*(r\delta) + \frac{2\nu\rho U}{L}\frac{1}{r}\left(rU_+(r)\right)_r\right\}$$

The stress in the stratified layer at $+\delta$ is

$$\underline{\sigma}(r\delta) = \frac{\nu\rho U}{H}\left\{-\underline{r}_1\left(V_+^{(1)}(r) - U_+^{(1)}(r)\right) + \underline{\theta}_1\left(V_+^{(1)}(r) + U_+^{(1)}(r)\right) + \tfrac{1}{2}p_{rz}(r\delta)\,\underline{\theta}_1\right\} -$$
$$- \underline{k}\left\{p_*(r\delta) + \frac{2\nu\rho U}{L} E^{-1/2}\frac{1}{r}\left(rU_+^{(1)}(r)\right)_r\right\}$$

Matching components:

$$-\frac{\nu\rho\mathcal{U}}{H}\left(V_+^{(1)}(r)-U_+^{(1)}(r)\right)=-\frac{\nu\rho\mathcal{U}}{L}E^{-1/2}\left(\tfrac{1}{2}(V_1-V_+)+\tfrac{3}{2}U_+\right)$$

$$\tfrac{1}{2}p_{rz}(r\delta^-)+\frac{\nu\rho\mathcal{U}}{H}\left(V_+^{(1)}(r)+U_+^{(1)}(r)\right)=\frac{\nu\rho\mathcal{U}}{L}E^{-1/2}\left(\tfrac{1}{2}(V_1-V_+)-\tfrac{1}{2}U_+\right)$$

$$\Rightarrow \frac{L}{H}E^{1/2}\left(V_+^{(1)}(r)-U_+^{(1)}(r)\right)=\tfrac{1}{2}\left(V_1-V_+\right)+\tfrac{3}{2}U_+$$

$$\frac{L}{H}E^{1/2}\left(V_+^{(1)}(r)+U_+^{(1)}(r)\right)=\tfrac{1}{2}\left(V_1-V_+\right)-\tfrac{1}{2}U_+-\frac{L}{H}E^{1/2}\tfrac{1}{2}p_{rz}(r\delta)$$

Now $E^{1/2}/\delta=\bar{E}^{1/2}$ and $\bar{E}^{1/2}U_+^{(1)}(r)=U_+(r)$ so that

$$\bar{E}^{1/2}V_+^{(1)}(r)=\tfrac{1}{2}(V_1-V_+)+5/2\,U_+(r)$$

$$\bar{E}^{1/2}V_+^{(1)}(r)=\tfrac{1}{2}(V_1-V_+)-3/2\,U_+(r)-\tfrac{1}{2}\bar{E}^{1/2}p_{rz}(r\delta)$$

which gives

$$\bar{E}^{1/2}U_+^{(1)}(r)=-\frac{\bar{E}^{1/2}}{8}p_{rz}(r\delta)=-\frac{E^{1/2}}{8\delta}p_{rz}(r\delta)=-\frac{E^{1/2}}{8\delta}p_{rz}(r\delta)$$

$$\bar{E}^{1/2}V_+^{(1)}(r)=\tfrac{1}{2}(V_1-V_+)-5/16\,\bar{E}^{1/2}p_{rz}(r\delta)=\tfrac{1}{2}\left(V_1(r)-V_+(r)\right)_r-5/16\,\bar{E}^{1/2}p_{rz}(r\delta)$$

Similarly at $z=-\delta$

$$\bar{E}^{1/2}U_-^{(1)}(r)=-\frac{\bar{E}^{1/2}}{8}p_{rz}(r_-\delta)$$

$$\bar{E}^{1/2}V_-^{(1)}(r)=-\tfrac{1}{2}\left(V_- - V_-\right)-5/16\,\bar{E}^{1/2}p_{rz}(r-\delta)$$

We can easily see that if there is a thermal wind at the interfaces, that is, a horizontal variation in the density field $T_r=p_{rz}$, then there must be a radial component of the interfacial velocity. This is required if the Ekman layer in the homogeneous fluid is to produce an interfacial stress to balance that in the stratified layer. If the temperature field is held constant at the interfaces, there is no thermal wind and no radial velocity at the interfaces. In the analysis that follows we assume that the temperature field vanishes at the two interfaces. This is certainly an assumption about the physical properties of the fluids involved. We must assume that the two homogeneous fluids are capable of maintaining the interfacial temperatures at a constant value.

Making this assumption we have the conditions

$$\bar{E}^{1/2} V_+^{(1)}(r) = \tfrac{1}{2}\left(V_1(r) - V_+(r)\right) = \tfrac{1}{2}\left(V_1(r) - V_+^{(0)}(r)\right) - \tfrac{1}{2}\bar{E}^{1/2} V_+^{(1)}(r)$$

$$\bar{E}^{1/2} V_-^{(1)}(r) = -\tfrac{1}{2}\left(V_-(r) - V_{-1}(r)\right) = \tfrac{1}{2}\left(V_{-1}(r) - V_-^{(0)}(r)\right) - \tfrac{1}{2}\bar{E}^{1/2} V_-^{(\ )}(r)$$

We must calculate $V_+(r)$ and $V_-(r)$ in terms of the prescribed velocities $V_1(r)$ and $V_{-1}(r)$. We can obtain a pair of linear equations by using the above relations and the equation relating $V_\pm^{(1)}(\ )$ to the pressure:

$$-(k_n/a)^2 V_\pm^{(1)}(n) = \begin{pmatrix}\alpha_n \\ \beta_n\end{pmatrix} \frac{\mu^2 (k_n\delta/a)^3 (1-\sigma s)}{2\delta^2(\sigma s)} \sinh \frac{\delta\mu^2 k_n}{a} \sinh \frac{\delta k_n}{a}$$

where $V_\pm^{(1)}(n)$ is the Bessel transform of $V_\pm^{(1)}(r)$ and

$$\alpha_n = \left[\frac{2a}{k_n}\right] \left\{ \frac{V_-^{(\ )}(n)\,\phi_n(0) - V_+^{(0)}(n)\,\phi_n(r)}{\phi_n^2(0) - \phi_n^2(\ )} \right.$$

$$\beta_n = \frac{2a}{k_n} \left\{ \frac{V_+^{(0)}(n)\,\phi_n(0) - V_-^{(0)}(n)\,\phi_n(\ )}{\phi_n^2(0) - \phi_n^2(\ )} \right.$$

and $\phi_n(z) = \mu^2 \sinh \frac{\delta\mu^2 k_n}{a} \cosh \delta k_n z/a - \sinh k_n\delta/a \cosh \delta\mu^2 k_n z/a$

Thus $$V_+^{(1)}(n) = \frac{(\sigma s - 1)}{2\sigma s} \mu^2 \left(\frac{k_n}{a}\delta\right) \sinh \frac{k_n}{a} \sinh \frac{k_n\delta\mu^2}{a} \left(\frac{2a}{k_n}\right) \frac{V_+^{(0)}(n)\,\phi_n(1) - V_-^{(0)}(n)\,\phi_n(0)}{\phi_n^2(1) - \phi_n^2(0)}$$

$$V_-^{(1)}(n) = \frac{(\sigma s - 1)}{2\sigma s} \mu^2 \frac{k_n\delta}{a} \sinh \frac{k_n\delta}{a} \sinh \frac{k_n\delta\mu^2}{a} \left(\frac{2a}{k_n}\right) \frac{V_-^{(0)}(n)\,\phi_n(1) - V_+^{(0)}(n)\,\phi_n(0)}{\phi_n^2(1) - \phi_n^2(0)}$$

Thus we have two pairs of linear equations for the Bessel transform of $V_\pm^{(1)}(r)$

$$\tfrac{3}{2}\bar{E}^{1/2} V_+^{(1)}(n) = \tfrac{1}{2}\left[V_1(n) - V_+^{(0)}(n)\right] = \tfrac{3}{2}\bar{E}^{1/2} K_n \cdot \frac{V_+^{(0)}(n)\,\phi_n(1) - V_-^{(0)}(n)\,\phi_n(0)}{\phi_n^2(1) - \phi_n^2(0)}$$

$$\tfrac{3}{2}\bar{E}^{1/2} V_-^{(1)}(n) = \tfrac{1}{2}\left[V_{-1}(n) - V_-^{(0)}(n)\right] = \tfrac{3}{2}\bar{E}^{1/2} K_n \cdot \frac{V_-^{(0)}(n)\,\phi_n(1) - V_+^{(0)}(n)\,\phi_n(0)}{\phi_n^2(1) - \phi_n^2(0)}$$

where $K_n = \left(\frac{\sigma s - 1}{\sigma s}\right) \delta\mu^2 \sinh \frac{k_n\delta}{a} \sinh \frac{k_n\delta\mu^2}{a}$. Written as a matrix equation this becomes

$$\begin{pmatrix} 1 + \dfrac{3\bar{E}^{1/2} K_n \phi_n(1)}{\phi_n^2(1) - \phi_n^2(0)} & -\dfrac{3\bar{E}^{1/2} K_n \phi_n(0)}{\phi_n^2(1) - \phi_n^2(0)} \\ -\dfrac{3\bar{E}^{1/2} K_n \phi_n(0)}{\phi_n^2(1) - \phi_n^2(0)} & 1 + \dfrac{3\bar{E}^{1/2} K_n \phi_n(1)}{\phi_n^2(1) - \phi_n^2(0)} \end{pmatrix} \begin{pmatrix} V_+^{(0)}(n) \\ V_-^{(0)}(n) \end{pmatrix} = \begin{pmatrix} V_1(n) \\ V_{-1}(n) \end{pmatrix}$$

Retaining only the terms to $O(\bar{E}^{1/2})$ we have

$$V_+^{(0)}(n) = V_1(n) - 3\bar{E}^{1/2} K_n \left\{ \frac{\phi_n(1) V_1(n) - \phi_n(0) V_{-1}(n)}{\phi_n^2(1) - \phi_n^2(0)} \right\}$$

$$V_-^{(0)}(n) = V_{-1}(n) - 3\bar{E}^{1/2} K_n \frac{\phi_n(1) V_{-1}(n) - \phi_n(0) V_1(n)}{\phi_n^2(1) - \phi_n^2(0)}$$

and using the relations between $V_\pm^{(1)}(n)$ and $V_\pm^{(0)}(n)$ we can write

$$V_+(n) = V_+^{(0)}(n) + \bar{E}^{1/2} V_+^{(1)}(n) = V_1(n) - 2\bar{E}^{1/2} K_n \left\{ \frac{\phi_n(1) V_1(n) - \phi_n(0) V_{-1}(n)}{\phi_n^2(1) - \phi_n^2(0)} \right\}$$

$$V_-(n) = V_-^{(0)}(n) + \bar{E}^{1/2} V_-^{(1)}(n) = V_{-1}(n) - 2\bar{E}^{1/2} K_n \left\{ \frac{\phi_n(1) V_{-1}(n) - \phi_n(0) V_1(n)}{\phi_n^2(1) - \phi_n^2(0)} \right.$$

Simplifying the notation,

$$\bar{E}^{1/2} K_n \frac{\phi_n(1,0)}{\phi_n^2(1) - \phi_n^2(0)} = \bar{E}^{1/2} \delta \left(\frac{\sigma S - 1}{\sigma S} \right) \mu^2 \sinh \frac{\delta k_n \mu^2}{a} \sinh \frac{\delta k_n}{a} \frac{\phi_n(1,0)}{\phi_n^2(1) - \phi_n^2(0)}$$

$$= E^{1/2} \cdot a_{1,0}(n) \qquad \text{III. 1}$$

we can now write

$$V_+(n) = V_1(n) - 2E^{1/2} \left[a_1(n) V_1(n) - a_0(n) V_{-1}(n) \right] \qquad \text{III. 2}$$

$$V_-(n) = V_{-1}(n) - 2E^{1/2} \left[a_1(n) V_{-1}(n) - a_0(n) V_1(n) \right]. \qquad \text{III. 3}$$

IV. The purpose of the calculation has been to show that the homogeneous layers of fluid are shielded from each other by the stratified layer in between. The interior velocity field in the homogeneous fluid is given as the average of the top and bottom velocities. Thus we have (Eqn's. II. 1, III. 2, 3)

$$(\text{upper})\ v(r) = \tfrac{1}{2}\left[V_1(r) + V_+(r) \right] = V_1(r) - E^{1/2} \{ \text{terms in } V_1 \text{ and } V_{-1} \}$$

$$(\text{lower})\ v(r) = \tfrac{1}{2}\left[V_-(r) + V_{-1}(r) \right] = V_{-1}(r) - E^{1/2} \{ \text{terms in } V_{-1} \text{ and } V_1 \}.$$

It is clear that the upper layer knows only to $O(E^{1/2})$ what the lower layer is doing. This is precisely what we set out to show.

