Notes on the 1965

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

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Contents of the Volumes

1965

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Editors' Preface

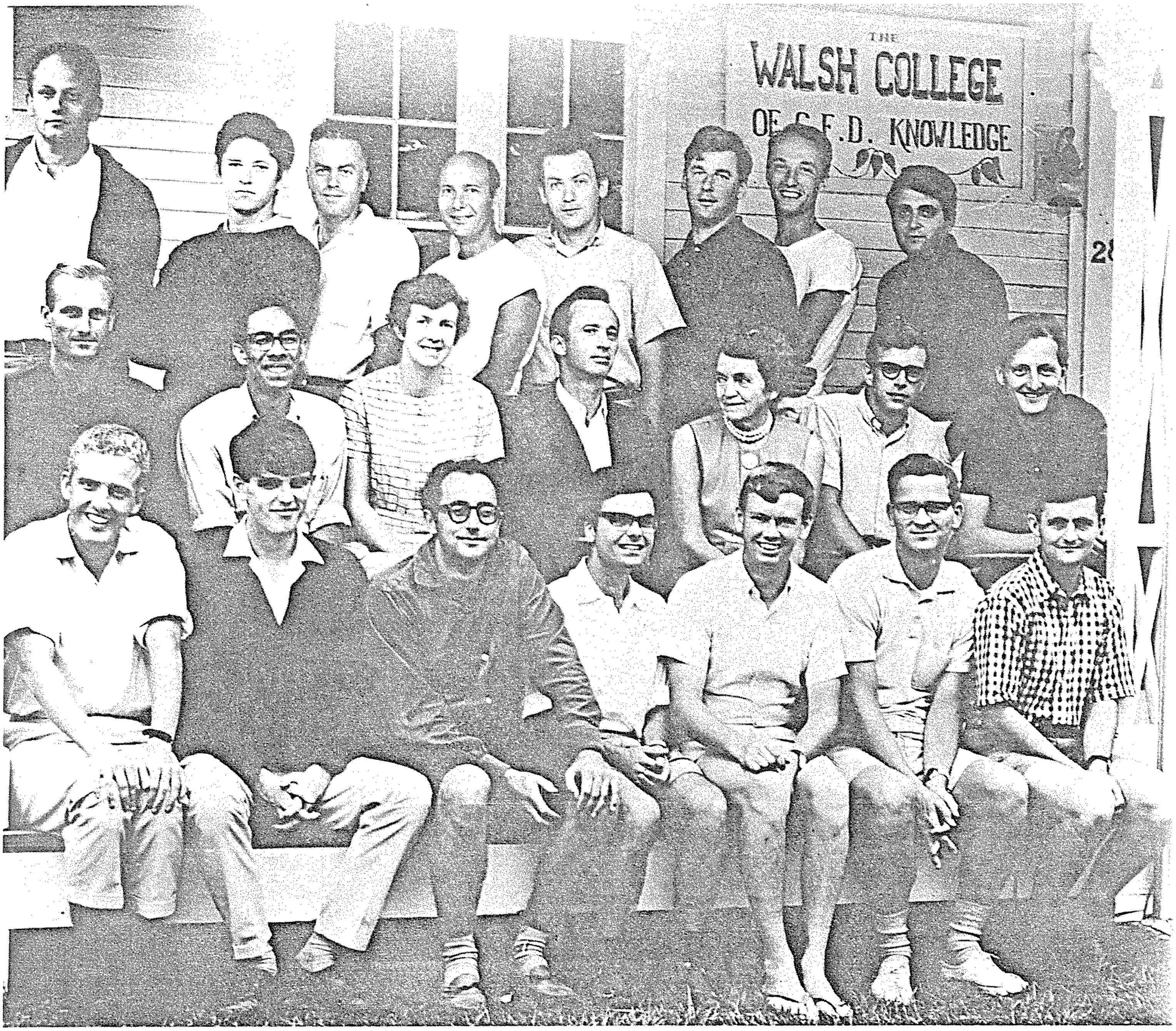
This volume contains the manuscripts of research lectures by pre-doctoral participants in the summer program. The staff guided the selection of the students' topics with several goals in mind. One goal was to isolate that part of a problem which might prove to be tractable in an effort of eight weeks or so. The more important goal was to find "open-ended" problems which would continue to challenge the student after his return to the university.

The degree of direction by a senior participant varied a great deal. In a few cases, there were frequent conferences and discussions about fruitful avenues of approach. In other cases, there was essentially no contact except one of encouragement and interest. The efforts cover a wide spectrum in originality also. Some of the reports represent a more extended study of material presented in the course of lectures - others are original contributions which are being prepared for publication.

Because of time limitations it was not possible for the notes to be edited and reworked. The reports may contain errors the responsibility for which must rest on the shoulders of the participant-author. It must be emphasized that this volume in no way represents a collection of reports of completed and polished work.

All those who took part in the summer program are grateful to the National Science Foundation for its encouragement and financial support of the program.

Mary C. Thayer
Willem V.R. Malkus



Back row: Maxworthy, Bisshopp, Turner, Keller, Spiegel, Welander, Ingersoll, A.Robinson

Middle row: Tomczak, Philander, K.Trustrum, Malkus, Thayer, Pedlosky, Bretherton

Bottom row: True, Crow, J.Robinson, Gough, Booker, Bowman, B.Trustrum

Absent: Howard, Jacobs, Kraichnan, O.Phillips, Stern, Veronis.

Contents of Volume II

Student Lectures

Transmission through an Atmospheric Critical Layer

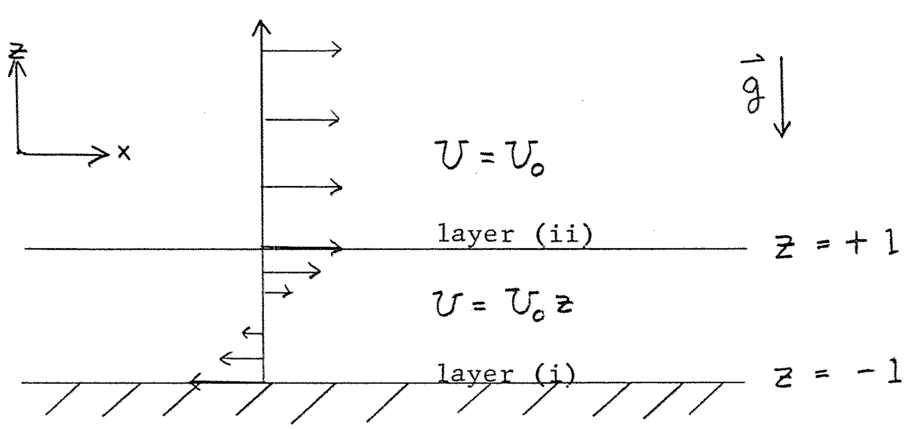
John R. Booker

I Introduction

Prompted by the desire to know whether gravity waves at the Earth's surface can significantly affect processes in the ionosphere, I will consider transmission of disturbances through a shear layer. In particular, I will look at the effect of a critical layer where the wave phase velocity equals the local wind velocity. Since the initial value problem leads to a non-invertible Laplace transform, the problem will be treated as steady. However, there will be ambiguities in the steady state solution. Therefore, the problem will be formulated initially as a time dependent one, pointing out how the initial value problem will resolve the ambiguities, and then returning to the steady state problem.

II Formulation of the Problem

Consider the following two-dimensional model of the atmosphere:



with the assumptions that:

(1) $\vec{u} = (U(z) + u, w)$

(2) u, w are small quantities

(3) $\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u \gg u\frac{\partial u}{\partial x}$

(4) viscosity and rotation may be neglected

(5) $w(z=-1) = \theta e^{ikx}$

(6) $w(t \leq 0) = 0$

(7) the fluid is Boussinesq

(8) $N^2 =$ constant

(9) $N/U_0 \gg 1$

Then the problem is to find, as $t \to \infty$, the ratio between W at the bottom boundary and W in layer (ii).

The model that I have chosen is advantageous because it locates the critical layer (i.e. at $z = 0$), has a tabulated mathematical function as a solution in layer (i), and has a readily interpreted exponential solution in layer (ii). The fluid has been made incompressible, with the belief that none of the essential physics will be changed. Only the exponential increase of velocity with height changes.

Now, with the preceding assumptions, the Navier-Stokes equations become:

(1) Momentum ——

x component

$$u_t + Uu_x + wU_z + \frac{1}{\rho_0} P_x = 0$$

z component

$$w_t + Uw_x + \sigma + \frac{1}{\rho_0} P_z = 0$$

(2) Continuity ——

$$u_x + w_z = 0$$

(3) Conservation of potential temperature ——

$$\sigma_t + U\sigma_x - N^2 w = 0$$

where σ is the buoyancy force per unit mass. Eliminating u, p, and σ, these equations reduce to:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 (w_{zz} + w_{xx}) - U_{zz}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) w_x + N^2 w_{xx} = 0$$

Now let $w = \tilde{w}e^{ikx}$ and Laplace transform the above equation.

$$\hat{w} = \int_0^\infty \tilde{w}\, e^{-pt}\, dt$$

where: $R(p) > 0$ and $\tilde{w}(t \le 0) = 0$

This operation leads to:

$$\hat{w}_{zz} - \left[\frac{k^2 N^2}{(p+ikU)^2} + \frac{ikU_{zz}}{(p+ikU)} + k^2\right]\hat{w} = 0$$

For $p \to 0$, this equation reduces to Scorer's equation for the steady state.

$$\tilde{W}_{zz} + \left[\frac{N^2}{U^2} - k^2 - \frac{U_{zz}}{U}\right]\tilde{W} = 0$$

To complete the formulation of the problem, boundary and matching conditions are needed. I have already given the boundary condition at $z = -1$. At $z = \infty$, it is required that the energy radiation be upwards. This is equivalent to $\hat{W} = 0$ at ∞ for the initial value problem. The matching conditions are that the streamlines are tangent at the interface and the pressure is continuous.

The tangency of the streamlines implies that:

$$\hat{W}_1(+1) = \hat{W}_2(+1)$$

where $\hat{W}_1$ and $\hat{W}_2$ are the velocities in layers (i) and (ii) respectively.

The second matching condition may be easily derived by integrating the differential equation across the boundary:

$$\int_1^2 \left[\hat{W}_{zz} - \left(\frac{k^2N^2}{(p+ikU)^2} + \frac{ikU_{zz}}{(p+ikU)} + k^2\right)\hat{W}\right] dz = 0$$

$$\hat{W}_z\Big|_1^2 - \frac{ikU_z\hat{W}}{p+ikU}\Bigg|_1^2 = \int_1^2 k^2\left(\frac{N^2}{(p+ikU)^2} + 1\right) dz$$

Everything on the right-hand side is continuous across the interface, so the integral is 0 . I might add that the right-hand side is essentially the integral of the pressure, the matching condition is therefore:

$$W_{2z} - W_{1z} = \left[\frac{ik}{p+ikU_0}\right]\hat{W}\left(U_{2z} - U_{1z}\right)$$

But,
$$U_{2z} = 0$$
$$U_{1z} = U_0$$

Finally:
$$W_{2z}(1) - W_{1z}(1) = \left[\frac{-ikU_0}{p+ikU_0}\right]\hat{W}(1)$$

In the steady case, this condition reduces to:
$$\hat{W}_{1z}(1) - \tilde{W}_{2z}(1) = \tilde{W}(1)$$

III. Solutions

(a) layer (i): $U = U_0 z$; $U_{zz} = 0$.

The differential equation becomes:
$$\hat{W}_{zz} - \left[\frac{N^2 k^2}{(p+ikU_0 z)^2} + k^2\right]\hat{W} = 0 .$$

If I let $\hat{W} = (p+ikU_0 z)^{\frac{1}{2}} F\left(\frac{p}{U_0} + ikz\right)$, the differential equation transforms into Bessel's equation and the solution is:
$$W_1 = (p+ikU_0 z)^{1/2}\left[A J_\nu\left(\frac{p}{U_0} + ikz\right) + B J_{-\nu}\left(\frac{p}{U_0} + ikz\right)\right]$$
where: $\nu = \left(\frac{1}{4} - \frac{N^2}{U_0^2}\right)^{\frac{1}{2}}$, which is a pure imaginary number.

For the time independent case, this solution becomes:
$$\tilde{W}_1 = z^{\frac{1}{2}}\left[A J_\nu(ikz) + B J_{-\nu}(ikz)\right]$$

(b) layer (ii) $U = U_0$; $U_{zz} = 0$.

The differential equation is:
$$\hat{W}_{zz} - \left[\frac{N^2 k^2}{(p+ikU_0)^2} + k^2\right]\hat{W} = 0$$

Let $\beta = \left(\frac{N^2 K^2}{p + ikU_0^2} + K^2\right)^{1/2}$

The solution is then:

$$\hat{W}_z = Ce^{\beta z} + De^{-\beta z}$$

As $p \to 0$, the exponent becomes:

$$\beta = \left(k^2 - \frac{N^2}{U_0^2}\right)^{1/2}$$

Theoretically one could now plug the time dependent solutions into the boundary and matching conditions, solve the system of equations for A, B, C, D, and invert the expression for $\hat{W}_z$ to obtain W_z. However, the Laplace inversion is very sticky, and all that one can hope to do is to look at the asymptotics as $t \to \infty$.

Except in a region around $z = 0$ where the solution is foully singular, and takes a very long time to settle down, one can use the steady state solution. Looking at the steady state solutions, one can see that two ambiguities arise. First, $z^{1/2} J_\nu(ikz)$ has a branch point at $z = 0$. Second, $\tilde{W}_z$ has two parts, only one of which represents a wave whose energy propagates upwards. However, one can easily resolve these ambiguities by appealing to the initial value problem.

IV Resolution of the Ambiguities.

(a) Branch point at $z = 0$:

For small values of its argument, A bessel function may be represented as: $J_\nu(x) = x^\nu$

Thus, for small p , k and z small, $\hat{W}_1(z)$ is made up of terms of the form:

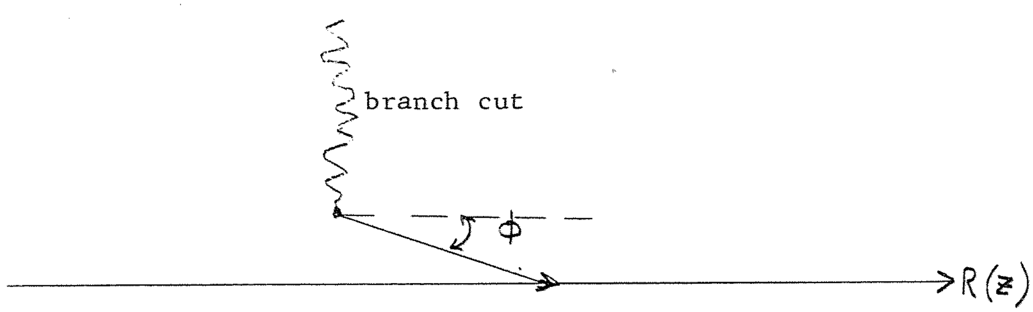
$$\left(z - \frac{ip}{U_0 k}\right)^{\eta}$$

where η is some number. This expression has a branch point at $z = \frac{ip}{U_0 k}$ which approaches the real z-axis as $p \to 0$. What I want to do is discover, for $p \to 0$, which way to go around the branch point if $\tilde{W}_1$ is to be a continuous function of z . This is equivalent to finding which way the branch point leaves the real z axis as p becomes non-zero. Now:

$$z = \frac{ip}{U_0 k}$$

$$R(p) > 0$$

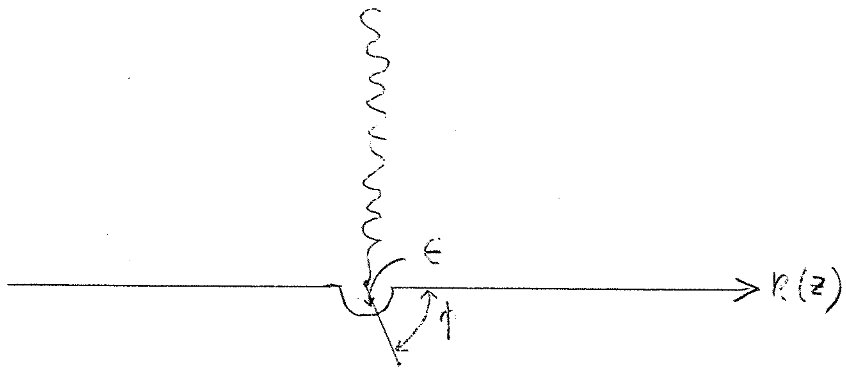
therefore: $I(z) > 0$ and the branch point is above the real z-axis. If the branch is drawn cut to $+i\infty$, there is no discontinuity in $\left(z - \frac{ip}{U_0 k}\right)^{\eta}$ as $R(z)$ goes from $+\infty$ to $-\infty$.



Let: $\quad z - \dfrac{ip}{U_0 k} = r e^{i\phi}$

then: $\quad (re^{i\phi})^{\eta} = e^{\eta(\ln|r| + i\phi)}$

and as z goes from $+\infty$ to $-\infty$, ϕ goes from 0 to $-\pi$. As $p \rightarrow 0$, the branch point approaches the real axis, but ϕ must still go from 0 to $-\pi$ as z goes from $+\infty$ to $-\infty$.



Therefore, if I restrict myself to $z > \epsilon$,

$$z > 0 \qquad \phi = 0$$
$$z < 0 \qquad \phi = -\pi$$

Hence:

$$(+z)^{\eta} = e^{\eta \ln|z|}$$
$$(-z)^{\eta} = e^{\eta \ln|z| - i\eta\pi}$$

Thus the following prescription for continuing z^{η} around $z = 0$ is:

$$(-z)^{\eta} = e^{-i\eta\pi}(+z)^{\eta}$$

(b) Upward travelling wave:

In the initial value problem, the assumption is made that for all $t < \infty$, $W_2(\infty) = 0$. Thus the question of which solution represents an upward-going wave is answered by determining, for

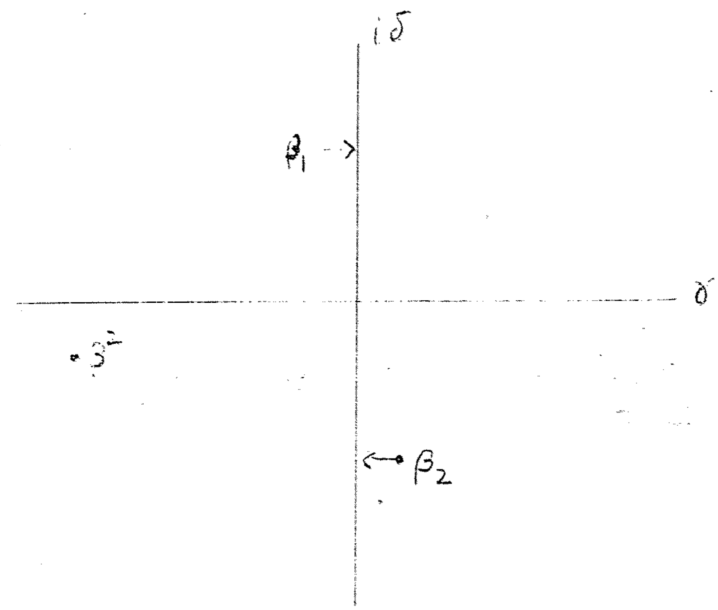
$R(p) > 0$, which part of $\hat{W}_2$ goes to 0 at $z = \infty$.

$$\beta^2 = k^2\left(1 + \frac{N^2}{(p + ikU_0)^2}\right)$$

Separating this expression into its real and imaginary parts:

$$\beta^2 = -\gamma - i\delta$$

where γ and δ are both real and positive.



Obviously only β_1 has a negative real part. As $p \to 0$, β_1 approaches the imaginary axis and the solution I want is:

$$W_z = Ce^{+ik\left[\frac{N^2}{U_0^2} - 1\right]^{1/2} z}$$

V <u>Calculation of the Transmission Coefficient</u>:

My program now is to plug the steady state solutions into the bottom boundary condition and the matching conditions, solve the linear system for A, B, and C, and compare $|\tilde{W}_2|$ to the amplitude of the input disturbance, $|\Theta|$.

The set of equations found are:

(i) $(-1)^{\frac{1}{2}} J_{\nu}(-ik)A + (-1)^{\frac{1}{2}} J_{-\nu}(-ik)B = \Theta$

(ii) $(+1)^{\frac{1}{2}} J_{\nu}(ik)A + (+1)^{\frac{1}{2}} J_{-\nu}(ik)B - e^{\beta}C = 0$

(iii) $(+1)^{\frac{1}{2}}\left[ik J_{\nu}'(ik) + \frac{1}{2} J_{\nu}(ik)\right]A +$

$$+ (+1)^{\frac{1}{2}}\left[ik J_{-\nu}'(ik) + \frac{1}{2} J_{-\nu}(ik)\right]B - [\beta + 1]e^{\beta}C = 0$$

The solution for C is:

$$C = \frac{ik(-1)^{-\frac{1}{2}}(+1)^{\frac{1}{2}} e^{-\beta}\left[J_{\nu}(ik)J_{-\nu}'(ik) - J_{-\nu}(ik)J_{\nu}'(ik)\right]\Theta}{ik\left[J_{\nu}(-ik)J_{-\nu}'(ik) - J_{-\nu}(-ik)J_{\nu}'(ik)\right] + (\beta+\frac{1}{2})\left[J_{-\nu}(-ik)J_{\nu}(ik) - J_{-\nu}(ik)J_{\nu}(-ik)\right]}$$

Now, assuming k is small, letting $\nu = i\mu$ and $\beta = ib$, and using the prescribed continuation around $z = 0$,

$$|C| = \frac{\mu|\Theta|}{\left[(\mu\cosh\mu\pi + b\sinh\mu\pi)^{\frac{1}{2}} + \frac{1}{4}\sinh^2\mu\pi\right]^{\frac{1}{2}}}$$

$$\equiv T_{r}|\Theta|$$

I have already assumed that $\frac{N^2}{U_c^2} \gg 1$. Therefore $\mu = b = \frac{N}{U_c}$, $\cosh\mu\pi = \sinh\mu\pi = \frac{1}{2}e^{\mu\pi}$, and T_r reduces to:

$$T_r = e^{-\mu\pi}$$

VI Discussion

It is extremely interesting to note that the presence of the factor $e^{-\mu\pi}$ in the transmission coefficient is a direct result of going around the branch point at $z = 0$. I am very tempted to conclude that T_r is the transmission coefficient of the critical layer. I can examine this conclusion by looking at the reflection of a sinusoidal wave by an interface with the same matching conditions as those between layers (i) and (ii) in my model:

$$W_2 = \uparrow \, Ce^{i\mu z}$$

$$z = 0$$

$$W_1 = \uparrow \, Ae^{i\mu z} + \downarrow \, Be^{-i\mu z}$$

(i) $W_{1z}(0) - W_{2z}(0) = W(0)$

(ii) $W_1(0) = W_2(0)$

Plugging W_1 and W_2 into (i) and (ii) and solving for $|B|/|A|$,

$$\frac{|B|}{|A|} = \frac{1}{2\mu}$$

If, as has been already assumed, $\mu \gg 1$, I will have only minor reflection at the interface. The conclusion reached above is therefore strengthened. I might add that, for large enough μ, one can identify $z^{1/2}J_{i\mu}$ as an upward moving wave and $z^{1/2}J_{-i\mu}$ as a downward moving wave. Then if one computes $|W_1 \text{ (downward)}| / |W_1 \text{ (upward)}|$ above the critical layer, the answer is the same as for the pure sine wave, i.e. $1/2\mu$.

In conclusion, I would like to add several comments. First, since μ is 5 or greater in the atmosphere, T_r is a very small number, and thus very little energy ever gets through the critical layer.

Second, it is easy to show that $|W_1 \text{ (downward)}| / |W_1 \text{ (upward)}|$ below the critical layer is just the value one expects if a wave travelling upward loses a factor $e^{-\mu\pi}$ going up through the critical layer, $1/2\mu$ at the interface reflection, and another, $e^{-\mu\pi}$ coming down through the critical layer. In other words, the critical layer is an absorber not a reflector.

Finally, I have not been able as yet to fully understand the physical process involved in the critical layer absorption. That is, I have found no way to predict from the physics of the problem, that a wave will decrease its amplitude by a factor $e^{-\mu\pi}$ as it passes through the critical layer. More important though, I have found no way to predict that this decrease in amplitude is independent of critical layer thickness and only depends on the source and receiver being on <u>opposite</u> sides of the layer and far enough away from the layer so that the <u>local</u> solution is nearly steady state.

Acknowledgment: I wish to thank Dr. Francis Bretherton for his patient guidance throughout this project.

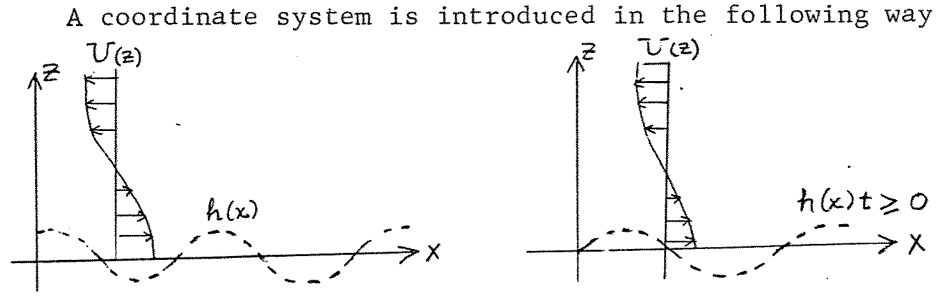
Energy and Momentum Transport in Lee Waves

Hans C. G. True

This work deals with an investigation of the transport of energy and momentum in winds blowing over mountains. It is assumed that the wind velocity is a function of the vertical coordinate only, and that the Boussinesq approximation is valid. Furthermore, it is assumed that the Brunt-Väisälä frequency is constant. The aim is to achieve a better understanding of the physics of the flow.

The problem considered is the following: Given a steady two-dimensional flow* with N constant over an infinite plane. To the time $t=0$ we "switch on" a mountain range on the bottom. This mountain range disturbs the fluid, and we assume that the characteristic dimensions of the mountains are such that the disturbances in the fluid are small compared with characteristic quantities in the basic flow, i.e. that $\frac{u}{U}$, $\frac{p_1}{p_0}$ and $\frac{\rho_1}{\rho_0}$ all are small compared to unity. Here small letters and subscript 1 denote disturbance quantities and capital letters or subscript 0 denote basic flow quantities. We can then linearize our equations.

A coordinate system is introduced in the following way:



*of an ideal gas

We neglect viscous effects and heat conduction and assume furthermore, that the flow is subsonic and substitute therefore $U + u$, w, $p_o + p_1$ and so on into the Euler equations and obtain under the assumptions mentioned above the so-called Scorer's equations for non-steady motion:

$$(1) \quad u_t + U u_x + w U_z + \frac{1}{\rho_o} p_{1x} = 0$$

$$(2) \quad w_t + \sigma + w_x U + \frac{1}{\rho_o} p_{1z} = 0$$

$$(3) \quad u_x + w_z = 0$$

$$(4) \quad \sigma_t + \sigma_x U - w N^2 = 0$$

where u, w, p_1 and $\sigma = g \frac{\rho_1}{\rho_o}$ all are continuous functions with continuous derivations of time and the space coordinates x and z. The Brunt-Väisälä frequency $N^2 = \frac{1}{\vartheta} \vartheta_z$, where ϑ denotes the potential temperature. ρ is the density, p the pressure and g the gravitational constant.

By eliminating u, p_1 and σ we obtain the following linear partial differential equation for w:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(w_{xx} + w_{zz}\right) - U_{zz}\left(\frac{\partial}{\partial t} + \left[U - \frac{N^2}{U_{zz}}\right]\frac{\partial}{\partial x}\right) w_x = 0$$

If we, in this equation - as it is normally done - substitute a disturbance function which is sinusoidal in x and t, we easily see that the differential equation has a singularity for that or those values of z, which make $U = \frac{\omega}{k}$, where $\frac{\omega}{k}$ is the phase velocity in x-direction of our disturbance. This value of z is called the critical layer (subs. the critical layers). Concerning the important effects of that layer, the reader is referred to the paper by John Booker in this volume.

From now on it is assumed that our disturbance functions are periodic in X. We arrive at our energy equation by multiplying equation (1) by u, (2) by w and (4) by σ and integrating over a volume which conviently can be chosen so as to have the length of a period in x-direction. By doing this we realize that the contribution stemming from multiplying (1) by U will vanish, because all terms in that expression will be periodic functions in X integrated over their period.

We then obtain the following expression:

$$\iiint_\Omega \rho_0 \left[\frac{\partial}{\partial t}\left(\frac{u^2+w^2}{2}\right)+U\frac{\partial}{\partial x}\left(\frac{u^2+w^2}{2}\right)+uwU_z+w\sigma\right]d\Omega+\iiint_\Omega (up_{1x}+wp_{1z})\,d\Omega = 0$$

By substituting $W=\frac{\sigma_t}{N^2}+U\frac{\sigma_x}{N^2}$ we obtain:

$$\iiint_\Omega \frac{\rho_0}{2}\left[\frac{\partial}{\partial t}\left(u^2+w^2+\frac{\sigma^2}{N^2}\right)+U\frac{\partial}{\partial x}\left(u^2+w^2+\frac{\sigma^2}{N^2}\right)+uwU_z\right]d\Omega+\iiint_\Omega (up_{1x}+wp_{1z})\,d\Omega = 0$$

Using the fact that $\operatorname{div} \underline{v} = 0$, where $\underline{v}=(u, 0, w)$ we obtain the following equation:

$$\iiint_\Omega \frac{\rho_0}{2}\frac{\partial}{\partial t}\left(u^2+w^2+\frac{\sigma^2}{N^2}\right)d\Omega = -\iint_F \frac{\rho_0}{2}\left(u^2+w^2+\frac{\sigma^2}{N^2}\right)\underline{U}\cdot \underline{df} - \iint_F p_1\underline{v}\cdot\underline{df} - \iiint_\Omega \rho_0 U_z\, uw\, d\Omega$$

In this equation $d\Omega$ is a volume element and $\underline{df}$ an outgoing normal vector on the element $\underline{df}$ of the closed surface F of the volume Ω.

If we now define the quantity $\frac{\rho_0}{2}\left(U^2+W^2+\frac{\sigma^2}{\gamma^2}\right)$ as the energy density E, we can write the equation in the following way:

$$\iiint_\Omega \frac{\partial E}{\partial t}\, d\Omega = -\iint_F E\underline{V}\cdot d\underline{f} - \iint_F p_1 \underline{v}\cdot d\underline{f} - \iiint_\Omega \rho_0 U_z\, uw\, d\Omega$$

The vector $E\underline{V}$ is the energy density flux vector. The energy balance has been established only for the disturbance quantities. The part of the energy density and the energy density flux stemming from the basic flow are however not important in connection with this problem and have therefore been left out.

Equation (5) states that the change of energy density with time because of the disturbance, is equal to the net energy flow into the volume plus the work done by the pressure forces on the volume plus an energy term, which consists of the work done by the momentum flux in z-direction and the work done by the basic flow on the disturbance or conversely.

By integrating the last term by parts over z, we obtain:

$$\iiint_\Omega \frac{\partial E}{\partial t}\, d\Omega = -\iint_F E\underline{V}\cdot d\underline{f} - \iint_F p_1 \underline{v}\cdot d\underline{f} - \left[U\iint_F \rho_0 uw\, dx\, dy\right]_{z_1}^{z_2} + \iiint_\Omega U\frac{\partial}{\partial z}(\rho_0 uw)\, d\Omega$$

We from now on assume that the disturbance functions depend sinusoidally on x and that the dependence on t can be written on the form of an exponentially growing or decaying trigonometric function:

$$w = \frac{1}{2}\left[\hat{w}(z)e^{-i(kx-\omega_r t)} + \hat{w}^*(z)e^{i(kx-\omega_r t)}\right]e^{-\omega_i t}$$

$$u = \frac{1}{2ik}\left[\hat{w}_z e^{-i(kx-\omega_r t)} - \hat{w}_z^* e^{i(kx-\omega_r t)}\right]e^{-\omega_i t}$$

$$uw = -\frac{1}{4ik}\left[\hat{w}\hat{w}_z^* - \hat{w}^*\hat{w}_z\right]e^{-2\omega_i t} + \frac{1}{4ik}\left[\hat{w}\hat{w}_z e^{-2i(kx-\omega_r t)} - \hat{w}^*\hat{w}_z^* e^{2i(kx-\omega_r t)}\right]e^{-2\omega_i t}$$

where asterisks denote the complex conjugate function. ω_r is the real part of ω and ω_i is the imaginary part of ω .

