Notes on the 1964

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

Reference No. 64-46

Contents of the Volumes

Volume I   Course Lectures and Seminars

Volume II   Student Lectures

Refererence No. 64-46

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*Participated in less than one-half of the program.

Dr. John G. Pierce and Dr. Shoji Kato should have been included in the post-doctoral participants.

Editors' Preface

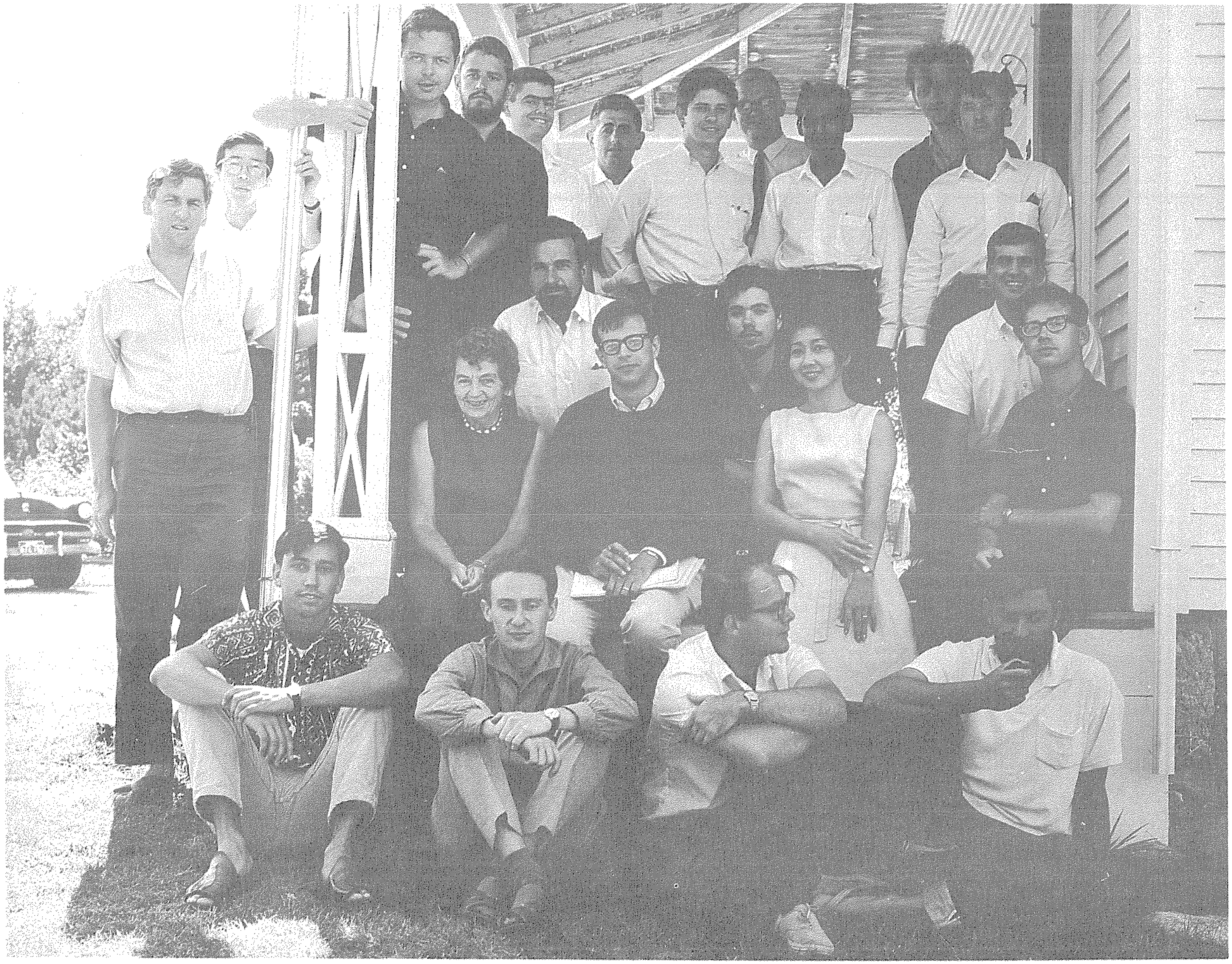
This volume contains the manuscripts of research lectures by pre-doctoral participants in the summer program. The staff guided the selection of the students' topics with several goals in mind. One goal was to isolate that part of a problem which might prove to be tractable in an effort of eight weeks or so. The more important goal was to find "open-ended" problems which would continue to challenge the student after his return to the university.