An interesting problem that arises from this work is that of trying to formulate general jump conditions for the velocities between two layers of homogeneous fluid, supposed to be separated by a thin layer of stratified fluid. It might be expected that these conditions are independent of many of the exact details of the stratified layer as long as the layer is thicker than the Ekman depth.

In view of the rather complicated dependence upon δ in Eqn. III. 1, the jump conditions are not easily found. These questions must be left to further investigations.

Acknowledgment

I want to thank Prof. L. N. Howard for many stimulating conversations about this problem out of which came much of the solution.

Appendix: One question still remains: What happens to the interfaces between the layers of homogeneous and stratified fluids? To answer this question we assume that there is a slight deflection of the interface $\zeta(r)$. We must look at the matching of the vertical components of the stress across the interface. Let $z = \delta + \zeta(r)$, then the vertical component of stress is

$$p_*(r\delta+\zeta(r)) + \frac{2\nu\rho U}{L}\frac{1}{r}\left(rU_+(r)\right)_r$$

in both the homogeneous and stratified layers. $U_+(r)$ is the same in both layers so that the continuity of stress implies the matching of the dimensional pressure. In the homogeneous layer

$$p_*(r\delta+\zeta(r)) = \rho_H UL\Omega\, p^H(r\delta+\zeta(r)) - \rho_H g\,\zeta(r) \cong \rho_H UL\Omega\, p^H(r\delta) - \rho_H g\,\zeta(r)$$

and in the stratified layer

$$p_*(r\delta+\zeta(r)) = \rho_S U\Omega L\, p^S(r\delta+\zeta(r)) - \rho_S g\,\zeta(r) \cong \rho_S U\Omega L\, p^S(r\delta) - \rho_S g\,\zeta(r)$$

Equating these we have

$$\rho_s U\Omega L p^s(r\delta) - \rho_H U\Omega L p^H(r\delta) \cong (\rho_s - \rho_H) g \zeta(r)$$

On the left-hand side we can neglect the difference in density between the layers so that

$$\Delta p(r\delta) \equiv \frac{g}{U\Omega}\left(\frac{\Delta\rho}{\rho}\right)\left(\zeta(r)/L\right)$$

Using the Froude number $F = \frac{\Omega^2 L}{g}$ and the Rossby number $\epsilon = U/L\Omega$ we have

$$\Delta p = \frac{1}{F\cdot\epsilon}\left(\frac{\Delta\rho}{\rho}\right)\left(\frac{\zeta(r)}{L}\right)$$

$\Delta p(r)$ is of the order of the difference of the interior velocity in the homogeneous and stratified layers and is of order $E^{1/2}$. Thus

$$\left(\frac{\Delta\rho}{\rho}\right)\left(\frac{\zeta(r)}{L}\right) \cong F\cdot\epsilon\cdot E^{1/2}$$

We have throughout assumed that both F and ϵ are $\ll E$ so that even a very small $\left(O(E^{1/2})\right)$ density jump at the interface will not require a large distortion of the interface.

References

V. Barcilon and J. Pedlosky (1967) J.Fluid Mechanics 29: 1 ; 29: 609.

CROSS-STREAM FLOWS IN CONTINUOUS f-PLANE FRONTAL MODELS, WITH APPLICATION TO COASTAL UPWELLING FRONTS

Christopher N. K. Mooers

Oceanographic Motivation

Due to alongshore (equatorward) winds, coastal upwelling is generally the predominant coastal phenomenon occurring at subtropical latitudes in the eastern boundary current regimes of the world ocean during the summer season, Wooster and Reid (1963). Observations off Oregon indicate (Fig. 1) that:

(i) the "surface layer" flows equatorward and offshore, while the "lower layer" flows poleward and onshore;

(ii) a strong, quasi-steady frontal structure, viz. an inclined frontal layer (pycnocline), exists in the vertical plane normal to the coastline;

(iii) the creation of a temperature inversion at the base of the frontal layer close to the coast, Pattullo and McAlister (1962), and the maintenance of the upwelled frontal layer on a seasonal time scale, are evidence for significant cross-stream flow and mixing in the coastal region.

Currie (1953) reports similar patterns off Southwest Africa.

Most hydrodynamical studies of coastal upwelling phenomena have employed layered models, with the exception of Latun (1962) and Leetma (1968) who have used continuous models with linearized perturbation density fields. With the use of layered models, Yoshida (1967) has been successful in explaining qualitatively and quantitatively many of the observed characteristics of coastal upwelling, but layered models are of limited value for the detailed study of the transverse (cross-stream) circulation and mixing. (The importance of the Gulf Stream's cross-stream flow, and its frontal character, has been recognized by Neumann (1952).) The present study proceeds with the assumptions that the non-linear nature of the mass diffusion equation and the continuous structure of the hydrodynamic fields are essential in understanding cross-stream flow. The nature of the eddy diffusion processes is crucial in the models considered in this paper. Since we do not know proper values for the eddy diffusivities and viscosities, nor their functional forms in the likely event that they are not constant, any conclusions that can be drawn are of limited physical value except to give guidance to further measurement.

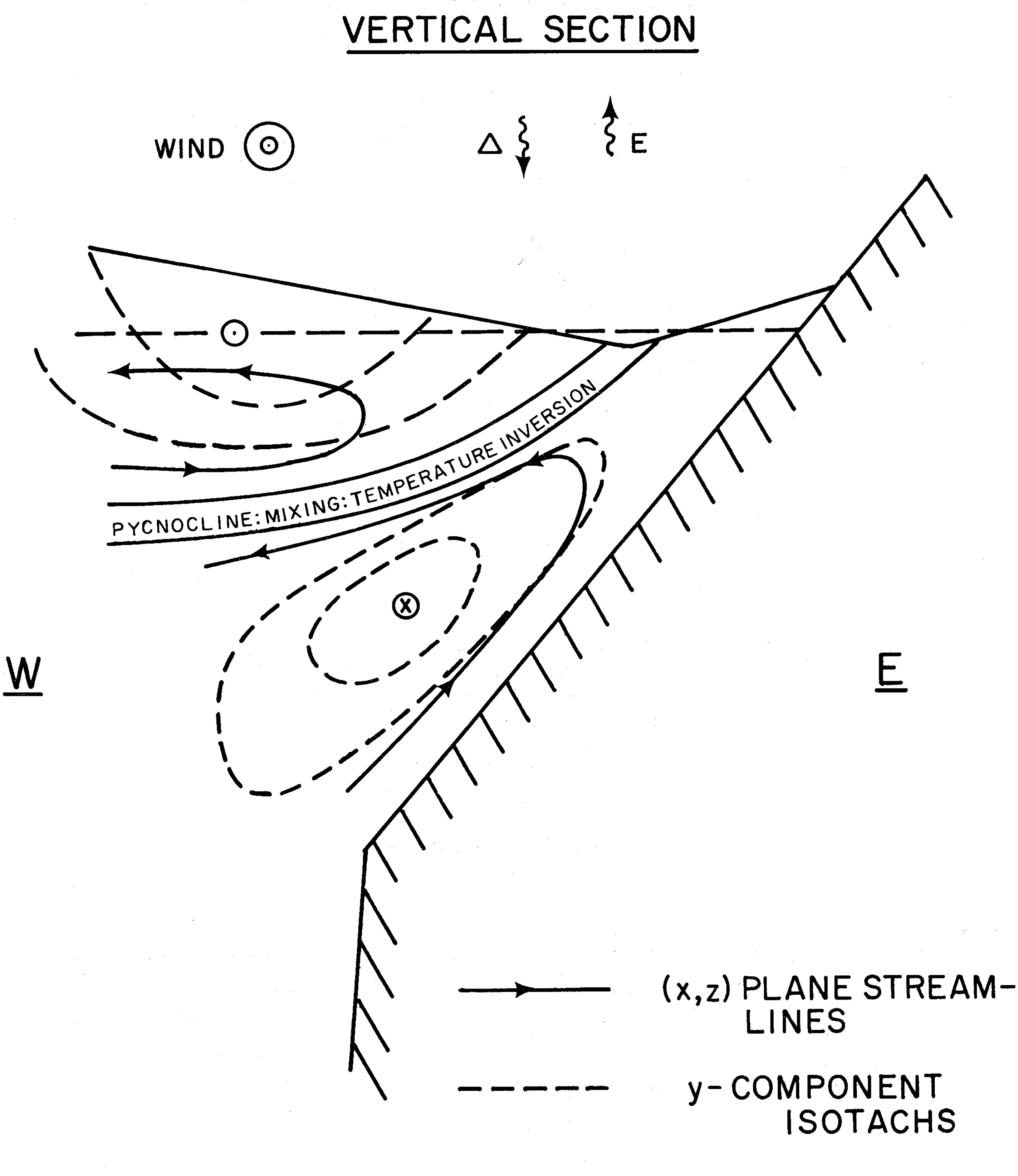

VERTICAL SECTION
WIND
Δ
E
PYCNOCLINE: MIXING: TEMPERATURE INVERSION
W
E
(x,z) PLANE STREAM-LINES
y-COMPONENT ISOTACHS

Figure 1

FRONTAL VARIETIES and VORTICITY

(LOOKING TO THE NORTH; EAST TO THE RIGHT)