We shall now calculate the two surface integrals

$$\iint_F p_1 \underline{v} \cdot d\underline{f} + \left[\rho_0 U \iint_F u w \, dx \, dy \right]_{z_1}^{z_2} \tag{6}$$

From equation (1) we have:

$$p_1 = \frac{\rho_0}{2} \left[\left(\frac{\omega_r}{k} - U \right) \left(\frac{\hat{w}_z}{ik} e^{-i(kx-\omega_r t)} - \frac{\hat{w}_z^*}{ik} e^{i(kx-\omega_r t)} \right) + U_z \left(\frac{\hat{w}}{ik} e^{-i(kx-\omega_r t)} - \frac{\hat{w}^*}{ik} e^{i(kx-\omega_r t)} \right) \right] e^{-\omega_i t} +$$

$$+ \frac{\rho_0}{2} \omega_i \left[\frac{\hat{w}_z}{k^2} e^{-i(kx-\omega_r t)} + \frac{\hat{w}_z^*}{k^2} e^{i(kx-\omega_r t)} \right] e^{-\omega_i t}$$

If we consider this expression we see that it contains a term

$$- \frac{\rho_0}{2} U \left(\frac{\hat{w}_z}{ik} e^{-i(\;)} - \frac{\hat{w}_z^*}{ik} e^{i(\;)} \right) = -\rho_0 U u .$$

By performing the surface integral, a term $-\rho_0 U u w \, dx \, dy$ will develop, which has to be integrated over x and y in both heights z_1 and z_2 . This term added to the last term in equation (6) gives zero.

By calculating the remaining terms, we obtain:

$$\iint_F p_1 \underline{v} \cdot d\underline{f} + \left[\rho_0 U \iint_F u w \, dx \, dy \right]_{z_1}^{z_2} =$$

$$\left[\int_{z_1}^{z_2} \frac{\rho_0}{4} \left(\frac{\omega_r}{k} - U \right) \left(\frac{\hat{w}_z}{ik} e^{-i(\;)} - \frac{\hat{w}_z^*}{ik} e^{i(\;)} \right) \left(\frac{\hat{w}_z}{ik} e^{-i(\;)} - \frac{\hat{w}_z^*}{ik} e^{i(\;)} \right) e^{-2\omega_i t} dz \right]_{x_1}^{x_2} +$$

$$+ \left[\int_{z_1}^{z_2} \frac{\rho_0}{4} U_z \left(\frac{\hat{w}}{ik} e^{-i(\;)} - \frac{\hat{w}^*}{ik} e^{i(\;)} \right) \left(\frac{\hat{w}_z}{ik} e^{-i(\;)} - \frac{\hat{w}_z^*}{ik} e^{i(\;)} \right) e^{-2\omega_i t} dz \right]_{x_1}^{x_2} +$$

$$+ \frac{\omega_i}{4} \left[\int_{z_1}^{z_2} \rho_0 \left(\frac{\hat{w}_z}{k^2} e^{-i(\;)} + \frac{\hat{w}_z^*}{k^2} e^{i(\;)} \right) \left(\frac{\hat{w}_z}{ik} e^{-i(\;)} - \frac{\hat{w}_z^*}{ik} e^{i(\;)} \right) e^{-2\omega_i t} dz \right]_{x_1}^{x_2} +$$

$$+ \left[\frac{\rho_0}{4} \frac{\omega_r}{k} \int_{x_1}^{x_2} \left(\frac{\hat{w}_z}{ik} e^{-i(\;)} - \frac{\hat{w}_z^*}{ik} e^{i(\;)} \right) \left(\hat{w} e^{-i(\;)} + \hat{w}^* e^{i(\;)} \right) e^{-2\omega_i t} dx \right]_{z_1}^{z_2} +$$

$$+\left[\frac{\rho_0}{4}U_z\int_{x_1}^{x_2}\left(\frac{\hat{w}}{ik}e^{-i(\,)}-\frac{\hat{w}^*}{ik}e^{i(\,)}\right)\left(\hat{w}e^{-i(\,)}+\hat{w}^*e^{i(\,)}\right)e^{-2\omega_i t}dx\right]_{z_1}^{z_2}+$$

$$+\left[\frac{\rho_0\omega_i}{4}\int_{x_1}^{x_2}\left(\frac{\hat{w}_z}{k^2}e^{-i(\,)}+\frac{\hat{w}_z^*}{k^2}e^{i(\,)}\right)\left(\hat{w}e^{-i(\,)}+\hat{w}^*e^{i(\,)}\right)e^{-2\omega_i t}dx\right]_{z_1}^{z_2}$$

The first three integrals will vanish if we choose x_1, and x_2 such that their difference is a period. When we integrate over the same length in the last three integrals, all terms containing $e^{-2i(kx-\omega_r t)}$ and $e^{2i(kx-\omega_r t)}$ will also vanish. We are left with:

$$\left[\frac{\rho_0\omega_r}{4ik^2}(x_2-x_1)\left(\hat{w}^*\hat{w}_z-\hat{w}\hat{w}_z^*\right)+\frac{\rho_0\omega_i}{4k^2}(x_2-x_1)\left(\hat{w}\hat{w}_z^*+\hat{w}^*\hat{w}_z\right)\right]_{z_1}^{z_2}e^{-2\omega_i t}$$

Our energy expression now is:

$$\iiint_\Omega\frac{\partial E}{\partial t}d\Omega=-\iint_F E\underline{V}\cdot d\underline{f}-|x_2-x_1|\left(\frac{\omega_r+\omega_i}{k}\right)\left[\rho_0 uw\right]_{z_1}^{z_2}+\iiint_\Omega\rho_0 U\frac{\partial}{\partial z}(uw)d\Omega$$

We have now found that the energy producing terms can be split up into two parts - one which is proportional to the difference in Reynold's stresses at two different heights with a factor of proportionality equal to $\frac{\omega_r+\omega_i}{k}$ and another term which expresses the work done by the basic flow on the disturbance or conversely.

If we have a steady flow purely sinusoidal in x and $t(\omega_i=0)$, it can be shown that $\frac{\partial}{\partial z}(uw)$ identically vanishes by integrating over a period in x-direction.

$$\frac{\partial}{\partial z}(uw)=\frac{1}{4ik}\left(\hat{w}\hat{w}_{zz}^*-\hat{w}^*\hat{w}_{zz}\right)+$$ something which vanishes by integrating over x.

If we apply Scorer's equation, which is in the form: $W_{zz} + [F]w = 0$ and correspondingly for the complex conjugate: $W^*_{zz} + [F]^* w^* = 0$ where F is $F(\omega, k, U, N^2)$ we see that the products $\hat{w}\,\hat{w}^*_{zz}$ and $\hat{w}^*\,\hat{w}_{zz}$ are equal provided $[F] = [F]^*$ i.e. provided ω is a real number, which we assumed. This means that the Reynold's stress is independent of height.

The equation then reduces to the familiar result for steady flow:

$$\iint_F E \underline{V} \cdot d\underline{f} = 0 \, .$$

It is interesting to notice that the influence of the mountain range will not be expressed explicitly in this form of the energy equations. If we however consider the energy transport in a moving reference frame we must add a term which expresses the work done by the mountains on the fluid flow. This term will be of the form: Drag x velocity of reference frame, where the drag is assumed to be a very small quantity. If we choose a reference frame, where the waves are steady, i.e. a frame moving with velocity $\frac{\omega_r}{k}$ we obtain the following relation (ω_r in our new system will be equal to zero):

$$\iiint_\Omega \frac{\partial E}{\partial t} d\Omega = -\iint_F E \underline{V}' \cdot d\underline{f} - |x_2 - x_1| \frac{\omega_i}{k} \left[\rho_0 u w\right]_{z_1}^{z_2} + \iiint_\Omega \rho_0 U' \frac{\partial}{\partial z}(uw) d\Omega + W \quad (7)$$

where $U' = U - \frac{\omega_r}{k}$ and W is the work done by the mountain per unit time. The energy equation does not tell us anything about the energy transport in the vertical direction - it only informs us about the amount of the energy which stays within the volume and contributes to an increase of energy density with time.

We now finally apply this result to John Booker's calculations, and we choose our limits z_1 and z_2 close to and on both sides of the critical zone. Within that region of course the work done by the mountain range W will not be included in the equation, because the lower limit z_1 normally will be above the peak of the mountains.

John Booker found that the amplitude of the disturbance functions within the critical zone decreased by a factor $e^{-\mu \pi}$, where μ is a large number, which means that the Reynold's stress will decrease by a factor $e^{-2\mu\pi}$.

If we look at equation (7) we see that the term determining the exchange of energy between the basic flow and the disturbance flow field is very large, because $\frac{\partial}{\partial z}(UW)$ will take a large numerical value in the critical layer. Furthermore, the energy production term $\left[\frac{\omega_i}{K} UW\right]_{z_1}^{z_2}$ will be large because of the large difference in Reynold's stress at the two layers z_1 and z_2 . As we have no energy transport in the vertical direction, this increase of energy within the critical zone must result in an increase of disturbance energy density and eventually also in an increase of the basic velocity U . This leads to a rapid increase of the amplitudes of the disturbance quantities and therefore to a breakdown of the linear approximations within the critical zone in Booker's model.

Although the energy equation in this form can not explain the energy propagation in the vertical direction, it indicates what happens to the energy of the disturbance field in the critical zone while this zone is developing. We still do not know, however, why any energy at all goes through the critical zone.

The Shake-Up Problem

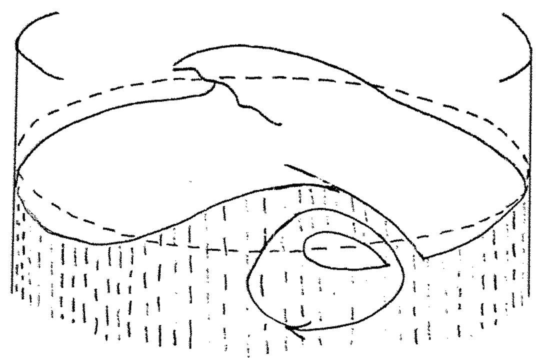
Steven C. Crow

Introduction

When an ordinary man wants to excite a vortex - in a cocktail glass, say - he shakes the glass around in a tight circle. The fluid swirls around, splashes, and soon begins to spin. The experiment can be repeated more decorously with a chemical flask. Anyone who has taken a chem lab course knows about the intensely concentrated vortex that can be formed by shaking a bulbous wash-bottle. The phenomenon is striking and accessible, yet it has only recently been noticed in a fluid mechanics laboratory and has received no theoretical attention at all.

How can I shake up a vortex from scratch anyway? I don't spin the container. I hold it in my hand and move my hand in a circle. Viscous friction against the container walls can only cause a vortex to decay. I should be able to do the same thing with an inviscid fluid. But doesn't Kelvin's theorem (conservation of circulation around closed fluid lines) forbid the introduction of vorticity then?

The splash is the mechanism which drives vorticity into the fluid. The shaking doesn't generate vorticity directly - it generates surface waves which carry angular momentum irrotationally. When a wave breaks over, the fluid volume ceases to be simply connected, and circulation develops as shown:



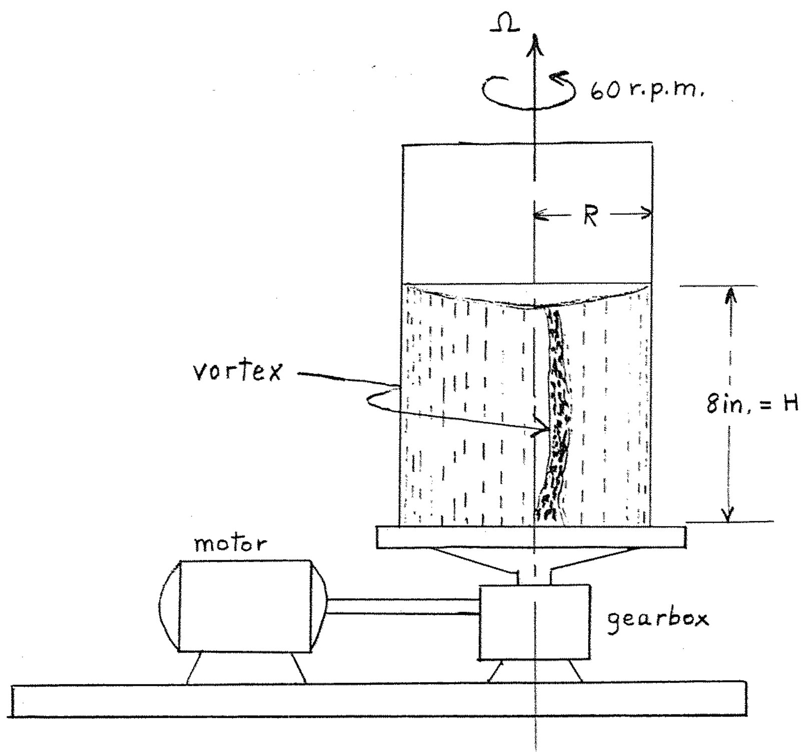
When the waves break and collapse the fluid ceases to carry its angular momentum irrotationally and spins up.

But why does further shaking concentrate vorticity and leave a compact vortex column near the center of the container? Apparently a rotating acceleration field can concentrate vorticity already present in a fluid with a free surface. An easy way to get a rotating acceleration field is to tilt the axis of rotation of a spinning vessel of fluid slightly off vertical. The fluid will reach nearly solid body rotation at an angular velocity $\underset{\sim}{\Omega}$ directed on angle θ off vertical. The acceleration of gravity will not be constant in a coordinate system fixed to the vessel, but will have a horizontal component of magnitude $\sim g\theta$ rotating at an angular speed $-\Omega$. That small, rotating acceleration generates inertial-gravity waves which concentrate vorticity into vertical sheets and filaments. The resulting non-steady flow is usually considered a great nuisance. Turntables for free surface experiments must be leveled very accurately to avoid it.

Fultz (1965) is the first to have found the phenomenon worthy of special experimental work. He produced intense vortices near the axis of a tilted rectangular tank and speculated on the wave mechanism for their generation. I conducted similar experiments in a cylindrical tank to make sure the vortices depend on surface wave motion and to see their structure and manner of formation.

The Experiments

The experimental setup is shown below:



The tank was 8 5/8 in. in diameter and was run at 60 r.p.m.; the speed was not continuously variable. I tried all kinds of flow visualization techniques - hydrogen bubbles, a dilute suspension of aluminum flakes, permanganate dye, fluorescein dye, pepper. Fluorescein dye, bright yellow, brightly illuminated against a black background, was far the best for most purposes, but pepper lying on the bottom showed how the boundary layer behaves there, and aluminum showed a violent surging up and down the vortex column.

Deliberately tilting the platform 1 or 2 degrees produced shear flow at all heights H. To produce a clean columnar vortex I had to level the platform as accurately as possible and adjust the water height (the only physical quantity I could vary easily and measure) to give resonance in an inertial-gravity wave mode. By making sure the water line matched a mark on the side of the vessel within .02 in. or so, no matter which way the vessel was turned, I could be sure the axis of rotation was within .004 rad. of the vertical. The condition for resonance in the first accessible inertial-gravity mode is

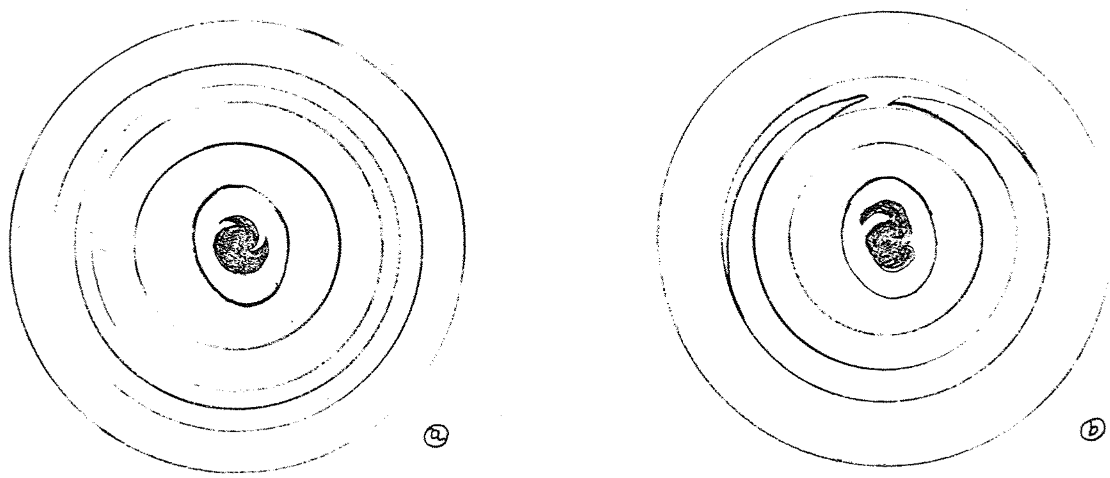
$$\tan\left(1.58\,\frac{H}{R}\right) = -\frac{\Omega^2 R}{1.58\,g} \qquad (14)$$

That gives H = 7.85 in. The experimental vortex appeared strongest when the level surface of the water was 8.2 in. above the bottom and was altogether absent above 9.2 in. and below 7.2 in. (it is easy to tell when a vortex is not there - the dye descends in globules and does not shear out). The concavity of the free surface, about 1 in. deep in the experiments, is eliminated in the theory by an artificial surface pressure distribution, so the experiment and theory are not quite matched. But the fact that a vortex shows up only near an inertial-gravity wave resonance when the axis of rotation is nearly vertical strongly suggests that the waves somehow sustain the vortex.

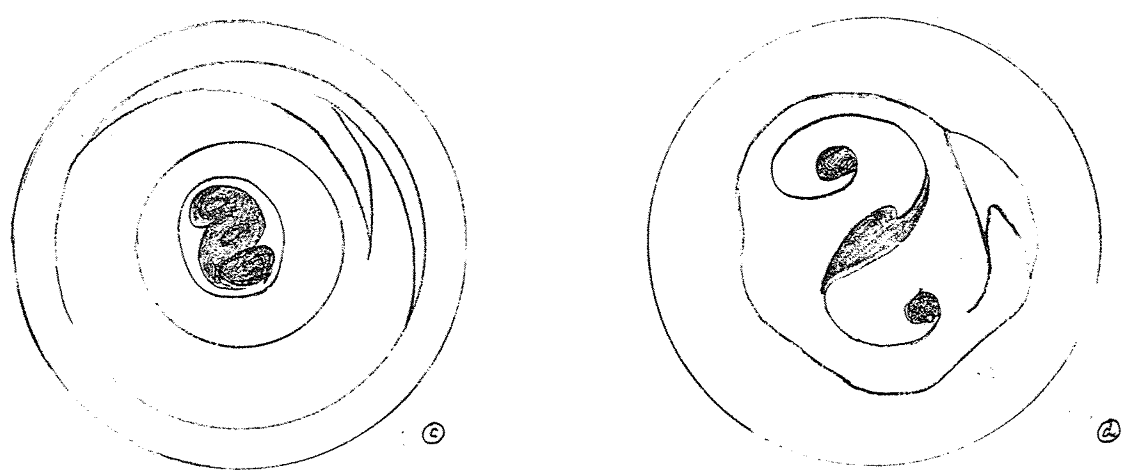
Some of the gross features of the fully evolved vortex: it is always slightly bowed and never rides dead center, but about 1 in. off the axis of the container; it lags the container slightly, moving at 55 r.p.m. around the axis; its core velocity is $\sim$100 r.p.m.— the core spins nearly twice as fast as the container; there is a strong vertical wave-like motion in the core; the dye does not diffuse through the core, but lies in twisted sheets.

Drops of dye added shortly before spin up is completed are smeared into circular cylinders as they sink. By looking down on these rings the evolution of the vortex could be traced. The following drawings are taken from a sequence of photographs of the process*. At zero time the motor is turned on and the fluid is spun up from rest. At four min. the spin up is almost complete, and an instability appears near the axis (a) . A jet soon appears near the periphery (b) .

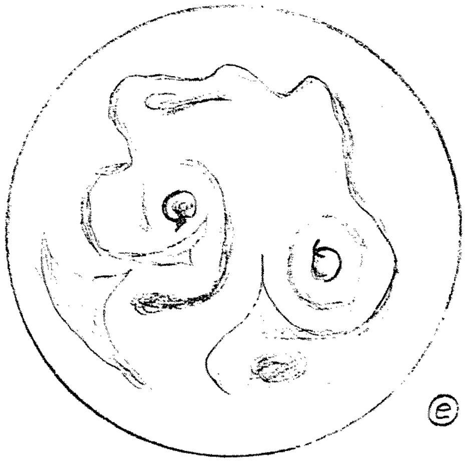
*See the photographs at the end of the paper.



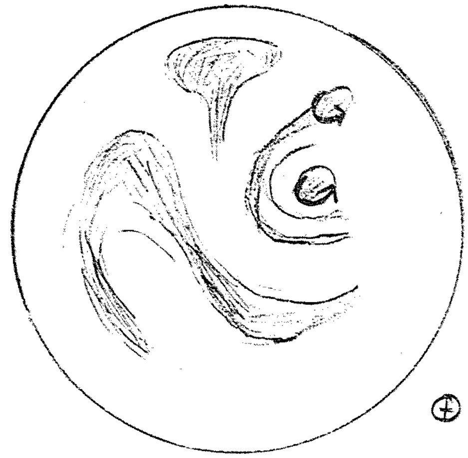
The instability at the center resolves itself into two vortices of positive sign c which move outward d



One of the vortices wraps up the dye which was left at the center e . After seven or eight min. one vortex wraps the other up to form a single loosely packed residue f .

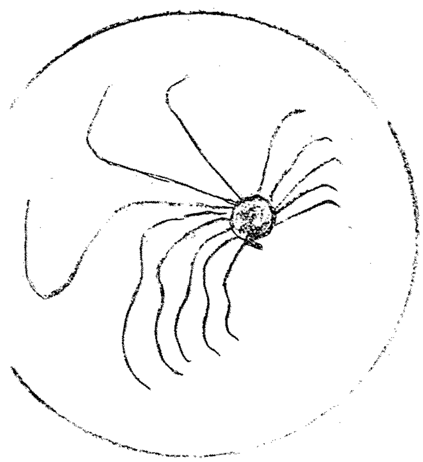


e



f

Fresh dye injected at the vortex core is pulled into thin sheets, and the whole surface becomes striated g .



g

After the spin up the flow is almost two-dimensional. Patterns a - g are top views of sheets and filaments of dye which extend through the whole depth of the fluid.

The vortex thus begins as a complicated shear instability. The rotating acceleration field breaks Taylor columns loose, and the resulting vortex sheets finally wrap up into a column. During its formation, the dye patterns suggest two-dimensional turbulence rather than a predictable evolution. But once a column vortex is formed, the driving acceleration sustains it against viscous dissipation. That balance might be accounted for analytically.

Theoretical Work

1. Equations of Motion and Statement of the Problem

The Navier-Stokes equations for an incompressible fluid and a rotating and accelerating reference frame are

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} + \underset{\sim}{a}(t) + 2\underset{\sim}{\Omega}\times\underset{\sim}{u} - g + \nabla\left(P - \tfrac{1}{2}\Omega^2 r^2\right) = \nu\nabla^2\underset{\sim}{u}$$

$$\nabla\cdot\underset{\sim}{u} = 0$$

$\underset{\sim}{\Omega}$ is the rotation, $\underset{\sim}{a}(t)$ the acceleration, g gravity, r the distance from the axis of rotation, P the kinematic pressure and ν the kinematic viscosity. Experssions in physical variables are marked with an (*) on the left. The boundary conditions are

$$\underset{\sim}{u} = 0 \quad \text{on solid boundaries}$$

$$* \qquad P = \tfrac{1}{2}\Omega^2 r^2 \quad \text{at surface}$$

The second condition is an artifice to remove the concavity of the upper surface due to solid body rotation. It might be important experimentally, it might have a focusing effect, but it is not the fundamental phenomenon, and there is no point in distorting the geometry to include it. Write

$$\underset{\sim}{a}(t) = \nabla(\underset{\sim}{a}\cdot r)$$

$$* \qquad \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = \nabla\xi + \underset{\sim}{\omega}\times\underset{\sim}{u}$$

where

$$\mathcal{E} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{u}}{2} \qquad \underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$$

and

$$-\underset{\sim}{g} = \nabla\left[g(z-H)\right]$$

where z is the vertical coordinate and H is the mean height of the fluid. Then for

$$* \qquad p = P + \mathcal{E} + \underset{\sim}{a}\cdot\underset{\sim}{r} + g(z-H) - \tfrac{1}{2}\Omega^2 r^2$$

the equations of motion and boundary conditions are

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{\omega}\times\underset{\sim}{u} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u} + \nabla p = \nu\nabla^2\underset{\sim}{u}$$

$$* \qquad \nabla\cdot\underset{\sim}{u} = 0 \tag{1}$$

$$\underset{\sim}{u} = 0 \quad \text{on solid boundaries}$$

$$p = \mathcal{E} + \underset{\sim}{a}\cdot\underset{\sim}{r} + g\zeta \quad \text{at surface } z = H + \zeta$$

A periodic solution with the period of $\underset{\sim}{a}(t)$ is sought.

Set $2\Omega = f$ and non-dimensionalize as follows:

$$\underset{\sim}{a}(t) = a\underset{\sim}{\alpha}(t) \qquad \text{where } \underset{\sim}{\alpha} \text{ is a unit vector}$$

lengths $\longrightarrow H$

speeds $\longrightarrow af^{-1}$

$p \longrightarrow aH$

$t \longrightarrow f^{-1}$

$\zeta \longrightarrow af^{-2}$

Eqns. 1 become

$$\left.\begin{aligned} &\frac{\partial \underset{\sim}{u}}{\partial t} + \varepsilon \underset{\sim}{\omega} \times \underset{\sim}{u} + \underset{\sim}{k} \times \underset{\sim}{u} + \nabla p = \frac{1}{Re} \nabla^2 \underset{\sim}{u} \\ &\qquad\qquad \text{etc.} \\ &p = \varepsilon \xi + \underset{\sim}{\alpha} \cdot \underset{\sim}{r} + \Gamma \zeta \quad \text{at } z = 1 + \varepsilon \zeta \end{aligned}\right\} \tag{2}$$

where

$$\Gamma = \frac{g}{f^2 H}$$

$$\varepsilon = \frac{a}{f^2 H}$$

$$Re = \frac{H^2 f}{\nu}$$

In the experiments Γ was about 0.3, R_ε was 5×10^5, and ε was between 10^{-2} and 10^{-3}. The viscous term is thus very small except near solid boundaries or interior shear surfaces. An interior solution can be sought without that term if the transient part of the flow can be recognized and dropped. The final statement of the problem is as follows:

$$\left.\begin{aligned} &\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{k} \times \underset{\sim}{u} + \nabla p = \varepsilon \underset{\sim}{u} \times \underset{\sim}{\omega} \\ &\nabla \cdot \underset{\sim}{u} = 0 \\ &\underset{\sim}{u} \cdot \underset{\sim}{n} = 0 \text{ on boundaries } z = 0 \text{ and } r = R \\ &p = \underset{\sim}{\alpha} \cdot \underset{\sim}{r} + \Gamma \zeta + \varepsilon \xi \quad \text{at } z = 1 + \varepsilon \zeta \end{aligned}\right. \tag{3}$$

and viscosity or an equivalent will be invoked to eliminate transients.

2. Perturbation Expansion

It is useful to change the boundary condition at $z = 1 + \varepsilon \zeta$ to a set of conditions at $z = 1$. Integrate the vertical momentum equation from $z = 1$ to $z = 1 + \varepsilon \zeta$:

$$p(z-1) = \underset{\sim}{\alpha} \cdot \underset{\sim}{r} + \Gamma \zeta + \varepsilon \xi (1) + \varepsilon \zeta \frac{\partial w}{\partial t}(1) + O(\varepsilon^2) \tag{4}$$

where $w = \underset{\sim}{u} \cdot \underset{\sim}{k}$. At the surface w is the convected derivative of ζ, so

$$w(z=1) = \frac{\partial \zeta}{\partial t} + \varepsilon \underset{\sim}{u}(1) \cdot \nabla \zeta - \varepsilon \zeta \frac{\partial w}{\partial z}(1) + O(\varepsilon^2)$$

$$= \frac{\partial \zeta}{\partial t} + \varepsilon \nabla_1 \cdot (\underset{\sim}{u} \zeta) + O(\varepsilon^2) \qquad (5)$$

where $\nabla_1 \cdot$ is the horizontal divergence. ζ can be eliminated between eqns. (4) and (5) and a boundary condition on $\underset{\sim}{u}$, p set up at $z=1$. Now expand all quantities in powers of ε :

$$p = p_0 + \varepsilon p_1 + \dots$$
$$etc.$$

Eqns. (3), (4) and (5) yield the following zeroth and first order problems:

<u>zeroth order</u>

$$\left.\begin{aligned} \frac{\partial \underset{\sim}{u}_0}{\partial t} + \underset{\sim}{k} \times \underset{\sim}{u}_0 + \nabla p_0 &= 0 \\ \nabla \cdot \underset{\sim}{u}_0 &= 0 \\ \underset{\sim}{u}_0 \cdot \underset{\sim}{n} = 0 \text{ at } z &= 0 \text{ and } r = R \\ \frac{\partial p_0}{\partial t} = \frac{\partial \underset{\sim}{\alpha}}{\partial t} \cdot \underset{\sim}{r} + \Gamma w_0 &\text{ at } z = 1 \end{aligned}\right\} \qquad (6)$$

<u>first order</u>

$$\left.\begin{aligned} \frac{\partial \underset{\sim}{u}_1}{\partial t} + \underset{\sim}{k} \times \underset{\sim}{u}_1 + \nabla p_1 &= \underset{\sim}{u}_0 \times \underset{\sim}{\omega}_0 \\ &etc. \\ p_1 = \Gamma \zeta_1 + \mathcal{E}_0 &+ \zeta_0 \frac{\partial w_0}{\partial t} \\ w_1 = \frac{\partial \zeta_1}{\partial t} &+ \nabla_1 \cdot (\underset{\sim}{u}_0 \zeta_0) \end{aligned}\right\} \qquad (7)$$

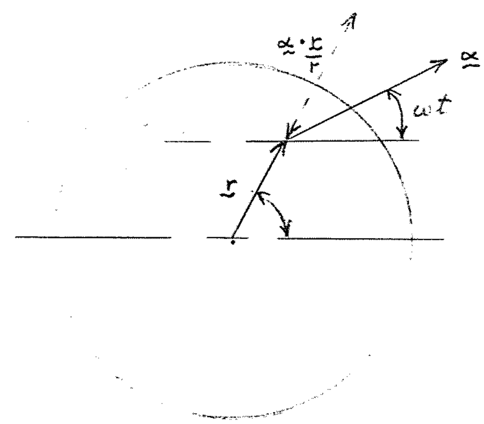
The curl of the zeroth order momentum equation is

$$\frac{\partial \underset{\sim}{\omega}_0}{\partial t} = \frac{\partial \underset{\sim}{u}_0}{\partial z} \tag{8}$$

a form which will be useful later.

3. Solution of the Zeroth Order Problem

The inertial-gravity wave response to a rotating acceleration $\underset{\sim}{\alpha}$ is sought, where $\underset{\sim}{\alpha}$ is perpendicular to the axis of rotation $\underset{\sim}{k}$. Homogeneous solutions to eqns.(6) should be dropped since they would decay under the action of viscosity. Viscosity can modify only slightly the forced response in the interior of the fluid if the forcing is done off resonance.