The degree of direction by a senior participant varied a great deal. In a vew cases, there were frequent conferences and discussions about fruitful avenues of approach. In other cases, there was essentially no contact except one of encouragement and interest. The efforts cover a wide spectrum in originality also. Some of the reports represent a more extended study of material presented in the course of lectures - others are original contributions which are being prepared for publication.

Because of time limitations it was not possible for the notes to be edited and reworked. The reports may contain errors the responsibility for which must rest on the shoulders of the participant-author. It must be emphasized that this volume in no way represents a collection of reports of completed and polished work.

All those who took part in the summer program are grateful to the National Science Foundation for its encouragement and financial support of the program.

Mary C. Thayer
Willem V.R. Malkus



Back Row: Welander, Kato, Pierce, Wälin, Halpern, Greenspan, Bisshopp, Toomre, Krishnamurti, Bretherton, Malkus.

Middle Row: Thayer, Veronis, Pedlosky, Devine, R.Krishnamurti, Orszag, Pond.

Bottom Row: Kenyon, Frisch, Davis, Hasselmann.

Unphotographable: Hide, Robinson, Stern, Stommel.

List of Seminars

1964

June 29	Dr. Willem V.R. Malkus	"Boussinesq Equations"
June 30	Dr. Willem V.R. Malkus	"Problem Discussion, and Boussinesq Energetics"
July 1	Dr. Willem V.R. Malkus	"Convection: Stability, the Problems of Realizability and Uniqueness"
July 2	Dr. Frederic Bisshopp	"Problem Discussions and Variational Procedures"
July 3	Dr. Willem V.R. Malkus	"Thermal Convection: Experiments and Finite Amplitude Effects. 'Role' of Non-linear Processes"
	Dr. Deitrich Lortz	"Instability of Finite Amplitude Solutions of the Convection Problem"
	Dr. George Veronis	"Finite Amplitude Stability in Water Stratified by Both Salt and Heat"
July 6	Dr. Robert Kraichnan	"Turbulence: Aims and Tools"
	Dr. Willem V.R. Malkus	"Turbulence: Modeling in Statistical Hydrodynamics"
July 7	Dr. Jackson Herring	"The Role of the Mean Field in Turbulence Theory"
July 8	Dr. Robert Kraichnan	"Turbulence: Foundations for a Deductive Theory"
July 9	Dr. Klaus Hasselmann	"Turbulence: Lagrangian Basis for Expansion"
July 10	Dr. Robert Kraichnan	"Turbulence: Recent Achievements"
July 13	Dr. Allan Robinson	"Rotating Fluids and the General Oceanic Circulation" (Flow in Rotating Systems)
July 14	Dr. Melvin Stern	"On the Instability of Quasigrostrophic Waves in a Baroclinic Current"
July 15	Dr. Allan Robinson	"Friction Modification of Geostrophy"

List of Seminars (continued)

July 16	Dr. Nicholas Fofonoff	"Oceanic Observations"
July 17	Dr. Allan Robinson	"Frictional Modification of Geostrophy: Problems"
July 20	Dr. Allan Robinson	"Wind-driven Flow"
July 21	Dr. Tiruvalam Krishnamurti	"On the Theory of Air Flow over Mountains"
July 22	Dr. Allan Robinson	"Double Boundary Layers"
	Dr. Henry Stommel	"Theories and Ocean Currents" (general lecture)
July 23	Dr. Pierre Welander	"Some Integral Constraints for the Ocean Circulation"
July 24	Dr. Allan Robinson	"Ocean Circulation"
July 27	Dr. Francis P. Bretherton	"Time Dependent Motions in the Ocean"
July 28	Dr. Francis P. Bretherton	"Resonant Interactions between Waves"
July 30	Dr. Shoji Kato	"Response of an Unbounded Isothermal Atmosphere to Point Disturbances"
July 31	Dr. Kirk Bryan	"The Ocean Circulation"
Aug. 4	Dr. Harvey Greenspan	"Contained Rotating Fluids"
Aug. 5	Dr. Harvey Greenspan	"Contained Rotating Fluids"
Aug. 6	Dr. Joseph Pedlosky	"Unsteady Ocean Circulations"
Aug. 7	Dr. Harvey Greenspan	"Contained Rotating Fluids"
Aug. 11	Dr. Albert Barcilon	"Diffusion of a Semi-infinite Line Vortex Normal to a Stationary Plane"
Aug. 12	Dr. Frederic Bisshopp	"Rapidly Rotating Convection"
Aug. 14	Dr. J. Stewart Turner	"Coupled Convection of Heat and Salt"