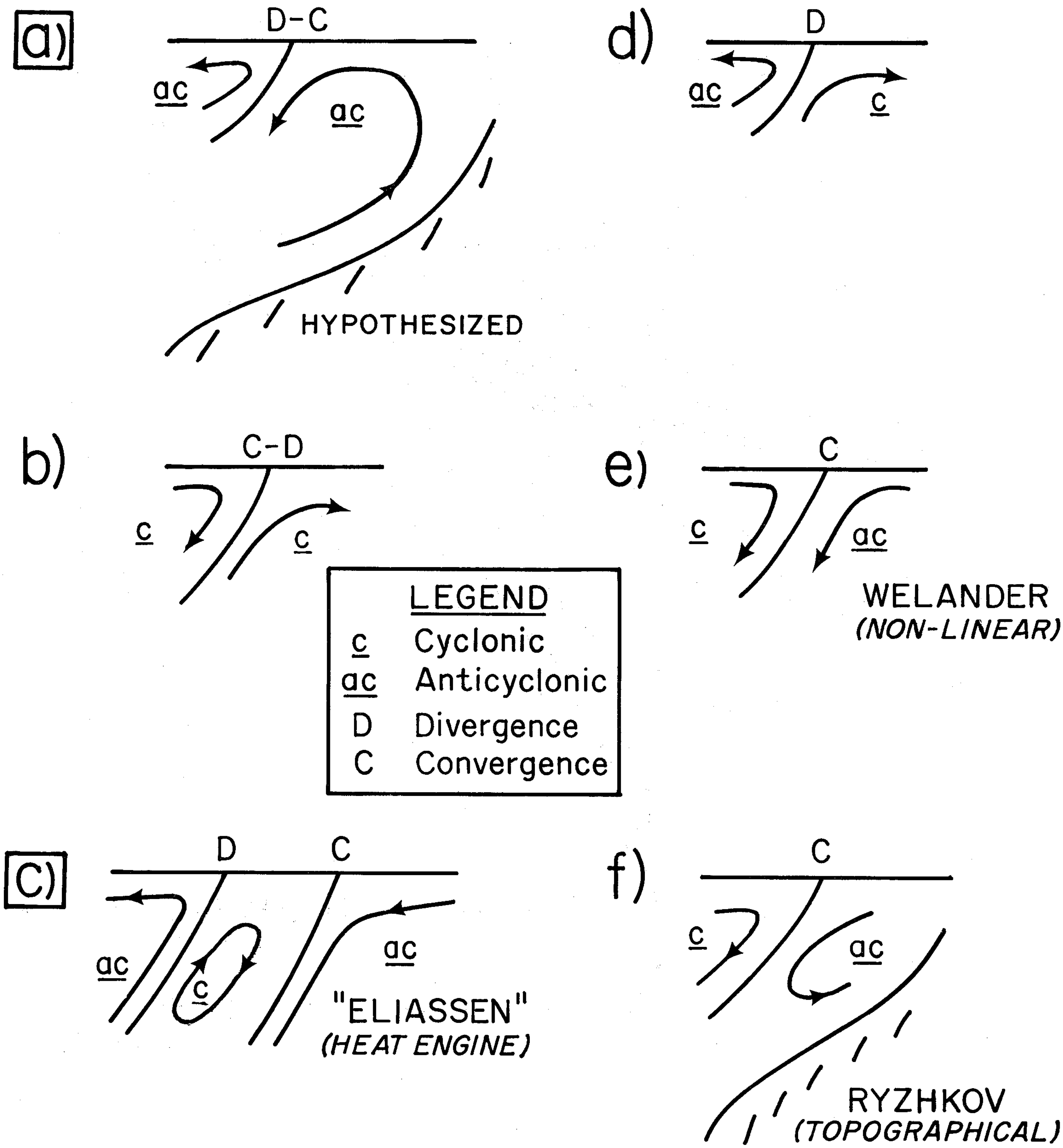


Figure 2

There are several competing mechanisms in the problem considered:

(i) effects of wind stress (spatial structure of the wind field can play a crucial role);

(ii) effects of stratification (the density field is assumed derived from an open ocean reservoir and modified in the coastal region prior to return to the reservoir);

(iii) effects of turbulent friction and mixing (vertical but not lateral diffusion of mass and momentum is included; the choice is rather arbitrary, but it does admit turbulent processes);

(iv) effects of rotation (cases of uniform rotation and non-uniform rotation alter the dynamics of the problem significantly).

The x, y, z, coordinates are the standard oceanographic right-handed coordinates. A straight coastline of infinite extent and parallel to the y-axis is assumed. All time-dependent effects are neglected. The hydrostatic and Boussinesq approximations are used. Frictional, inertial, and frictional-inertial frontal cases are considered, each with an isentropic subcase. Heat and mass transfers at the free surface are neglected. The objective of this study is to explore the formulation of the physical problem in several stages of complexity in order to learn about its analytical structure, and in order to determine how to proceed further numerically and observationally. Most of this study only has relevance to f-plane models, or equivalently, relatively small horizontal scales, of the order of 10 kilometers, with a vertical scale of the order of 100 meters.

One could view these studies as frontal models which are somewhat different from those usually considered in the ocean and atmosphere. In Fig. 2, several frontal flow patterns are shown. Fig. 2a shows the pattern hypothesized from observations in advance of this study. Welander (1963) used a surface Ekman layer plus a non-linear advective frontal boundary layer to explain the pattern of Fig. 2e. Eliassen (1959) used a surface Ekman layer plus a heat engine, located below the tropopause, and operating on the latent heat of evaporation, to drive the atmospheric equivalent to the frontal regime of Fig. 2c. Ryzhkov and Koleshnikov (1963) studied the effects of variable topography on frontal flows in a stratified fluid which had the sturcture shown in Fig. 2f. The dynamical studies of Pettersson and Austin (1942) indicated the significance of $\zeta^{(y)}$, the horizontal component of vorticity parallel to fronts, which is why its sense has been indicated in Fig. 2.

General formulation and reduction

The complete coastal upwelling problem requires a β-plane model. The analysis commences with the heuristic formulation, which can be supported formally, of a fully consistent β-plane model; the problem is then reduced to solving several approximate (f-plane) problems exactly in order to understand qualitatively something of the larger problem. The "lesser" problems are of some intrinsic theoretical interest too because they emphasize the frontal, or small scale, aspect of the complete problem.

A stratified, Boussinesq, gravitating, rotating, turbulent, and incompressible fluid is considered. First, the alongshore flow is assumed highly geostrophic, but frictional drive from the onshore component of the wind and frictional drag from the lateral boundary may also be significant:

$$-fv = -\frac{p_x}{\bar{\rho}} + (N_V u_z)_z + (N_H u_x)_x . \qquad \text{(i)}$$

Second, it is assumed that the cross-stream (onshore-offshore) flow is only weakly geostrophic, that the same frictional terms may be significant, and that the inertial terms may contribute:

$$u v_x + v v_y + w v_z + f u = -\frac{p_y}{\bar{\rho}} + (N_V v_z)_z + (N_H v_x)_x \qquad \text{(ii)}$$

Third, the flow is assumed hydrostatic:

$$0 = -p_z - \rho g . \qquad \text{(iii)}$$

Fourth, the three-dimensional equation of continuity is used:

$$u_x + v_y + w_z = 0 . \qquad \text{(iv)}$$

Fifth, the non-linear mass diffusion equation is assumed to apply:

$$u \rho_x + v \rho_y + w \rho_z = (K_V \rho_z)_z + (K_H \rho_x)_x . \qquad \text{(v)}$$

This system of equations is very similar to the system considered by Veronis (1960) and Robinson (1960) for the Cromwell Current. As noted by Yoshida (1959), the basis of the analogy between (i) equatorial undercurrent and (ii) coastal upwelling is that, in the cross-stream flow, for (i) $fv = 0$ at the equator, since $f = 0$, and, for (ii), $fu = 0$ at the coast, since $u = 0$; another aspect of the analogy is the occurrence of strong vertical mixing in both flow regimes.

Imposition of an Ekman layer(s), yields the minimal, consistent set of equations for the interior, and applicable to the β-plane:

$$-fv = -\frac{p_x}{\bar{\rho}} , \qquad \text{(ia)}$$

$$fu = -\frac{p_y}{\bar{\rho}} + N_V v_{zz} , \qquad \text{(iia)}$$

$$0 = -p_z - \rho g \qquad \text{(iiia)}$$

$$u_x + v_y + w_z = 0, \quad \text{and} \qquad \text{(iva)}$$

$$u \rho_x + v \rho_y + w \rho_z = K_V \rho_{zz} \qquad \text{(va)}$$

with the right-hand side of (va) optional, constant eddy coefficients, neglect of horizontal turbulence, neglect of inertial accelerations in (iia), and neglect of friction in (ia). The problem is still complicated for strictly analytic treatment.

If we are willing to consider problems on the f-plane, uniformity in the y-dimension for all variables can be logically imposed, reducing the full problem to:

$$-fv = -\frac{p_x}{\bar{\rho}} + (N_V u_z)_z + (N_H u_x)_x , \qquad \text{(ib)}$$

$$u v_x + w v_z + fu = (N_V v_z)_z + (N_H v_x)_x , \qquad \text{(iib)}$$

$$0 = -p_z - \rho g , \qquad \text{(iiib)}$$

$$u_x + w_z = 0, \quad \text{and} \qquad \text{(ivb)}$$

$$u \rho_x + w \rho_z = (K_V \rho_z)_z + (K_H \rho_x)_x . \qquad \text{(vb)}$$

Again, neglecting lateral friction, assuming constant eddy coefficients, and imposing a surface Ekman layer,(and potentially a bottom Ekman layer), a minimal, consistent formulation for the interior regime is then:

$$-fv = -\frac{p_x}{\bar{\rho}} , \qquad \text{(ic)}$$

$$u v_x + w v_z + fu = N_V v_{zz} , \qquad \text{(iic)}$$

$$0 = -p_z - \rho g , \qquad \text{(iiic)}$$

$$u_x + w_z = 0 , \quad \text{and} \qquad \text{(ivc)}$$

$$u \rho_x + w \rho_z = K_V \rho_{zz} . \qquad \text{(vc)}$$

This system of equations forms the physical basis for the subsequent discussion of f-plane fronts. The entire system governs what is defined to be inertial-frictional fronts; reduced systems will lead to inertial and frictional fronts.

The vertical component of vorticity is a central property in the problem, which includes the effects of rotational wind fields, variable topography, stratification, and non-uniform rotation. From the full system of equations on the β-plane, (i), (ii), and (iv) yield the approximate equation for the vertical component of vorticity, $\zeta^{(z)} = v_x - u_y$,

$$w_x v_z + \vec{v}\cdot\nabla v_x - (f+v_x)w_z + \beta v = N_V \zeta^{(z)}_{zz} + N_H \zeta^{(z)}$$

where constant eddy viscosity has been assumed for convenience. Neglecting lateral friction and non-linear terms, we have the approximate relation which is essential for determining the limits of applicability for the f-plane models:

$$-f w_z + \beta v \approx N_V \zeta^{(z)}_{zz}.$$

If frictional effects are to dominate the β-effect, it follows that

$$N_V \zeta_{zz} \approx N_V v_{xzz} \gg \beta V \quad \text{or} \quad \frac{N_V}{L D^2} \gg \beta,$$

where L and D are the horizontal and vertical scales, respectively. For example,

let $N_V = 20\ \mathrm{cm}^2/\mathrm{sec}$ and $D = 10^4\,\mathrm{cm}$ (pycnocline thickness),

then, since $\beta = 2 \times 10^{-13}\ \mathrm{cm}^{-1}\mathrm{sec}^{-1}$,

$$L \ll \frac{N_V}{\beta D^2} = 10^6\,\mathrm{cm}.$$

Conservative estimates for N_V and D have been made for two reasons: (i) we are ignorant of the proper value for N_V and (ii) it is important to realize that β-effects are probably significant in the oceanographic problem considered prior to neglecting them.

The boundary conditions realizable in the "lesser" problems differ from those desired, of course, due to the lower order dynamics accepted. Satisfaction of the following boundary conditions is desired:

<u>at sea surface</u> $(z = h)$

$$\tau^{w}_{(x)} = N_v u_z, \; \tau^{w}_{(y)} = N_v v_z, \quad w = 0;$$

$\vec{\tau}^{w}$ and ρ arbitrarily specified.