If $\underset{\sim}{\alpha}$ rotates at an angular velocity ω (seen from the rotating coordinate system, with ω non-dimensionalized by f), then the sketch shows

$$\underset{\sim}{\alpha}\cdot\underset{\sim}{r} = r\cos(\phi - \omega t)$$

within a phase factor. Switching to complex notation, we look for solutions to eqns.(6) whose ϕ and t dependence is given by the factor $\exp i(\phi - \omega t)$. In cylindrical coordinates eqns.(6) are

$$\frac{\partial u_0}{\partial t} - v_0 + \frac{\partial p_0}{\partial r} = 0$$

$$\frac{\partial v_0}{\partial t} + u_0 + \frac{1}{r}\frac{\partial p_0}{\partial \phi} = 0$$

$$\frac{\partial w_0}{\partial t} + \frac{\partial p_0}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_0) + \frac{1}{r}\frac{\partial v_0}{\partial \phi} + \frac{\partial w_0}{\partial z} = 0$$

$$w_0 = 0 \quad at \quad z = 0$$

$$u_0 = 0 \quad at \quad r = R$$

$$\frac{\partial p_0}{\partial t} = -i\omega r e^{i(\phi-\omega t)} + \Gamma w_0 \quad at \quad z = 1$$

where u_0, v_0, w_0 are the radial, tangential and vertical components of velocity. Write

$$u_0 = \tilde{u}_0 e^{i(\phi-\omega t)}$$

$$etc.$$

Then the momentum equations give

$$\tilde{u}_0 = \frac{i}{1-\omega^2}\left(\omega\frac{\partial \tilde{p}_0}{\partial r} - \frac{\tilde{p}_0}{r}\right)$$

$$\tilde{v}_0 = \frac{1}{1-\omega^2}\left(\frac{\partial \tilde{p}_0}{\partial r} - \frac{\omega}{r}\tilde{p}_0\right) \tag{9}$$

$$\tilde{w}_0 = -\frac{i}{\omega}\frac{\partial \tilde{p}_0}{\partial z}$$

and the continuity equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{p}_0}{\partial r}\right) - \frac{\tilde{p}_0}{r^2} - \frac{(1-\omega^2)}{\omega^2}\frac{\partial^2 \tilde{p}_0}{\partial z^2} = 0$$

A solution satisfying this equation and the boundary condition at $z = 0$ is

$$\tilde{p}_0 = A J_1(\beta r)\cos kz$$

for

$$\beta^2 = k^2\frac{(1-\omega^2)}{\omega^2} \tag{10}$$

By the first of eqns.(9), the boundary condition at $r = R$ is

$$\frac{\partial \tilde{p}_0}{\partial r} = \frac{\tilde{p}_0}{\omega r}$$

That is satisfied if

$$\beta R J_0(\beta R) = \left(\frac{1}{\omega}+1\right) J_1(\beta R) \tag{11}$$

and a set of eigenvalues $\{\beta n\}$ is specified. Thus

$$\tilde{p}_0 = \Sigma A_n J_1(\beta_n r) \cos k_n z \tag{12}$$

where k_n is given in terms of β_n by eqn. (10). The functions $J_1(\beta_n r)$ are orthogonal in the sense that

$$\int_0^R r J_1(\beta_m r) J_1(\beta_n r)\, dr = 0 \qquad m \neq n$$

Thus the quantities A_n can be found from the boundary condition at $z = 1$,

$$\tilde{p}_0 - \frac{\Gamma}{\omega^2} \frac{\partial \tilde{p}_0}{\partial z} = r$$

or

$$\Sigma A_n J_1(\beta_n r)\left(\cos k_n + \frac{\Gamma k_n}{\omega^2} \sin k_n\right) = r$$

$$A_n = \frac{\int_0^R r^2 J_1(\beta_n r)\, dr}{\left(\cos k_n + \frac{\Gamma k_n}{\omega^2} \sin k_n\right) \int_0^R r J_1^2(\beta_n r)\, dr} \tag{13}$$

A rotating acceleration field is simulated experimentally by tilting the axis of rotation of the vessel a small angle θ off the vertical. In that case

*

$$a = g\theta$$

$$\epsilon = \Gamma\theta$$

and the field rotates with an angular velocity $-\Omega$ in a reference frame fixed to the vessel —

$$\omega = -\frac{1}{2}$$

Then eqn.(11) gives $\beta_1 R$ = 2.74, and eqns. (10) and (13) yield a condition for resonance in the first mode. In dimensional form,

$$* \qquad \tan\left(1.58\,\frac{H}{R}\right) = -\frac{\Omega^2 R}{1.58\,g} \qquad (14)$$

4. <u>The First Order Problem</u>

The forcing functions of eqns. (7) can now be evaluated in terms of the zeroth order solution, eqns. (9)-(13). The zeroth-order quantities have the form

$$u_0 = \tilde{u}_0(r,z)\,e^{i(\phi-\omega t)} + \tilde{u}_0^*(r,z)\,e^{-i(\phi-\omega t)}$$

so their products will contain terms independent of ϕ and t and terms with the factor $\exp \pm 2i(\phi-\omega t)$. At this stage we can hope to find a steady circulation driven by the steady part of the non-linear coupling. Since the motions of frequency 2ω are of no interest, the problem will be formulated with the steady parts only of the forcing functions included.

Let $\mathcal{S}\{\ \}$ mean "steady part of $\{\ \}$". By eqn. (8)

$$\mathcal{S}\{\underset{\sim}{u}_0 \times \underset{\sim}{\omega}_0\} = \frac{1}{i\omega}\frac{\partial}{\partial z}\left(\underset{\sim}{\tilde{u}}_0^* \times \underset{\sim}{\tilde{u}}_0\right) = (\kappa, \lambda, \mu)$$

where κ, λ, μ are the components of that steady body force.

Since $\dfrac{\partial \zeta_0}{\partial t} = w_0\,(z=1)$,

$$\mathcal{S}\left\{\xi_0 + \zeta_0 \frac{\partial w_0}{\partial t}\right\} = \mathcal{S}\left\{\xi_0 + \frac{\partial}{\partial t}(\zeta_0 w_0) - w_0^2\right\}$$

$$= \mathcal{S}\left\{\frac{u_0^2 + v_0^2 - w_0^2}{2}\right\} = \sigma(r)$$

at $z = 1$.

Finally, since $\mathcal{S}\left\{\dfrac{\partial G}{\partial \phi}\right\} = 0$ for any forcing function term,

$$\mathcal{S}\{\nabla_1\cdot(\underset{\sim}{u}_0\zeta_0)\} = \mathcal{S}\{\frac{1}{r}\frac{\partial}{\partial r}(ru_0\zeta_0) + \frac{1}{r}\frac{\partial}{\partial\phi}(v_0\zeta_0)\}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\mathcal{S}\{u_0\zeta_0\}\right) = \frac{1}{i\omega r}\frac{\partial}{\partial r}\left[r(\tilde{u}_0^*\tilde{w}_0 - \tilde{w}_0^*\tilde{u}_0)\right]$$

Since $\tilde{P}_0$ is real, eqns. (9) give

$$\tilde{u}_0^*\tilde{w}_0 - \tilde{w}_0^*\tilde{u}_0 = 0$$

Hence the steady part of $\nabla_1\cdot(\underset{\sim}{u}_0\zeta_0)$ is zero, and the body force component λ is zero as well. In cylindrical coordinates, eqns.(7) are then

$$\left.\begin{aligned}
&\frac{\partial u_1}{\partial t} - v_1 + \frac{\partial p_1}{\partial r} = K(r,z)\\
&\frac{\partial v_1}{\partial t} + u_1 = 0\\
&\frac{\partial w_1}{\partial t} + \frac{\partial p_1}{\partial z} = \mu(r,z)\\
&\frac{1}{r}\frac{\partial}{\partial r}(ru_1) + \frac{\partial w_1}{\partial z} = 0\\
&w_1 = 0 \text{ at } z = 0\\
&u_1 = 0 \text{ at } r = R\\
&\left.\begin{aligned} p_1 &= \Gamma\zeta_1 + \sigma(r)\\ w_1 &= \frac{\partial\zeta_1}{\partial t}\end{aligned}\right\} \text{at } z = 1
\end{aligned}\right\} \quad (15)$$

where

$$\left.\begin{aligned}
&K = \frac{1}{i\omega}\frac{\partial}{\partial z}\left(\tilde{v}_0^*\tilde{w}_0 - \tilde{w}_0^*\tilde{v}_0\right)\\
&\mu = \frac{1}{i\omega}\frac{\partial}{\partial z}\left(\tilde{u}_0^*\tilde{v}_0 - \tilde{v}_0^*\tilde{u}_0\right)\\
&\sigma = \left(\tilde{u}_0\tilde{u}_0^* + \tilde{v}_0\tilde{v}_0^* - \tilde{w}_0\tilde{w}_0^*\right) \text{ at } z = 1
\end{aligned}\right\} \quad (16)$$

Since the steady forcing functions do not depend on ϕ ,

ϕ -derivatives have been dropped from the equations.

The obvious thing to do now is set the time derivatives in eqns. (15) equal to zero and solve for the steady flow. The result is

$$u_1 = w_1 = 0$$

$$(v_1 - \Gamma \zeta_{1_r}) = \sigma_r - K(z, r) - \int_z^1 \mu_r(r, \xi)\, d\xi \qquad (17)$$

but it is impossible to find v_1 and ζ_1 separately. Any axially symmetric geostrophic flow is a solution to the homogeneous equations and can be added on. This is an embarrassing ambiguity, since the vortex we wanted in the first place is such a geostrophic flow. There was the same ambiguity in the zeroth order problem, of course, but there we knew how to identify and reject transient terms. Here we don't. The only legitimate way to resolve the trouble is to retain viscosity explicitly. Another way might be to solve the non-steady problem, to turn on non-linearity at $t = 0$. At first glance this seems inappropriate (if not absurd). But a similar ambiguity is found for rotating flow about obstacles, and in the cases where both non-steady and viscous problems have been worked out, it is found that the asymptotic steady flow obtained by Laplace transforms is the same as the steady viscous flow outside of thin boundary layers. (Moore, 1963). The Laplace transformed equations of motion are formally identical to the steady equations with a Rayleigh viscosity term added, a term proportional to the velocity rather than second space derivatives of the velocity. This is a cheap way of buying friction without raising the order of the differential equations.

If eqns. (15) are Laplace transformed for zero initial conditions, then the momentum equations yield $\bar{u}_1, \bar{v}_1, \bar{w}_1$ in terms of $\bar{p}_1$:

$$\bar{u}_1 = \frac{1}{1+s^2}\left(K - s\frac{\partial \bar{p}_1}{\partial r}\right)$$

$$\bar{v}_1 = \frac{1}{1+s^2}\left(\frac{\partial \bar{p}_1}{\partial r} - \frac{K}{s}\right)$$

$$\bar{w}_1 = \frac{\mu}{s^2} - \frac{1}{s}\frac{\partial \bar{p}_1}{\partial z}$$

where

$$\bar{p}_1(r, z; s) = \int_0^\infty p_1(r, z, t)e^{-st}dt$$

etc. These expressions for $\bar{u}_1$ and $\bar{w}_1$ can be put into the continuity equation, and the boundary conditions expressed in terms of $\bar{p}_1$. Write

$$\bar{p}_1 = \frac{F(r, z; s)}{s}$$

$$\mu = \frac{\partial M}{\partial z} \qquad \text{(see eqn.(16))}$$

The problem is then

$$\left.\begin{aligned}
&\frac{s^2}{1+s^2}\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial F}{\partial r} - K\right)\right] + \frac{\partial^2}{\partial z^2}(F - M) = 0 \\
&z = 0 \quad \frac{\partial}{\partial z}(F - M) = 0 \\
&z = 1 \quad \frac{\partial}{\partial z}(F - M) = \frac{s^2}{\Gamma}(\sigma - F) \\
&r = 0 \quad F \text{ finite} \\
&r = R \quad \left(\frac{\partial F}{\partial r} - K\right) = 0
\end{aligned}\right\} \qquad (18)$$

The forces K, σ and M are complicated functions of r and z .

But let us approximate $\hat{P}_0$ by the first term of eqn. (12) (that term can be made to dominate experimentally by exciting near its resonance). Eqns. (9) and (16) show

$$M = \mathcal{M}(r)(1+\cos lz)$$
$$K = \mathcal{K}(r)\cos lz$$

under that approximation, where $l = 2k$, and $\mathcal{M}$ and $\mathcal{K}$ are multiples of low-order Bessel functions. Set

$$F = H(r;s)\cos lz + \eta(z,r;s)$$

Then the problem posed in eqns. (18) can be split:

$$\left.\begin{aligned}
&\text{(A)}\quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dH}{dr}\right) - l^2\frac{(1+s^2)}{s^2}H = \frac{1}{r}\frac{d}{dr}(r\mathcal{K}) - l^2\frac{(1+s^2)}{s^2}\mathcal{M}\\
&\qquad r=0 \quad H \text{ finite}\\
&\qquad r=R \quad \frac{dH}{dr} = \mathcal{K}(R)
\end{aligned}\right\}\quad (19)$$

$$\left.\begin{aligned}
&\text{(B)}\quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\eta}{\partial r}\right) + \frac{1+s^2}{s^2}\frac{\partial^2\eta}{\partial z^2} = 0\\
&\qquad r=0 \quad \eta \text{ finite}\\
&\qquad r=R \quad \frac{\partial\eta}{\partial r} = 0\\
&\qquad z=0 \quad \frac{\partial\eta}{\partial z} = 0\\
&z=1 \quad \frac{\partial\eta}{\partial z} + \frac{s^2}{\Gamma}\eta = l\sin l\,(H-\mathcal{M}) + \frac{s^2}{\Gamma}(\sigma - H\cos l)
\end{aligned}\right\}\quad (20)$$

(A) can be solved by variation of parameters and the result put into the top boundary condition of (B). (B) can be solved by separation of variables and the condition at $z = 1$ met by expanding the right-hand side in the same series used for η. The result is opaque even in the limit $s \to 0$. In the next section, the essential consequences of eqns. (19) and (20) are illuminated by a simple model of the mathematics.

5. The Model Problem

The following substitutions transform the original problem into the model:

(i) $H \rightarrow Y, \quad \eta \rightarrow y$

(ii) $\frac{\partial}{\partial r}, \; \frac{1}{r}\frac{\partial}{\partial r}(r \quad) \longrightarrow \frac{\partial}{\partial x}$

(iii) $\mathcal{K}, \mathcal{M} \longrightarrow$ constants

(iv) boundary conditions at $x = 0$ are chosen to avoid artificial singularities there as $s \longrightarrow 0$.

(v) $1 + s^2 \longrightarrow 1$; that eliminates branch cuts at $s = \pm i$ which would add to the inverted result a term decaying algebraically with time.

The models of (A) and (B) are

$$\text{(A)} \qquad \left. \begin{array}{l} s^2 Y_{xx} - \ell^2 Y = -\ell^2 \mathcal{M} \\ x = 0 \quad Y = \mathcal{M} \\ x = R \quad Y_x = \mathcal{K} \end{array} \right\} \tag{21}$$

$$\text{(B)} \qquad \left. \begin{array}{l} s^2 y_{xx} + y_{zz} = 0 \\ x = 0 \quad y = 0 \\ x = R \quad y_x = 0 \\ z = 0 \quad y_z = 0 \\ z = 1 \quad y_z + \frac{s^2}{\Gamma} y = \ell \sin \ell \, (Y - \mathcal{M}) + \frac{s^2}{\Gamma}(\sigma(x) - Y \cos \ell) \end{array} \right\} \tag{22}$$

The solution to (A) is

$$Y = \mathcal{M} + \frac{sK}{\ell} \frac{\sinh \frac{\ell x}{s}}{\cosh \frac{\ell R}{s}} \tag{23}$$

A solution to (B) satisfying the boundary condition at $z = 0$ is

$$y = A \sin \lambda x \cosh s\lambda z$$

The boundary condition at $x = R$ requires

$$\cos \lambda R = 0 \qquad \lambda_n = \frac{(2n-1)\pi}{2R}$$

Thus

$$y = \sum_{n=1}^{\infty} A_n \sin \lambda_n x \cosh s\lambda_n z \tag{24}$$

The series is orthogonal in x from 0 to R. By eqns. (23) and (24), the boundary condition at $z = 1$ is

$$\sum A_n \sin \lambda_n x \left[s\lambda_n \sinh s\lambda_n + \frac{s^2}{\Gamma} \cosh s\lambda_n \right]$$
$$= s \sin \ell K \frac{\sinh \frac{\ell x}{s}}{\cosh \frac{\ell R}{s}} + \frac{s^2}{\Gamma}(\sigma(x) - \mathcal{M} \cos \ell)$$
$$- s^3 \frac{\cos \ell}{\ell \Gamma} K \frac{\sinh \frac{\ell x}{s}}{\cosh \frac{\ell R}{s}}$$

But $\sinh \frac{\ell x}{s} = -\frac{2\ell}{Rs} \cosh \frac{\ell R}{s} \sum \frac{(-)^n \sin \lambda_n x}{\lambda_n^2 + \frac{\ell^2}{s^2}}$

Suppose

$$\sigma(x) - \mathcal{M} \cos \ell = \sum a_n \sin \lambda_n x$$

Then

$$A_n \left[s\lambda_n \sinh s\lambda_n + \frac{s^2}{\Gamma} \cosh s\lambda_n \right] = -(-)^n \frac{2\ell \sin \ell K}{R\left(\frac{\ell^2}{s^2} + \lambda_n^2\right)} + \tag{25}$$
$$+ \frac{s^2}{\Gamma} a_n + (-)^n \frac{2 \cos \ell \, K s^2}{\Gamma R \left(\frac{\ell^2}{s^2} + \lambda_n^2\right)}$$

The analog F is

$$F_{model} = y \cos l z + y$$

Thus the analog Laplace-transformed pressure is

$$\bar{P}_{1model} = \frac{F_{model}}{s}$$

$$= \left[\frac{M}{s} + \frac{K}{l} \frac{\sinh \frac{lx}{s}}{\cosh \frac{lR}{s}} \right] \cos l z + \Sigma \frac{A_n(s)}{s} \sin \lambda_n x \cosh s \lambda_n z \tag{26}$$

The result of inverting this expression is a model for p_1 .

The completely steady part of p_1 is the residue of eqn. (26) at $s = 0$. As mod $s \to 0$,

$$A_n(s) \longrightarrow \frac{\frac{a_n}{\Gamma} - (-)^n 2 \frac{\sin l}{Rl} K}{\lambda_n^2 + \frac{1}{\Gamma}}$$

from eqn. (25). Thus the steady part of the analog pressure is

$$\boxed{p_{1model}(\text{steady}) = M \cos l z + \sum_{n=1}^{\infty} \left[\frac{\frac{a_n}{\Gamma} - (-)^n 2 \frac{\sin l}{Rl} K}{\lambda_n^2 + \frac{1}{\Gamma}} \right] \sin \lambda_n x} \tag{27}$$

The first term is analogous to the indefinite integral of the vertical momentum eqn. (15) for the steady flow: $p_1 = M$. The series is the pressure field of the analog geostrophic flow, the ambiguous term in the purely steady, inviscid problem. The first part of the series converges very rapidly, since the series in a_n must already converge without the λ_n^2 denominator. That part represents a very smooth field, nothing like a concentrated vortex. But the part of the geostrophic series proportional to K does not converge so rapidly. In fact, as $\Gamma \to \infty$,

$$P_{1\,mod.}(\text{geostr.}) \Longrightarrow \left(\frac{2K}{R\ell}\sin\ell\right)x$$

for $0 \leq x < R$, and the slope of P_1 (geostr.) goes to zero abruptly at $x = R$. Some such slowly converging term in the solution to the real problem may show the kind of vorticity concentration one sees experimentally.

The second term of eqn. (26)

$$\frac{K}{\ell}\,\frac{\sinh\frac{\ell x}{s}}{\cosh\frac{\ell R}{s}}\cos\ell z$$

has an accumulation point of poles at the origin of the complex s-plane. The inversion can be performed by summing the residues inside the usual integration contour, and the result is

$$-\frac{2K}{R}\cos\ell z\sum\frac{(-)^n}{\lambda_n^2}\cos\frac{\ell t}{\lambda_n}\sin\lambda_n x$$

The higher the spatial frequency of a mode, the slower it oscillates in time. At $t = 0$ the expression sums to

$$\left(\frac{2K}{R}\cos\ell z\right)x$$

for $0 \leq x < R$, and again the slope goes abruptly to zero at $x = R$. As t grows, the phase correlations among the lower modes are broken, and the function oscillates about zero. Only for high λ_n, for which $\cos \ell t/\lambda_n$ is about 1, and near the wall $x = R$ are the phase correlations left intact. Thus for large t the function represents a pressure boundary layer at the wall. That could have been anticipated from the original first order eqns. (15). At the wall $u_1 = 0$, so $\partial v_1/\partial t = 0$. Thus $v_1 = 0$ also. Then at the wall

$$\frac{\partial p_1}{\partial r} = K(R, z)$$

$$\frac{\partial w_1}{\partial t} + \frac{\partial p_1}{\partial z} = \mu(r, z)$$

and by cross differentiation,

$$\left\{ \frac{\partial}{\partial t}\left(\frac{\partial w_1}{\partial r}\right) = \frac{\partial \mu}{\partial r} - \frac{\partial K}{\partial z} \right\}_{r=R}$$

From eqns. (9) and (16) it is easy to show $\frac{\partial \mu}{\partial r} - \frac{\partial K}{\partial z} \neq 0$, so $\frac{\partial w_1}{\partial r}$ must increase linearly with time at the wall.

The model of this section is not a physical approximation, but a sketch of the mathematics of the problem posed by eqns. (19) and (20). Those equations in turn are only an approximation to eqns. (18), and the relevance of eqns. (18) depends on the assumed equivalence of Rayleigh viscosity and physical viscosity for the steady interior flow. The time dependent part of the flow found by Laplace transformation has nothing to do with the original problem, although it may be interesting in itself. In particular, the boundary layer sheet at the wall would be smoothed and steadied by viscosity. While confessing these sins I should mention again the artificial surface pressure condition assumed right at the beginning. Much work remains before any real theoretical appreciation of the shake-up vortex can be achieved.

References

Fultz, David — in review of symposium on concentrated vortices by Küchemann, J.F.M. 21, pg.1, 1965.

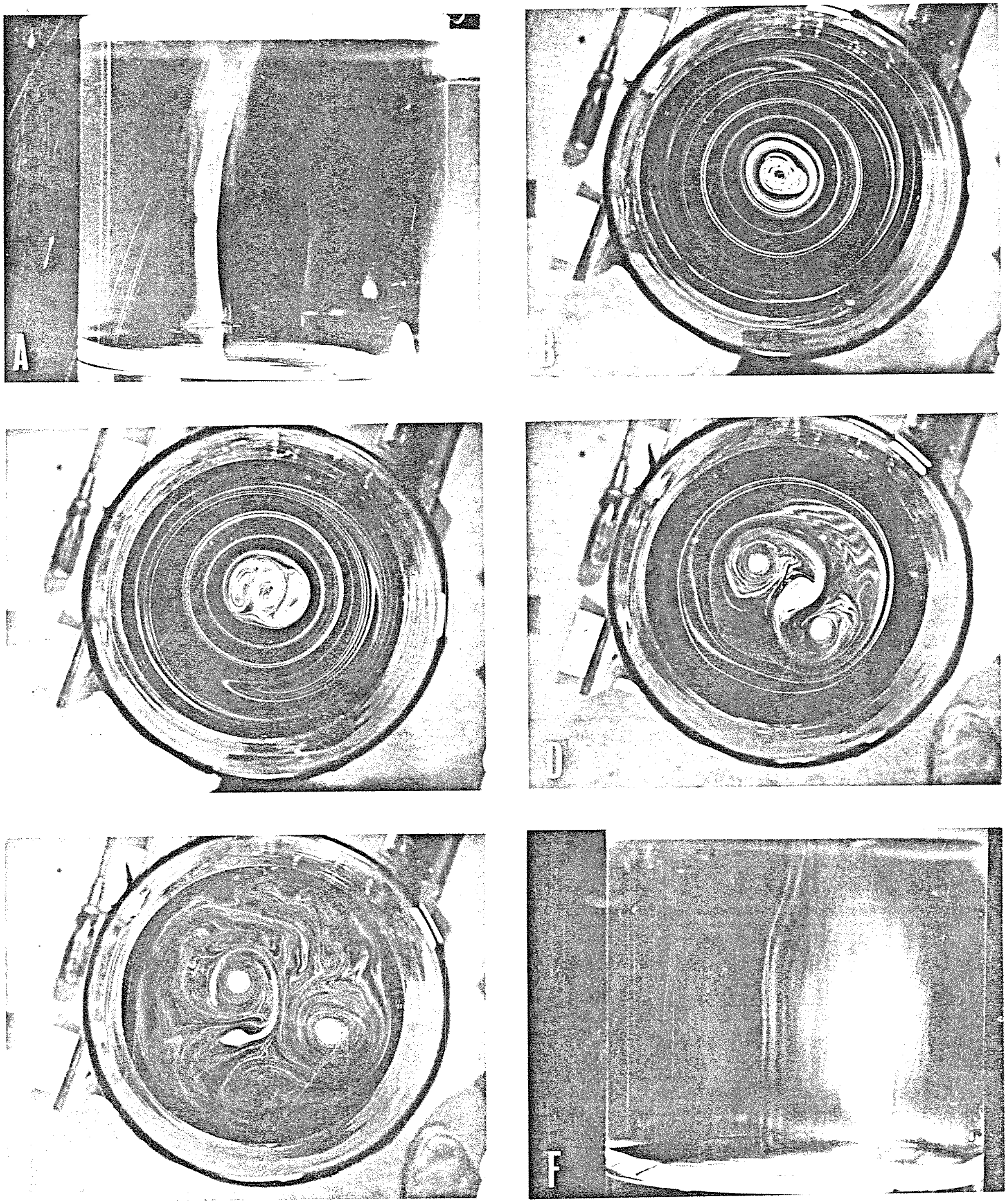
Moore, Derek — Lectures on Rotating Fluids, G.F.D. course notes, Vol.I, 1963.

Photographs

(a) a column vortex sustained by gravity waves — Fluorescein dye has been injected at the core.

(b), (c), (d), (e) stages in the development of the vortex — same as (b) $\longrightarrow$ (d) in the text.

(f) side view of the phenomenon illustrated in (g) of the text — sheets of dye continue to be sheared out of the vortex core.


A
B
D
F

The Pulsational Stability of a Convective Atmosphere

Douglas O. Gough

Introduction

As a star evolves it may pass through phases where it can no longer remain in stable quasi-static equilibrium. The instability arises from the existence of growing oscillatory disturbances (overstability) when the conditions inside the star are favourable. The star then begins to pulsate with increasing amplitude; eventually the structure of the star will be sufficiently changed so that the pulsational mode no longer grows, and the star continues to vibrate with finite amplitude. Recently numerical calculations have been made to determine when a star is pulsationally unstable to infinitesimal radial perturbations (e.g. Baker and Kippenhahn, 1962) but the results do not agree well with observation. There are two major reasons why this could be so. The first is that equilibrium structure about which the perturbation analysis was made may not have been correct, and the second is that convection was disregarded in the stability analysis, although it was taken into account to compute the equilibrium model.

It has been observed that in the computed unstable stars which lie farthest from the observed variable stars, convection plays an increasingly important role in providing the energy flux in the equilibrium model, and it seems likely, therefore, that its neglect in the stability analysis may have led to serious errors. The calculations of Baker and Kippenhahn were made to determine the positions of the so-called Cepheid Variable stars. There is another group of variable stars, however, with very long periods, in which convective energy transport is probably important. Stability calculations on these have found that they are stable. Could convection have the opposite effect for these stars, and cause them to pulsate?

It is the purpose of this study to try to understand how convection can affect the stability of a star. For a star in radiative equilibrium, the instability arises in the atmosphere, to which the pulsation is also confined. The detailed structure of the deep interior is of little importance (Zhevakin, 1953; Cox, 1963), except in so far as it provides the boundary conditions for the base of the atmosphere. It is plausible that this will still be the case when convection is taken into account; the weak convection which may take place in the deep interior is unlikely to affect appreciably the high damping of the pulsation mode there.

In order to understand the physics of the pulsational instability the simple one-zone model of Baker (1963) will be used. The model was invented specifically for this purpose and concentrates attention on a single spherical shell through which a flux of energy flows. It can be clearly seen how the shell can tap the thermal energy and convert it into energy of pulsation. Baker used his model only to consider the stability of a star in radiative equilibrium, however, and his analysis will now be extended to take convection into account. In the absence of anything better, the mixing-length theory of convection will be employed, and for completeness a brief description of the theory as usually used in astrophysics will first be given. Later it will be found necessary to modify this theory to take account of the time dependence of the mean properties of the atmosphere.

I. Mixing-length Theory of Turbulent Convection

The starting point of the mixing-length theory is the equation

$$H_c = \overline{\rho C_p w \theta} \tag{1.1}$$

where H_c is the convective heat flux, C_p the specific heat at constant pressure, w the vertical velocity and ρ the density. In the atmosphere we neglect the effects of curvature and restrict ourselves to a plane parallel layer with the z-axis vertically upward. The overbar denotes an average over the horizontal coordinates x and y. In a statistically steady state such an average is a function only of z. θ is the deviation of the temperature from the horizontal mean defined by the equation

$$T(x,y,z,t) = \overline{T}(z) + \theta(x,y,z,t). \tag{1.2}$$

The mixing-length theory assumes that the temperature fluctuation at (x, y, z) is determined by the arrival of a convective element which started life at (x, y, z_0) with approximately the local ambient temperature. If we expand the temperature and its horizontal average in a Taylor series and subtract we obtain

$$\theta(x,y,z,t) = \left(\frac{dT}{dz} - \frac{d\overline{T}}{dz}\right)(z - z_0) + \ldots \tag{1.3}$$

where $\frac{dT}{dz}$ is the temperature gradient actually experienced by the convective element. The next approximation is to replace the right-hand side of (1.1) by the product of the mean values of each quantity and multiply by a phase factor φ. It is further assumed that each convective element travels only a finite distance $\tilde{\ell}$, the mixing-length, after which it loses its identity, owing to its break-up by the ambient turbulent field. Since, on the average, at any given time an element will have traversed about one-half of its mixing-length, we replace $z - z_o$ in (1.3) by $\frac{1}{2}\tilde{\ell}$ to obtain the mean value of θ. The convective heat flux is then given by:

$$H_c = \frac{1}{2}\varphi\rho C_p \bar{\omega}\tilde{\ell}\left(\frac{dT}{dz} - \frac{d\bar{T}}{dz}\right)$$

The phase factor φ is usually taken to be unity and we shall adopt this procedure here.

The mean velocity $\bar{\omega}$ is determined by equating the work done by the buoyancy force on an element to its final kinetic energy. Since the final velocity just before break-up is about $2\bar{\omega}$ we obtain

$$\frac{1}{2}\rho(2\bar{\omega})^2 \simeq \frac{1}{2}g\tilde{\ell}^2\left(\frac{d\bar{\rho}}{dz} - \frac{d\rho}{dz}\right) \qquad (1.4)$$

where $\frac{1}{2}g\left(\frac{d\bar{\rho}}{dz} - \frac{d\rho}{dz}\right)\tilde{\ell}$ is the mean buoyancy force and g the gravitational acceleration. If the element always remains in pressure balance with its surroundings

$$\frac{d\bar{\rho}}{dz} - \frac{d\rho}{dz} = \frac{\delta\bar{\rho}}{\bar{T}}\left(\frac{dT}{dz} - \frac{d\bar{T}}{dz}\right)$$

where $\delta = -\left.\frac{\partial \log\rho}{\partial \log T}\right/_P$.

Equation (1.4) then becomes

$$\overline{\omega}^2 = \frac{1}{4} \frac{g \tilde{\ell}^2 \delta}{T}\left(\frac{dT}{dz} - \frac{d\overline{T}}{dz}\right) \tag{1.5}$$

and the heat flux is

$$H_c = \frac{1}{4} \rho C_p \tilde{\ell}^2 \sqrt{\frac{g\delta}{T}} \left(\frac{dT}{dz} - \frac{d\overline{T}}{dz}\right)^{\frac{3}{2}} \tag{1.6}$$

If we assume that there is no loss of heat by radiation during the lifetime of the convective element, the derivative $\frac{dT}{dz}$ is simply the adiabatic temperature gradient which is calculated assuming that the element remains always in pressure balance with the ambient medium.

$$\left(\frac{dT}{dz}\right)_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{P} \frac{dP}{dz} = \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{\overline{P}} \frac{d\overline{P}}{dz}$$

where Γ_2 is the second adiabatic exponent of Chandrasekhar (1938) defined by

$$\left(\frac{d \log T}{d \log P}\right)_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2}$$

Thus we may define the "adiabatic heat flux" $H_{c_{ad}}$ by

$$H_{c_{ad}} = \frac{1}{4} \rho C_p \tilde{\ell}^2 \sqrt{\frac{g\delta}{T}} \left[\left(\frac{dT}{dz}\right)_{ad} - \frac{d\overline{T}}{dz}\right]^{3/2}$$

This formula for H_c may be improved by estimating the loss of heat by radiation (Vitense, 1953). The horizontal gradients in temperature are, in order of magnitude, equal to the vertical gradients in temperature fluctuation, and hence we may estimate the rate of loss of heat by an element by

$$\tilde{K}\left[\frac{dT}{dz} - \frac{d\overline{T}}{dz}\right] S$$

where S is the surface area of the element and $\tilde{K}$ is the radiative conductivity.

But this is simply the difference between the heat flux produced by the element if it were to rise adiabatically and the actual heat flux of the element. Hence

$$\tilde{K}\left(\frac{dT}{dz}-\frac{d\bar{T}}{dz}\right)S = S_h\left(H_{c_{ad}}-H_c\right)$$
$$= \frac{1}{2} S_h \rho C_p \bar{\omega}\tilde{\ell}\left[\left(\frac{dT}{dz}\right)_{ad}-\frac{dT}{dz}\right]$$

where S_h is the horizontal cross-sectional area of an element. Following Vitense, and taking $S_h/S = 2/3$, this can be rewritten

$$\frac{\frac{dT}{dz}-\frac{d\bar{T}}{dz}}{\left(\frac{dT}{dz}\right)_{ad}-\frac{dT}{dz}} = \frac{1}{3}\,\frac{\bar{\omega}\tilde{\ell}}{\tilde{K}}\rho C_p . \tag{1.7}$$

The factor $\frac{1}{3}$ is, of course, rather arbitrary. Equations (1.5) to (1.7) may now be combined to yield a quadratic in $H_c^{1/3}$.

$$(\rho C_p)^{2/3}\left(\frac{g\delta\tilde{\ell}^4}{16T}\right)^{2/3} H_c^{2/3} + \frac{3}{2}k\left(\frac{g\delta\tilde{\ell}^4}{16T}\right)^{1/3}(\rho C_p)^{-1} H_c^{1/3} - \frac{\delta g\beta\tilde{\ell}^4}{16T} = 0$$

where $k = \frac{\tilde{K}}{\rho C_p}$ and $\beta = \left(\frac{dT}{dz}\right)_{ad} - \frac{d\bar{T}}{dz}$ is the magnitude of the superadiabatic temperature gradient.

The solution is

$$H_c = T\rho C_p \frac{16\left(\frac{3}{4}k\right)^3}{g\delta\tilde{\ell}^4}\left\{-1+\left(1+\frac{g\delta\beta\tilde{\ell}^4}{9Tk^2}\right)^{\frac{1}{2}}\right\}^3 \tag{1.8}$$

The parameter $\frac{g\delta(1/T)\beta\tilde{\ell}^4}{k^2}$ is the product of the Prandtl number and a Rayleigh number based on the mixing-length. If this is large, as is the case in the middle of a convective zone, the motions of the elements are nearly adiabatic and (1.8) reduces to:

$$H_c \simeq \frac{1}{4}\rho C_p \bar{l}^2 \sqrt{\frac{g\delta}{T}}\, \beta^{3/2} = H_{c_{ad}} \tag{1.9}$$

It should be emphasized that this theory holds only when the conditions in the ambient medium do not change appreciably over a mixing length, so that there is some meaning in retaining only the first order terms in the Taylor expansions (1.2), (1.3). The mixing length is usually taken to be one or two pressure or density scale heights, and so this condition is not really fulfilled, but the situation is even worse near the boundary of an ionisation zone where the physical properties of the gas vary rapidly. In this case the path of the element should be followed in greater detail (Spiegel, 1963) but this added complication will not be considered in this initial investigation.

II. Formulation of the Linearised Stability Problem.

We take for the independent variable the mass M lying inside the radius r in the equilibrium model; this is then a Lagrangian variable which moves with the gas in a radial pulsation. The equation of motion of the star is then given by

$$\frac{\partial P}{\partial M} = -\frac{1}{4\pi r^2}\left(\frac{GM}{r^2} + \frac{\partial^2 r}{\partial t^2}\right), \tag{2.1}$$

which is supplemented by the equation of continuity:

$$\frac{\partial r}{\partial M} = \frac{1}{4\pi r^2 \rho} \tag{2.2}$$

where t is time and G is the gravitational constant. We are assuming spherical symmetry throughout, and the effects of viscosity and turbulent pressure have been neglected. The equation of state is that of a perfect gas in a radiation field.

$$P = \frac{\mathcal{R} \rho T}{\beta \mu}, \text{ where } \beta = \frac{P_{gas}}{P} = 1 - \frac{aT^4}{3P} \tag{2.3}$$

Here μ is the molecular weight of the gas, $\mathcal{R}$ is the gas constant and a is Stefan's constant. The contribution to the luninosity by radiative diffusion L_r, is:

$$L_r = - \frac{64\pi^2 a c r^4 T^3}{3 \kappa} \frac{\partial T}{\partial M} \tag{2.4}$$

where κ is the mean opacity and c the velocity of light. The energy equation is obtained by equating the difference between the luminosities on either side of a spherical shell with the rate of change of thermal and turbulent energies.

$$\frac{\partial L}{\partial M} = - \frac{\partial}{\partial t}\left(Q + E_t\right) = - C_p \frac{\partial T}{\partial t} + \frac{\delta}{\rho} \frac{\partial P}{\partial t} - \frac{\partial}{\partial t}\left(\frac{1}{2} \bar{u}^2\right) \tag{2.5}$$

where $\bar{u}^2$ is the mean-square turbulent velocity, which we shall approximate by $\bar{w}^2$. L is the total luminosity. Since we are concentrating attention on the atmosphere there is no nuclear energy generation and so no source term appears in equation (2.5). The energy exchange during ionisation is taken into account by the variability of δ and C_p. Finally we require an expression for the convective contribution to the luminosity, L_c; this will be provided by the mixing-length theory. The total luminosity is the sum of this and the radiative contribution.

$$L = L_r + L_c \tag{2.6}$$

Equations (2.1) to (2.6) together with an expression for L_c provide a set of first order differential equations in M for the dependent variables $r,\ \rho,\ P,\ T,\ L,\ L_r,\ L_c$. We shall now assume that a static equilibrium solution for these equations is known which will be denoted by $r_0,\ \rho_0,\ P_0,\ T_0,\ L_0,\ L_{r0},\ L_{c0}$, and set

$$r(M,t) = r_0(M)\left\{1 + r'(M,t)\right\},$$

with similar equations for the other quantities. The equations, linearised in the time dependent perturbation quantities become, after ρ has been eliminated, using (2.3):

$$\frac{\partial p'}{\partial M} = \frac{1}{4\pi r_0 P_0}\left\{\sigma_0^2(4r' + p') - \ddot{r}'\right\}, \tag{2.7}$$

$$\frac{\partial r'}{\partial M} = \frac{-1}{4\pi r_0^3 \rho_0}\left\{3r' + \alpha p' - \delta t'\right\}, \tag{2.8}$$

$$\frac{\partial t'}{\partial M} = \frac{1}{T_0}\frac{dT_0}{dM}\left\{\ell_r' - 4r' + \kappa_p p' + (\kappa_T - 4)t'\right\}, \tag{2.9}$$

$$\frac{\partial \ell'}{\partial M} = \frac{-P_0\delta_0}{L_0\rho_0}\left\{c\dot{t}' - \dot{p}' + \nu\dot{E}'\right\} \tag{2.10}$$

$$\ell' = \frac{L_{c0}}{L_0}\ell_c' + \frac{L_{r0}}{L_0}\ell_r' \tag{2.11}$$

and we shall assume the perturbed convective luminosity to be expressible in the form:

$$\ell_c' = L_{cr}r' + L_{cp}p' + L_{cr}t' + g \tag{2.12}$$

where g is a function which disappears when $\frac{\partial t'}{\partial M}\frac{\partial p'}{\partial M} = 0$, and L_{cr}, L_{cp}, L_{cT} depend on the equilibrium state, and are independent of time.

Here we have used:

$$\sigma_o^2 = \frac{g_o}{r_o} = \frac{GM}{r_o^3}, \qquad C = \frac{C_p \mu \beta}{\varkappa \delta}, \qquad \dot{r}' \equiv \frac{\partial r'}{\partial t}, \text{ etc.},$$

$$\varkappa_p = \left.\frac{\partial \log \varkappa}{\partial \log P}\right|_T, \qquad \varkappa_T = \left.\frac{\partial \log \varkappa}{\partial \log T}\right|_\rho, \qquad \alpha = \left.\frac{\partial \log \rho}{\partial \log P}\right|_T,$$

$$\nu = \frac{\rho_o E_{to}}{\delta_o P_o}$$

and we have set

$$\frac{1}{2}\overline{W}^2 \equiv E_t = E_{to}\left\{1 + E'(M,t)\right\}.$$

C is a non-dimensional specific heat and σ_o^{-1} is a characteristic mechanical time scale for the portion of the star within a radius r_o.

III. Baker's One-Zone Model

Since the coefficients of equations (2.7) to (2.12) are time independent it is possible to perform a separation of time and "space" variables. If it is assumed that the time dependence is of the form $r'(M,t) = r'(M)e^{st}$ there remains in general a fifth order system of differential equations in the space variable M which, together with appropriate boundary conditions, provides an eigenvalue problem for the time constant s. The essential simplification of the one-zone model is to ignore, as far as possible, the space dependence of the coefficients and the variables. We consider a single spherical shell of gas having mass m lying somewhere in the interior of the star.

The fluctuations $r', p', \dot{t}'$ are assumed to be constant throughout the layer so that

$$\frac{\partial r'}{\partial M}, \frac{\partial p'}{\partial M}, \frac{\partial t'}{\partial M} = 0 \qquad (3.1)$$

We cannot make such an assumption about the luminosity fluctuation, because its variability is an essential feature of the physics. Only if ℓ' varies can energy be taken from the thermal field and converted into pulsational energy. If ℓ_L' and ℓ_U' are the non-dimensional luminosity perturbations at the lower and upper boundaries, Baker sets

$$\ell' = \frac{\ell_U' + \ell_L'}{2} \quad , \quad \frac{\partial \ell'}{\partial M} = \frac{\ell_U' - \ell_L'}{m}$$

and then simplifies to the special case $\ell_L' = 0$. The latter approximation would be quite good for a zone at the base of the lowest pulsationally unstable zone, since the centre of the star, as has been mentioned above, does not vary much and provides an almost constant luminosity. Even if the zone considered were nearer the surface, this assumption would lead, in some sense, to the contribution of the layer to the overall damping or excitation of the radial mode. It might be possible to allow ℓ_L to vary in order to obtain a more realistic model, but this would complicate the mathematics, and we seek as simple a model as possible. Thus we set

$$\frac{\partial \ell'}{\partial M} = \frac{2\ell'}{m} \qquad (3.2)$$

Applying the simplifications (3.1) and (3.2) to the equations (2.7) to (2.12) we obtain

$$
\left.\begin{aligned}
&\sigma_0^2(4r'+p') - \ddot{r}' = 0, \\
&3r' + \alpha p' - \delta t' = 0, \\
&\ell_r' - 4r' + \kappa_p p' + (\kappa_T - 4)t' = 0, \\
&C\dot{t}' - \dot{p}' = -K(\varepsilon \dot{E}' + \sigma_0 \ell'), \\
&\ell' = \frac{L_{c0}}{L_0}\ell_c' + \frac{L_{r0}}{L_0}\ell_r', \\
&\ell_c' = L_{cr} r' + L_{cp} p' + L_{cT} t'.
\end{aligned}\right\} \qquad (3.3)
$$

where $K = \dfrac{2L_0\rho_0}{mP_0\delta_0\sigma_0}$ and $\varepsilon = \nu/K$. Within the framework of the one-zone model it will be assumed that we may write:

$$E' = E_r r' + E_p p' + E_T t'. \qquad (3.4)$$

Again E_r, E_p, E_T are to be determined from the mixing-length theory. K is essentially a coupling constant between the mechanical and thermal behaviour of the system. As $K \to 0$ the coupling disappears and the star pulsates adiabatically with its characteristic frequency $\sqrt{B_0}\,\sigma_0$ where

$$B_0 = \frac{3c - 4(\alpha c - \delta)}{\alpha c - \delta}.$$

The period of pulsation is proportional to $\sigma_0^{-1} \propto r_0^{3/2} M^{-1/2}$. Since we expect the instability to be confined to the outer layers of the star r_0 and M are approximately the total stellar radius and mass, and we are led to the usual period-mean density relation. It should be pointed out that this law arises from the assumed form of the disturbance and essentially from choosing r_0 as the length scale and M as the mass scale. The period-mean density relation is not in accord with observation (Fernie, 1964), but it is known that it is only the atmosphere of the star which pulsates. If this is taken into account it can be shown that the period is proportional to $r_0^2 M^{-1}$ for a radiative atmosphere where the helium II ionisation zone, the seat of pulsational instability (Cox, 1963), has been taken as the base of the atmosphere. The one-zone model could be adjusted to take this into account but since it is unlikely to reveal any new physics this will not be done, so that the model retains maximum simplicity.

Equations (3.3) and (3.4) may now be combined to yield a single third order differential equation for r' :

$$\dddot{r}' + K\sigma_0 A\ddot{r}' + \sigma_0^2 B\dot{r}' + K\sigma_0^3 D r' = 0 \tag{3.5}$$

where

$$A = \frac{L_{r0}}{L_{c0}} \frac{(\widetilde{\mathcal{H}}_T - 4)\alpha + \delta\widetilde{\mathcal{H}}_p}{\alpha\widetilde{C} - \delta\widetilde{D}}$$

$$B = \frac{3\widetilde{C} - 4(\alpha\widetilde{C} - \delta\widetilde{D}) + \nu\delta E_r}{\alpha\widetilde{C} - \delta\widetilde{D}}$$

$$D = \frac{L_{r0}}{L_{c0}} \frac{(4\alpha - 3)(\widetilde{\mathcal{H}}_r - 4) + 4\delta(\widetilde{\mathcal{H}}_p + 1) + \delta L_{cr}\frac{L_{c0}}{L_{r0}}}{\alpha\widetilde{C} - \delta\widetilde{D}} .$$

The form of Baker's non-convective equations has been preserved as much as possible. The new coefficients introduced are defined as follows:

$$\widetilde{C} = C + \nu E_T , \quad \widetilde{D} = 1 - \nu E_p ,$$

$$\widetilde{\mathcal{H}}_p = \mathcal{H}_p - \frac{L_{c0}}{L_{r0}} L_{cp} , \quad \widetilde{\mathcal{H}}_T = \mathcal{H}_T - \frac{L_{c0}}{L_{r0}} L_{cT} .$$

If we now assume r' to have an exponential time dependence

$$r'(m,t) = r'(M)e^{st} \tag{3.7}$$

equation (3.5) reduces to an algebraic equation in S :

$$S^3 + K\sigma_0 A s^2 + \sigma_0^2 B s + K\sigma_0^3 D = 0 \tag{3.8}$$

which is the eigenvalue equation for the one-zone model. It must be emphasized that this analysis only holds in the present form in virtue of the assumptions (2.12) and (3.4) for the form of the convective energy flux and the turbulent energy per unit mass. However it is found that under the restriction (3.7) these forms are possible if mixing-length theory is employed. In general A, B, D are functions of S and equation (3.8) would be difficult to solve. We shall consider two special cases which produce considerable simplification: In the first we shall assume that the pulsation is so slow that the convective motions adjust themselves to their new conditions in a time short compared with the pulsation period so that the formulae derived in section I may be applied directly. In the second case we shall assume that the star pulsates nearly adiabatically and expand about this situation.

IV. Immediate Adjustment of Convective Motions

If the period of pulsation is long compared with the mean lifetime of a convective element we may, as a first approximation, neglect the dynamical interaction between the convection and the pulsation and assume that the convection adjusts itself immediately to the changing conditions as the star pulsates. In other words, we assume that the functional forms (1.9) and (1.5) for the convective heat flux and the turbulent energy density are preserved, and we merely substitute the forms $r = r_0 (1 + r' e^{st})$, etc. into these equations and retain only the zero order and first order terms. If the mixing-length is assumed to be either the pressure or density scale height, the coefficients A, B, D defined in the previous section are real and independent of S. Equation (3.5) is now a cubic in S with real coefficients and the discussion of the stability parallels that given by Baker for the radiative case.

With $K \neq 0$, the condition that all three roots of (3.5) have negative real parts is:

a) $\sigma_0^2 B > 0$

b) $K \sigma_0^3 D > 0$

c) $K \sigma_0^3 (AB - D) > 0.$

Condition (a) is necessary for dynamical stability and condition (b) for secular stability. The condition for secular stability is not satisfied by the model because there is no nuclear energy generation, but the whole star is secularly stable for most of its life. For a more detailed discussion the reader is referred to Baker's paper. Condition (c) is the condition for pulsational stability and may be written:

$$\frac{4\tilde{D}}{\tilde{C}} - \left[\frac{1+\frac{1}{3}\alpha\nu E_r}{\tilde{C}}\tilde{\mathcal{K}}_T + \left(1+\frac{1}{3}\delta\nu E_r\right)\tilde{\mathcal{K}}_p\right] + \frac{4\alpha\nu E_r}{3\tilde{C}} - \tag{4.1}$$

$$- \frac{4(\alpha\tilde{C} - \delta\tilde{D})}{3\tilde{C}}\left(1 + \frac{1}{4} L_{cr} \quad \frac{L_{co}}{L_{r_0}}\right) > 0 \quad \text{for stability.}$$

If the pressure scale height is taken for the mixing length, the criterion (4.1) written in full is:

$$\frac{4\left\{1-\nu\left[1-\alpha-\delta_P-(\eta-1)\Gamma_P\right]\right\}}{c+\nu\left[\delta+\delta_T-(\eta-1)\Gamma_T\right]}-\left\{\frac{\mathcal{K}_T-f\left[1-\frac{1}{2}\delta+\frac{1}{2}\delta_T+C_{PT}-\frac{3}{2}(\eta-1)\Gamma_T\right]}{c+\nu\left[\delta+\delta_T-(\eta-1)\Gamma_T\right]}+\right.$$

$$\left.+\mathcal{K}_P-f\left[\tfrac{1}{2}(\alpha+1)+\tfrac{1}{2}\delta_P+C_{PP}-\tfrac{3}{2}(\eta-1)\Gamma_P\right]\right\}-$$

$$-\frac{4\left\{\alpha c-\delta+\nu\left[2\alpha\delta-\delta+(\alpha\delta_T+\delta\delta_T)+(\eta-1)(\delta\Gamma_P-\alpha\Gamma_T)\right]\right\}}{3\left\{c+\nu\left[\delta+\delta_T-(\eta-1)\Gamma_T\right]\right\}}\left\{1+\frac{1}{2}f\right\}>0 \qquad (4.2)$$

for stability.

where

$$C_{PP}=\left.\frac{\partial\log C_P}{\partial\log P}\right|_T,\quad \delta_P=\left.\frac{\partial\log\delta}{\partial\log P}\right|_T,\quad \Gamma_P=\left.\frac{\partial\log c}{\partial\log P}\right|_T=\frac{1}{\Gamma_2-1}\left.\frac{\partial\log\Gamma_2}{\partial\log P}\right|_T$$

$$C_{PT}=\left.\frac{\partial\log C_P}{\partial\log T}\right|_P,\quad \delta_T=\left.\frac{\partial\log\delta}{\partial\log T}\right|_P,\quad \Gamma_T=\left.\frac{\partial\log c}{\partial\log T}\right|_P$$

and $f=\dfrac{L_{co}}{L_{ro}}$.

The radiative terms each have a clear physical interpretation. The gas can modulate the flux passing through it producing a variable outward flux at the top. If the flux is decreased on compression, the zone absorbs heat at a high temperature and re-emits it at a lower temperature. The zone acts as a heat engine and feeds energy into the pulsation. If, on the other hand, the flux increases on compression, the pulsation is damped. It was first noted by Eddington (1926) that this may be the cause of the instability. In the absence of convection condition (4.2) becomes

$$\frac{4}{c}-\left(\frac{\mathcal{K}_T}{c}+\mathcal{K}_P\right)-\frac{4}{3}\,\frac{\alpha c-\delta}{c}>0 \qquad \text{for stability}$$

The first term arises from the fact that the temperature gradient is increased on compression, thus increasing the heat flux and stabilising the pulsation. The last term is geometrical and is destabilising; as the zone contracts its radius decreases and the mass per unit area increases and thus offers additional resistance to the heat flux. The middle term arises from the variation of opacity with pressure and temperature and can be either stabilising or destabilising. $\mathcal{H}_p$ is positive and $\mathcal{H}_T$ is negative. For Kramers' opacity $\mathcal{H}_p = 1$ and $\mathcal{H}_T = -9/2$ and in general the middle term is stabilising. In an ionisation zone, however, c can become very high and the stabilising effect of $\mathcal{H}_T$ is reduced and the opacity variation can then have a strong destabilising effect.

One may attempt to discuss the convective terms in a similar way, but since the mixing-length formalism of the first section is not valid in regions where the physical properties vary rapidly, we shall discuss the criterion (4.2) only in regions where the derivatives of the physical properties may be neglected. The stability criterion may then be written

$$\frac{4}{c+\nu} - \left\{\frac{\mathcal{H}_T - \frac{1}{2}f}{c+\nu} + \mathcal{H}_p - f\right\} - \frac{4(c+\nu-1)}{3(c+\nu)}\left(1+\tfrac{1}{2}f\right) > 0 \qquad (4.3)$$

where we have set α and δ equal to unity.

Since the "internal" energy in the zone is now shared between thermal energy and turbulent energy the effective specific heat increases. This reduces the stabilising influence of $\mathcal{H}_T$ and also the relative effect of the diffusion term. The effect of the variation in the convective heat flux due to changes in pressure and temperature is seen in the middle term. On compression the density and the superadiabatic temperature gradient increase, thus causing an increase in heat flux and a destabilisation. The mixing length, on the other hand, decreases, which works in the other direction. The last term contains the geometrical destabilisation together with the variation of mixing length with r . The combined effect due to variations in pressure and temperature is to stabilise, but the total effect is not yet clear. It is convenient to multiply (4.3) by $C+\nu$ and separate the convective and radiative terms.

$$4-(\mathcal{H}_T+C\mathcal{H}_p)-\tfrac{4}{3}(C-1)+\tfrac{1}{3}\left\{f\left(C+\nu+\tfrac{7}{2}\right)-\nu(4+3\mathcal{H}_p)\right\}>0.$$

Convection has a stabilising influence unless

$$f<\frac{\nu(4+3\mathcal{H}_p)}{C+\nu+\frac{7}{2}},$$

which is most likely to be satisfied in the ionisation zone where C is high. For a perfect gas, with Kramers' opacity, $C = 5/2$ and $\mathcal{H}_p = 1$ and this reduces to

$$f<\frac{7\nu}{6+\nu}$$

Such a condition will be satisfied only in a very narrow region at the edge of the convection zone, and so convection would exert a damping effect in most regions, if immediate adjustment were plausible.

We must therefore conclude that if the functional form of the standard mixing-length theory can be applied with any validity to the case of a pulsating atmosphere, the convection will provide a damping of the pulsation mode, except possibly in the ionisation zones, the seat of the radiative pulsational instability and the very place where the theory is least applicable. Immediate adjustment is likely if the lifetime of an element, τ is small compared with the period of pulsation, P_e .

Their ratio can easily be estimated to be about

$$\frac{\tau}{P_e} \approx \frac{F}{\sqrt{\nabla - \nabla_{ad}}}$$

where $\nabla = \dfrac{d\log P}{d\log T}$ and ∇_{ad} is the adiabatic value of ∇ . F is a factor of order unity. If F is not too large and $\nabla - \nabla_{ad} \approx 1$, then the immediate adjustment calculation has some meaning, but if $\nabla - \nabla_{ad} << 1$, the results are clearly not valid. In this case, however, the convective heat flux will not be so large and its effect on the stability will be felt less strongly. However, the balance between damping and excitation in a star is delicate, and it seems worth investigating the effect convection may have if it does not adjust itself immediately to the pulsation. It seems plausible that if the phase of the oscillating convective flux could lag sufficiently far behind the pulsation, its effect on the stability of the star could be reversed. Perhaps this could explain the instability of the long-period variables. To this end the mixing-length theory has been modified to try to take this effect into account, but now attention is restricted to pulsations which are nearly adiabatic.

V. <u>A Modified Mixing-length Theory</u>

In order to investigate the phase difference between the convection and the pulsation it is necessary to consider in more detail the motions of convective elements. Of course, at present this is an impossible task, even for statistically steady convection, but a qualitative feeling for the nature of at least some aspects of the interaction can be obtained within the formalism of a mixing-length theory.

We shall fix our attention on a mass zone in the star and set up rectangular space coordinates locally within it, measuring ζ vertically upward. Since the zone is in the stellar atmosphere we can with good accuracy regard it as being plane. We now express the horizontally-averaged temperature, pressure and density as a static value, plus a small perturbation which is harmonic in time

$$\rho(M,t) = \rho_0(M)\left[1+\rho'(M)e^{i\omega t}\right], \text{ etc.} \tag{5.1}$$

and consider the motion of a convective element in this field. ω is complex and will be determined eventually from the stability analysis. We shall restrict ourselves once more to a short mixing-length approximation; that is, to cases where the physical properties of the medium do not vary considerably over a mixing length, so that Taylor expansions may be made about a point as was done in section I. The approximation of the one-zone model will be made at the outset, so that

$$\frac{\partial \rho'}{\partial M} = 0 \tag{5.2}$$

Furthermore, for the purposes of calculating the required convective heat flux and energy density, it will be assumed that the star is pulsating adiabatically. It is then consistent to assume that the convective elements rise and fall adiabatically. The formulae obtained at the end can be used in Baker's model to investigate small departures from adiabacy. If this approximation were not made it would be necessary to consider the variation of the radiative flux through the convective element as the star pulsates, and this would complicate the model considerably. We shall also neglect radiation pressure and the effects of ionisation.

For simplicity it will also be assumed that the element starts life at (z_0, t_0) with zero velocity, relative to the ambient medium, and a temperature difference

$$\theta(z_0, t_0) = \Delta_T \bar{T}(z_0, t_0) \tag{5.3}$$

where z_0 is the initial value of ζ. It will be assumed that Δ_T is small and independent of z_0; that this is plausible will be evident when the results are compared with those of section I in the case of no pulsation. Δ_T need not be considered constant, and may be expanded in the form (5.1).

Although, in reality, it will have a continuous probability distribution, we shall approximate this by regarding half the elements to be created with a temperature difference $\bar{T}|\Delta_T|$ and the other half with $-\bar{T}|\Delta_T|$. The variation of the gravitational acceleration g with ζ over a mixing length will be neglected, but its variation arising from changes in r, the radius of the mass zone, and $\ddot{r}$ will, of course, be included.

It is then possible to calculate, to first order in perturbed quantities, the trajectory of the convective element:

$$\zeta - z_0 = f(t, t_0) \tag{5.4}$$

The calculation is outlined in the appendix. In order to estimate the life of the element it will be assumed that the probability of its break-up is determined by the value of the mixing length which prevails at that instant. If the mixing length is taken to be a function only of position and time, this means that the probability of survival does not depend on the past history of the element, nor directly on the detailed structure of the velocity field in its vicinity. Thus if $p(\zeta, t; z_0, t_0)$ is the probability that an element which originated at (z_0, t_0) has survived until (ζ, t) :

$$\frac{\partial p}{\partial \zeta} = -\mathcal{H}(\zeta, t)\, p \qquad (5.5)$$

where $\mathcal{H}(\zeta, t) = \dfrac{1}{\ell(\zeta, t)}$. ζ and t are, of course, related through (5.4). Since $p(z_0, t_0; z_0, t_0) = 1$, the solution of (5.5) is

$$\begin{aligned} p(\zeta, t; z_0, t_0) &= e^{-\int_{z_0}^{\zeta} \mathcal{H}(\eta, s)\, d\eta} \\ &= e^{\int_{t_0}^{t} \mathcal{H}(\eta, s)\, w(s; z_0, t_0)\, ds} \end{aligned}$$

where the integral is taken over the trajectory of the particle. w is the vertical velocity of the element and is defined by:

$$w(t; z_0, t_0) = \left.\frac{\partial \zeta}{\partial t}\right/_{t_0} \qquad (5.6)$$

Since we are making a short mixing-length approximation we may set $\eta = \zeta$ in the integral and write

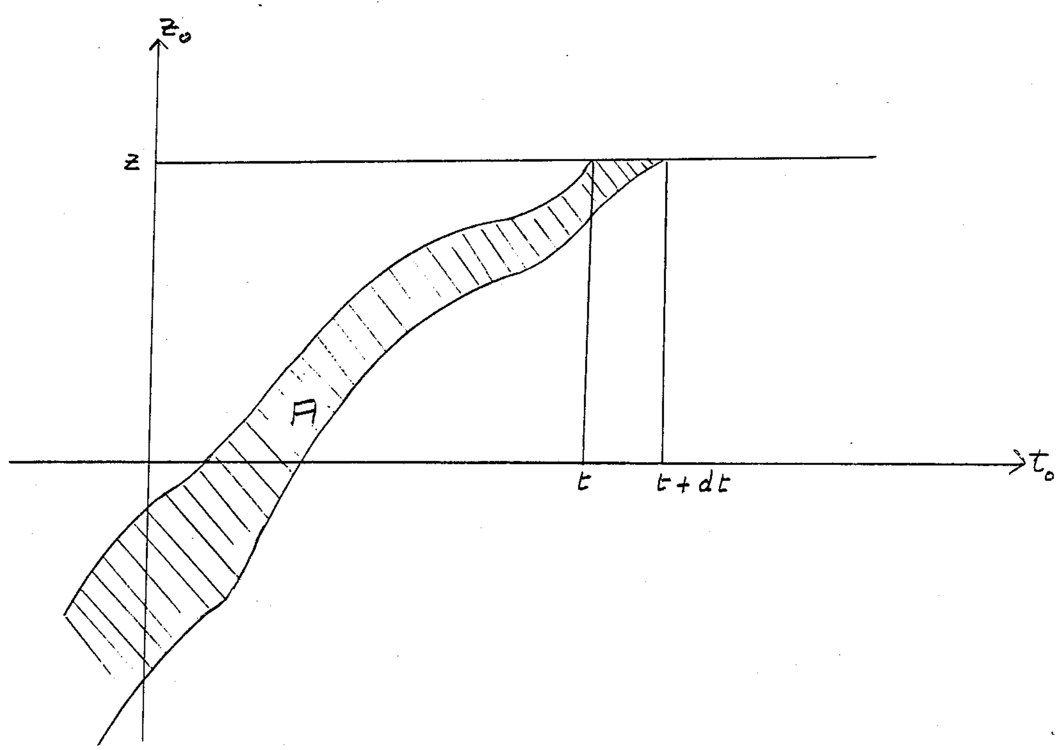
$$p(\zeta, t; z_0, t_0) = e^{-\int_{t_0}^{t} \mathcal{H}(\zeta, s)\, w(s; z_0, t_0)\, ds} \qquad (5.7)$$

If we introduce a source function $n(\zeta, t)$, such that

$n\,d\zeta\,dt$ is the number of elements created per unit area between ζ and $\zeta + d\zeta$ in the time interval $(t, t+dt)$ we can now write down an expression for the convective heat flux $H_c(z, t)$. If dQ_r is the heat transported by convection across the plane $\zeta = z$ in $(t, t+dt)$ by rising elements we may write:

$$dQ_r \simeq C_p \int_A m(z_o, t_o) \tfrac{1}{2} n(z_o, t_o)\, \theta(z, t; z_o t_o)\, p(z, t; z_o, t_o)\, dA$$

where $m(z_o, t_o)$ is the mass of an element created at (z_o, t_o) and A is the area between the two trajectories $z_o = z_o(t_o; z, t)$; $z_o = z_o(t_o; z, t+dt)$, (see diagram). The trajectories are defined by (5.1). The factor ½ appears because only one-half of the elements are created with a positive temperature fluctuation $T/\Delta_T/$.



Thus
$$dQ_r = \tfrac{1}{2} C_p \int_{-\infty}^{t} \left\{ \int_{z_0(t+dt;\,z_0,t_0)}^{z_0(t,\,z_0,t_0)} m\,n\,\theta p\, dz_0 \right\} dt_0$$
$$= -\tfrac{1}{2}\, dt\, C_p \int_{-\infty}^{t} m\,n\,\theta p \left.\frac{\partial z_0}{\partial t}\right|_{t_0} dt_0 .$$

The convective heat flux by rising elements is simply $\dfrac{dQ_r}{dt}$. It is easy to show that to the accuracy of the short mixing-length approximation the contribution to the heat transport from falling elements is the same as that from the rising ones. Hence

$$H_c(z,t) = C_p \int_{-\infty}^{t} m\,n\,\theta\, w\, p\, dt_c \tag{5.8}$$

since $\left.\dfrac{\partial z_0}{\partial t}\right|_{t_0} = -\left.\dfrac{\partial f}{\partial t}\right|_{t_0} = -w.$

Although the integral in (5.8) is along a trajectory (5.1) we may once more neglect the space dependence of the variables and evaluate them all at z .