<u>at sea bottom</u> $(z = -D(x))$ (may be variable depth)

$$u = v = w = 0.$$

<u>at coast</u> $(x = L)$

$$u^{*} = v = 0.$$

<u>at boundary to open ocean reservoir</u> $(x = x_0)$

ρ arbitrary function of depth.

(*In the β-plane models, we may have coastal convergence of flow which provides an efflux or influx to the alongshore flow so that this is the appropriate boundary condition; in the f-plane models, the equivalent boundary condition is $m_x \equiv \int_{-D(x)}^{h} u \, dz = 0$ for all x.)

There are several observed features which a fully adequate model would reproduce:

(i) the dimensions of the flow are expected to be of the order of 100 kilometers wide, 0.1 to 1 kilometer deep, and several megameters long;

(ii) an inclined pycnocline(s), or frontal surface, which rises to the surface at about 10 kilometers offshore from a depth of about 125 meters in the deep ocean;

(iii) a reversal in the alongshore flow near the base of the pycnocline, equatorward above and poleward below; the subsurface countercurrent will be referred to as the undercurrent;

(iv) cross-stream flow which may have one or more reversals as a function of depth; the reversals are expected to be connected with the details of the pycnocline structure.

The dependence of these features on the magnitude, sense, and spatial structure of the wind field should be evidenced. It would be instructive to know what qualitative and quantitative effects the various processes have on the structure of frontal regimes.

Several Classes of f-plane Fronts

There are three major frontal classes within our basic system of equations for the f-plane: inertial-frictional, and its two sub-classes, inertial and frictional, cross-stream flows. In each class, there are two further sub-cases: isentropic and non-isentropic, i.e. with mixing and without mixing, respectively. Though only solutions for the frictional case will be examined extensively, the formulations of the other cases are derived, classified, described, etc., since they are intrinsically interesting and since they will have ranges of applicability depending primarily upon the relative values of the eddy coefficients, or eddy Prandtl number.

Non-dimensionalization and scale analysis

In order to clarify the range of applicability of the various types of fronts, the system (ic) - (vc) is non-dimensionalized and scaled:

let
$$u = \mathcal{U} u'$$
$$v = \mathcal{V} v'$$
$$w = \mathcal{W} w'$$
$$p = \mathcal{P} p'$$
$$\rho = \bar{\rho} + \Delta\rho\, \rho' ,$$

D: vertical scale, and L: horizontal scale,

then $\mathcal{W} = \frac{D}{L}\mathcal{U}$, from (ivc)

$P = \Delta\rho g D$, from (iiic)

$\mathcal{V} = \frac{\Delta\rho g D}{\bar{\rho} f L}$, from (ic),

Dropping the primes, the system of equations becomes,

$$V = p_x, \tag{i}$$

$$\gamma\,(u V_x + w V_z) + u = \delta V_{zz}, \tag{ii}$$

$$0 = -p_z - \rho, \tag{iii}$$

$$u_x + w_z = 0, \text{ and} \tag{iv}$$

$$\epsilon\,(u\rho_x + w\rho_z) = \rho_{zz}, \tag{v}$$

where

$$\gamma \equiv \frac{\mathcal{V}}{Lf} = \left(\frac{\Delta\rho}{\bar{\rho}}\ \frac{g}{f^2 L}\right)\left(\frac{D}{L}\right),$$

$$\delta \equiv \frac{\mathcal{V} N_v}{\mathcal{U} f D^2} = \left(\frac{\Delta\rho}{\bar{\rho}}\ \frac{g}{f^2 L}\right)\left(\frac{N_v}{D\mathcal{U}}\right), \text{ and}$$

$$\epsilon \equiv \frac{\mathcal{U} D^2}{K_v L} = \left(\frac{D\mathcal{U}}{K_v}\right)\left(\frac{D}{L}\right).$$

The non-dimensional physical parameters of importance are

$\frac{D}{L}$, the aspect ratio $\sim$ frontal slope,

$\frac{\Delta\rho}{\bar{\rho}}\ \frac{g}{f^2 L}$, the internal Froude number,

$\frac{D\mathcal{U}}{N_v}$, the eddy Reynolds number, and

$\sigma \equiv \frac{N_v}{K_v}$, the eddy Prandtl number.

For inertial-frictional fronts $O(\gamma) = O(\delta) = O(\epsilon) = 1$,

so $$\mathcal{U} = \frac{\Delta\rho}{\bar{\rho}}\ \frac{g}{f^2 L}\ \frac{N_v}{D}$$

and $\epsilon = \sigma \left(\frac{\Delta\rho}{\bar\rho} \frac{g}{f^2 L}\right)\left(\frac{D}{L}\right) = \sigma\gamma$,

or $O(\sigma) = 1$; ϵ can also be written as $\left(\frac{D}{L_D}\right)^2$, where L_D is the Lineykin depth.

If $O(\gamma) = 1$ but $O(\delta) << 1$, then $\frac{D^2}{L} >> \frac{N_v}{\mathcal{V}}$, so, if $O(\epsilon) = 1$, then $O(\sigma) << 1$; thus, inertial fronts occur for small eddy Prandtl numbers only. If $O(\delta) = 1$ but $O(\gamma) << 1$, then $O(\sigma)$ (must) $>> 1$ if $O(\epsilon) = 1$; thus, frictional fronts occur for large eddy Prandtl numbers only. The physically most interesting cases correspond to $O(\sigma) = 1$, i.e., to inertial-frictional fronts. Most of the results of this study pertain to the frictional fronts, which is one of the "lesser of the lesser" problems. (The neglect of the term $N_v u_{zz}$ in simplifying from (ib) to (ic) can now be rationalized by comparing $N_v u_{zz}$ to fv:

$$O(N_v u_{zz}/f_v) = \left(\frac{N_v}{D^2 f}\right)^2 = 4 \times 10^{-6}.)$$

Inertial-frictional fronts $(O(\sigma) = 1)$

The system of equations was formulated in the previous part. Since a two-dimensional stream function exists, equations (iic) and (vc) can be rewritten in dimensionful form:

$\nabla(\ )$ is the x, z gradient operator,

$$\nabla v \times \nabla\psi - f\psi_z = N_v v_{zz} \qquad (ii)'$$

and

$$\nabla\rho \times \nabla\psi = K_v \rho_{zz} \qquad (v)'$$

And, with (ic) and (iiic), v and ρ are expressed as functions of p so that, with $A = \bar\rho f^2/2$,

$$\nabla p_x \times \nabla\psi - 2A\psi_z = N_v p_{xzz} \qquad (ii)''$$

and

$$\nabla p_z \times \nabla\psi = K_v p_{zzz} . \qquad (v)''$$

(ii)'' and (v)'' are a system of two third-order, quasi-linear partial differential equations in two dependent and independent variables; they are both third-order in p only and are thus parabolic. They can be rewritten in expanded form to facilitate the next step:

$$p_{xx}\psi_z - p_{xz}\psi_x - 2A\psi_z = N_v p_{xzz} \qquad \text{(ii)'''}$$

and

$$p_{xz}\psi_z - p_{zz}\psi_x = K_v p_{zzz} \,. \qquad \text{(v)'''}$$

Cross-differentiating (ii)''' and (v)''' , we have

$$(p_{xx}\psi_z - p_{xz}\psi_x)_z - 2A\psi_{zz} - (p_{xz}\psi_z - p_{zz}\psi_x)_x = (N_v - K_v)p_{xzz}$$

or, if $N_v = K_v$,

$$2p_{xz}\psi_{xz} - (p_{xx}\psi_{zz} + p_{zz}\psi_{xx}) + 2A\psi_{zz} = 0;$$

reinterpreting, we have

$$2(\bar{\rho}v_z f)w_z - [(\bar{\rho}v_x f)(-u_z) + (-g\rho_z)(w_x)] + 2A(-u_z) = 0$$

so,

$$2(v_z w_z - f u_z) + (v_x + f)u_z - \frac{E}{f}w_x = 0,$$

which is a vorticity equation.

Anticipating the type of boundary conditions desired, e.g., p and ψ specified at $x = x_0$, and assuming a power law dependence in x for p and ψ , it is found that, with

$$p = x^2 g(z) \text{ and } \quad \psi = x\,h(z) \quad \text{then for } N_v \neq K_v$$

$$gh' - g'h - Ah' = N_v g'' \qquad \text{(ii)}^{IV}$$

and

$$2g'h' - g''h = K_v g''' \qquad \text{(v)}^{IV}$$

(or,

$$-h^2\left(\frac{(g-A)}{h}\right)' = N_v g'' \qquad \text{(ii)}^{V}$$

and

$$-h^3\left(\frac{g'}{h^2}\right)' = K_v g'''.) \qquad \text{(v)}^{V}$$

We can reduce the system to a single equation:

$$(*)\; N_v K_v g''' - \left[2N_v h' + h^2\right] g' + hh'(g-A) = 0$$

A general way to proceed is to assume a form for g or h and find the other from (*). For example,

if $g = h$, (*) becomes: $N_v K_v g''' - 2N_v (g')^2 - A g g' = 0.$

In the isentropic case $(K_v = 0)$, the governing equations (GE's) become:

$$P_{xx}\Psi_z - P_{xz}\Psi_x - 2A\Psi_z = N_v P_{xzz} \qquad \text{(ii)}^{VI}$$

and

$$P_{xz}\Psi_z - P_{zz}\Psi_x = 0 \qquad \text{(v)}^{VI}$$

Again, let $P = x^2 g(z)$ and $\Psi = x$

so

$$(g-A)h' - g'h = N_v g'' \quad \text{(as before)} \qquad \text{(ii)}^{VII}$$

and

$$2g'h' - g''h = 0 \qquad \text{(v)}^{VII}$$

Then, h can be found as a function of g' from (v)^{VII} and substituted into (ii)^{VII} to produce a single (GE):

since $h^2 = C_o g'$,

and $h' = C_o g''/2h$

then $g''\left[\frac{(g-A)}{2} - N_v\sqrt{\frac{g'}{C_o}}\right] = (g')^2.$

Inertial fronts $(O(\sigma) << 1)$

With $N_v = 0$ but $K_v \neq 0$,

the (GE's) for inertial fronts are in dimensionful form:

$$\nabla V \times \nabla \psi - f \psi_z = 0 \qquad \text{(ii)}^{VIII}$$

and

$$\nabla \rho \times \nabla \psi = K_v \rho_{zz} \qquad \text{(v)}^{VIII}$$

or,

$$\rho_{xx} \psi_z - \rho_{xz} \psi_x - 2A \psi_z = 0 \qquad \text{(ii)}^{IX}$$

and

$$\rho_{xz} \psi_z - \rho_{zz} \psi_x = K_v \rho_{zzz} \qquad \text{(v)}^{IX}$$

Again let $p = x^2 g(z)$ and $\psi = x h(z)$ so that

$$(g - A) h' - g' h = 0 \qquad \text{(ii)}^{X}$$

and

$$2g'h' - g''h = K_v g'''. \qquad \text{(v)}^{X}$$

Thus, $h = C_o (g - A)$

and $h' = C_o g'$ from (ii)^{X}

so

$$2(g')^2 - g''(g - A) = \frac{K_v}{C_o} g''', \qquad \text{(v)}^{XI}$$

or,

$$\frac{K_v}{C_o} g''' + g g'' - 2(g')^2 - A g'' = 0,$$

which is very similar to the Falkner-Skan Equation for laminar boundary layers,

viz.,

$$f''' + f f'' + B(1 - (f')^2) = 0.$$

In the isentropic case, $K_v = 0$ and $\psi = \psi(\rho)$ then the (GE's) are:

$$(g - A) h' - g h = 0, \qquad \text{(ii)}^{XII}$$

so $h = C_o (g - A)$, as above,

and

$$2 g'h' - g''h = 0, \qquad \text{(v)}^{XII}$$

so $h^2 = C_1 g'$, i.e., $\psi = \psi(\rho)$.