Before (5.8) can be evaluated it is necessary to specify m and n . It is usual to set the scale of the convective elements equal to the mixing length. Hence we write:

$$m(z_0,t_0) = \rho(z_0,t_0)\,\hat{\ell}^{\,3}(z_0,t_0) \tag{5.9}$$

The choice of the source function is more difficult, and several possible prescriptions present themselves. It may be argued that it will be determined primarily by the unstable stratification, and is only weakly dependent on the detailed velocity field. Whether this is a good approximation is not clear, but since the same has been assumed of the mixing length it is unlikely to generate any further errors. Accordingly we shall identify the value for the heat flux obtained from (5.8) in the limit $\frac{\tau}{P_e} << 1$ with the expression obtained from (1.9) and the assumption of immediate adjustment (when pulsation is nearly adiabatic) and assume that the form so obtained for the source function is universally true. In standard mixing-length theory it is assumed that the elements fill space. Since the characteristic scale of an element is equal to a mixing length, this is expressed, in the case of no oscillations, by the equation:

$$n_o \tilde{\ell}_o^3 = \tau_o^{-1} \tag{5.10}$$

If this equation is used for n_o in (5.8) for the case of no oscillations, the functional form for H_c is found to agree with (1.9). Identification of the two formulae now determines Δ_T, and it is confirmed that Δ_T is small. In fact it is found that

$$\frac{\Delta_T}{\theta(z_o+\tilde{\ell}, t)} \simeq \lambda \simeq 2e^{-4} \simeq 0.035 .$$

It is not suggested that this number has much physical significance because it arises from equating two different methods of averaging. The calculation is outlined in the appendix and yields, for the convective luminosity, if a pressure scale-height is taken for the mixing length:

$$L_c = L_{co}(1 + \ell_c' e^{i\omega t}), \tag{5.11}$$

where

$$\ell_c' = L_{cp} p' + L_{cr} r', \tag{5.12}$$

and $$L_{cr} = \frac{4}{4+\alpha^2}\left\{6+\alpha^2 - i\alpha - \frac{1}{2}\alpha(2i-\alpha)(1-i\alpha)\tilde{B}\right\} -$$
$$-2\,\frac{(2-i\alpha)(1-i\alpha)}{1+\alpha^2}(1-i\alpha)\tilde{B}, \tag{5.13}$$

$$L_{cp} = \frac{\gamma+1}{\gamma(4+\alpha^2)}\left\{2-i\alpha-\frac{1}{2}\alpha(2i-\alpha)(1-i\alpha)\tilde{B}\right\} +$$
$$+1+\frac{1}{\gamma}(1-2i\alpha)\tilde{B}-\frac{1}{\gamma}\cdot\frac{(2-i\alpha)(1-i\alpha)}{1+\alpha^2}(1-i\alpha)\tilde{B} \tag{5.14}$$

We have used

$$\alpha = \omega\sqrt{\frac{T_0}{g_0\beta_0}} \tag{5.15}$$

$$\tilde{B} = e^{-4i\alpha}\Gamma(1-i\alpha) \tag{5.16}$$

where Γ is the gamma function, and γ is the ratio of principal specific heats.

The other quantity required is the turbulent energy per unit mass. If we assume that this arises solely from the vertical motions of the elements it is easy to show that it is given by the expression

$$E_t = \frac{1}{2}\rho^{-1}(z,t)\int_{-\infty}^{t} m(z_0,t_0)\,n(z_0,t_0)\,W^2(z;z_0,t_0)\,p(z,t;z_0,t_0)\,dt_0 .$$

This turns out to be:

$$E_t = E_{t_0}\left(1+E'e^{i\omega t}\right), \tag{5.17}$$

$$E' = E_p p' + E_r r' \tag{5.18}$$

where $$E_r = \frac{4}{4+\alpha^2}\left\{2(2-i\alpha)+i\alpha(1-i\alpha)\left[1+i\alpha\left(1+\frac{2}{\alpha^2}\right)\right]\tilde{B}\right\}+2(1-2i\alpha)\tilde{B}+\frac{4(1-i\alpha)}{1+\alpha^2}. \tag{5.19}$$

$$E_p = \frac{\gamma+1}{\gamma(4+\alpha^2)}\left\{2(2-i\alpha)+i\alpha(1-i\alpha)\left[1+i\alpha\left(1+\frac{2}{\alpha^2}\right)\right]\tilde{B}\right\} +$$
$$+\frac{1}{2\gamma}\left\{\gamma+3-(\gamma+5)i\alpha\right\}\tilde{B}-\frac{3+\alpha^2-2i\alpha}{\gamma(1+\alpha^2)} \tag{5.20}$$

VI The Stability of the Atmosphere

Equations (5.11) to (5.20) provide the necessary information to specify equation (3.8) after the identification $s = i\omega$ has been made. The case $\alpha \longrightarrow 0$ corresponds to immediate adjustment, the implications of which have already been discussed. For α not zero, the coefficients A, B, D now depend strongly on s and the equation is no longer a cubic. However, we have insisted that the pulsation is nearly adiabatic and so we can expand about the adiabatic solution. If B_o is defined by

$$B_o = \frac{4-c}{c-1}$$

the adiabatic solution is given by

$$s_o = i\sqrt{B_o}\,\sigma_o .$$

If we now set $s = s_o + s_1$, and neglect terms of higher order in s_1 than the first, we obtain, on substitution into (3.8):

$$s_1 = \frac{-K\sigma_o}{3B_o - B}\left\{AB_o - \frac{i}{K}\sqrt{B_o}\,(B - B_o) - D\right\} \qquad (6.1)$$

The condition for stability is $\mathcal{R}e\,(s_1) < 0$.

The author has not had time to study the expression in detail. It is hoped that if values of the required parameters characteristic of Opheid variables are substituted into (6.1) a stabilisation will be found. Possibly a destabilisation might be found for long-period variables.

In the limit $\alpha \rightarrow \infty$ the effect is to stabilise, as might be expected. As an example in the region of interest, the case $\alpha = 1$

was evaluated. The result is, approximately,

$$S_1 \propto \left(\frac{2\varkappa_T}{5} + \varkappa_\rho\right)(1+\nu) - \frac{1}{5}f\left(1+\frac{\nu}{K}\right) \tag{6.2}$$

If Kramer's opacity is used, the effect of convection is to stabilise still further. When α is large, Stirling's approximation can be used for the gamma-function and an asymptotic formula can be obtained for S_1 . As might be expected, the effect in this limit is also to add further stabilisation; but since the convection is weak for large α , the effect is very small.

VII Further Developments

The mixing-length formalism involves many assumptions, and although it may clarify some of the more obvious ways in which convection and pulsation can interact, the validity of theory, especially when stretched to apply to oscillating atmospheres, is in doubt. The theory as presented in section V can be improved, but it is probably more worthwhile to generalise Spiegel's approach (1963) to include time dependence.

A more direct attack on the stability problem has also begun. The mean-field equations have recently been shown to predict a convective heat transport for a Boussinesq fluid between two horizontal plates which agrees well with experiment, (Herring 1964). Although the result cannot be applied for very low Prandtl numbers which are relevant to stellar atmospheres, it seems worthwhile to study the solutions of these equations for a compressible plane parallel atmosphere with Prandtl number of order unity, and to study the stability of the result to pulsation-like perturbations. Recently Spiegel (1965) has developed a method of obtaining asymptotic solutions to these equations for a Boussinesq fluid, and it is probably possible to generalise this.

The mean-field equations arise from what has been called the "Weak Coupling Approximation" (Spiegel, 1965) and using this prescription, equations can be derived for a compressible gas. In the special case of very weak convection it is easy to write down W.K.B. solutions to the equations. Using the Weak Coupling prescription, the perturbation equations arising from the stability analysis can be considerably simplified, although they are still very complicated. One interesting property can be seen immediately, however. If the pulsational perturbation is required to be a function only of the vertical coordinate, the mean equations reduce almost to the ones one would obtain if one neglected convective motions altogether and considered the stability of a static atmosphere with the equilibrium pressure, density and temperature fields obtained from the convection equations. However, there appears to be a strong coupling between the convection and a pulsation-like mode which also has a horizontal variation with the same wavelength as the principal mode of the convection itself. It may be that a mode such as this can be unstable and that the variation in luminosity of some stars may be due to a highly non-radial velocity perturbation.

VIII Conclusion

The analysis has not revealed with certainty the effect of convection on the pulsational stability of a star, but it does seem likely that convection can stabilise pulsation modes in some cases. The mixing-length formalism has been found useful in explaining the physics of at least part of the interaction, with the help of Baker's one-zone model. It may be that convection can destabilise some atmospheres but this has not been found. A brief study of the mean-field equations suggests that variability in the luminosity of a star may be the result of highly non-radial modes in some circumstances. This is an interesting idea which is worth pursuing.

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Appendix

In the short mixing-length approximation, the temperature and density of a rising element and of the ambient medium may be expanded in a Taylor series as in section I. If it is assumed that the element starts at (z_0, t_0) with $\Theta(z_0,t_0)=\bar{T}(z_0,t_0)\Delta_T$, and that it rises adiabatically in an adiabatically-oscillating atmosphere, it is easy to show that the buoyancy force at (ζ, t) is

$$\rho_0\left[\mu^2(1+2r')(\zeta-z_0)+g_0\Delta_T\left(1-\frac{p'}{\gamma}\right)\right](1+p'+2r')$$

if the approximation of the one-zone model is made, where $g_0=-\frac{1}{\rho_0}\frac{dp_0}{d\zeta}$ and $\mu^2=\frac{g_0\beta_0}{T_0}$, $\beta_0=\left(\frac{\gamma-1}{\gamma}\right)\frac{T_0}{p_0}\frac{dp_0}{d\zeta}-\frac{dT_0}{d\zeta}$. The equation of motion is, therefore,

$$\ddot{\zeta}=\left\{\mu^2\left(1+\frac{p'}{\gamma}+2r'\right)(\zeta-z_0)+g_0\Delta_T\right\}(1+p'+2r')\,. \qquad \text{(A1)}$$

We have assumed Δ_T small and have therefore neglected terms like $\Delta_T\frac{dp_0}{d\zeta}(\zeta-z_0)$. Setting $\zeta-z_0=y_0+y_1$, where y_0 is independent of perturbed quantities, we may separate equation (A1) into zero-order and linearised first order parts:

$$\left.\begin{aligned}&\ddot{y}_0-\mu^2y_0=g_0\Delta_T\,,\\&\ddot{y}_1-\mu^2y_1=\mu^2\left(\frac{\gamma+1}{\gamma}p'+4r'\right)y_0+g_0\Delta_T(p'+2r')\end{aligned}\right\} \qquad \text{(A2)}$$

Assuming the exponential time dependence $p'(\zeta,t) = p'(\zeta)e^{i\omega t}$, etc., the solutions of (A2) satisfying the initial conditions $y=0$, $\dot{y}=0$ at $t=t_0$ lead to the following expression for ζ :

$$
\begin{aligned}
\zeta - z_0 = {} & \frac{g_0\Delta_T}{\mu^2}(\cosh x - 1) - \\
& - \frac{g_0\Delta_r}{2(4\mu^2+\omega^2)}\left(\gamma \pm \frac{1}{\gamma}p' + 4r'\right)\Big\{2\mu\cosh x\left(e^{i\omega t} - e^{i\omega t_0}\right) - i\omega\sinh x\left(e^{i\omega t} + e^{i\omega t_0}\right) + \\
& + \frac{1}{\omega}\left(4\mu^2+\omega^2\right)\sinh x\left(e^{i\omega t} - e^{i\omega t_0}\right)\Big\} + \\
& + \left(\frac{p'}{\gamma} + 2r'\right)\frac{g_0\Delta_r}{\mu(\omega^2+\mu^2)}\left(\mu e^{i\omega t} - \mu\cosh x\, e^{i\omega t_0} - i\omega e^{i\omega t_0}\sinh x\right) \\
\equiv {} & f(t, t_0), \qquad \text{(A3)}
\end{aligned}
$$

where $x = \mu(t - t_0)$.

The velocity is

$$
\begin{aligned}
W(\zeta,t;z_0,t_0) = {} & \frac{g_0\Delta_T}{\mu}\sinh x - \left(\gamma \pm \frac{1}{\gamma}p' + 4r'\right)\frac{g_0\Delta_r}{4\mu^2+\omega^2}\Big\{\left(1+\frac{2\mu^2}{\omega^2}\right)i\omega\cosh x\left(e^{i\omega t} - e^{i\omega t_0}\right) - \\
& - \mu\sinh x\left(e^{i\omega t} + e^{i\omega t_0}\right)\Big\} + \left(\frac{p'}{\gamma} + 2r'\right)\frac{g_0\Delta_r}{\omega^2+\mu^2}\left(i\omega e^{i\omega t} - \mu\sinh x\, e^{i\omega t_0} - i\omega\cosh x\, e^{i\omega t_0}\right).
\end{aligned}
\qquad \text{(A4)}
$$

The temperature excess $\theta(\zeta, t; z_0, t_0)$ may be obtained by retaining the first order terms in the Taylor series and substituting for $\zeta - z_0$ using (A3). Now

$$
\begin{aligned}
p(\zeta,t;z_0,t_0) &= e^{-\int_{t_0}^{t}\kappa w\,ds} \\
&\simeq e^{-\kappa_0(\zeta - z_0)}\, e^{-\kappa_0\kappa'\int_{t_0}^{t} w_0 e^{i\omega s}\,ds}
\end{aligned}
$$

This may be written

$$p \simeq e^{-\Lambda_0-\Lambda_1} \qquad \text{where } \Lambda_0 = \varkappa_0 y_0 = \frac{\varkappa_0 g_0 \Delta_r}{\mu^2}(\cosh x - 1)$$

$$\simeq e^{-\Lambda_0}(1-\Lambda_1)$$

$$\simeq e^{-\lambda(\cosh x-1)}(1-\Lambda_1) \qquad \text{where } \lambda = \frac{\varkappa_0 g_0 \Delta_r}{\mu^2} .$$

We shall assume that $\lambda << 1$; this is what we mean by Δ_T being small. In that case the major contribution to the integrals in (5.8) and the expression for E_t comes from large values of X . Hence we may neglect terms of order unity compared with $\cosh x$ or $\sinh x$ and set $\cosh x \approx \sinh x \approx \frac{1}{2}e^x \equiv \frac{1}{2}y$. To this approximation we then obtain:

$$p(\xi,t) \approx e^{-\frac{1}{2}\lambda y}\Big[\Big[1+\left(\frac{p'}{\gamma}+2r'\right)\frac{1}{1+\alpha^2}\tfrac{1}{2}\lambda y(1+i\alpha)e^{i\omega t_0}+$$
$$+\left(\frac{\gamma+1}{\gamma}p'+4r'\right)\frac{1}{4+\alpha^2}\tfrac{1}{2}\lambda y\left\{\left(1+\frac{2i}{\alpha}\right)e^{i\omega t}-\left[1+i\alpha\left(1+\frac{2}{\alpha^2}\right)\right]e^{i\omega t_0}\right\}+$$
$$+\frac{\tilde{\ell}'}{1+\alpha^2}\tfrac{1}{2}\lambda y\,(1-i\alpha)e^{i\omega t}\Big]\Big] \tag{A5}$$

where we have set $\alpha = \omega/\mu$, Δ_T = constant.

$$\tilde{\ell} = \tilde{\ell}_0(1+\tilde{\ell}'e^{i\omega t})$$

and used the fact that $\varkappa' = -\tilde{\ell}'$. It is a simple matter to include the effect of variations in Δ_T . If we set $n = n_0(1+n'e^{i\omega t})$ the integral in equation (5.8) can be evaluated, and an expression for the convective luminosity can then be written down. The result is, to first order in perturbation quantities:

$$L_c = 4\pi r_0^2 (1+2r' e^{i\omega t}) C_p \beta_c \int_{-\infty}^{t} \rho(z_0,t_0)\, \tilde{\ell}^{3}(z_0,t_0)\, n(z_0,t_0)\, w\, \theta_R\, dt_0$$

$$= 4\pi r_0^2 \rho_0 n_0 C_p \tilde{\ell}_0^{5} \beta_0 (1+\ell_c' e^{i\omega t}) \qquad \text{(A6)}$$

where

$$\ell_c' = n'(1-i\alpha)\tilde{B} + \left(\gamma\tfrac{+1}{\gamma} p' + 4r'\right)\frac{1}{4+\alpha^2}\left\{2 - i\alpha + i\alpha\left[1+i\alpha\left(1+\frac{2}{\alpha^2}\right)\right](1-i\alpha)\tilde{B}\right\} -$$

$$- i\alpha\left(\frac{p'}{\gamma}+2r'\right)\tilde{B} + p' + 2r' + \frac{\ell'(2-i\alpha)(1-i\alpha)^2 B}{1+\alpha^2} + \left(\frac{p'}{\gamma}+3\ell'\right)(1-i\alpha)\tilde{B}. \qquad \text{(A7)}$$

Here, integrations by parts have been made, and

$$\tilde{B} = \tfrac{1}{2}\lambda\mu \int_{-\infty}^{t} e^{\mu(t-t_0) - i\omega(t-t_0)}\, e^{-\frac{1}{2}\lambda \exp[\mu(t-t_0)]}\, dt_0$$

$$= \left(\frac{\lambda}{2}\right)^{i\alpha} \Gamma\left(1-i\alpha, \tfrac{1}{2}\lambda\right),$$

where $\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1}\, dt$ is an incomplete gamma function. Since λ is small, this may be written

$$\tilde{B} \approx \left(\frac{\lambda}{2}\right)^{i\alpha} \Gamma(1-i\alpha). \qquad \text{(A8)}$$

$\Gamma(a)$ is the complete gamma function.

The convective luminosity for a static atmosphere is obtained by combining (5.10) with (A6) when ℓ_c' is set equal to zero. This yields

$$L_{c0} = 4\pi r_0^2 \rho_c \tau_0^{-1} C_p \tilde{\ell}_0^{2} \beta_c. \qquad \text{(A9)}$$

τ_0 may be estimated by setting $t - t_c = \tau_0$ and $\zeta - z_c = \tilde{\ell}_0$, in (A3), specialised to the case of a static atmosphere. This yields

$$\tilde{\ell}_c = \frac{g_0 \Delta_r}{\mu^2}(\cosh \mu\tau_0 - 1)$$

from which we obtain:

$$\tau_o \approx \frac{-1}{\mu} \log \lambda/2 \, .$$

Substituting into (A9) and using the definition of μ yields:

$$L_{co} = \frac{-1}{\log \lambda/2} \cdot 4\pi r_o^2 \rho_o C_p \tilde{\ell}_o^2 \sqrt{\frac{g_o}{T_o}} \beta_o^{3/2} ,$$

which is to be compared with the value obtained from section I:

$$L_c = \pi r^2 \rho C_P \tilde{\ell}^2 \sqrt{\frac{g}{T}} \beta^{3/2} \, . \tag{A10}$$

Hence $\log \lambda/2 = -4$ and so $n_o = \frac{1}{4}\mu \tilde{\ell}_o^{-3}$. Thus $\lambda = 2e^{-4} \approx 0.035$ which verifies that it is small.

In order to specify n' we consider the limit of (A7) as $\alpha \to 0$.

$$\ell_c' \sim n' + 4r' + 5\tilde{\ell}' + \frac{\gamma+1}{\gamma} p' ,$$

and identify it with $6r' + 2\tilde{\ell}' + \frac{3}{2}\frac{\gamma+1}{\gamma} p'$, the value obtained by expanding (A10) assuming immediate adjustment. We thus obtain:

$$n' = -3\tilde{\ell}' + 2r' + \frac{\gamma+1}{2\gamma} p' \tag{A11}$$

which we assume holds for all α . We finally obtain, after a little rearrangement:

$$\ell_c' = \frac{\gamma+1}{\gamma} \frac{p'}{4+\alpha^2} \left\{ 2 - i\alpha - \tfrac{1}{2}\alpha(2i-\alpha)(1-i\alpha)B + \frac{4+\alpha^2}{\gamma+1}\Big[(1-2i\alpha)B + \gamma\Big] \right\} + $$
$$+ \frac{4r'}{4+\alpha^2}\left\{ 6 + \alpha^2 - i\alpha - \tfrac{1}{2}\alpha(2i-\alpha)(1-i\alpha)B \right\} + \frac{(2-i\alpha)(1-i\alpha)^2}{1+\alpha^2} \tilde{\ell}' B \, .$$

If $\tilde{\ell}$ is assumed to be a pressure scale height, $\tilde{\ell}' = -2r' - \frac{p'}{\gamma}$, and the expressions (5.13) and (5.14) follow immediately.

The turbulent energy density can easily be shown to be given by

$$E_t = \frac{1}{2\rho(z,t)} \int_{-\infty}^{t} m(z,t_0)\, n(z,t_0)\, w^2(z,t;z_0,t_0)\, p(z,t;z_0,t_0)\, dt_0 \, .$$

This can be evaluated by the same methods as used for L_c and leads to:

$$E_t = E_{t_0}\left(1 + E' e^{i\omega t}\right),$$

where

$$E' = \left(\frac{\gamma \pm 1}{\gamma} p' + 4r'\right) \frac{1}{4+\alpha^2} \left\{2(2-i\alpha) + i\alpha(1-i\alpha)\left[1 + i\alpha\left(1 + \frac{2}{\alpha^2}\right)\right] B\right\} -$$

$$- i\alpha\left(\frac{p'}{\gamma} + 2r'\right) B + \frac{p'}{\gamma}\left[\gamma \frac{+3}{2}(1-i\alpha) B - 1\right] + 2r'(1-i\alpha) B + \frac{2\tilde{\ell}'(1-i\alpha)}{1+\alpha^2} \, ,$$

and if $\tilde{\ell}' = -2r' - \frac{p'}{\gamma}$ the results (5.19) and (5.20) follow.

On the Circumpolar Current

George Philander

This investigation attempts at providing a mathematical expression for streamlines as drawn by Stommel[1] in his model of the circumpolar current.

β -plane equations at high latitudes:

We write down the equations of motion in spherical coordinates and transform to local coordinates defined by

$$z = r - R$$
$$y = R(\theta - \theta_0)$$
$$x = R \sin\theta_0 \varphi$$

θ is measured from the axis of rotation, is zero at the south pole, π at the north pole. R is the radius of the earth.

Let $\ell\, L\, H$ denote characteristic length scales in the x-, y-, z-directions, respectively.

Let U , V , W denote characteristic velocity magnitudes.

(a) Continuity equation:

$$\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w) = 0$$

becomes

$$\frac{1}{1+\alpha y}u_x + \frac{1}{1+\alpha y}\frac{\partial}{\partial y}\left\{v(1+\alpha y)\right\} + \frac{1}{r^2}\left(2rw + r^2 w_z\right) = 0$$

where
$$\alpha = \frac{\cot\theta_0}{R}$$
$$\beta = \frac{\tan\theta_0}{R}$$

We have expanded $\cos\theta$ and $\sin\theta$ about θ_0 , have assumed θ_0 small (approximately 30° in our case) and have assumed

$$\text{(i)} \quad L^2/R^2 << 1$$

If in addition $\quad$ (ii) $H/R << 1$

we get

$$u_x + (hv)_y + (hw)_z = 0$$

where $\quad h = 1 + \alpha y$

(b) Coriolis acceleration:

$$2\underset{\sim}{\Omega} \wedge \underset{\sim}{u} = 2\Omega(w \sin\theta + v\cos\theta, -u\cos\theta, -u\sin\theta)$$

$$x \text{ component} \quad : 2\Omega\cos\theta_0 \{w\tan\theta_0 + \alpha y w \tan\theta_0 + v - \beta y v\}$$

If we assume that

$$\text{(iii)} \quad \frac{W}{V} \leq H/L$$

i.e. that the vertical velocity attains its upperbound when the vertical divergence is of the same order of magnitude as the horizontal divergence; and in addition, that

$$\text{(iv)} \quad H/L \tan\theta_0 << 1$$

Then we may neglect $w\tan\theta_0(1+\alpha y)$ when compared to v and the Coriolis acceleration may be written

$$2\underset{\sim}{\Omega} \wedge \underset{\sim}{u} \sim (fv, -fu, \cdot)$$

where $\quad f = 2\Omega\cos\theta_0(1 - \beta y)$

(c) $\quad \nabla P = \left(\frac{1}{h}\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}\right)$

provided assumptions (i) - (iv) are valid.

(d) Inertial terms may be neglected under assumptions (i) - (iv).

(e) Viscosity terms:

x-component of momentum: $\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{h^2}\frac{\partial^2 u}{\partial x^2}-\frac{\cot\theta_0}{hR}\frac{\partial u}{\partial y}$

y-component of momentum: $\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial y^2}+\frac{1}{h^2}\frac{\partial^2 v}{\partial x^2}-\frac{\cot\theta_0}{hR}\frac{\partial v}{\partial y}$.

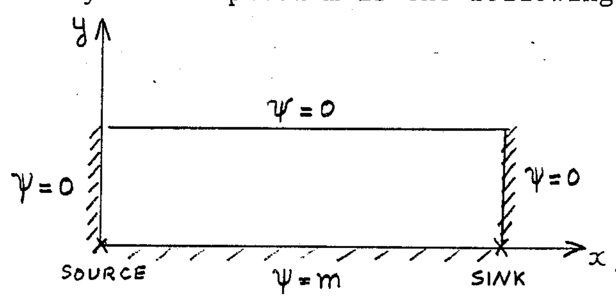
We shall further assume (v) that the motion in the vertical is hydrostatic. The relevant differential equation:

$$f v + \frac{1}{h} p_x = \text{viscosity terms}$$

$$-f u + p_y = \text{viscosity terms}$$

$$U_x + (hv)_y + (hw)_z = 0$$

The geometry of the problem is the following:



The eastern and western boundaries of the ocean basin are the western and eastern shores of the South American continent. The southern boundary is the Antarctic and the northern boundary is the streamline on which the 'windstress curl' vanishes. The 'gaps' between Cape Horn and Graham Land have been replaced by a source and sink. The wind-stress is assumed to be in the x-direction only and to have y variations only.

We adopt the Stommel model in assuming frictional forces to be of the form $(\mu u, \mu v)$ and to be effective at the bottom of the ocean only.

Assuming an undistorted sea surface and a level sea bottom, integrating the equations of motion over z from $-H$ to 0, we get

$$f v + \frac{1}{h} P_x = -Ru + \bar{\tau}$$
$$-fu + P_y = -Rv - R$$
$$u_x + (hv)_y = 0$$

Let $\quad hv = \psi_x$

$$u = -\psi_y$$

Introduce non-dimensional variables

$$\psi' = \frac{\partial f}{\partial y} \psi / \tau^x$$
$$x' = x/\ell$$
$$y' = y/L$$

where $\bar{\tau} = \tau^x \tau$ and $\tau = o(1)$

Put $\quad h = e^{\alpha L \eta}$

and note that $f(\eta) = 2\,\Omega \cos\theta_0 \left(1 - \beta/_\alpha\, e^{\alpha \eta L} + \beta/_\alpha\right)$

$$\sim -2\,\Omega \cos\theta_0 \beta L \eta \quad \text{under (i)}$$

Eliminate p by cross differentiation. Then

$$\psi_x + \varepsilon\left\{\psi_{\eta\eta} + \frac{L^2}{\ell^2}\psi_{xx}\right\} = -\frac{\ell}{L}(h\tau)_\eta$$

where $\quad \varepsilon = -\dfrac{R\ell}{\frac{\partial f}{\partial y} L^3}$

If

$$\ell \sim 3R, \quad L \sim R/3, \quad \mu \sim 10^{-6}, \quad \theta_0 \sim 30^\circ$$

Then $\quad \varepsilon \sim .35$

$L/\ell \sim .1$

Boundary Conditions: $\quad \psi = 0$ on $x = 0 \quad x = \ell, \quad y = L,$

$\psi = m$ on $y = 0$

Geostrophic Flow:

$$\psi^G = -(h\tau)_\eta x \frac{\ell}{L} + g(y)$$

Let $\quad \tau = \sin y/h$

Then $\quad \psi^G = h \cos y \,(\ell - x)\, \ell/L$

This stream function does not satisfy b. conditions on $x = 0$ or $y = 0$ and it is necessary to introduce boundary layers.

Boundary layer on $y = 0$:

Let $\quad \psi = \psi' + \psi^G$

where ψ' decays exponentially away from $\eta = 0$. We stretch the coordinate in the η-direction so as to make the $\epsilon\psi_{\eta\eta}$ term of the same order of magnitude as the ψ_x term. The resulting differential equation and boundary condition:

$$-\psi'_{\bar{x}} + \psi'_{\xi\xi} = 0$$

$$\psi' = rn - \frac{\ell}{L}\bar{x} \quad \text{on } \xi = 0$$

$$\psi' \to 0 \text{ as } \xi \to \infty$$

$$\psi' = 0 \text{ on } \bar{x} = 0, 1$$

where $\quad \bar{x} = 1 - x$

$$\xi = \eta/\sqrt{\varepsilon}\,.$$

As stated above, the differential equation is over-specified. Hence the Ψ_{xx} term must become important near $x = 0$ and will be effective in a region $O(\sqrt{\varepsilon}\, L/\ell)$. If we assume the $x = 0$ boundary to be absent, we may use Laplace transform methods to solve the problem and obtain a solution valid to the right of $x = \sqrt{\varepsilon}\, L/\ell$.

$$-p\bar{\Psi} + \varepsilon \bar{\Psi}_{\eta\eta} = 0$$

$$\Psi = m/p - \ell/L \frac{1}{p^2} \text{ on } \eta = 0$$

$$\Psi = \left(\frac{m}{p} - \ell/L \frac{1}{p^2}\right) e^{-\sqrt{\frac{p}{\varepsilon}}\,\eta}$$

$$\Psi' = m\,\mathrm{erfc}\,\frac{\eta}{2\sqrt{\varepsilon(1-x)}} - 4\frac{\ell}{L}(1-x)\,\mathrm{erfc}\,\mathrm{erfc}\,\frac{\eta}{2\sqrt{\varepsilon(1-x)}}$$

This boundary layer width decreases as x increases and at $x = 0$ is $O(\sqrt{\varepsilon}) \sim .6$.

Boundary layer on $x = 0$:

We assume Ψ to be of the form

$$\Psi = \Psi^6 + \Psi' + \Psi^2$$

where Ψ^2 decays exponentially with increasing x the relevant differential equations and boundary conditions:

$$\Psi_x + \varepsilon \frac{L^2}{\ell^2} \Psi_{xx} = 0$$

$$\Psi = -\Psi^6(x=0) - \Psi'(x=0) \text{ on } x = 0$$

$$= F(y) \text{ say.}$$

We can immediately tell that the boundary layer width will be of order $(\varepsilon\, L^2/\ell^2) \sim .035$

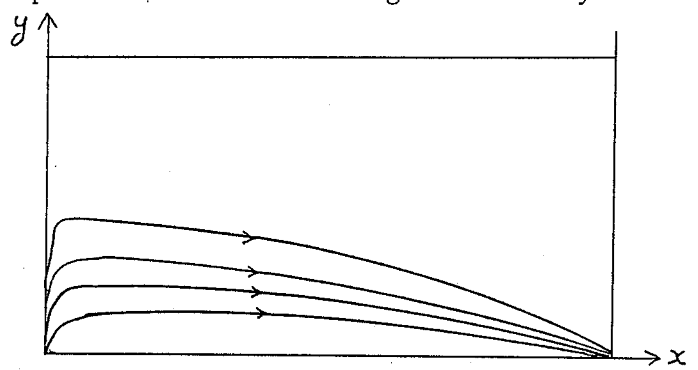
Hence $\Psi^2 = F e^{-x/\varepsilon\, L^2/\ell^2}$

and the complete solution to the problem is

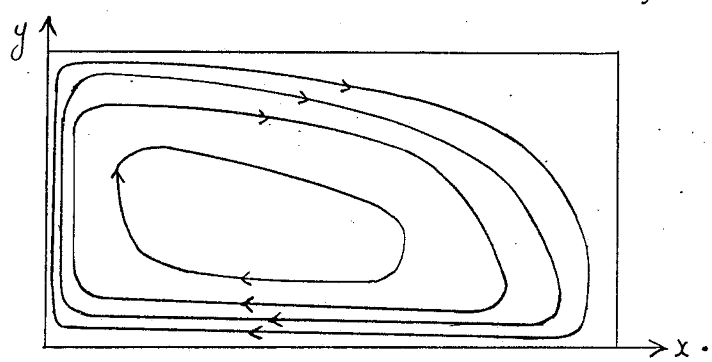
$$\Psi = \Psi^6 + \Psi' + \Psi^2$$

where Ψ' Ψ^2 Ψ^6 have the values stated previously.

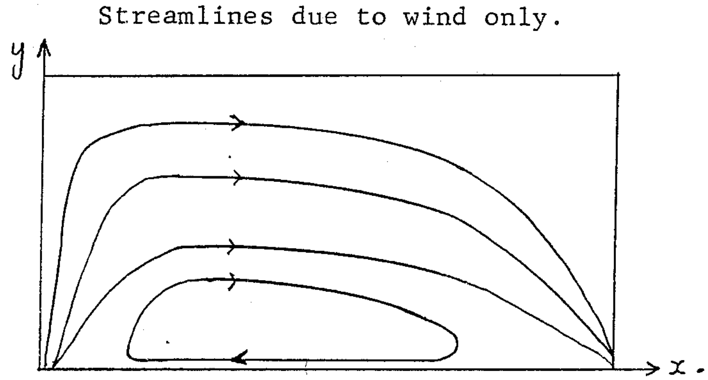
Sketches of streamlines are interesting in that we have a northward deflection of streamlines on the east coast of South America without having taken high ridges to the right of our source into account. It may be concluded that bottom topography plays a role less important that that assigned to it by Stommel[1].



Streamlines due to a source and sink only.



Streamlines due to wind only.



Streamlines due to a source, sink and wind.

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[1]Stommel, H. 1962, J.Mar.Res. 20: 92-96.

A Boundary Layer Convection Cell°

John L. Robinson

I. Introduction

Consider a horizontally infinite layer of fluid heated from below. This layer becomes unstable for a Rayleigh number $\left(R = \frac{g\alpha\Delta T d^3}{\varkappa\nu}\right)$ of about 1000, depending on the boundary conditions, and a steady cellular motion is set up. For a Prandtl number $\left(\sigma = \frac{\nu}{\kappa}\right)$ of the order of unity it is found that the motion becomes turbulent at a Rayleigh number of about 50,000[1].

In June of this year on the suggestion of the author, T. Rossby carried out this experiment for several Prandtl numbers of the order of 200 and it was found that the cellular motion, which was three-dimensional, remained for Rayleigh numbers of over 10^6, although slow-motion films showed that the cells pulsated.

In this work we make the assumption that for an infinite Prandtl number the cellular motion will not break down as the Rayleigh number is increased. It is thought that the introduction of a finite Prandtl number will give the critical value of the Rayleigh number for breakdown of the cellular motion into turbulence.

The problem is formulated for two-dimensional cells with free horizontal boundary conditions and approximate calculations are carried out to determine the Nusselt number. It is shown that the formulation for rigid horizontal boundaries is more complex and this problem has not been solved.

The Governing Equations

The equations of the Boussinesq approximation are:

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)T = \kappa\nabla^2 T \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}\left(\frac{\tilde{p}}{\rho_0}\right) + \nu\nabla^2\vec{v} - g\alpha(T-T_0)\hat{k} \tag{2}$$

$$\vec{\nabla}\cdot\vec{v} = 0 \tag{3}$$

where

κ : thermometric conductivity

α : coefficient of thermal expansion

$\tilde{p}$: $p - \rho_0 g z$

The boundary conditions are : $T = T_0 + \Delta T$ on $z = 0$

$T = T_0 - \Delta T$ on $z = 1$

$v_z = 0$ on all boundaries,

$v_x = 0$ on rigid boundaries

$\frac{\partial v_x}{\partial z} = 0$ on free boundaries.

Non-dimentionalization:

$$\vec{r} = d\,\vec{r}', \quad t = \frac{d^2}{\kappa}t', \quad \vec{v} = \frac{\kappa}{d}\vec{v}',$$

$$T = T_0 + \Delta T\,\theta, \quad \tilde{p} = \frac{\kappa\nu\rho_0}{d^3}p'.$$

The equations become:

$$\left(\frac{\partial}{\partial t'} + \vec{v}'\cdot\vec{\nabla}'\right)\theta = \nabla'^2\theta \tag{4}$$

$$\frac{1}{\sigma}\left(\frac{\partial}{\partial t'} + \vec{v}'\cdot\vec{\nabla}'\right)\vec{v}' = -\vec{\nabla}'p' + \nabla'^2\vec{v}' + R\theta\hat{k} \tag{5}$$

where $R = \frac{g\alpha\Delta T d^3}{\kappa\nu}$, $\sigma = \frac{\nu}{\kappa}$ (Prandtl number)

For steady motion and $\sigma \longrightarrow \infty$:

$$(\vec{v}'\cdot\vec{\nabla}')\theta = \nabla'^2\theta \tag{6}$$

$$\vec{\nabla}'p' = \nabla'^2\vec{v}' + R\theta\hat{k} \tag{7}$$

Since $\vec{\nabla}'\cdot\vec{v}' = 0$ the velocity may be expressed in terms of a stream function: $\vec{v}' = \vec{\nabla}\times\psi\hat{j}$. After eliminating the pressure the final equations are (dropping the primes):

$$J(\psi,\theta) = \nabla^2\theta \qquad (8)$$

$$\nabla^4\psi = R\frac{\partial\theta}{\partial x} \qquad (9)$$

The boundary conditions are: $\theta = 1$ on $z = 0$

$\theta = -1$ on $z = 1$

$\psi = 0$ on all boundaries

$\frac{\partial\psi}{\partial z} = 0$ on rigid boundaries

$\frac{\partial^2\psi}{\partial z^2} = 0$ on free boundaries (since $\frac{\partial^2\psi}{\partial x^2} = 0$ on the boundaries, this latter condition may be written $\nabla^2\psi = 0$).

II. Formulation of the Problem

Consider two-dimensional motion in which narrow jets of heated or cooled fluid travel from one boundary to the other. The streamlines within each cell are taken to be closed.

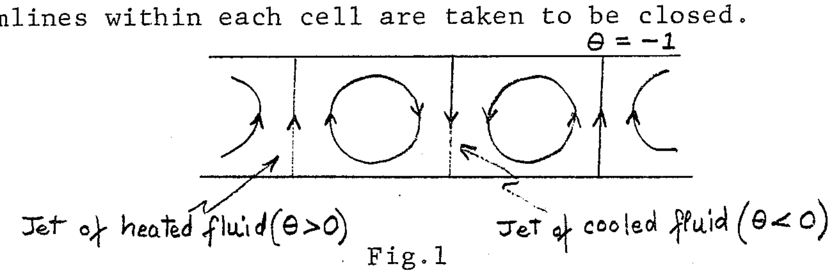

Fig.1

The basic assumption for the existence of the cellular motion is that there are thermal boundary layers at the vertical boundaries of each cell.

a) <u>The Vertical Boundary Layer</u>

Assumptions: 1. There exists a thermal boundary layer at $x = 0, L$.

2. There is no velocity shear at these boundaries.

The symmetries of the motion will also be taken into account. An implicit assumption here, based on the rigid boundary experiments, is that the length of a cell is of the order of magnitude of the separation of the plates $(L \sim 1)$.

Let the width of the boundary layer by δ and $\xi = \delta x$. Introduce interior and boundary stream functions $\psi_0(x, z)$, $\psi_{BL}(\xi, z)$ so that $\psi = \psi_0 + \psi_{BL}$, $R\frac{\partial\theta}{\partial x} = \nabla^4 \psi_{BL}$ and $\psi_{BLxx} + \psi_{0xx} = 0$ at $x = 0$. From these

$$[\psi_{BL}] \sim R\delta^3 \tag{10}$$

$$[\psi_{BL}] \sim [\psi_0]\delta^2 \tag{11}$$

Note that the velocities introduced by the boundary layer stream function are small compared with the interior velocities.

(10) and (11) show that $[\psi_0] \sim R\delta$ (12)

Since $J(\psi, T) = \nabla^2 T$, then $[\psi_0] \sim \delta^{-2}$ (13)

An implicit assumption here, based on the rigid boundary experiments, is that the length of a cell is of the order of magnitude of the separation of the plates $(L \sim 1)$.

Therefore <u>$\delta \sim R^{1/3}$, $\psi_0 \sim R^{2/3}$ and heat flux $\sim R^{1/3}$</u>.

Note that these results are independent of the horizontal boundary conditions and are implied by our basic assumptions.

The interior velocities are of the order of $\frac{\kappa}{d} R^{2/3}$. For one experiment[2] we find that: $\frac{\kappa}{d} R^{2/3} \approx 2\ \mathrm{cm/sec}$.

Observed Velocity $\approx \frac{1}{2}\ \mathrm{cm/sec}$.

The ordering of the interior velocity is thus in good agreement with experiment. We note however that if $\delta \sim R^{-\frac{1}{4}}$, $Vel \sim \frac{K}{d} R^{\frac{1}{2}} \approx .2$ cm/sec.

The equations for the $x = 0$ boundary layer are:

$$\varphi_{ox}(0,z)\,\theta_z(\xi,z) - \varphi_{oxz}(0,z)\,\xi\,\theta_\xi(\xi,z) = \theta_{\xi\xi}(\xi,z) \tag{14}$$

$$\varphi_{BL\,\xi\xi\xi\xi}(\xi,z) = \theta_\xi(\xi,z) \tag{15}$$

where $\psi_o = R^{2/3}\varphi_o$, $\psi_{BL} = \varphi_{BL}$, $\xi = R^{1/3}x$.

The boundary conditions are: $\theta_\xi(0,z) = 0$,

$\theta \to 0$ as $\xi \to 0$ (this is shown in the next section),

$\varphi_o(0,z) = 0, \varphi_{oxx}(0,z) + \varphi_{BL\,\xi\xi}(0,z) = 0$

Integrating (15) twice with respect to ξ from 0 to ∞ gives:

$$\varphi_{BL\,\xi\xi}(0,z) = \int_0^\infty \theta(\xi,z)\,d\xi = -f(z) \tag{16}$$

Then

$$\varphi_{oxx}(0,z) = f(z) \tag{17}$$

This is a boundary condition for the interior wave function on the vertical boundary.

Integrating the equation of motion (14) with respect to ξ gives:

$$\frac{d}{dz}\left\{\varphi_{ox}(0,z)\int_0^\infty \theta(\xi,z)\,d\xi\right\} =$$

$$= -\frac{d}{dz}\left\{\varphi_{ox}(0,z)\,f(z)\right\} = 0$$

This expresses conservation of vertical heat flux in this region.

$$\text{i.e.} \quad \varphi_{ox}(0,z)\,f(z) = -A \tag{18}$$

where $R^{1/3}A$ is the vertical heat flux.

b) <u>The Interior Region</u>

Let $\psi_o = R^{2/3}\varphi_o(x,z) + R^{\frac{2}{3}-a}\varphi_1(x,z) + \cdots$

$\theta_o = T_o(x,z) + R^{-a}T_1(x,z) + \cdots$

To highest order in R $J(\varphi_0, T_0) = 0$, i.e. T_0 is constant along streamlines, which are closed

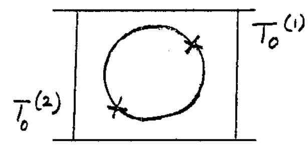

Fig. 2

If $T_0^{(1)} \leqslant 0$ then $T_0^{(2)} \geqslant 0$ by the symmetry of the problem. However $T_0^{(1)} = T_0^{(2)}$ and therefore $T_0 = 0$. The next order equation is $J(\psi_0, T_1) = 0$ and so $T_1 = T_2 = \ldots 0$.

The interior equations are therefore

$$\theta_0 = 0, \quad \nabla^4 \varphi_0 = 0 \tag{19}$$

The boundary conditions on φ_0 are:

$$\varphi_0 = 0 \quad on \quad x = 0, L \; ; \; z = 0, 1 .$$

$$\varphi_{0xx} = \nabla^2 \varphi_0 = f(z) \; on \; x = 0$$

$$\varphi_{0xx} = \nabla^2 \varphi_0 = f(1-z) \; on \; x = L$$

$\varphi_{0zz} = \nabla^2 \varphi_0 = 0$ on $z = 0, 1$ for the free boundaries.

c) The Horizontal Boundary Layer

The nature of the horizontal boundary layer is determined by the requirements that the difference in the heat convected into and out of this layer be conducted into the fluid there and that this should be an order $R^{1/3}$ quantity.

Simple arguments showed the following pictures to be incorrect for free horizontal boundary conditions:

(i) A rectangular corner with or without a velocity boundary layer.

(ii) A rectangular temperature boundary layer with relaxation of the requirement that the heat flux is convected through this region.

Consider a region of width $\sim R^{-b}$ and length R^{-a} near the origin with fluid impinging from above. In order that $J(\psi_0, \theta) \sim \nabla^2\theta$ and incoming flux $\sim R^{1/3}$, $b = \frac{1}{3}$, $a = 1$.

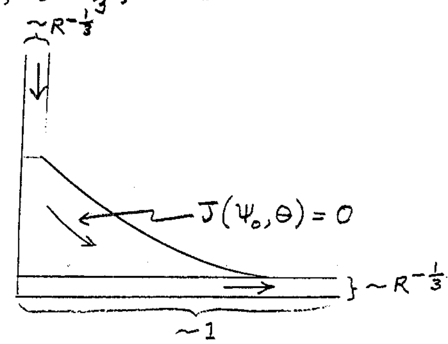

Fig.3

The fluid is convected into the horizontal boundary layer from above and within this layer conduction is important.

The temperature equation and boundary conditions are:

$$\varphi_{0z}(x,0)\,\theta_x(x,\zeta) - \varphi_{0zx}(x,0)\,\zeta\,\theta_\zeta(x,\zeta) = \theta_{\zeta\zeta}(x,\zeta) \tag{20}$$

$\theta(x, 0) = 1$, $\theta = \theta_{IN}$ on $\zeta = \ell$ (see below)

$\theta_z(0,\zeta) = 0$. where $\zeta = R^{\frac{1}{3}} z$.

In order to deal with the difficulty introduced in taking the boundary layer coordinate to infinity we assume that the horizontal boundary layer is of constant thickness. Later it is shown that this boundary layer thickness may be calculated.

<u>Assumption</u> 3. The horizontal boundary layer is of constant thickness.

III. Interior Region Calculations

The equation and boundary conditions are:

$$\nabla^4 \varphi_0 = 0 \qquad \varphi_0 = 0 \text{ on } x = 0, L \ ; z = 0, 1$$

$$\nabla^2 \varphi_0 = f(z) \text{ on } x = 0$$

$$\nabla^2 \varphi_0 = f(1 - z) \text{ on } x = L$$

$$\nabla^2 \varphi_0 = 0 \text{ on } z = 0, 1 .$$

This problem is solved in two parts:

(i) $\nabla^2(\nabla^2 \varphi_0) = 0$, $\nabla^2 \varphi_0$ boundary conditions.

(ii) $\nabla^2 \varphi_0$ is known from (i), φ_0 boundary conditions.

(i) Let $f(z) = \sum_1^\infty a_n \sin n\pi z$ (This does not imply that $\lim_{z \to 0} f(z) = 0$)

Then $$\nabla^2 \varphi_0 = \sum_1^\infty a_n \sin n\pi z \left(\frac{\sinh n\pi(L-x) - (-1)^n \sinh n\pi x}{\sinh n\pi L} \right) \tag{21}$$

(ii) The Green's function of this problem is:

$$G(x,z;x',z') = \begin{cases} 2\sum_1^\infty \dfrac{\sinh n\pi L}{n\pi} \sin n\pi z \sin n\pi z' \sinh n\pi(L-x') \sinh n\pi x & (x < x') \\ 2\sum_1^\infty \dfrac{\sinh n\pi L}{n\pi} \sin n\pi z \sin n\pi z' \sinh n\pi x' \sinh n\pi (L-x) & (x > x') \end{cases} \tag{22}$$

and the solution is:

$$\varphi_0(x,t) = -\sum_1^\infty \frac{a_n}{2\pi n} \sin n\pi z \left[\frac{x}{\sinh n\pi L} \left\{ \cosh n\pi(L-x) + (-1)^n \cosh n\pi x \right\} - \right. \\ \left. - L \frac{\sinh n\pi x}{\sinh^2 n\pi L} \left\{ 1 + (-1)^2 \cosh n\pi L \right\} \right] \tag{23}$$

Vertical Heat Flux

$$\varphi_{0x}(0,z) = -\sum_1^\infty a_n C_n \sin n\pi z$$

where $$C_n = \frac{1}{2\pi n} \frac{1}{\sinh^2 n\pi L} \left[\sinh n\pi L \left\{ \cosh n\pi L + (-1)^n \right\} - n\pi L \left\{ 1 + (-1)^n \cosh n\pi L \right\} \right] \tag{24}$$

Since $\psi_{0x}(0,z)f(z) = -A$ (18)

then $\left(\sum_1^\infty a_n c_n \sin n\pi z\right)\left(\sum_1^\infty a_n \sin n\pi z\right) = A$ (25)

From this $\sum_1^\infty c_n a_n^2 = A$ (26)

$$\sum_{n=N+1}^{\infty} a_n c_n a_{n-N} + \sum_{n=1}^{\infty} a_n c_n a_{n+N} - \sum_{n=1}^{N-1} a_n c_n a_{N-n} = 0 \qquad (27)$$

$N = 1, 2, 3 \ldots$.

These determine the as yet unknown constants up to an arbitrary factor $\sqrt{A}$ where $R^{\frac{1}{3}}A$ is the Nusselt number of the problem.

IV. An Approximate Solution

In this section an approximate value of the Nusselt number is obtained. The stream function and $f(x)$ are approximated by the first terms of their sine expansions.

The method is as follows:

(i) The temperature profile at the corner $x = 0, z \sim R^{-\frac{1}{3}}$ is approximated. This gives an expression for the thickness of the horizontal boundary layer.

(ii) An approximate value of the stream function at the edge of the vertical boundary region is equated to an approximate value of the stream function at the edge of the horizontal boundary layer. This assumes that the heat flux is bounded by a definite stream line.

(iii) A reasonable form for the temperature of the fluid flowing into the vertical boundary region is chosen at $z = 1$. Using the approximation of a constant vertical velocity, the temperature at $z = 0$, $x = 0$ is calculated.

(iv) The heat flux from the vertical boundary layer is considered to be convected into the top of the horizontal boundary layer. This gives a relation between the width and the length of this layer.

(v) A reasonable form of the boundary conditions for temperature on the upper boundary of the horizontal boundary layer is chosen and the outgoing heat flux is calculated. It is assumed that this is approximately equal to the vertical heat flux. Approximations to this heat flux and thus the Nusselt number are then obtained.

Velocity Approximations

$$A = \varphi_{ox}(0,z) f(z) \approx a_1^2 c_1$$

If $\alpha = \dfrac{1 + \frac{\pi L}{\cosh \pi L}}{2\pi} \approx c_1$, then $a_1 \approx \sqrt{\dfrac{A}{\alpha}}$

The vertical velocity at $x = 0$ is $\varphi_{ox}(0,z) \approx -\sqrt{\alpha A}$

Near the corner $x = 0$, $z = 0$: $\varphi_{ox} \approx -\pi\sqrt{\alpha A}\, z$

$\varphi_{ox} \approx -\pi\sqrt{\alpha A}\, x$

In the horizontal boundary layer $\varphi_{oz}(x,0) \approx -\pi\sqrt{\alpha A}\, x$ and this agrees well with the approximate value at $x = \frac{L}{2}$.

Notation

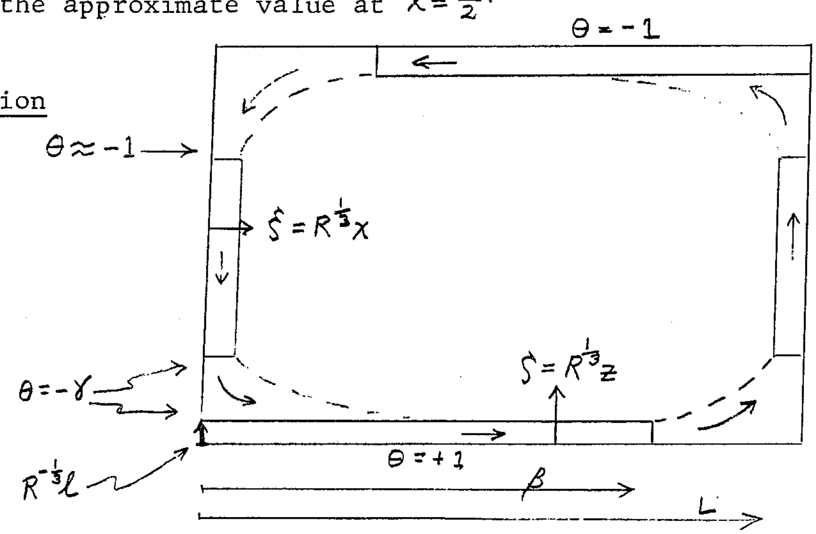

Fig.4

(i) Temperature profile for $x=0,\ z \sim R^{-\frac{1}{3}} (\zeta \sim 1)$

Since $\theta_z = 0$ at $x = 0$, the approximate equation is

$$\theta_{\zeta\zeta}(0,\zeta) = -\pi\sqrt{\alpha A}\ \zeta\,\theta_{\zeta}(0,\zeta) \tag{28}$$

with boundary conditions $\theta = 1$ at $\zeta = 0$

$$\theta \longrightarrow -\gamma \text{ as } \zeta \to \infty$$

The solution is $\theta(0,\zeta) = 1 - (\gamma + 1)\ \operatorname{erf}\left(\zeta\sqrt{\tfrac{3}{2}}(\alpha A)^{\frac{1}{4}}\right)$ (29)

and thus

$$\ell = \ell'\sqrt{\frac{2}{3}}\ \frac{1}{(\alpha A)^{1/4}} \tag{30}$$

with $\ell' \sim 1$.

(ii) The Edge Stream-line

(a) $x \sim R^{-\frac{1}{3}},\ z \simeq \frac{1}{2}$

$$\varphi_0 \approx \frac{a_1}{2\pi}\ \frac{(\pi L + \sinh \pi L)(\cosh \pi L - 1)}{\sinh^2 \pi L}\ \zeta$$

In this region $\varphi_{0x} \approx -\sqrt{\alpha A}$.

If $\theta \simeq e^{-\frac{\zeta^2}{m^2}}$ then, since $\varphi_{0x} f(z) = -A$ (18)

then $m \simeq \frac{2}{\sqrt{\pi}}\sqrt{\frac{A}{\alpha}}$ and thus:

$$\varphi_0 \approx \frac{a_1}{\pi\sqrt{\pi}}\ \frac{(\pi L + \sinh \pi L)(\cosh \pi L - 1)}{(\sinh^2 \pi L)}\sqrt{\frac{A}{\alpha}} \tag{31}$$

is the value of the stream function at the edge of this boundary layer.

(b) $z \sim R^{-\frac{1}{3}},\ x \approx \frac{L}{2}$

$$\varphi_0 \approx \frac{a_1}{2}\, L\ \frac{\sin \frac{\pi L}{2}}{\sinh^2 \pi L}\left(\cosh \pi L - 1\right)\zeta \tag{32}$$

Putting $\zeta = \ell$ in (32) and equating this with (31) gives:

$$\ell \approx \sqrt{\frac{A}{\alpha}}\,\frac{2}{\pi\sqrt{\pi}}\left(\frac{\pi L+\sinh \pi L}{\sinh\frac{\pi L}{2}}\right)\frac{1}{L} \tag{33}$$

Equating the values of ℓ given by (30), (33) gives:

$$A^{3/4} \approx \ell'\alpha^{\frac{1}{4}}\sqrt{\frac{2}{3}}\;\frac{\pi\sqrt{\pi}}{2}\left(\frac{L\sinh\frac{\pi L}{2}}{\pi L+\sinh\pi L}\right) \tag{34}$$

If at this stage we set $\ell' = 1$, $L = 1$ we obtain $A \approx .17$, $N \approx .17$. However it is shown in (iv) that $\ell' \approx \frac{1}{\beta}$ and thus a possibly more reasonable estimate for ℓ' is $\ell' \approx 2$ corresponding to $\beta \approx \frac{1}{2}$. For this latter choice of parameters $A \approx .38$, $N \approx .38$.

(iii) <u>The Vertical Boundary Layer</u>

The vertical velocity is approximated by a constant and the horizontal velocity is approximately zero in this region. We take an incoming temperature of $\theta = -1$ over a region which is determined by the heat flux.

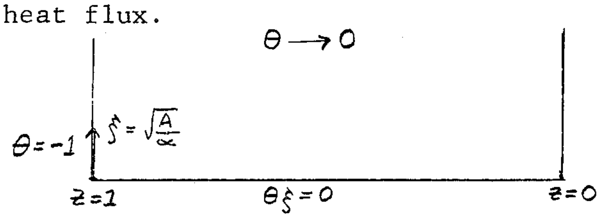

Fig.5

The equation is $\theta_{\xi\xi} = -\sqrt{\alpha A}\,\theta_z$ and the solution is

$$\theta(\xi, z) \approx -\int_0^\infty \frac{2}{\pi}\frac{1}{k}\sin\left(k\sqrt{\frac{A}{\alpha}}\right)\cos k\xi \times e^{-\frac{k^2}{\sqrt{\alpha A}}(1-z)}\,dk \tag{35}$$

We wish to approximate the temperature at $z = 0, \xi = 0$,

i.e. $\gamma = \theta(0,0) \simeq -\int_0^\infty \frac{2}{\pi}\frac{1}{k}\sin\left(k\sqrt{\frac{A}{\alpha}}\right)e^{-\frac{k^2}{\sqrt{\alpha A}}}\,dk$

For $A \approx .2$ (this is consistent with our final result), the major contribution to this integral is from $k\sqrt{\frac{A}{\alpha}} \ll \frac{\pi}{2}$.

Therefore $\gamma \simeq -\int_0^\infty \frac{2}{\pi}\sqrt{\frac{A}{\alpha}}\, e^{-\frac{k^2}{\sqrt{\alpha A}}}\, dk = -\frac{1}{\sqrt{\pi}}\left(\frac{1}{\alpha}\right)^{1/4} A^{3/4}$ (36)

(iv) <u>The Heat Influx to the Horizontal Boundary Layer</u>.

We consider the heat influx into the horizontal boundary layer to take place over a distance β and choose a reasonable form, $-\gamma\left(1-\frac{x^2}{\beta^2}\right)$, for the temperature profile on the upper boundary of this region.

Vertical Velocity $\approx \pi\sqrt{\alpha A}\, \ell$

Heat influx (scaled) $\approx \pi\sqrt{\alpha A}\, \ell \int_0^\beta \gamma\left(1-\frac{x^2}{\beta^2}\right)dx = \frac{2}{3}\pi\sqrt{\alpha A}\,\gamma\beta\ell$ (37)

If this is set equal to A using the values of ℓ, γ given by (30), (36) then $\beta\ell' \approx 1$. (38)

A more realistic form of the upper temperature profile; $-\gamma e^{-\frac{x^2}{\beta^2}}$, gives $\beta\ell' \approx \frac{3}{4}$. However the above form was chosen to simplify the calculations in (v) for the horizontal boundary layer.

(v) <u>The Horizontal Boundary Layer</u>.

Let $\theta(x,\zeta) = 1-(\gamma+1)\,\mathrm{erf}\left(\sqrt{\frac{3}{2}}\sqrt{\alpha A}\,\zeta\right) + \theta_1(x,\zeta)$

Then $B x\,\theta_{1x} - B\zeta\,\theta_{1\zeta} = \theta_{1\zeta\zeta}$ where $B = \pi\sqrt{\alpha A}$

and $\theta_1 = 0$ on $x = 0,\ \zeta = 0$

$\theta_1 \approx +\gamma\frac{x^2}{\beta^2}$ on $\zeta = \ell$.

The solution is:

$$\theta(x,\zeta) = (\gamma+1)\,\mathrm{erfc}\left(\sqrt{\tfrac{3}{2}}\sqrt{\alpha A}\,\zeta\right) - \gamma + \gamma\frac{x^2}{\beta^2}\left[\mathrm{ierfc}\left(\sqrt{\tfrac{B}{2}}\,\zeta\right) - \mathrm{ierfc}\left(-\sqrt{\tfrac{B}{2}}\,\zeta\right)\right] \times$$
$$\times\left[\mathrm{ierfc}\left(\sqrt{\tfrac{B}{2}}\,\ell\right) - \mathrm{ierfc}\left(-\sqrt{\tfrac{B}{2}}\,\ell\right)\right]^{-1} \qquad (39)$$

where[3] $\quad i\,\mathrm{erfc}\, z = \int_z^\infty \mathrm{erfc}\, t\, dt$

We wish to equate the heat efflux at $x=\beta$ to A:

i.e. $\quad B\int_0^\infty \theta(\beta,\zeta)\, d\zeta \approx A$

With the values of $i^2\mathrm{erfc}\, z$ given in reference 3, we find:

$$A^{3/4}\left\{.6\,\ell' - .25\right\} \approx .5\,\alpha^{\frac{1}{4}} \tag{40}$$

If now we set $L=1,\ \beta=\frac{1}{2}$ (and therefore $\ell'\approx 2$), then $N = A \approx .26$.

Rewriting (40) and (34):

$$A^{3/4}\alpha^{-1/4} \approx \frac{.5}{.6\ell' - .25} \approx \ell'\sqrt{\frac{2}{3}}\ \frac{\pi\sqrt{\pi}}{2}\left(\frac{L\sin\frac{\pi L}{2}}{\pi L + \sinh \pi L}\right) \tag{41}$$

In (ii) we assumed that the edge stream line met the upper boundary of the horizontal boundary layer at $x=\frac{L}{2}$. For consistency we must take $\beta=\frac{L}{2}$, the horizontal extent of this region in which cool fluid impinging from above is heated by the lower horizontal plate. Thus $\ell'\approx\frac{2}{L}$ and (41) gives an equation for L .

The result of this calculation is:

$$L \simeq .8\ ;\ \ell' \approx 2.5\ ;\ \ell \approx 4.0\ ;\ A \approx .2\ ;\ N \approx .25\ ;\ \gamma \simeq -.23\ .$$

The advantage of this final result over those previously given is that none of the parameters is guessed.

Note: a) Since $\ell' = 2.5$, the width of the horizontal boundary layer is 2.5 times the scale length for the temperature at $x=0$, $z \sim R^{-\frac{1}{3}}$. Thus despite Assumption 3 that the horizontal boundary layer is of constant width the temperature profile has a longer scale length at $x = \frac{L}{2}$ than at $x = 0$, as is to be expected; i.e. Assumption 3 has not constricted the temperature profile unrealistically.

(b) The horizontal length of the cell is determined by the requirement that the heat be convected into the fluid over a region of length $\frac{L}{2}$. Should the cell be of length greater than this L there would be two regions: 1. a region where fluid impinges from above and the horizontal heat efflux is less than A; 2. a region with no fluid impinging from above where the fluid is heated further to give a horizontal heat efflux A. Our picture of the flow indicates that this second region is expected to be unstable.

If the cell is of length less than this L the picture of the flow is more difficult to envisage. Perhaps in this case no cellular motion is possible.

Stability: The Rayleigh number is $R = \frac{g\alpha \Delta T d^3}{\kappa \nu}$. The temperature difference for the horizontal boundary layer is about ΔT and the width is $\ell R^{-\frac{1}{3}}$. Therefore the boundary layer Rayleigh number is:

$$R_\delta \approx \frac{R}{2}\left(R^{-\frac{1}{3}}\ell\right)^3 \approx 32 .$$

If the horizontal boundary layer is thought of as a fluid layer heated from below with the impinging fluid forming an upper boundary or lid then this is stable (i.e. $R_\delta << R_{crit.}$).

Once the region in which there is cooler fluid impinging from above is passed by any fluid particle, then this particle is effectively in an infinite fluid heated from below, an unstable situation (removal of the "lid"). We may thus formulate the followind dynamical picture of the fluid motion:

The fluid flowing along the lower horizontal boundary becomes unstable when the downdraft of colder fluid no longer acts as a lid. It then forms a narrow upward jet of heated fluid (cf. thermals). The center of this jet is cooled by outward conduction and convection as it travels upwards while the vertical heat flux in the jet remains constant. On impinging on the upper layer this jet spreads out along the plate after the heat has followed the streamlines round the corner and conduction takes place within a narrow horizontal layer which is stable to small perturbations. The above process is then repeated.

This picture indicates that the extent of the horizontal boundary layers should be approximated by L rather than by $\frac{L}{2}$ as we have done in the calculations. However the stream function is zero for $x = L$ and thus the value of the edge stream function was estimated for $x = \frac{L}{2}$. For consistency β had to be taken equal to $\frac{L}{2}$ also.

V. <u>Comparison with other Work</u>.

(i) The estimate of the Nusselt number obtained here is $N \approx .25R^{\frac{1}{3}}$ with a large possible error.

For turbulent motion at infinite Prandtl number and free boundary conditions Herring[4] has found that for maximum heat flux:

$N \approx .31\, R^{\frac{1}{3}}$ numerically, and

$N \approx .27\, R^{\frac{1}{3}}$ analytically.

In the 1964 G.F.D. lecture notes[5] Orszag gives values of $N \approx .36 R^{\frac{1}{3}}$ and $N \approx .29 R^{\frac{1}{3}}$ by two slightly different methods, again for turbulent motion and free horizontal boundary conditions.

(ii) For Prandtl numbers of the order of unity Weinbaum[6] gives values of $N \approx .35 R^{\frac{1}{4}}$ and $N \approx .24 R^{\frac{1}{4}}$ for natural convection in a horizontal circular cylinder. Since he takes rigid boundary conditions all round it is improbable that his work approximates cellular convection phenomena in an infinite horizontal region.

He compares his result with the experimental work of Schmidt and Saunders[7] (1937) and states that they find $N = .23 R^{\frac{1}{4}}$. In these experiments the ratio of the cell widths to the horizontal extent of the equipment varies from $\frac{1}{20}$ to $\frac{1}{4}$ and therefore the experimental results cannot be compared with any confidence to theories relating to either an infinite horizontal region or a totally enclosed cell.