Combining the above relations, the (GE) becomes:

$$C_0^2(g-A)^2 = C_1 g',$$

so, $g(z) = A - \frac{1}{\gamma z}$, where $\gamma = \frac{C_0^2}{C_1}$;

and $h(z) = -\frac{C_0}{\gamma z}$.

Thus,

$$\psi = -\frac{(x+x_0)}{\mu(z+z_0)}, \quad \left(\mu = \frac{C_0}{C_1}\right)$$

$$p = (x+x_0)^2\left(A - \frac{1}{\gamma(z+z_0)}\right),$$

$$\rho = -\frac{(x+x_0)^2}{\gamma(z+z_0)^2}, \quad \text{and}$$

$$V = -2(x+x_0)\left(A - \frac{1}{\gamma(z+z_0)}\right).$$

More general results can be found more directly; since $\psi = \psi(\rho)$,

then $\nabla v \times \nabla \rho - f\rho_z = 0$,

so $\nabla p_x \times \nabla p_z = A\, p_{zz}$.

Now the (GE) can be recognized as a Monge-Ampere equation:

$$(\#) \quad p_{xx}\, p_{zz} - (p_{xz})^2 = A\, p_{zz},$$

which is of the general parabolic classification. Not knowing anything further about the general solution to (#), a particular solution is sought of the form:

$p = x^2 Z(z)$, then

$$2ZZ'' - 4(Z')^2 = AZ''$$

or

$$Z = \tfrac{1}{2}\left[A + \frac{1}{C_0 z + C_1}\right].$$

Thus,

$$p = \frac{x^2}{2}\left[A + \frac{1}{C_0 z + C_1}\right] \quad \text{, as before,}$$

but, since $\rho \propto p_z$, then $\psi = F\left(\frac{x^2}{(C_0 z + C_1)^2}\right)$, where F is arbitrary. The earlier solution corresponds to $F(\cdot) = (\cdot)^{1/2}$. Other solutions can be found by allowing $P = p_x$ and $Q = p_z$, then the (GE) becomes

$$Q_z = P_z^2/(P_x - A).$$

One way to proceed is to assume a form for P, find Q and P, and check to ensure that the solution is valid. For example,

let $\quad P = ax + bz$

so $\quad Q_z = \dfrac{b^2}{a-A}$,

then $\quad Q = \dfrac{b^2 z}{a-A} + f(x)$.

Solving for p

$$p = \int P\,dx = \frac{ax^2}{2} + bxz + g(z)$$

and

$$p = \int Q\,dz = \frac{b^2}{a-A}\,\frac{z^2}{2} + zf(x) + h(x),$$

then

$$p = \frac{ax^2}{2} + bxz + \frac{b^2}{(a-A)}\,\frac{z^2}{2},$$

$$\rho = -\frac{b}{g}\left[x + \frac{b}{(a-A)}z\right], \text{ and}$$

$$v = (\tilde{\rho} f)^{-1}(ax + bz).$$

Frictional Fronts $(O(\sigma) >> 1)$

General structure of problem

This case will be studied extensively; it corresponds to: $\sigma >> 1$ or $\Upsilon << 1$. Thus, the dimensionless equations in the interior are:

$$-v = -p_x, \qquad \text{(id)}$$

$$u = \delta v_{zz}, \qquad \text{(iid)}$$

$$0 = -p_z - \rho, \qquad \text{(iiid)}$$

$$u_x + w_z = 0, \text{ and} \qquad \text{(ivd)}$$

$$\epsilon\left[u\rho_x + w\rho_z\right] = \rho_{zz}. \qquad \text{(vd)}$$

From (id) and (iiid), the thermal wind relation holds, viz., $v_z = -\rho_x$.
With the thermal wind relation, we have from (iid) that $u = -\delta\rho_{xz}$; then, from (4) we have that

$$\psi = \delta\rho_x + G(x)$$

and

$$w = \delta\rho_{xx} + G'(x) \,,$$

where $G(x)$ is a function of integration to be determined from boundary conditions. Substituting the values for u and w into (v), we then have the (GE) written in terms of a single dependent variable, ρ :

$$\text{(GE)}: \quad \alpha\left[\rho_{xx}\rho_z - \rho_{xz}\rho_x\right] + G'(x)\rho_z = \rho_{zz} \,,$$

$$\text{where } \alpha = \delta\epsilon = \left(\frac{\Delta\rho g}{\bar{\rho} f^2 L}\right)\sigma\left(\frac{D}{L}\right).$$

The (GE) has a Jacobian, $J\left(\frac{\cdot\,,\,\cdot}{x,z}\right)$, structure: $\alpha J(\rho_x, \rho) + G'(x)\rho_z = \rho_{zz}$.

It can be noted that (Fig. 3):

1) $v_z \sim -\rho_x$, so ρ_x must change sign if an undercurrent is to occur, i.e., there must be downwarping in the lower layer as well as upwarping of isopycnals in the upper layer; if the undercurrent is a maximum near the base of the pycnocline, a horizontal isopycnal must exist there.
2) $u \sim -\rho_{xz}$ and $w \propto \rho_{xx}$, so they'll change signs across a frontal surface.
3) $\psi \sim \rho_x$, so it will have an extremum in an inclined frontal layer.
4) $\zeta^{(y)} = \nabla^2\psi \propto (\rho_{xx} + \rho_{zz})_x \neq 0$, so the y-component of vorticity will have an extremum near an inclined frontal layer.

The (GE) can be rendered into various physical forms which are of some qualitative significance. Let $s = -\frac{\rho_x}{\rho_z}$ be the slope of an isopycnal and $E_0 = \frac{-g\Delta\rho}{D\bar{\rho}}$ be the scale static stability, then, dividing through by $(\rho_z)^2$, we have, first, that

$$(\text{GE}): \quad \alpha \left(\frac{\rho_x}{\rho_z}\right)_x + \frac{G'(x)}{\rho_z} = \frac{\rho_{zz}}{(\rho_z)^2} = -\left(\frac{1}{\rho_z}\right)_z ,$$

and, second, that

$$(\text{GE}): \quad \alpha S_x - \frac{E_o G'(x)}{E} = \frac{E_o E_z}{E^2} .$$

Neglecting the $G'(x)$ term, we note that the horizontal variation in the slope of isopycnals is balanced by vertical diffusion of mass. Above a pycnoline, $E_z < 0$ so $S_x < 0$ i.e., the isopycnals become less positively sloped in the positive x-direction --- they tend to be downwarped. Figure 4 shows the consequent tendency of a pycnocline and of vertical friction and diffusion to produce elevation and focusing of isopycnals. This relation leads to an integral property:

since, in a domain D, the (GE) can be written in divergence form $\nabla \cdot \vec{G} = 0$, for $(x,z) \varepsilon D$, where $G \equiv (\alpha S, E_o E^{-1})$, then $\int_\Gamma \vec{G} \cdot d\vec{r} = 0$, where Γ is the boundary of D and $d\vec{r}$ is an element of arc length. If we assume D is a rectangular domain parallel to the (x, z) axes,

$$\alpha \int_{z_B}^{z_T} (S_I - S_o)\, dz + E_o \int_{x_I}^{x_o} \left(\frac{1}{E_T} - \frac{1}{E_B}\right) dx = 0,$$

where $z = z_T$ is the top surface, $z = z_B$ is the bottom surface, $x = x_o$ is the offshore boundary, and $x = x_I$ is the inshore boundary. In words, the integral relation states that the difference in vertical averages of isopycnal slopes taken along bounding verticals is proportional to the difference in horizontal averages of reciprocal static stability taken along bounding horizontals.

Another frontal property can be used to relate S and E ; it is the frontal Richardson Number; written in terms of dimensionful variables:

$$\text{FRN} \equiv \frac{E}{(V_z)^2} = \frac{f^2}{S^2 E} , \text{ and then}$$

$$(\ln E)_z = \frac{E_z}{E} = \alpha \frac{E}{E_o} S_x = \left(\frac{\sigma}{\text{FRN}}\right)\left(\frac{S_x}{S^2}\right).$$

The (GE) is a second-order, quasi-linear partial differential equation with two independent variables and one dependent variable. If it can be