The author finds that Schmidt and Saunders do not quote any form of the dependence of the Nusselt number on the Rayleigh number but the results for water given in their figure 3 are equally well fitted by $N \approx .11 R^{\frac{1}{3}}$ and $N \approx .30 R^{\frac{1}{4}}$.

We may conclude that Weinbaum has shown that for a cell enclosed by rigid boundaries $N \approx .3 R^{\frac{1}{4}}$, while this paper shows that for cellular motion between free horizontal boundaries $N \approx .25 R^{\frac{1}{3}}$ for infinite Prandtl number. The dependence of the Nusselt number on the Rayleigh number can be shown to be independent of the Prandtl number if $\sigma \gtrsim 1$.

i.e. all boundaries rigid $N = A(\sigma)\, R^{\frac{1}{4}}$

all boundaries free $N = B(\sigma)\, R^{\frac{1}{3}}$

Neither of the theoretical results may be compared with confidence to the available experimental results as these concern a fluid of large horizontal extent between rigid horizontal boundaries.

We will now indicate a general method which leads to the above dependences of the Nusselt number on the Rayleigh number.

Let $\xi = R^a x$

$\theta = \theta_0(x, z) + \theta_1(\xi, z) + \ldots$

$\psi = R^b \varphi_0(x,z) + R^{b-ma} \varphi_1(x,z) + \ldots$

If it is required that all terms in the equations are important and that for rigid boundaries $\frac{\partial \varphi_0}{\partial x} \sim R^{-ma} \frac{\partial \varphi_1}{\partial x}$, for free boundaries $\frac{\partial^2 \varphi_0}{\partial x^2} \sim R^{-ma} \frac{\partial^2 \varphi_1}{\partial x^2}$ in order to satisfy the boundary conditions, then for both finite (~ 1) and infinite Prandtl number:

	Rigid boundary	Free boundary
Horizontal boundary	$m=1,\ a=\frac{1}{5},\ b=\frac{2}{5}$	$m=2,\ a=\frac{1}{4},\ b=\frac{1}{2}$
Vertical boundary	$m=1,\ a=\frac{1}{4},\ b=\frac{1}{2}$	$m=2,\ a=\frac{1}{3},\ b=\frac{2}{3}$

We then arrive at Weinbaum's result for rigid vertical boundaries. In this paper the values indicated by the free vertical boundary conditions are chosen. On the horizontal boundaries the condition that $\frac{\partial^2 \varphi_0}{\partial z^2} \sim R^{-m} \frac{\partial^2 \varphi_1}{\partial z^2}$ is relaxed and instead $\frac{\partial^2 \varphi_0}{\partial z^2} = 0$ is the boundary condition. As indicated in the next section the method appears to fail for rigid horizontal boundaries.

It has been noted that for a horizontal thermal boundary layer $R_\delta \sim R\,\delta^3$ and that this is of order unity for $\delta = R^{-\frac{1}{3}}$. However for $\delta = R^{-\frac{1}{4}}$, $R_\delta \sim R^{\frac{1}{4}}$ and thus $R_\delta \to \infty$ as $R \to \infty$ and R_δ will reach the critical value for some finite, though large, value of R.

VI. Rigid Horizontal Boundary Conditions.

We require the heat conducted in at the boundaries to be $\sim R^{\frac{1}{3}}$ over a region ~ 1. This indicates a $\delta \sim R^{-\frac{1}{3}}$ thermal boundary layer. The vertical heat flux into this region must be ~ 1 and thus $m = 1$ (previous notation).

i.e. $$\psi = R^{\frac{2}{3}}\varphi_0(x,z) + R^{\frac{1}{3}}\varphi_1(x,\zeta) + \dots$$
$$\theta = \theta_1(x,\zeta) + \dots \qquad\qquad \zeta = R^{\frac{1}{3}}z$$

Then $$J(\psi,\theta_1) = \nabla^2\theta_1 \quad \text{and}$$
$$\varphi_{1\zeta\zeta\zeta\zeta} = 0 \tag{42}$$

Since equation (42) has no decaying solutions we take $\psi = \psi_{BL} = R^{\frac{1}{3}}\varphi_{BL}$ within the boundary region and attempt to match this solution to $\varphi_0(x,z)$. i.e. $\varphi_{BL\zeta\zeta\zeta\zeta} = 0$; $\varphi_{BL} = \varphi_{BL\zeta} = 0$ at $\zeta = 0$.

Then $$\varphi_{BL} = a(x)\,\zeta^3 + b(x)\,\zeta^2$$
i.e. $$\psi_{BL} = a(x)\,R^{\frac{4}{3}}z^3 + b(x)\,R\,z^2 .$$

Since $\psi_0 \sim R^{2/3}$ we are unable to match the two solutions.

We have therefore considered the following possibilities:

a) Velocity boundary layer of different thickness than the thermal boundary layer.

b) Two velocity boundary layers. We require $\nabla^4\psi \sim R\,T_x$ in order to change the above boundary layer solution. However when matching is attempted between the two boundary layers we find $\psi >> R^{2/3}$.

Three other possibilities may be suggested:

a) $N \sim R^{\alpha}$ with $\frac{1}{4} \leqslant \alpha \leqslant \frac{1}{3}$ and possibly $\alpha = \alpha(\sigma)$. (As noted at the end of V, if $\alpha < \frac{1}{3}$ then $R_{\delta} \to \infty$ as $R \to \infty$).

b) We have here assumed a length scale of order unity for the convection cell, based on the experimental evidence available. However this may not be valid for our limiting process and perhaps $L \sim R^{-a}$, i.e. $\psi_0 = \psi_0 (R^{a} x, z)$.

c) In mixing-length turbulence theory Kraichnan[8] shows that for $1 \geq \sigma > 0.15$ and very large R, $N \sim 9.10^{-3} \sigma^{-\frac{1}{4}} \left[\frac{R}{(\ln R)^3}\right]$.

Perhaps for rigid horizontal boundaries the Nusselt number has a more complex form of Rayleigh number dependence than that chosen here.

VII. <u>Turbulence</u>.

We have assumed the cell to be stable for an infinite Prandtl number and wish to find at what value of R the motion becomes turbulent for finite large Prandtl number. This will be when the shear term ($u . \nabla u$) becomes important. The full equation is:

$$\frac{1}{\sigma}(\vec{u}.\vec{\nabla})\vec{u} = \nabla^2 \vec{u} - R\theta\vec{k} - \frac{1}{\rho}\nabla p \qquad (5)$$

In the boundary layer $\vec{u}.\nabla\vec{u} \sim R^{4/3}$

$\nabla^2 \vec{u} \sim R\theta \sim R$ and thus the shear term becomes important in the boundary layer for $\frac{R^{4/3}}{\sigma} / R = R^{\frac{1}{3}}\sigma^{-1} \sim 1$.

In the interior $\vec{u}.\nabla\vec{u} \sim R^{\frac{4}{3}}$

$\nabla^2 \vec{u} \sim R^{2/3}$, $\theta \sim 0$

and thus the shear term becomes important in the interior for

$$\frac{\mathcal{R}^{4/3}}{\sigma} \Big/ \mathcal{R}^{2/3} = \mathcal{R}^{2/3}\sigma^{-1} \sim 1 .$$

Our condition for the onset of turbulence is thus

$$\mathcal{R}_T^{2/3}\ \sigma^{-1} = \text{constant}. \qquad (43).$$

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Wind-generated Internal Waves

Matthias Tomczak

The following work tries to answer the question whether a wind which blows over the surface of a stratified ocean will give rise to internal waves and how the vertical amplitude distribution of these waves looks. In order to make the problem considerably tractable, the calculation presented here is done under the following assumptions.

1. The mean density $\bar{\rho}$ is independent of horizontal position and increases exponentially with depth, so that the Brunt-Väisälä frequency $\sqrt{g\Gamma}$ is a constant:
$$\Gamma = \frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dz} \equiv \Gamma_0 .$$
2. The mean velocity is $\underset{\sim}{\bar{u}} \equiv 0$.
3. The depth of the ocean is constant.
4. The Coriolis parameter f is constant.

The basic equations

If we define $P' = \frac{p}{\bar{\rho}}$, $R' = \frac{\rho}{\bar{\rho}}$, where p , ρ are the pressure and density fluctuations respectively, we get by using the Boussinesq and the hydrostatic assumptions the following set of linearized equations:

$$u_t + fv + P'_x - \mu u_{zz} = 0 \qquad (1)$$

$$v_t - fu + P'_y - \mu v_{zz} = 0 \qquad (2)$$

$$-gR' + P'_z = 0 \qquad (3)$$

$$R_t + \Gamma_0 W = 0 \qquad (4)$$

$$u_x + v_y + w_z = 0 \text{ (indices denote differentiation)} \tag{5}$$

where (3) and (4) can be combined to

$$g\Gamma_0 w + P'_{zt} = 0 \tag{6}$$

Here, x is directed towards the east, y towards the north and z downwards.

We define the operator $D = \frac{\partial}{\partial t} - \mu \frac{\partial^2}{\partial z^2}$, multiply (1) by f, (2) by D and add the resulting equations, which leads to

$$(D^2+f^2)v = -f P'_x - D P'_y \tag{7}$$

Similarly the equation

$$(D^2+f^2)u = f P'_y - D P'_x \text{ can be obtained.} \tag{8}$$

We multiply (5) by (D^2+f^2), substitute u_x and v_y by the aid of (7) and (8) and w_z from (6) and end up with an equation for P':

$$D(P'_{xx} + P'_{yy}) + (D^2+f^2)\frac{P_{zzt}}{g\Gamma_0} = 0 \tag{9}$$

The boundary conditions are

$$\left.\begin{aligned} w &= 0 \\ -\mu \frac{\partial u}{\partial z} &= \tau_x \\ -\mu \frac{\partial v}{\partial z} &= \tau_y \end{aligned}\right\} \text{ at the surface } z = 0$$

where τ_x, τ_y are the wind stress in x- and y-direction, respectively, and $u = v = w = 0$ at the bottom.

All boundary conditions can be expressed in terms of P' with the aid of (6) - (8).

Equation (9) for harmonic waves

We expect the fluctuations of the wind stress to cause wave motion in the interior of the ocean, hence, to deal with the most general case in respect to that, we assume for

$$\psi = u, v, w, P', R', \tau_x, \tau_y$$

$$\psi(x,y,z,t) = \sum_{\kappa} \sum_{\eta} \sum_{\omega} \psi(z) e^{i(\kappa x + \eta y + \omega t)} \tag{10}$$

$$(\tau_x, \tau_y \text{ no functions of } z)$$

So, (9) becomes

$$-D(\kappa^2+\eta^2)P(z) + \frac{i\omega}{g\Gamma_0}(D^2+f^2)P_{zz}(z) = 0 \tag{11}$$

Furthermore we set $P(z) = \sum_j A_j e^{i\zeta_j z}$ and get

$$D = i\omega + \mu \zeta_j^2$$

$D^2 = -\omega^2 + 2i\omega\mu\zeta_j^2 + \mu^2\zeta_j^4$, which yields (after having divided through $\frac{i\omega\mu^2}{g\Gamma_0}$)

$$\zeta^6 + \frac{2i\omega}{\mu}\zeta^4 - \left(\frac{\omega^2 - f^2}{\mu^2} + i\,\frac{(\kappa^2+\eta^2)g\Gamma_0}{\omega\mu}\right)\zeta^2 + \frac{(\kappa^2+\eta^2)g\Gamma_0}{\mu^2} = 0 \tag{12}$$

as the equivalent for (9).

To obtain the roots of (12) approximately, we make use of the scaling due to the Boussinesq approximation. If L is a typical horizontal length scale (i.e. $\left(\frac{1}{\kappa^2+\eta^2}\right)^{\frac{1}{2}}$), H a typical vertical one (i.e. the depth of the ocean) and d the thickness of the boundary layer, we have $g\Gamma_0 = \omega^2 \frac{L^2}{H^2}$ (see F.Bretherton, 4th lecture, Vol.I) provided $\omega >> f$. The terms of (12) therefore are of the order of magnitude

$$O(1)\cdot\zeta^6+O\left(\frac{1}{d^2}\right)\cdot\zeta^4+O\left(\frac{1}{d^4}\right)\cdot\left(1-\frac{f^2}{\omega^2}\right)\zeta^2+O\left(\frac{1}{d^2H^2}\right)\zeta^2+O\left(\frac{1}{d^4H^2}\right)\doteq 0$$

$\frac{f^2}{\omega^2}$ is a very small quantity, but will not be neglected in the following calculations so that all results are of first order with respect to $\frac{f^2}{\omega^2}$. If μ is very small, $\frac{d^2}{H^2}$ is a very small quantity, too, and we proceed to find zero and first order solutions with respect to $\frac{d^2}{H^2}$.

Because we are looking for internal waves we expect to find solutions ζ_j of the order of $\frac{1}{H}$. In that case the significant terms are those of the order of $\frac{1}{d^4H^2}$, and (12) becomes (to zero order)

$$-\frac{\omega^2-f^2}{\mu^2}\,\zeta^2+\frac{(k^2+\eta^2)g\Gamma_0}{\mu^2}=0, \tag{13}$$

the roots of which are

$$\zeta_j=\pm\sqrt{\frac{(k^2+\eta^2)g\Gamma_0}{\omega^2-f^2}} \qquad j=1,2$$

If ζ_j is of the order of $\frac{1}{d}$, it is easy to see that the important terms of (12) to zero order are

$$\zeta^6+\frac{2i\omega}{\mu}\,\zeta^4-\frac{\omega^2-f^2}{\mu^2}\,\zeta^2=0 \tag{14}$$

which has the roots

$$\zeta_j=\pm(1-i)\sqrt{\frac{\omega\mp f}{2\mu}} \qquad j=3,4,5,6$$

So $P(z)$ is the sum of six waves, four of which only occur in the surface or bottom boundary layer. The amplitudes of the waves have to be calculated by means of the boundary conditions. However, u and v are expressed in terms of P' after dividing (7) and (8) through D^2+f^2, and for the zero order solutions

$$D_j^2 + f^2 = -(\omega^2 - f^2) + 2i\omega\mu\, \zeta_j^2 + \mu^2 \zeta_j^4$$

is identically zero for j = 3, 4, 5, 6. Therefore, in the boundary layer a first order solution has to be determined. This is done by setting $\zeta^2 = \frac{x_0}{\mu} + x_1$ where $x_0 = -i\frac{\omega \mp f}{\mu}$ is the zero order solution. (12) then leads to

$$3x_0^2 x_1 + 4i\omega x_0 x_1 - (\omega^2 - f^2)x_1 - i\frac{(\kappa^2 + \eta^2) g\Gamma_0}{\omega} x_0 + (\kappa^2 + \eta^2) g\Gamma_0 = 0 \qquad (15)$$

and we have

$$x_1 = \frac{i(\kappa^2+\eta^2)g\Gamma_0 x_0 - \omega(\kappa^2+\eta^2)g\Gamma_0}{\omega} \cdot \frac{1}{3x_0^2 + 4i\omega x_0 - (\omega^2 - f^2)}$$

or, using (14) for $\zeta^2 = x_0$

$$x_1 = \frac{i(\kappa^2+\eta^2)g\Gamma_0 x_0 - \omega(\kappa^2+\eta^2)g\Gamma_0}{\omega} \cdot \frac{1}{x_0^2 + (\omega^2 - f^2)}$$

which finally, after substituting x_0, gives

$$x_1 = -\frac{(\kappa^2+\eta^2)g\Gamma_0}{2\omega(\omega \mp f)} \qquad (16)$$

or

$$\zeta_j^2 = -i\frac{\omega \mp f}{\mu} - \frac{(\kappa^2+\eta^2)g\Gamma_0}{2\omega(\omega \mp f)} \qquad j = 3, 4, 5, 6 \qquad (17)$$

Solutions (17) enable us to determine the amplitudes from the boundary conditions. The set of equations to be solved is (sums over j, j = 1, 2, ...6)

$$W(0) = 0 \quad \Longleftrightarrow \quad \sum \zeta_j A_j = 0 \qquad (18)$$

$$W(H) = 0 \quad \Longleftrightarrow \quad \sum \zeta_j e^{i\zeta_j H} A_j = 0 \qquad (19)$$

$$U(H) = 0 \quad \Longleftrightarrow \quad \sum \frac{-i\kappa f - i\eta D_j}{D_j^2 + f^2} e^{i\zeta_j H} A_j = 0 \qquad (20)$$

$$V(H) = 0 \quad \Longleftrightarrow \quad \sum \frac{i\eta f - i\kappa D_j}{D_j^2 + f^2} e^{i\zeta_j H} A_j = 0 \qquad (21)$$

$$\left.\frac{dU}{dz}\right|_{z=0} = -\frac{\tau_x}{\mu} \iff \sum i\zeta_j \cdot \frac{-ikf - i\eta D_j}{D_j^2+f^2} \cdot A_j = -\frac{\tau_x}{\mu} \tag{22}$$

$$\left.\frac{dV}{dz}\right|_{z=0} = -\frac{\tau_y}{\mu} \iff \sum i\zeta_j \cdot \frac{i\eta f - ikD_j}{D_j^2+f^2} \cdot A_j = -\frac{\tau_y}{\mu} \tag{23}$$

The coefficients in (18) - (23) are rather complicated, and it is not possible to simplify the occurring determinants very much. The solution will therefore not be presented here. It is possible, however, to get a solution for an infinitely deep ocean which might be of considerable significance.

Zero order solution for an infinitely deep ocean.

In this case one might start with equations (1), (2), (5) and (6) and use assumption (10). The resulting system can be transformed by the aide of

$$\Psi(\zeta) = \int_0^\infty \Psi(z)\cos\zeta z\, dz \quad \text{for } \Psi = U, V, P \tag{24}$$

$$W(\zeta) = \int_0^\infty W(z)\sin\zeta z\, dz \tag{25}$$

We then have

$$(i\omega + \mu\zeta^2)U + fV + ikP = \tau_x \tag{26}$$

$$(i\omega + \mu\zeta^2)V - fU + i\eta P = \tau_y \tag{27}$$

$$-\zeta P + \frac{g\Gamma_0}{i\omega} W = 0 \text{ (All variables are} \tag{28}$$

$$ik U + i\eta V + \zeta W = 0 \text{ variables of } \zeta \text{ now)} \tag{29}$$

We solve the system for W using Cramer's rule and obtain the equation

$$W(\zeta) = \frac{-i\mu(k\tau_x + \eta\tau_y)\zeta^3 + (\omega(k\tau_x+\eta\tau_y) + if(\eta\tau_x - k\tau_y))\zeta}{\mu^2\zeta^6 + 2i\omega\mu\zeta^4 - ((\omega^2-f^2) + i\frac{(k^2+\eta^2)g\Gamma_0\mu}{\omega})\zeta^2 + (k^2+\eta^2)g\Gamma_0} = \frac{g(\zeta)}{P(\zeta)} \tag{30}$$

The formula for reverting the transform is

$$W(z) = \frac{2}{\pi}\int_0^\infty w(\zeta)\sin\zeta z \, d\zeta \qquad (31)$$

which can be replaced by

$$W(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} 2\cdot w(\zeta) e^{i\zeta z} dz\,, \qquad (32)$$

because $W(\zeta)$ is an odd function.

In order to determine the poles of $W(\zeta)$, we have to determine the roots of $p(\zeta)=0$ which are known already from solving (12). However, because no poles must occur on the real axis, it becomes important that ζ_1 and ζ_2 are approximate solutions only and that the exact solutions have a small imaginary part. We therefore check the sign of the imaginary part of the exact solutions.

To zero order in $\frac{f}{\omega}$, $f=0$ and $\zeta^2 = -i\frac{\omega}{\mu}$ is an exact solution of (12). After dividing through $\left(\zeta^2 + i\frac{\omega}{\mu}\right)$, (12) becomes

$$\zeta^4 + i\frac{\omega}{\mu}\,\zeta^2 - i\frac{(k^2+\eta^2)g\Gamma_0}{\omega\mu} = 0 \qquad (33)$$

which has the solution

$$\zeta^2 = -i\frac{\omega}{2\mu} \pm \sqrt{-\frac{\omega^2}{4\mu^2} + i\frac{(k^2+\eta^2)g\Gamma_0}{\omega\mu}}$$

$$= -i\frac{\omega}{2\mu}\left(1 \mp 1\left(1 - i\frac{4(k^2+\eta^2)g\Gamma_0\mu}{\omega^3}\right)^{1/2}\right) \qquad (34)$$

Keeping in mind that $\frac{4(k^2+\eta^2)g\Gamma_0\mu}{\omega^3} = \varepsilon$ is of the order of $\frac{d^2}{H^2} \ll 1$, (34) gives ζ_1 and ζ_2 for the special case $f=0$ (W. Krauss, 1965). Now, the point $1-i\varepsilon = r(\cos\varphi - i\sin\varphi)$ is located in the lower half plane, and its real part is $r\cos\varphi = \sqrt{1+\varepsilon^2}\cos\varphi = 1$. Hence $+\sqrt{1-i\varepsilon} = \sqrt{r}\left(\cos\frac{\varphi}{2} - i\sin\frac{\varphi}{2}\right)$ has a real part

$$\sqrt{r}\cos\frac{\varphi}{2} = \sqrt[4]{1+\varepsilon^2}\cdot\sqrt{\frac{1+\cos\varphi}{2}}$$
$$= \sqrt{\frac{\sqrt{1+\varepsilon^2}+\sqrt{1+\varepsilon^2}\cos\varphi}{2}} > 1$$

and we can therefore write

$$-1+\sqrt{1-i\varepsilon} = +z_1 - i z_2 \quad (z_1>0\ ,\ z_2>0) \qquad \text{so that}$$

$$\zeta_{1/2}^2 = i\frac{\omega}{2\mu}\left(-1+\sqrt{1-i\varepsilon}\right) = i\frac{\omega}{2\mu}\left(+z_1 - i z_2\right) = \frac{\omega}{2\mu}z_2 + i\frac{\omega}{2\mu}z_1$$

This shows that the root with the positive imaginary part is ζ_1.

The same procedure goes through for $f \neq 0$, if one again expects the imaginary part to be small and tries a solution for (12) of the form $\zeta^2 = \alpha\varepsilon + i\beta\varepsilon^2$ (α and β of the same order). ζ_1 and ζ_2 occur to be the approximate solutions, and ζ_1 is the root in the upper half plane.

If we now evaluate $W(z)$ according to (32), we close the path of integration in the upper half plane and get contributions from ζ_1, ζ_4 and ζ_6. The corresponding amplitudes are given by

$$A_j = \frac{2\cdot g(\zeta_j)}{p'(\zeta_j)} \qquad \left(j=1,4,6 \quad p' = \frac{dp}{d\zeta}\right)$$

$$\frac{g(\zeta)}{p'(\zeta)} = \frac{-i\mu(k\tau_x+\eta\tau_y)\zeta^2+\omega(k\tau_x+\eta\tau_y)+if(\eta\tau_x-k\tau_y)}{6\mu^2\zeta^4+8i\omega\mu\zeta^2-2(\omega^2-f^2)-2i\frac{(k^2+\eta^2)g\bar{\sigma}\mu}{\omega}} \tag{35}$$

With the scaling mentioned before the terms are of the following order of magnitude:

$$g(\zeta_1) = (k\tau_x+\eta\tau_y)\left[-i\omega\,O\left(\frac{d^2}{H^2}\right)+\omega\,O(1)+if\,O(1)\right]$$
$$p'(\zeta_1) = \omega^2\left[6\cdot O\left(\frac{d^4}{H^4}\right)+8i\cdot O\left(\frac{d^2}{H^2}\right)-2\left(1-\frac{f^2}{\omega^2}\right)\cdot O(1)-2_2\cdot O\left(\frac{d^2}{H^2}\right)\right]$$

Hence taking into account terms of order of unity only,

$$A_1 = -\left[\frac{\omega(\kappa\tau_x+\eta\tau_y)}{\omega^2-f^2} + i\,\frac{f(\eta\tau_x-\kappa\tau_y)}{\omega^2-f^2}\right] \tag{36}$$

Similarly

$$g(\zeta_{4,6}) = (\kappa\tau_x+\eta\tau_y)\left[-i\omega O(1)+\omega O(1)+ifO(1)\right]$$

$$p'(\zeta_{4;6}) = \omega^2\left[6\cdot O(1)+8i\cdot O(1)-2\left(1-\frac{f^2}{\omega^2}\right)\cdot O(1)-2\tfrac{i}{2}\, O\left(\frac{d^2}{H^2}\right)\right]$$

and therefore

$$A_{4;6} = \frac{-(\kappa\tau_x+\eta\tau_y)(\omega\mp f)+\omega(\kappa\tau_x+\eta\tau_y)+if(\eta\tau_x-\kappa\tau_y)}{-6(\omega\mp f)^2+8\omega(\omega\mp f)-2(\omega^2-f^2)}\cdot 2$$

$$= \frac{1}{2}\left[\frac{\kappa\tau_x+\eta\tau_y}{\omega\mp f} \pm i\,\frac{\eta\tau_x-\kappa\tau_y}{\omega\mp f}\right] \tag{37}$$

If we write A_1 in the form

$$A_1 = \frac{\frac{1}{2}(\kappa\tau_x+\eta\tau_y)}{\omega-f} + \frac{\frac{1}{2}(\kappa\tau_x+\eta\tau)}{\omega+f} + i\left[\frac{\frac{f}{2\omega}(\eta\tau_x-\kappa\tau_y)}{\omega-f} + \frac{\frac{f}{2\omega}(\eta\tau_x-\kappa\tau_y)}{\omega+f}\right]$$ we get

$$\begin{aligned} W(z) = {} & A\left(e^{-\varphi_4 z}\cos\varphi_4 z-\cos\varphi_1 z\right) + B\left(e^{-\varphi_4 z}\sin\varphi_4 z+\frac{f}{\omega}\sin\varphi_1 z\right) \\ & + C\left(e^{-\varphi_6 z}\cos\varphi_6 z-\cos\varphi_1 z\right) + D\left(-e^{-\varphi_6 z}\sin\varphi_6 z+\frac{f}{\omega}\sin\varphi_1 z\right) \\ & + i\Big[-A\left(e^{-\varphi_4 z}\sin\varphi_4 z+\sin\varphi_1 z\right) + B\left(e^{-\varphi_4 z}\cos\varphi_4 z-\frac{f}{\omega}\cos\varphi_1 z\right) \\ & -C\left(e^{-\varphi_6 z}\sin\varphi_6 z+\sin\varphi_1 z\right) - D\left(e^{-\varphi_6 z}\cos\varphi_6 z+\frac{f}{\omega}\cos\varphi_1 z\right)\Big] \end{aligned} \tag{38}$$

where

$$\left.\begin{matrix}A\\C\end{matrix}\right\} = \frac{\kappa\tau_x+\eta\tau_y}{2(\omega\mp f)} \qquad \left.\begin{matrix}B\\D\end{matrix}\right\} = \frac{\eta\tau_x-\kappa\tau_y}{2(\omega\mp f)}$$

$$\varphi_1 = \sqrt{\frac{(\kappa^2+\eta^2)g\Gamma_0}{\omega^2-f^2}} \qquad \left.\begin{matrix}\varphi_4\\\varphi_6\end{matrix}\right\} = \sqrt{\frac{\omega\mp f}{2\mu}} .$$

It is easily seen that the first five terms and the seventh term satisfy the boundary condition $W(0) = 0$ individually, and the sum of the remaining terms vanishes for $z = 0$, too:

$$B\left(1-\frac{f}{\omega}\right) - D\left(1+\frac{f}{\omega}\right) = \frac{\eta\tau_x - k\tau_y}{\omega}\left(\frac{\omega - f}{2(\omega - f)} - \frac{\omega + f}{2(\omega + f)}\right) = 0$$

For $z > d$, where the Ekman-layer type solutions are negligible, the complete solution is given by the sum of all components

$$W_{k,\eta,\omega}(x, y, z, t) = -\frac{\omega(k\tau_x + \eta\tau_y)}{\omega^2 - f^2}\cos\left(kx + \eta y + \omega t + \sqrt{\frac{(k^2+\eta^2)g\Gamma_0}{\omega^2 - f^2}}\, z\right) +$$

$$+ \frac{f(\eta\tau_x - k\tau_y)}{\omega^2 - f^2}\sin\left(kx + \eta y + \omega t + \sqrt{\frac{(k^2+\eta^2)g\Gamma_0}{\omega^2 - f^2}}\, z\right).$$

Discussion

Equation (39) describes travelling waves, which satisfy the boundary conditions for the surface when the other two waves, corresponding to ζ_4 and ζ_6, are added. This of course has to be expected, because no reflection can occur in an infinitely deep ocean. However, if a wave starts at $z = 0$ travelling downwards, the solution will be valid until the wave hits the bottom, in which moment the three other types of waves will occur and together with the obtained solution will add to a standing wave. That means that the amplitudes A_1, A_4 and A_6 will not change if the depth is finite, and the problem of solving (18) - (23) is reduced to the question of solving (19) - (21) only with known A_1, A_4 and A_6. Furthermore, if $f \equiv 0$ we might expect plane waves and set $v \equiv \eta \equiv 0$, which, together with the fact that $\zeta_3 = \zeta_5$ and $\zeta_4 = \zeta_6$, gives a fairly simple set of only two equations with the two unknowns A_2 and A_3 and makes the first order problem considerably tractable.

After having solved the first order problem for $f \equiv 0$, one might have some ideas about the first order solution both in μ and $\frac{f}{\omega}$ and the calculation can be carried through in that case also. The present work is a first step in this direction. The aim is to express the wave amplitude as a function of κ, η, ω and the spectral components $\tau_{x,\kappa\eta\omega}$ and $\tau_{y,\kappa\eta\omega}$ of the wind stress which finally might enable us to calculate the spectrum of the internal waves due to the known spectrum of the wind stress fluctuations.

Acknowledgements

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A Simple Problem of Mixed Barotropic-Baroclinic Instability

William A. Bowman

Introduction. Barotropic instability results when perturbations grow on a steady vertically uniform current. The energy for such a growth is derived from the kinetic energy manifested in the horizontal shear of the steady current. The vertical uniformity of the steady current, demanded by the barotropic problem, follows directly by assuming that the current considered is of planetary scale and therefore in geostrophic and hydrostatic balance. For a steady horizontally uniform current, it might be mentioned, the barotropic case yields only stable Rossby waves.

Baroclinic instability results from the potential energy of the cross-current horizontal temperature gradient. Here, in contrast with the barotropic case, the steady current is not uniform with height. Each of these cases leads to mathematically formidable problems, which although tractable under certain idealizations, neither has been solved in generality.

The mixed problem is no exception since it combines the two cases of barotropic and baroclinic instability by including both horizontal and vertical non-uniformity (shear) in the steady current. The mixed problem therefore includes both sources of energy: kinetic and potential. This paper represents an attempt to treat the mixed problem by considering the simplest physical system capable of supporting such instability. This system, which is described in the next section, as well as the formulation of the coupled equations, has been discussed by Pedlosky (J.Atmos.Sci. 1964).

Two-layer model. The first simplification will be to consider a two-layer hydrostatic gravitationally stable model on the β-plane as illustrated in Figure 1.

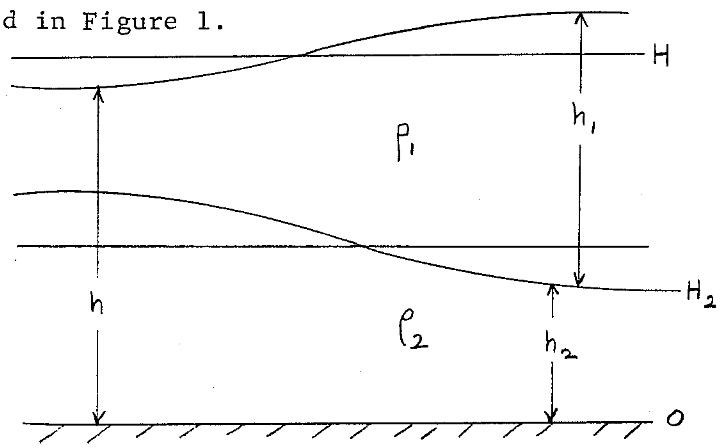

Fig. 1

Parameters of the upper layer are designated by subscript 1, and of the lower layer by subscript 2. Each layer is assumed to be homogeneous, incompressible and inviscid. The fluid is contained in an annular region $|y| \leq L$, $0 \leq z \leq z_2$ where z_2 is a free surface.