FRONTAL FLOWS

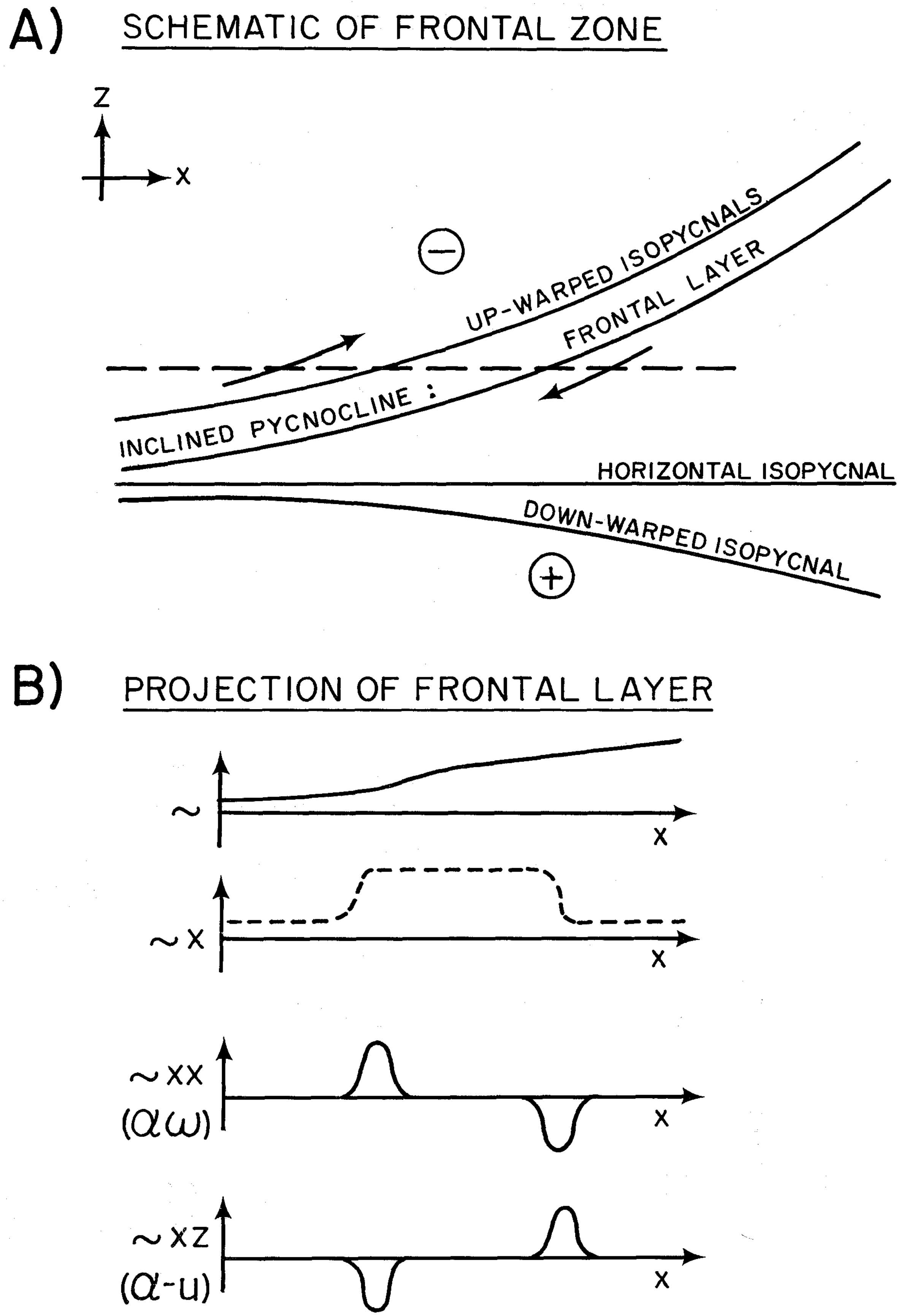
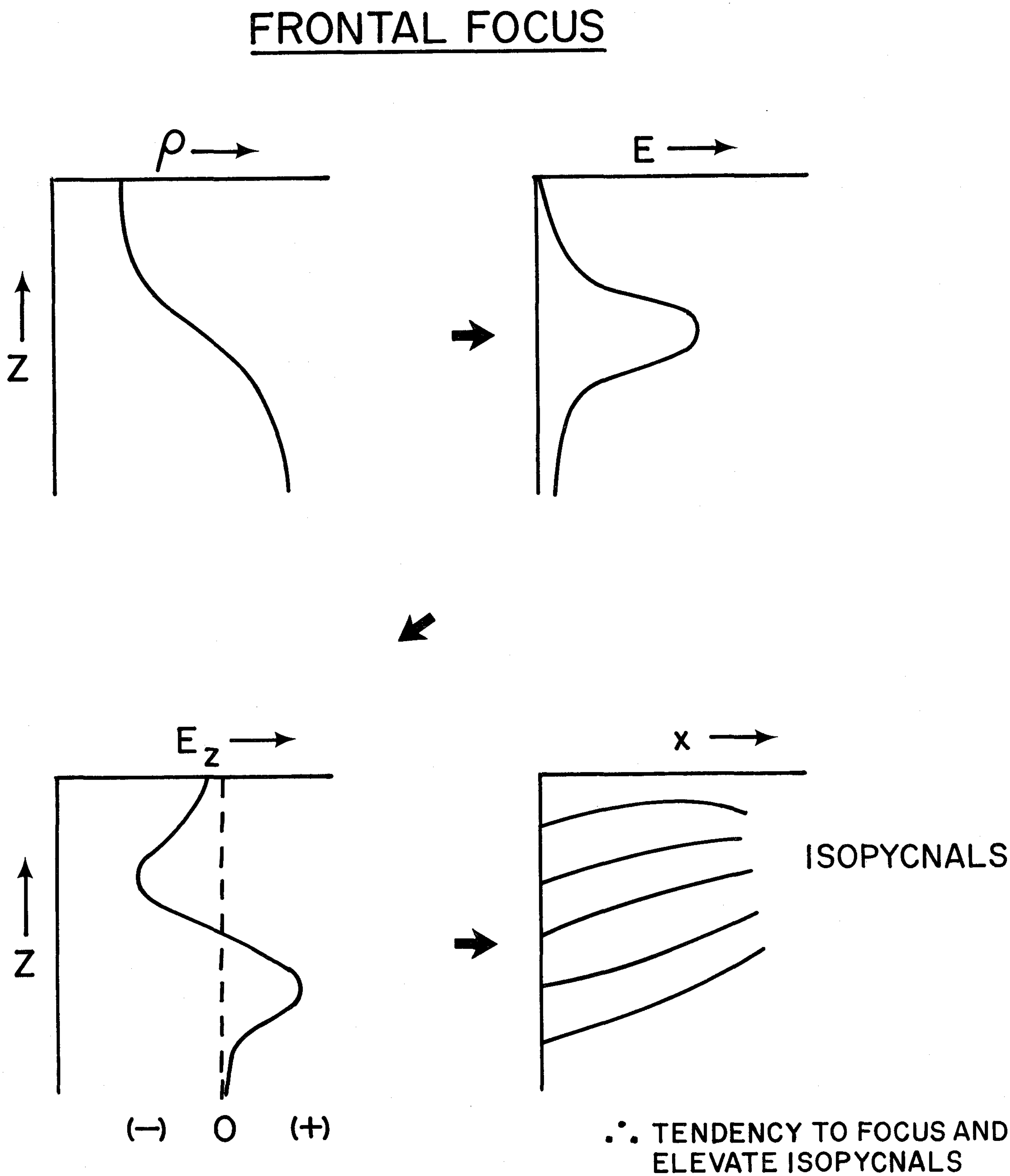


Figure 3



FRONTAL FOCUS
ρ →
E →
Z
E_z →
x →
ISOPYCNALS
(−) 0 (+)
∴ TENDENCY TO FOCUS AND ELEVATE ISOPYCNALS

Figure 4

classified as to type, one can readily determine what types of boundary conditions are permissible. The general, integral form of the solution for the (GE) is desired; that objective hasn't yet been achieved though a number of particular solutions have been found. (It is remarked that standard methods of integrating the (GE), e.g., the Monge method, have not yet been productive.) Thus, it is necessary to systematically investigate the character of the solution. The classification of the (GE) depends upon the sign of the discriminant, $\Delta = b^2 - 4ac$, calculated from the general form of the principal part: $a\rho_{xx} + b\rho_{xz} + c\rho_{zz}$.

In this case,

$$\Delta = \alpha^2(\rho_x)^2 + 4\alpha\rho_z = \rho_z\alpha\left[\alpha S^2\rho_z + 4\right],$$

where, again, S is the slope of an isopycnal. The a priori information available is that $S \sim \frac{D}{L} \sim 10^{-2} - 10^{-3}$, $\alpha \sim 1$, $\rho_z < 0$, and $|\rho_z| \sim 2 \times 10^{-7}$, so $\Delta \sim 4\alpha\rho_z < 0$. Therefore, the (GE) is elliptic for $0 < \alpha < \infty$; if $\alpha = 0$, the (GE) is parabolic; if $\alpha \to \infty$, the (GE) is hyperbolic. Thus, we have as typical, proper boundary conditions that for

(i) $\alpha = 0$, (diffusion only-parabolic degeneracy);

ρ and its normal derivative as functions of x at, say, $z = 0$.

(ii) $0 < \alpha < \infty$, (advection and diffusion-elliptic); ρ or its normal derivative along the boundary of a closed domain.

(iii) $\alpha \to \infty$, (advection only-hyperbolic degeneracy); ρ or its normal derivative along $x = 0$ or ρ along $x = 0$ and $z = 0$, etc.

It is anticipated that, with the low-order dynamics included in this model, that it will not be possible to satisfy very complicated boundary conditions. (Reference to Figure 5 will aid in visualizing the boundary conditions which can be satisfied.) Obviously, only inviscid boundary conditions can be satisfied in the x, z plane; logically, the bottom and free surface should be streamlines, and no net onshore volume flux is required through any vertical, i.e.,

BOUNDARY CONDITIONS

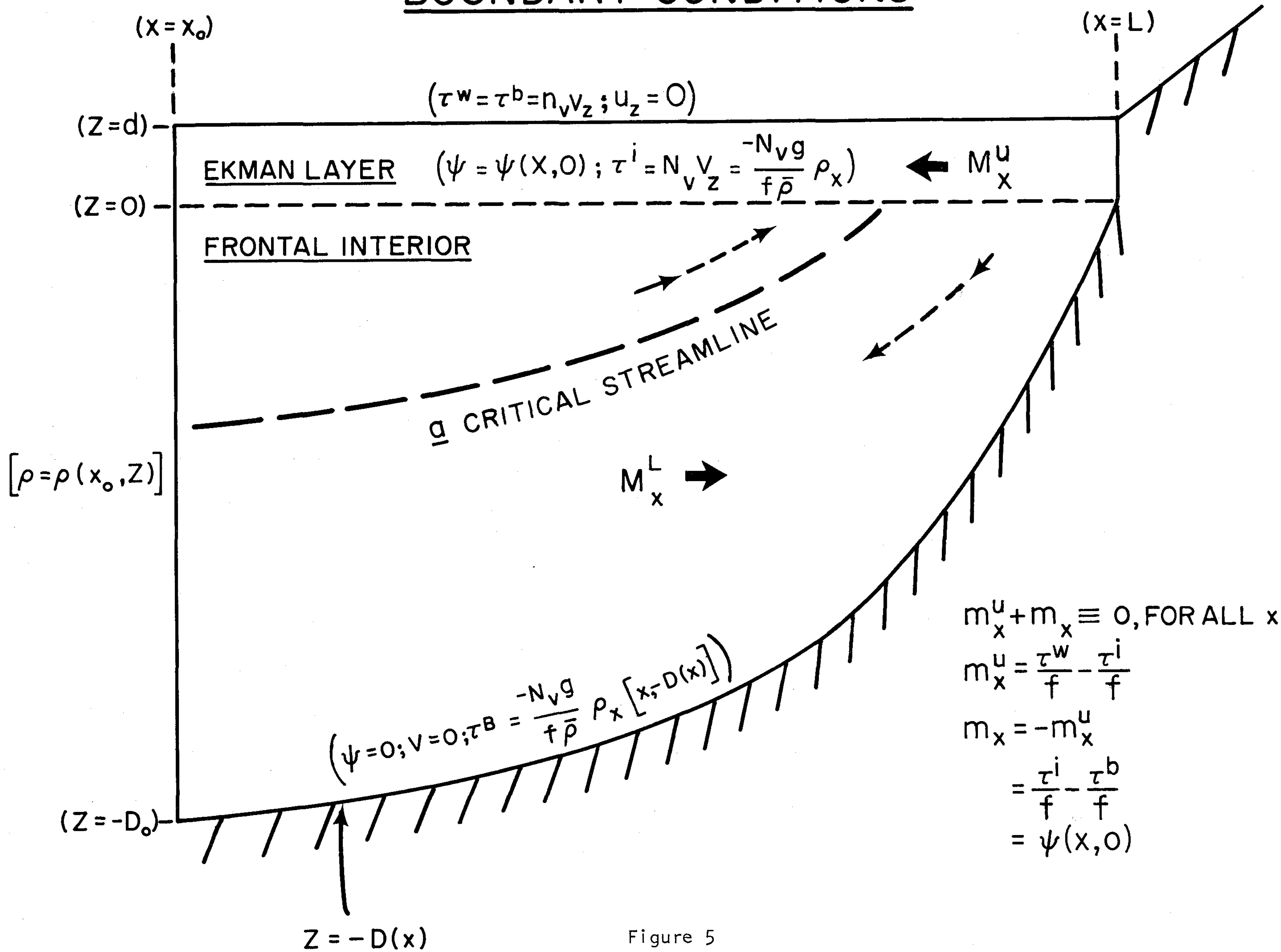


Figure 5

(1) $\psi\Big|_{z=-D(x)} = C_B$: constant

$\psi\Big|_{z=h} = C_S$: constant

(2) $C_B - C_S = 0.$

Condition (1) fixes the form of $G(x)$. Now note that

(3) $M_x^L \equiv \int_{-D(x)}^{0} u\, dz = C_B - \psi(x,0) \underset{=}{(also)} \frac{\tau^i}{f} - \frac{\tau^B}{f}$

(4) $M_x^u \equiv \int_0^h u\, dz = \psi(x,0) - C_S \underset{=}{(also)} \frac{\tau^W}{f} - \frac{\tau^i}{f}$

(5) since $M_x = M_x^u + M_x^L \equiv 0$,

then $\tau^W = \tau^B$ as is to be expected; this relation makes the horizontal density gradient at the bottom proportional to the wind stress and of the opposite sign; thus, as in the oceanographic case under consideration, if $\tau^W < 0$, $\rho_x > 0$ at the bottom, i.e., upwarping of isopycnals and so no undercurrent at the bottom.