Basic equations. The fluid motion within the upper layer is governed by the following set of equations:

$$\nabla' \cdot \left(\frac{\partial}{\partial t'} + \mathbb{V}_1' \cdot \nabla' + w_1' \frac{\partial}{\partial z'}\right)\mathbb{V}_1' + g' \nabla'^2 h' - \nabla' \cdot (f' \mathbb{K} \times \mathbb{V}_1') = 0 \qquad (1)$$

$$\left(\frac{\partial}{\partial t'} + \mathbb{V}_1' \cdot \nabla' + w_1' \frac{\partial}{\partial z'}\right)(\zeta_1' + f') + (\zeta_1' + f')\nabla' \cdot \mathbb{V}_1' + \mathbb{K} \cdot \nabla' w_1' \times \frac{\partial \mathbb{V}_1'}{\partial z'} = 0 \qquad (2)$$

$$\nabla' \cdot \mathbb{V}_1' + \frac{\partial w_1'}{\partial z'} = 0 \qquad (3)$$

$$w_1' = \left(\frac{\partial}{\partial t'} + \mathbb{V}_1' \cdot \nabla'\right) h', \quad at \quad z' = h' = h_1' + h_2' \qquad (4)$$

$$w_1' = \left(\frac{\partial}{\partial t'} + \mathbb{V}_1' \cdot \nabla'\right) h_2', \quad at \quad z' = h_2' \qquad (5)$$

where h' is related to p_1' through the hydrostatic equation $\rho_1 g'(h' - z') = p_1'$.

Non-dimensional equations. In order to isolate the scale of motion of interest the basic equations will be non-dimensionalized and expanded in a power series of an appropriate small parameter. For the planetary scale this small parameter is the Rossby number R_o. In the definitions below, the absence of a prime indicates a non-dimensional variable or constant.

$$(x', y', z') = L(x, y, \delta z) \;,\quad \delta = H/L$$
$$(\mathbb{V}', u', v', w') = U(\mathbb{V}, u, v, \delta R_o w) \;,\; R_o = \frac{U}{f_o L}$$
$$\left(\frac{1}{\rho'}, t\right) = L/U\left(1/\zeta, t\right)$$
$$f' = f_o\left(1 + \beta R_o y\right)$$
$$h' = H\left(1 + R_o \frac{f_o^2 L^2}{gH} h\right)$$
$$h_2' = H_2\left(1 + R_o \frac{f_o^2 L^2}{g^* H_2} h_2\right)$$

where $H = H_1 + H_2$ is the scale height and $g^* = \frac{\rho_2 - \rho_1}{\rho_2} g \ll g$ is reduced gravity. The non-dimensional equations are

$$\nabla^2 h - \nabla\cdot(\mathbb{K}\times\mathbb{V}_1) + R_o\nabla\cdot\left(\frac{\partial}{\partial t} + \mathbb{V}_1\cdot\nabla - \beta y \mathbb{K}\times\right)\mathbb{V}_1 + R_o^2\nabla\cdot w_1\frac{\partial \mathbb{V}_1}{\partial z} = 0 \tag{6}$$

$$\nabla\cdot\mathbb{V}_1 + R_o\left[\left(\frac{\partial}{\partial t} + \mathbb{V}_1\cdot\nabla\right)\zeta_1 + \beta_1 v_1 + (\zeta_1 + \beta y)\nabla\cdot\mathbb{V}_1\right] + R_o^2\left[w_1\frac{\partial \zeta_1}{\partial z} + \mathbb{K}\cdot\nabla w_1 \times \frac{\partial \mathbb{V}_1}{\partial z}\right] = 0 \tag{7}$$

$$\nabla\cdot\mathbb{V}_1 + R_o\frac{\partial w_1}{\partial z} = 0 \tag{8}$$

$$R_o w_1 = R_o\frac{f_o^2 L^2}{gH}\left(\frac{\partial}{\partial t} + \mathbb{V}_1\cdot\nabla\right)h \;,\; \text{at } z = \left(1 + R_o\frac{f_o^2 L^2}{gH} h\right) \tag{9}$$

$$R_o w_1 = R_o\frac{f_o^2 L^2}{g^* H}\left(\frac{\partial}{\partial t} + \mathbb{V}_1\cdot\nabla\right)h_2 \;,\; \text{at } z = \left(1 + R_o\frac{f_o^2 L^2}{g^* H_2} h_2\right)\frac{H_2}{H} \tag{10}$$

The coupled equations. Expanding in R_o yields the zero order equations:

$$\nabla^2 h^{(0)} - \nabla \cdot (\mathbb{K} \times \mathbb{V}_1^{(0)}) = 0 \tag{11}$$

$$\nabla \cdot \mathbb{V}_1^{(0)} = 0 \tag{12}$$

and the first order equations

$$\nabla^2 h^{(1)} - \nabla \cdot (\mathbb{K} \times \mathbb{V}_1^{(1)}) + \nabla \cdot \left(\frac{\partial}{\partial t} + \mathbb{V}_1^{(0)} \cdot \nabla - \beta y\, \mathbb{K} \times\right) \mathbb{V}_1^{(0)} = 0 \tag{13}$$

$$\nabla \cdot \mathbb{V}_1^{(1)} + \left(\frac{\partial}{\partial t} + \mathbb{V}_1^{(0)} \cdot \nabla\right) \zeta_1^{(0)} + \beta v^{(0)} = 0 \tag{14}$$

$$\nabla \cdot \mathbb{V}_1^{(1)} + \frac{\partial w_1^{(0)}}{\partial z} = 0 \tag{15}$$

$$w_1^{(0)} = \frac{f_0^2 L^2}{gH}\left(\frac{\partial}{\partial t} + \mathbb{V}_1^{(0)} \cdot \nabla\right) h^{(0)}, \text{ at } z = \left(1 + R_0 \frac{f_0^2 L^2}{gH} h^{(0)}\right) \tag{16}$$

$$w_1^{(0)} = \frac{f_0^2 L^2}{g^* H}\left(\frac{\partial}{\partial t} + \mathbb{V}_1^{(0)} \cdot \nabla\right) h_2^{(0)}, \text{ at } z = \left(1 + R_0 \frac{f_0^2 L^2}{g^* H_2} h_2^{(0)}\right) \frac{H_2}{H} \tag{17}$$

$\mathbb{V}_1^{(1)}$ independent of height implies (15) may be integrated over the layer. Performing this integration and substituting from (14), (16) and (17) while assuming

$$\frac{f_0^2 L^2}{gH} << 1 \text{ , } \frac{f_0^2 L^2}{g^* H_1} \sim 1$$

yields the following expression in zero order quantities

$$\left(\frac{\partial}{\partial t} + \mathbb{V}_1^{(0)} \cdot \nabla\right)\left(\frac{f_0^2 L^2}{g^* H_1} h_2^{(0)} + \zeta_1^{(0)} + \beta y\right) = 0 \tag{18}$$

Equations (11) and (12) imply the existence of a stream function having the form

$$\psi_1^{(0)} = h^{(0)} \tag{19}$$

Assuming $\Delta \rho / \rho_2 << 1$ the hydrostatic equation for layer 2,

$$p_2 = \rho_2 g \left(h_2 + \frac{\rho_1}{\rho_2} h_1 - z\right) \tag{20}$$

leads similarly to

$$\psi_2^{(0)} = \frac{\rho_1}{\rho_2} h_1^{(0)} + h_2^{(0)} , \tag{21}$$

the stream function for the lower layer. If the stream functions (19) and (21) are of the form

$$\psi = \overline{\psi}(y) + \psi'(x, y, t) , \tag{22}$$

where the bar designates the steady current and the prime the perturbation superimposed thereon, equation (18) may be written (after linearization)

$$\left(\frac{\partial}{\partial t} - \frac{\partial \overline{\psi}_1}{\partial y}\frac{\partial}{\partial x}\right)\left[\nabla^2 \psi_1' + F_1(\psi_2' - \psi_1')\right] + \frac{\partial \psi_1'}{\partial x}\frac{\partial}{\partial y}\left[\beta y + \frac{\partial^2 \overline{\psi}_1}{\partial y^2} + F_1(\overline{\psi}_2 - \overline{\psi}_1)\right] = 0 \tag{23}$$

for layer 1 where $F_1 = \frac{f_0^2 L^2}{g^* H_1}$, and

$$\left(\frac{\partial}{\partial t} - \frac{\partial \overline{\psi}_2}{\partial y}\frac{\partial}{\partial x}\right)\left[\nabla^2 \psi_2' + F_2(\psi_1' - \psi_2')\right] + \frac{\partial \psi_2'}{\partial x}\frac{\partial}{\partial y}\left[\beta y + \frac{\partial^2 \overline{\psi}_2}{\partial y^2} + F_2(\overline{\psi}_1 - \overline{\psi}_2)\right] = 0 , \tag{24}$$

for layer 2 where $F_2 = \frac{f_0^2 L^2}{g^* H_2}$. If the perturbation stream function has the form $Re\left[\varphi(y) e^{ik(x-ct)}\right]$ where k is the wave number and c is the phase speed, (23) and (24) become

$$(\overline{u}_1 - c)\left(\frac{d^2}{dy^2} - k^2\right)\varphi_1 + \varphi_1 \frac{d\overline{\zeta}_{a_1}}{dy} = F_1\left[\varphi_1 \overline{u}_2 - \varphi_2 \overline{u}_1 + c(\varphi_2 - \varphi_1)\right] \tag{25}$$

$$(\overline{u}_2 - c)\left(\frac{d^2}{dy^2} - k^2\right)\varphi_2 + \varphi_2 \frac{d\overline{\zeta}_{a_2}}{dy} = F_2\left[\varphi_2 \overline{u}_1 - \varphi_1 \overline{u}_2 + c(\varphi_1 - \varphi_2)\right] \tag{26}$$

where $\overline{\zeta}_{a_1}$, $\overline{\zeta}_{a_2}$ is the absolute vorticity of the steady current in layers 1, 2 respectively. Equations (25) and (26) form an energetically coupled set which governs the motion throughout the fluid. The terms on the right-hand side of the two equations are referred to as the coupling terms. When these terms are small, the motions in each layer become independent and the layers are said to be decoupled.

This decoupling can occur when the characteristic length scale, L, is small or when the static stability is large, i.e. whenever F is small. We shall make use of this fact in the discussion which follows.

Barotropic problem. For $F_1 = F_2 = F$, small enough for decoupling, consider a steady non-uniform current $\bar{u}_1$ in the upper layer. Specifically, let the horizontal shear (relative vorticity) be concentrated at the vertical interface at $y = 0$ as illustrated in Figure 2.

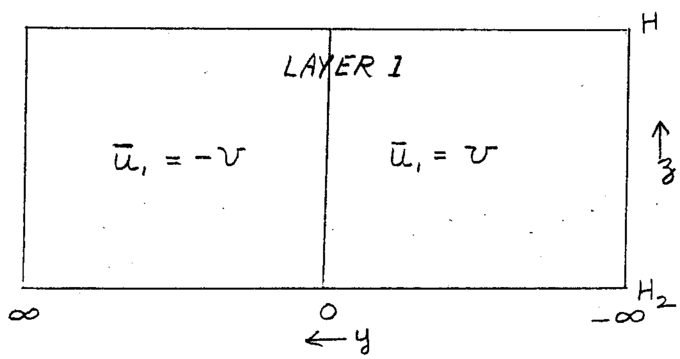

Fig. 2

Here, $\bar{u}_1$ is given by the delta-function

$$\bar{u}_1 = -\mathcal{V} \operatorname{sgn} y, \tag{27}$$

where $\mathcal{V} > 0$, and which has a zero order discontinuity at $y = 0$. This is the Helmholtz instability problem which has been discussed, for example, by Howard (G.F.D., 1960) in the case of a non-rotating system. Under the β-plane approximation, the boundary conditions derived by Howard are also applicable to this problem, viz.,

$$(\bar{u} - c)\frac{d\varphi}{dy} - \frac{d\bar{u}}{dy}\varphi \text{ and } \frac{\varphi}{\bar{u} - c} \quad \text{continuous at the interface} \tag{28}$$

and φ finite as $y \rightarrow \pm\infty$

for φ bounded on $0^- < y < 0^+$ and $c_i \neq 0$. Equation (24) reduces to

$$\left[D^2 - k^2\left(1 + \frac{U_R}{U \operatorname{sgn} y + c}\right)\right]\varphi_1 = 0 \tag{29}$$

where $D^2 = \dfrac{d^2}{dy^2}$ and $U_R = \beta/k^2$ is the Rossby phase speed. Equation (29) has a solution

$$\varphi = \Phi(\operatorname{sgn} y)\exp\left\{-\operatorname{sgn} y\left[k\left(1 + \frac{U_R}{U \operatorname{sgn} y + c}\right)^{\frac{1}{2}} y\right]\right\} \tag{30}$$

which must satisfy the jump conditions (28). For $c_i \neq 0$ and $c \neq -U \operatorname{sgn} y$ (28) requires that

$$(U-c)^2\left(1 - \frac{U_R}{U-c}\right)^{\frac{1}{2}} = (U+c)^2\left(1 + \frac{U_R}{U+c}\right)^{\frac{1}{2}} \tag{31}$$

which reduces to the cubic

$$c^3 + \tfrac{3}{4}U_R c^2 + U^2 c + \tfrac{1}{4}U_R U^2 = 0 \tag{32}$$

with roots $\quad p + q\,,\ \omega p + \omega^2 q\,,\ \omega^2 p + \omega q$

where $\quad \omega = \dfrac{-1 + i\sqrt{3}}{2}$

and

$$p = \frac{1}{4}\left\{-\frac{U^3}{6} + \left[\left(\frac{4U}{\sqrt{3}}\right)^6 - \left(\frac{16U^2U_R}{\sqrt{3}}\right)^2 + \left(4UU_R^2\right)^2 - \frac{2}{3}U_R^6\right]^{\frac{1}{2}}\right\}^{1/3}$$

$$q = \frac{1}{4}\left\{-\frac{U^3}{6} - \left[\left(\frac{4U}{\sqrt{3}}\right)^6 - \left(\frac{16U^2U_R^2}{\sqrt{3}}\right)^2 + \left(4UU_R^2\right)^2 - \frac{2}{3}U_R^6\right]^{\frac{1}{2}}\right\}^{1/3}$$

When $U_R = 0$ the solution reduces to $c = \pm Ui$, the solution to the Helmholtz instability problem in the non-rotating case. Equivalent results apply to layer 2.

<u>Baroclinic problem</u>. For $F_1 = F_2 = F$, sufficiently large that the layers remain coupled, consider a steady uniform current $\bar{u}_1 = U > 0$ in layer 1 and $\bar{u}_2 = 0$ in layer 2, as in Figure 3.

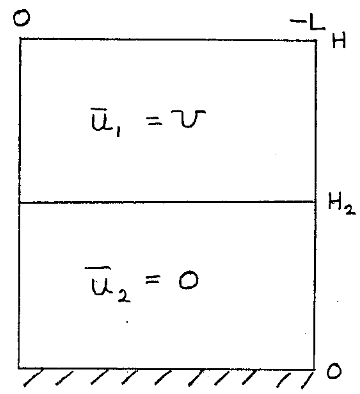

Fig.3

In the baroclinic problem the vertical shear is concentrated in the interface with mean height H_2. Equations (25) and (26) become

$$(U-c)(D^2-k^2)\varphi_1+\beta\varphi_1 = F\left[-\varphi_2 U + c(\varphi_2-\varphi_1)\right] \tag{33}$$

$$-c(D^2-k^2)\varphi_2+\beta\varphi_2 = F\left[\varphi_2 U + c(\varphi_1-\varphi_2)\right] \tag{34}$$

with boundary conditions $\varphi_1 = \varphi_2 = 0$ at $y = 0, -L$.

The problem has solutions[1]

$$\varphi_1 = A\sin ly, \quad \varphi_2 = B\sin ly, \quad l = \frac{2\pi n}{L}$$

which leads to a pair of homogeneous algebraic equations, the determinant of the coefficients of which must be zero. That is

$$\begin{vmatrix} (U-c)(l^2+k^2)-\beta-Fc & -F(U-c) \\ -Fc & c(l^2+k^2)+\beta-F(U-c) \end{vmatrix} = 0,$$

and the phase speed follows

$$c = U\left\{\left[\frac{1}{2} - \frac{y(x+1)}{2x+1}\right] \pm \frac{1}{2}\left[\frac{4x^2(y^2-1)+1}{(2x+1)^2}\right]^{1/2}\right\}, \tag{35}$$

1 l can assume any value as $L \to \infty$, i.e., get finite oscillation.

where $x = \frac{F}{\ell^2+k^2}$ and $y = \frac{\beta/U}{\ell^2+k^2}$. The discriminant of (35) yields the condition for instability

$$F^2 > \left(\frac{\ell^2+k^2}{2}\right)^2\left(1-\frac{U_R^2}{U^2}\right)^{-1} \tag{36}$$

where $U_R = \frac{\beta}{\ell^2+k^2}$ is the Rossby phase speed for this problem. It is interesting to consider the case of marginal stability for which

$$U^2 = \frac{(2F\beta)^2(\ell^2+k^2)^{-2}}{(2F)^2-(\ell^2+k^2)^2} \tag{37}$$

The curve for this equation is illustrated in Figure 4, in which

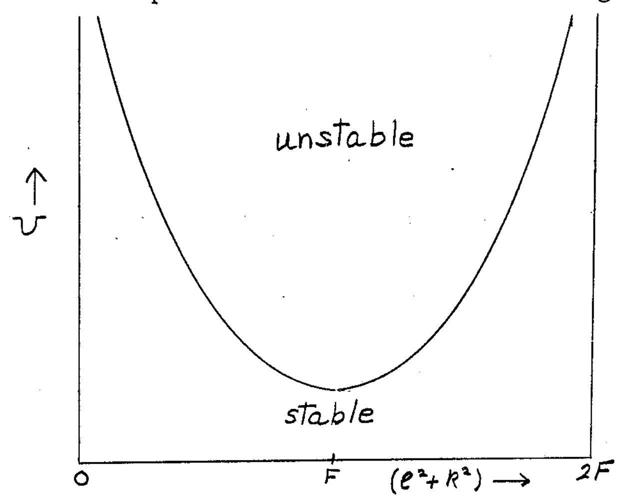

Fig.4

the abscissa is wave number and the ordinate is vertical shear. The point of minimum neutrality is given by the coordinates $(\sqrt{2}F, \beta/F)$, and the long- and short-wave cutoffs are represented by the asymptotes at $(\ell^2+k^2) = 0, 2F$. The figure also indicates that β has a stabilizing influence.

<u>Mixed barotropic-baroclinic problem</u>. For $F_1 = F_2 = F$, sufficiently large, let $\bar{u}_1 = U$ for $y < 0$; and $\bar{u}_1 = V \neq U$ for $y > 0$; also let $\bar{u}_2 = 0$ as indicated in Figure 5.

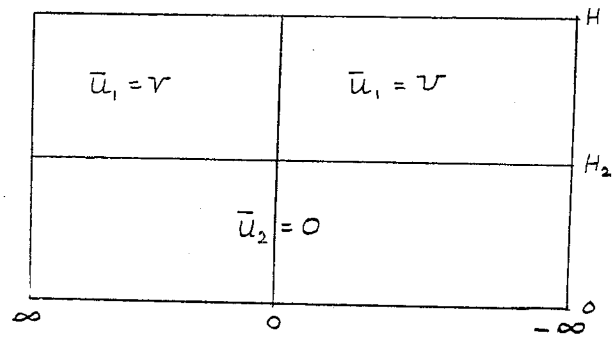

Fig. 5

Equations (33) and (34) apply directly for $y < 0$, viz.

$$[(U-c)(D^2-k^2)+\beta+Fc]\varphi^- + F[U-c]\varphi_2^- = 0 \tag{38}$$

$$[-c\,(D^2-k^2)+\beta-F(U-c)]\varphi_2^- + F(-c)\varphi_1^- = 0 \tag{39}$$

and for $y > 0$

$$[(V-c)(D^2-k^2)+\beta+Fc]\varphi_1^+ + F[V-c]\varphi_2^+ = 0 \tag{40}$$

$$[-c(D^2-k^2)+\beta-F(V-c)]\varphi_2^+ + F[-c]\varphi_1^+ = 0 \tag{41}$$

The boundary conditions for this problem are, for layer 1

$$\begin{gathered} (U-c)D\varphi_1^- = (V-c)D\varphi_1^+ \\ \frac{\varphi_1^-}{U-c} = \frac{\varphi_1^+}{V-c} \\ \varphi_1 \text{ finite as } y \to \pm\infty \end{gathered} \tag{42}$$

and for layer 2

$$\begin{gathered} D\varphi_2^- = D\varphi_2^+ \\ \varphi_2^- = \varphi_2^+ \\ \varphi_2 \text{ finite as } y \to \pm\infty \end{gathered} \tag{43}$$

where the superscripts refer to the sign of y , i.e. (+) refers to the left- and (-) to the right-hand side of the vertical interface at $y=0$ (c.f. Figure 5). The equations (38) through (41) have solutions

$$\varphi_1^- = \sum_1^4 a_i e^{r_i y}$$

$$\varphi_2^- = -\sum_1^4 a_i \left[\frac{(\nu-c)(r_i^2-k^2)+\beta+Fc}{F(\nu-c)}\right] e^{r_i y} \tag{44}$$

and

$$\varphi_1^+ = \sum_1^4 b_i e^{s_i y}$$

$$\varphi_2^+ = -\sum_1^4 b_i \left[\frac{(\nu-c)(s_i^2-k^2)+\beta+Fc}{F(\nu-c)}\right] e^{s_i y} \tag{45}$$

where each of the four roots are given by

$$r = \pm\left\{k^2 + \frac{-(\beta-F\nu)(2c-\nu)-2Fc^2 \pm \sqrt{[(\beta-F\nu)(2c-\nu)+2Fc^2]^2-4\beta[c(\nu-c)][F(\nu-2c)-\beta]}}{2c(\nu-c)}\right\}^{\frac{1}{2}}$$

$$s = \pm\left\{k^2 + \frac{-(\beta-F\nu)(2c-\nu)-2Fc^2 \pm \sqrt{[(\beta-F\nu)(2c-\nu)+2Fc^2]^2-4\beta[c(\nu-c)][F(\nu-2c)-\beta]}}{2c(\nu-c)}\right\}^{\frac{1}{2}}$$

The general functions (44) and (45) involve eight arbitrary constants which can be evaluated by means of the eight boundary conditions (42) and (43). Having closed our system let us now simplify it by assuming $\beta = 0$. Under this condition

$$r = \pm k\,,\ \pm k\eta$$

where $\eta = \left\{1 - \frac{F}{k^2}\left[\frac{(\nu-c)^2+c^2}{c(\nu-c)}\right]\right\}^{\frac{1}{2}}$ and

$$s = \pm k\,,\ \pm k\chi$$

where $\chi = \left\{1 - \frac{F}{k^2}\left[\frac{(\nu-c)^2+c^2}{c(\nu-c)}\right]\right\}^{\frac{1}{2}}$

Since the roots $\pm k$ are real the condition that the φ's remain finite requires $r_1 = k$, $s_1 = -k$. On the other hand, η and χ may be real in which case $r_2 = k\eta$ and $s_2 = -k\chi$; or η and χ may be complex and $r_{2,3} = \pm k\eta$, $s_{2,3} = \pm k\chi$ (complex conjugates). In either case the growing modes $r_4 = -k$, $s_4 = k$, are excluded by taking $a_4 = b_4 = 0$. If all roots are real then $a_3 = a_4 = b_3 = b_4 = 0$ and four arbitrary constants must be evaluated using the jump conditions at $y = 0$. If η and χ are complex we will choose $a_3 = b_3 = 0$ arbitrarily[2]. The problem then becomes that of evaluating the four arbitrary constants a_1, a_2, b_1 and b_2. With the problem thus defined (44) and (45) may be rewritten

$$\varphi_1^- = a_1 e^{ky} + a_2 e^{k\eta y}$$
$$\varphi_2^- = \frac{a_1 c}{\nu - c} e^{ky} + \frac{a_2(D-c)}{c} e^{k\eta y} \tag{46}$$

and

$$\varphi_1^+ = b_1 e^{-ky} + b_2 e^{-k\chi y}$$
$$\varphi_2^+ = -\frac{b_1 c}{\nu - c} e^{-ky} + \frac{b_2(\nu - c)}{c} e^{-k\chi y} \tag{47}$$

The jump conditions (42) and (43) yield the homogeneous system

$$\begin{aligned}
a_1\nu + a_2\eta\nu + b_1 + b_2\chi &= 0 \\
a_1 + a_2 - b_1\nu - b_2\nu &= 0 \\
a_1 c + a_2\sigma\eta + b_1 c\nu + b_2\tau\chi\nu &= 0 \\
a_1 c + a_2\sigma - b_1 c\nu - b_2\tau\nu &= 0
\end{aligned} \tag{48}$$

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[2]In such situations one can usually apply some radiation condition in the problem in order to decide which is the appropriate root, but in this case such a condition is not obvious.

where $\nu = \dfrac{U-c}{V-c} = \left(\dfrac{\sigma}{\tau}\right)^{\frac{1}{2}}$, $\sigma = -\dfrac{(U-c)^2}{c}$, $\tau = -\dfrac{(V-c)^2}{c}$

and $\eta = \left[1+\dfrac{F}{k^2}\left(\dfrac{\sigma-c}{U-c}\right)\right]^{\frac{1}{2}}$, $\chi = \left[1+\dfrac{F}{k^2}\left(\dfrac{\tau-c}{V-c}\right)\right]^{\frac{1}{2}}$.

From (48) we require

$$\begin{vmatrix} \nu & \eta\nu & 1 & \chi \\ 1 & 1 & -\nu & -\nu \\ c & \sigma\eta & c\nu & \tau\chi\nu \\ c & \sigma & -c\nu & -\tau\nu \end{vmatrix} = 0 \tag{49}$$

or

$$c(\sigma-\tau)^2 = \sigma(\tau^2+\sigma\tau-3c\tau+2c^2-c\sigma)\eta + \tau(\sigma^2+\sigma\tau-3c\sigma+2c^2-c\tau)\chi\,. \tag{50}$$

Equation (50) is a complicated twelfth-order expression in the phase speed c . Such an equation is normally soluble only by numerical techniques. However, there are several interesting results from (50) which do not require direct extraction of roots. First, by letting $V=U$, and $k\eta, k\chi = il$, two of the roots[3] of (50) are found to be those given by (35) when $y=0$. Similarly, if $V=-U$ and $F=0$ two of the roots[3] are $c=\pm Ui$. That is, (50) contains both the solutions of the baroclinic problem and the solutions of the barotropic problem. These solutions may be thought of as representing independent extremes of the mixed problem, with all intermediate solutions as yet undetermined. This idea suggests another approach. Namely, what is the effect of a small baroclinicity on the barotropic mode? This question is

[3]Other roots which appear are extraneous since they lead to trivial eigenfunctions (zero amplitude waves) when substituted into (48).

easily answered by differentiating (50) with respect to F and evaluating the result at $c_o = \pm \nu i$ as $F \to 0$. Performing the first two steps yields

$$16\frac{\partial c}{\partial F} = \frac{F}{k^2\eta}(4 \mp 3i)\frac{\partial c}{\partial F} - \frac{5\nu}{k^2\eta}(1 \mp i) + \frac{F}{k^2\chi}(4 \pm i)\frac{\partial c}{\partial F} + \frac{5\nu}{k^2\chi}(1 \pm i) +$$

$$+ 4\eta(7 \pm 4i)\frac{\partial c}{\partial F} + 4\chi(7 \mp 4i)\frac{\partial c}{\partial F}. \quad (51)$$

As $F \to 0$, η and $\chi \to 1$ and (51) reduces to

$$\left.\frac{\partial c}{\partial F}\right|_{\substack{c_o \pm \nu i \\ F=0}} = \mp \frac{\nu i}{(2k)^2} \quad (52)$$

Equation (52) states that the fractional change of phase speed with respect to F is inversely proportional to the square of the wave number. This implies that the effect of small baroclinicity on the barotropic mode is greater for smaller wave number, i.e. long waves. Furthermore, the baroclinicity has a stabilizing (destabilizing) effect on the growing (decaying) barotropic mode.

Further suggestions. The other approximation, namely that of determining the effect of a small horizontal shear on the baroclinic mode, can also be studied. Following this, equation (50) can be solved numerically with the results above, no doubt, providing some assistance. The simplifications $\beta \neq 0$ and $F_1 = F_2 = F$ should also be removed if possible. Finally it would be interesting to construct the eigenfunctions for the problem and then determine the circulation in the two layers as well as the energy conversions for the mixed problem. This can be done by evaluating the arbitrary constants (except for one) of (48) (or its equivalent for the complete problem) and substituting the eigenvalues from the equivalent of (46) and (47) into the energy equations which follows from (23) and (24).

Acknowledgment

I would like to thank Dr. J. Pedlosky for many helpful discussions.

ERRATA

p.79: Appendix, line 7, the buoyancy force should read:

$$\rho_0\left[\mu^2\left(1+2r'+\frac{p'}{\gamma}\right)(\xi-z_0)+g_0\Delta_T\right]\left(1+\frac{\gamma+1}{\gamma}p'+2r'\right)$$

p.71: The geometrical term has been omitted from the formula for the heat transport, which should read:

$$dQ_r=C\int_A m(z_0,t_0)\tfrac{1}{2}n(z_0,t_0)\frac{r^2(t_0)}{r^2(t)}\,\Theta(z,t;z_0,t_0)\,p(z,t;z_0,t_0)\,dA$$

and should be included in the integrals on page 72.

p.82: The convective luminosity then becomes:

$$L_c=4\pi r_0^2 C_p\beta_0\int_{-\infty}^{t}\left(1+2r'e^{i\omega t_0}\right)\rho(z_0,t_0)\,\ell^3(z_0,t_0)\,n(z_0,t_0)\,w\,\Theta\,p\,dt_0$$

and equation (A7) becomes:

$$\ell_c'=n'(1-i\alpha)\tilde{B}+\left(\frac{\gamma+1}{\gamma}p'+r'\right)\frac{1}{4+\alpha^2}\left\{2-i\alpha+i\alpha\left[1+i\alpha\left(1+\frac{2}{\alpha^2}\right)\right](1-i\alpha)\tilde{B}\right\}-$$
$$-i\alpha\left(\frac{p'}{\gamma}+2r'\right)\tilde{B}+p'+2r'+\frac{2\tilde{\ell}'(1-i\alpha)}{1+\alpha^2}+\left(\frac{p'}{\gamma}+3\tilde{\ell}'\right)(1-i\alpha)\tilde{B}+2r'(1-i\alpha)\tilde{B}$$

which includes a further correction of an algebraic error.

p.83: The equation following equation (A11) is now:

$$\ell_c'=\frac{\gamma+1}{\gamma}\frac{p'}{4+\alpha^2}\left\{2-i\alpha+\tfrac{1}{2}\alpha(2i-\alpha)(1-i\alpha)\tilde{B}+\frac{4+\alpha^2}{\gamma+1}\left[(1-2i\alpha)\tilde{B}+\gamma\right]\right\}+$$
$$+\frac{4r'}{4+\alpha^2}\left\{2-i\alpha+\left(4+\tfrac{1}{2}\alpha^2+i\alpha\right)(1-i\alpha)\tilde{B}\right\}+\frac{2(1-i\alpha)\tilde{\ell}'}{1+\alpha^2}$$

which means that equations (5.13) and (5.14) become

p.74: $$L_{cr}=\frac{4}{4+\alpha^2}\left\{2-i\alpha+\left(4+\tfrac{1}{2}\alpha^2+i\alpha\right)(1-i\alpha)\tilde{B}\right\}-\frac{4(1-i\alpha)}{1+\alpha^2},$$

and $$L_{cp}=\frac{\gamma+1}{\gamma(4+\alpha^2)}\left\{2-i\alpha-\tfrac{1}{2}\alpha(2i-\alpha)(1-i\alpha)\tilde{B}\right\}+1+\frac{1}{\gamma}(1-2i\alpha)\tilde{B}-\frac{1-i\alpha}{\gamma(1+\alpha^2)}$$

respectively.

The new formulae reduce to the same linit as $\alpha\to 0$ but for $\alpha=1$

p.76:

$$S_1 \propto -\frac{4}{5} + \left(\frac{2\mathcal{K}_I}{5} + \mathcal{K}_p\right) - 1.3f\left(\frac{1+0.9\nu}{1+2\nu}\right) + \frac{2\nu}{K}\,\frac{1-0.1\nu+0.1\nu^2}{(1-0.2\nu)^2(1+2\nu)}$$

approximately.

The effect of convection could be to destabilize if K is small.

p.108: Second paragraph of (ii): Delete complete sentence: "In these experiments the ratio of the cell widths ... or a totally enclosed cell".

p.108: Third paragraph of (ii): first sentence: delete "but the results for water given in their figure 3 are equally well fitted by $N \simeq .11 R^{\frac{1}{3}}$ and $N \approx .30 R^{\frac{1}{4}}$". Add here the following sentence: "Since the range of Rayleigh number for which this motion is cellular is 1,700 to 45,000 it is not possible using this data to differentiate between a $R^{\frac{1}{4}}$ Nusselt number dependence and a $R^{\frac{1}{3}}$ Nusselt number dependence".