(6) (1) above implies that

$$G(x) = C_B - \delta\rho_x(x, -D(x)),$$

so

$$\psi = \delta\left[\rho_x(x,z) - \rho_x(x,-D(x))\right] + C_B,$$

(7) with (6) and

$$\tau = -f\delta\rho_x$$

then $\psi = -f^{-1}\left[\tau(x,z) - \tau(x,-D(x))\right] + C_B$

$= \frac{\tau^B}{f} - \frac{\tau}{f}$, with $C_B = 0$ WLOG and $\tau(x,-D(x)) \equiv \tau^B$,

so

$$M_x^L = \frac{\tau^B - \tau^i}{f}$$

(8) v will satisfy viscid boundary conditions,

i.e., $v_z\Big|_{z=0} = \tau^i$ and $v\Big|_{z=-D(x)} = 0$ or $v = -\int_{-D(x)}^{z} \rho_x \, dz$.

(9) $G'(x) = \dfrac{\tau_x^w}{f}$, so it is the curl of the wind stress which enters the (GE) as a coefficient to the ρ_z term.

(10) $W\Big|_{z=0} = \dfrac{\tau_x^w - \tau_x^i}{f}$, the Ekman suction.

In short, the vertical shear of the alongshore geostrophic flow provides a (frictional) cross-stream flow, which flows into, and out of, the surface Ekman layer; the bottom stress equals the wind stress, and it is provided by the horizontal gradient of density at the bottom (this will serve as a constraint when we examine actual solutions); the interfacial stress depends upon the flow into, or out of, the surface Ekman layer from the interior and will in general have an x-dependence substantially different from, but related to, that of the wind stress; the stream function (alias shear stress or horizontal density gradient) can be a quite general function of x and z, with zeroes and extreme (critical streamlines) interior to the frontal zone. It is in fact an objective to determine necessary and sufficient conditions for (and consequences of) producing critical streamlines.

We will consider the cases $\alpha \longrightarrow \infty$ and $0 < \alpha < \infty$; the case $\alpha = 0$ is neglected because it is of limited interest, corresponding to geostrophic alongshore flow (with uniform vertical shear) which is uncoupled to vertical diffusion and cross-stream advection of mass.

<u>($\alpha \longrightarrow \infty$) isentropic flow</u>; <u>the hyperbolic case</u>.

Thus, the basis of the analysis is

$$\beta(\rho_{xx}\rho_z - \rho_{xz}\rho_x) + G'(x)\rho_z = 0, \quad \beta = K_v \alpha \xrightarrow[K_v \to 0]{} \text{finite}$$

Since, in this case $\psi = \psi(\rho)$, then the bottom must be a streamline with positive slope, i.e., there is no downwarping of isopycnals possible near the bottom, which is necessary to produce an undercurrent near the bottom geostrophically. First let $G'(x) = 0$, i.e., τ^w is uniform; then, the general solution to the (GE) can be readily found; rewrite (GE): $\rho_z \rho_{xx} - \rho_x \rho_{xz} = 0$

as $\left(\frac{\rho_x}{\rho_z}\right)_x = 0$

so, $\frac{\rho_z}{\rho_x} = g'(z)$, $\quad g(z)$ is arbitrary at this stage

and $g'(z)\,\rho_x - \rho_z = 0.$

Let H be defined such that $\nabla H \equiv (1, g'(z))$,

then $H = x + g(z)$

and $\nabla \rho \times \nabla H = 0$, or $\rho = f(H)$.

Thus, $\rho = f(x + g(z))$, where $f \varepsilon C^{(2)}$ and $g \varepsilon C^{(1)}$ are arbitrary functions within the limitations of reasonable boundary conditions and other _a priori_ properties, e.g.,

i) ρ is a monotonically decreasing function of z for x fixed, which implies f is a monotonically (increasing)/(decreasing) function of g and g is a monotonically (increasing)/(decreasing) function of z. (Also, this implies that f and g each have an inverse.)

ii) Since a stable density profile has been assumed, $\rho_z < 0$ or $f'g' < 0$, for all $x, z \varepsilon D$ and, since $\rho_x = f'$ has arbitrary sign, then f' and g' must be of opposite sign for all z. Since f (and g) is monotone, f' (and g') can't change sign and so neither, can ρ_x, which rules out an undercurrent.

Further properties of the general solution can be deduced when the stream function is examined. First, since $\psi = \delta\rho_x + G_o$, G_o: constant, $\psi = \delta f'(x+g(z)) + G_o$. ψ is constant $(=C_B)$ along $z = -D(x)$, so $f'(x-g(-D(x)) = \frac{C_B - G_o}{\delta}$ yields the implicit equation for the bottom, viz., $x - g(-D(x)) = C_o$ constant. Similarly, $x = C_i - g(z)$ is the equation of the i^{th} streamline. (In case a critical streamline(s) exists, a simple precaution of assigning values to C_i will be necessary to avoid ambiguities because ψ will be multiple-valued.) Second, if there is a reversal in the cross-stream flow with depth, i.e., if a critical streamline (cs) exists, it must occur at an extremum in the stream function, i.e., $\psi_x = 0$ and $\psi_z = 0$ along cs or, $f'' = 0$ and $g'f'' = 0$. Thus, $f'' = 0$ is a necessary and sufficient condition for reversal. Since an inflection point in the vertical density profile (or, "center of pycnocline" (cop)) occurs where $\rho_{zz} = 0$, or $f''(g')^2 + f'g'' = 0$, then, since $f' \not\equiv 0$, cop and cs coincide if and only if $g'' = 0$ along cs. The condition that $g'' = 0$ is trivially true if g is a linear function of z, otherwise cop's and cs's may behave quite independently.

An auxiliary relation between the stream function and the density is found from $\psi = \psi(\rho)$ and $\psi_z = \delta\rho_{xz}$, so $\frac{d\psi}{d\rho} = \delta(\ln \rho_z)_x = \delta(\ln E)_x$, which relates the change in volume transport in the x,z plane between isopycnals to the horizontal variation of static stability.

An oceanographically pertinent solution is $\rho = \tanh(ax + bz + c)$ with $(a>0; b<0)$ in $(0 \le x \le L; -D_{(x)} \le z \le 0)$, $D(x)$ to be determined; it satisfies the previously established criteria for physical acceptability. Obviously, a cs and a cop exist and coincide if $\rho = 0$ in the domain. Pick C such that ρ is zero at $x = x_o$, $z = z_o$ then $\rho = \tanh(a(x - x_o) + b(z - z_o))$. The cs and cop intersect the interface z = 0 at $x = x_o + \frac{b}{a} z_o$. Now

$$\rho = \tanh(a(x-x_o)+b(z-z_o))$$

$$\psi = \delta a\left[\operatorname{sech}^2(a(x-x_o)+b(z-z_o)) - \operatorname{sech}^2(B)\right]$$

$$z_B = -D(x) = z_o - \frac{a}{b}(x-x_o) + B,$$

$B = -(D_o + z_o)$, *where* D_o *is the depth at* $x = x_o$

$$\tau^w = \delta a \operatorname{sech}^2(B)$$

$$\tau^i = \delta a \operatorname{sech}^2(a(x-x_o) - bz_o)$$

$$v = \frac{a}{b}\left[\tanh(a(x-x_o)+b(z-z_o)) - \tanh(B)\right]$$

(With $\rho = \tan(ax+bz+c)$, the solutions are similar in form, but the cross-stream flow is of opposite sign because the corresponding stream functions are qualitatively different in this sense. The equivalent ambiguity is not present in the case of non-isentropic, frictional flow.)

This example illustrates that, once ρ is specified at $x=0$ and along $z=0$, it is fully determined, as is the "permissible" τ^w and bottom profile. Another way of expressing this is to say that, given $\tau^w_{(x)}$, $D_{(x)}$, and the flow model, the compatible density distribution is determined. Some other interesting solution forms are

1) $\rho = \exp(x + \ln(-z))$

$\rho(0,z) = z$; $\rho(x,0) = \exp(x)$

$\psi = \delta\left[\exp(x+\ln(-z)) - \exp(x+\ln(D_{(x)}))\right]$

$z_B = -D(x) = -e^{(B-x)}$

cs and cop do not exist

2) $\rho = \tanh(x + z^3)$

$$\rho(0, z) = \tanh(z^3);\ \rho(x, 0) = \tanh(x)$$

$$\psi = \delta\left[\operatorname{sech}^2(x + z^3) - \operatorname{sech}^2(x - D^3_{(x)})\right]$$

$$z_B = -D(x) = (B - x)^{1/3}$$

cs and cop exist but do not coincide;

cs is given by $x + z^3 = 0$,

while cop is given by $3z^3 \tanh(x + z^3) = 1$

($0 < \alpha < \infty$) non-isentropic flow; the elliptic case.

The basis of the analysis is the (GE): $\alpha(\rho_{xx}\rho_z - \rho_{xz}\rho_x) + G'(x)\rho_z = \rho_{zz}$. A number of solutions of varying usefulness have been found for this case. The solutions can be found by different procedures but the general approach is that of similarity solutions. (An elementary but comprehensive discussion of similarity analysis can be found in Hansen (1965).) The oceanographic study which most closely compares with the spirit of this analysis is that of Hansen and Rattray (1965) on certain non-linear estuarine circulation problems. As in their study, it is hoped that, knowing solutions to the (GE), the corresponding forms for bottom topography and wind stress are physically tenable. Initially, it is not clear whether "symmetry" is to be expected in the horizontal or vertical dimension, which will determine along which axis the boundary conditions are set.

First, the separation variable will be found by assuming $\rho = A x^{\mu} z^{\nu}$ and substituting this form for ρ into the (GE), with $G' = 0$, to find that $\rho = \eta \equiv \frac{x^2}{\alpha z}$. This is more than a particular solution because it discloses much of the symmetry of the problem. For instance, setting $\rho = H(\eta)$, another solution is found:

$$H = \frac{1}{2a} \ln\left(\frac{1 + a\eta}{1 - a\eta}\right).$$

As a generalization of the above let, for ν real, $\eta = \frac{x^2}{\alpha z}$

$$\rho = \eta^{\nu} H(\eta),\ \eta = e^{r},\ \text{and}\ H = e^{-\nu r} F(r),$$

then the (GE) becomes

$$F'' + 2F' - \nu F = 2e^{-r}(F')^2.$$

No further solutions have been found from it yet, but this equation is Lipschitzian so uniqueness to an initial value problem in η is assured if numerical solution is necessary.

Another particular solution can be found by taking the density to be of the form $\rho = x^2 f(z)$ so that the (GE) becomes $-2\alpha f f' + G'(x) f' = f''$, or, since $G'(x)$ must be a constant, say H_1, then $(H_1 - 2\alpha f)f' = f''$. Integrating once, we have $f' = C^2 + H_1 f - \alpha f^2$, where C^2 is an arbitrary constant.

With $$R \equiv \frac{1}{2}\left(\left(\frac{H_1}{a}\right)^2 + 4C^2\right)^{1/2};$$

then $$f = \frac{H_1}{2a^2} + \frac{R}{a}\tanh\left(aR\left(z + z_0\right)\right).$$

If $H_1 = 0$, $R = C$ so that $f = \frac{c}{a}\tanh(ac(z+z_0))$, assuming $c^2 > 0$; if $c^2 < 0$,

$$f = \frac{-c}{a}\tan\left(ac\left(z + z_0\right)\right);$$

and, if $c = 0$, returning to the original equation for f, $f = \frac{1}{\alpha(z+z_0)}$ as before. Assuming $c^2 < 0$ for a realistic depth dependence to the density, the solution for ρ is then:

$$\rho = (x + x_0)^2 f(z)$$
$$= \frac{(x+x_0)^2}{2\alpha}\left[H_1 - \left(H_1^2 + 4c^2\alpha\right)^{1/2} \tan\left(\frac{1}{2} H_1^2 + 4c^2\alpha\right)^{1/2}\left(z + z_0\right)\right],$$

where H_1, $|c|$, and z_0 are still arbitrary.

A set of exponential solutions exists which can be found either by the similarity transformations to be introduced below or directly. First, let $G'(x)$ = constant; then assume $\rho = e^{\alpha x + bz}$. The Jacobian portion of the (GE) vanishes, so that $G(x) = bx$.

Second, let $G'(x) = -k_o e^{\alpha x+b}$; then $\rho = e^{e^{(\alpha x + b \ln(k_o z) + c)}}$ is found.

Third, again with $G'(x) = b$, it is found that

$$\rho = ax^2 + bxe^{dz} + ce^{2dz}, C = \frac{\alpha b^2}{2[(d-1)+2a\alpha]} .$$

Similarity transformations with the similarity variables $\eta = zk(x)$ and $\eta = xk(z)$ have been examined and the details are summarized below.

For $\rho = A(x)\,B(\eta)$ and $\eta = zk(x)$,

$$A = A_o k^{2\nu+1} e^{\mu x},$$

$$[(2\nu+1)k^{2(\nu-1)} + \mu k^{2\nu-1}](k')^2 = \frac{C_3}{A_o} e^{-\mu x},$$

$$\frac{B_{\eta\eta}}{B_\eta} = \frac{[\alpha(C_1 B + C_2 \eta B_\eta) + C_4]}{[1+\alpha C_3 \eta B]},$$

$$C_1 + C_3 = \frac{A}{k}\left(\frac{A'}{A}\right)' \text{ and } C_2 - C_3 = \frac{A}{k}\left(\frac{k'}{k}\right)'; \quad \frac{C_1+C_3}{C_2-C_3} = 2\nu+1;$$

$$C_3 = \frac{A'k'}{k^2} \text{ and } C_4 = \frac{G'}{k}.$$

Several particular solutions have been found to this system but not the general solution.

For $\rho = A(z)\,B(\eta)$ and $\eta = x\,k(z)$,

$$H'(x) \overset{\text{(must)}}{=} \text{constant},$$

$$A = A_o k^{-\left(\frac{1+2\nu}{\nu}\right)} \quad (\text{also, } A = A_o k^{-2}),$$

$$k = (rz+s)^{\nu} \quad (\text{also, } k = e^{rz}),$$

$$B_{\eta\eta} = \frac{[C_1(B_\eta)^2 - C_2 \eta B_\eta - C_3 B]}{[B - C_4 \eta^2]}$$

$$C_1 = \left(\frac{\nu+1}{2\nu+1}\right),$$

$$C_2 = \frac{-3r\nu}{\alpha A_o}\left(\frac{\nu+1}{2\nu+1}\right),$$

$$C_3 = \frac{2r}{\alpha A_o}(\nu+1), \text{ and}$$

$$C_4 = -\frac{r\nu^2}{\alpha A_o}\frac{1}{(2\nu+1)}.$$

This system is more manageable than the previous because the equation for k has been solved, but the equation for B is still difficult to solve in general. Since the separation in $\eta = x\,k(z)$ is of simpler form than that in $\eta = z\,k(x)$, the problem is, in a pragmatic sense, more symmetric in x than in z. This suggests that the boundary conditions be given at $x = 0$, e.g.

$$\rho(0,z) = A(z)B(0) \quad \text{or} \quad \rho_z(0,z) = A'(z)B(0),$$

and that numerical methods be used to solve the equation for B.

Summary of Solutions

The particular solutions for non-isentropic, frictional fronts cited previously are summarized below in a uniform manner; the functional, dimensionless forms for ρ, ψ, $-D$, τ^w, and V are given, as well as succinct comments. It is premature to plot these solutions extensively until the results are considered more carefully in terms of boundary conditions and the physics of the problem. Two important considerations are: (i) to what extent can different solutions in different subdomains be combined? and (ii) what generalizations of boundary conditions are possible, e.g., can a bottom Ekman layer be added gracefully? These questions are not answered here.

Solution (1):

$$\rho = \frac{(x+x_0)^2}{\alpha(z+z_0)}$$

$$\psi = \frac{2}{\epsilon}\;\frac{(x+x_0)}{(z+z_0)} + G_0$$

$$z_B = -D(x) = -z_0 - \frac{G_0\,\epsilon}{2}(x+x_0)$$

$$\tau^w = G_0$$

$$V = -\frac{2}{\alpha}(x+x_0)\ln\left(\frac{z+z_0}{z_B+z_0}\right).$$

Comments: uniform bottom slope and wind stress; isopycnals have non-zero horizontal derivatives only to second order; V is a monotonic function of z, so there is no possibility of an undercurrent structure.

Solution (2):

$$\rho = \frac{1}{2a\alpha} \ln \left[\frac{z + a x^2}{z - a x^2} \right]$$

$$\psi = \frac{1}{2\epsilon} \left[\frac{4 x z}{(z^2 - a^2 x^4)} \right] + G_0$$

$$z_B = -D(x) = \frac{x}{\epsilon G_0} \mp \left(\left(\frac{x}{\epsilon G_0} \right)^2 + a^2 x^4 \right)^{1/2}$$

$$\tau^w = G_0$$

$$V = -\frac{2x}{\epsilon} \ln \left(\frac{z^2 - a^2 x^4}{z_B^2 - a^2 x^4} \right) .$$

Comments: uniform wind stress; non-uniform bottom slope; isopycnals have non-zero horizontal and vertical derivatives of all orders; V is a monotonic function of z.

Solution (3):

$$\rho = e^{ax + bz}$$

$$\psi = \delta a e^{ax+bz} + bx + G_0$$

$$z_B = -D(x) = \frac{1}{b} \left[-ax + \ln \left(\frac{-1}{\delta a} \left(bx + G_0 \right) \right) \right]$$

$$\tau^W = G_0 + bx$$

$$V = -\frac{a}{b} e^{ax} \left[e^{bz} - e^{-b D(x)} \right].$$

Comments: non-uniform wind stress and bottom slope; isopycnals have non-zero horizontal and vertical derivatives of all orders; V is a monotonic function of z.

Solution (4):

$$\rho = e^{e^{(ax + \ln(b(z+z_0)))}}$$

$$\psi = \delta a b (z + z_0) e^{ax} e^{e^{(ax+\ln(b(z+z_0)))}} + G_0 - \frac{b}{a} e^{ax+1}$$

$$z_B = -D(x) \text{ (rather Transcendental} \sim \psi|_{z=-D(x)} = 0)$$

$$\tau^w = G_0 - \frac{b}{a} e^{ax+1}$$

$$V = -\frac{a}{b}\left[e^{e^{(ax+\ln(b(z+z_0)))}} - e^{e^{(ax+\ln(b(-D(x)+z_0)))}}\right]$$

Comments: non-uniform wind stress and bottom slope; isopycnals have non-zero horizontal and vertical derivatives of all orders; V is a monotonic function of z .

Solution (5):

$$\rho = ax^2 + bxe^{dz} + ce^{2dz}, \quad c = \frac{\alpha b^2}{2[(d-1)+z a \alpha]}$$

$$\psi = \delta[2ax + be^{dz}] + bx + G_0$$

$$z_B = -D(x) = \frac{1}{d} \ln\left[-\frac{G_0}{\delta b} - \left(\frac{1}{\delta} + \frac{2a}{b}\right)x\right]$$

$$\tau^w = bx + G_0$$

$$V = -\left[2ax(z + D(x)) + b\left(e^{dz} - e^{-dD(x)}\right)\right]$$

Comments: non-uniform wind stress and bottom slope; isopycnals have only non-zero horizontal derivatives to second order; V may or may not be monotonic depending on signs of a , b and d , so there is the possibility of an undercurrent structure.

Solution (6): with $H_1 \equiv (4c^2\delta - G_1^2)^{1/2}$,

$$\rho = \frac{(x+x_0)^2}{2\delta}\left[G_1 - H_1 \tan\left(\tfrac{1}{2} H_1 (z+z_0)\right)\right]$$

$$\psi = (x+x_0)\left[2G_1 - H_1 \tan\left(\tfrac{1}{2} H_1 (z+z_0)\right)\right] + G_0$$

$$z_B = -D(x) = -z_0 + \frac{2}{H_1}\tan^{-1}\left[\left(\frac{G_0}{(x+x_0)} + 2G_1\right)\Big/ H_1\right]$$

$$\tau^w = G_0 + G_1 (x+x_0)$$

$$V = -(x+x_0)\left\{G_1(z+D(x)) + 2\ln\left[\frac{\cos\left(\frac{1}{2}H_1(z+z_0)\right)}{\cos\left(\frac{1}{2}H_1(-D(x)+z_0)\right)}\right]\right\}$$

Comments: non-uniform wind stress and bottom slope; isopycnals have only non-zero horizontal derivatives to second order; V may or may not be monotonic depending on H_1, z_0, and D_0, so there is the possibility of an undercurrent structure.

Conclusion

The inertial and frictional fronts have been studied qualitatively and analytically to determine that they produce some of the key features of cross-stream flow. The most physical expression of the problem's boundary conditions has been shown (implicitly) to be the specification of bottom topography, vertical density profile in the open ocean, and surface wind stress; from this, the magnitude and horizontal scale of the density variation is determined. The concepts of center of pycnocline, critical streamlines, and undercurrent extremum constitute the basic physical and analytical vocabulary for frontal cross-stream flows. The vertical "wiggles" of the density structure in the open ocean reservoir have been shown to be crucial for producing vertical structure in the cross-stream flow. It has been shown possible to produce critical streamlines for both isentropic and non-isentropic frictional flows and undercurrents in only the non-isentropic case and then not entirely satisfactorily.

Partial list of symbols (Only those symbols which are not "standard", defined in the text, or completely obvious in context are listed below.)

$\tau = N_V V_z$,	y-component of stress in dimensionful form
$\tau^W = \tau\big\vert_{z=h}$,	wind stress
$\tau^B = \tau\big\vert_{z=z_B}$,	bottom stress
$\tau^i = \tau\big\vert_{z=0}$,	interfacial stress, i.e., stress at interface between Ekman layer and frontal interior
$E = \frac{-g}{\bar{\rho}} \frac{d\rho}{dz}$,	static stability
$z_B = -D(x)$,	vertical coordinate of bottom

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