List of Seminars (continued)

Aug. 17 Dr. Raymond Hide "Detached Shear Layers in a Rotating Fluid"

Aug. 18 Dr. George Veronis "Generation of Mean Ocean Circulation by Fluctuating Winds"

Aug. 19 Dr. Klaus Hasselmann "Non-linear Processes in Random Wave Fields"

Aug. 20 Dr. Klaus Hasselmann "Non-linear Processes in Random Wave Fields"

Aug. 21 Dr. Robert Miller "Observations on Breaking Waves" Demonstration of Experiments on Nauset Beach by Dr. John Zeigler

Aug. 27 Dr. Tiruvalam Krishnamurti "Some Transformations for the Mountain Wave Problem in Compressible and Incompressible Atmospheres"

Student Seminars

Sept. 2 Patrick A. Davis "Inertial Oscillations in Magnetohydrodynamics"

Stephen Pond "Ocean Circulation Models for Regions of High Latitude"

Uriel Frisch "Microscopic Descriptions of Long-range Correlations in a Gas"

Benjamin Halpern "Upper Bound on Heat Transport by Turbulent Convection"

Sept. 3 Kern Kenyon "Non-linear Rossby-waves"

Ruby Krishnamurti "Finite Amplitude Instability in Quasi-steady Convection"

Steven Orszag "Analytic Approach to Some Problems of Turbulence"

Sten Gõsta Walin "Thermal Circulation in a Deep Rotating Annulus

Sept. 4 Michael F. Devine "A Two-layer Model of the Equatorial Undercurrent"

John G. Pierce "A Problem in Mountain Wave Theory"

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Modification of Inertial Oscillations by a Magnetic Field

Patrick A. Davis

A. Introduction

An attempt is made to understand the consequences of imposing a magnetic field upon an inertial oscillation.

In order to simplify the discussion and yet maintain a close connection with a physically meaningful situation, a particular model has been selected which it is hoped fairly closely resembles the situation in the earth's core, namely, a toroidal field superimposed on a contained rotating fluid. This problem is examined from three distinct points of view:

a) As an (β -plane) analogue. This might be a reasonable approach in the limit of small amplitude and more or less zonal motion.

b) When the amplitude can no longer be considered small, the full spherical problem must be posed. As one approach to this the equations are transformed to a new variable which under certain assumptions satisfies an equation identical in form to the inviscid problem already solved by Greenspan, with in addition a small perturbation term. In this approximation the ordinary inviscid modes when properly scaled should represent the possible inertial oscillations in the presence of a magnetic field.

c) In order to determine within what range of parameters the previous approach is valid, the full problem for this model is solved and the resultant eigenvalue equation is examined with a view to resolving this question.

B. The Equations of Motion

The form of the equations of motion and Maxwell's equation adopted in the rest of the work are as follows:

$$\left(\frac{d\vec{v}}{dt}\right) + \operatorname{grad} P + g\hat{k} + \vec{f}\times\vec{v} = \frac{1}{\rho}\vec{H}'\cdot\nabla\vec{H}' + \nu\nabla^2 v \qquad (1)$$

where $P = \left(\frac{p'}{\rho} + \frac{\mu|H'|^2}{8\pi\rho} - \frac{1}{2}|\vec{\Omega}\times\vec{r}|^2 + V\right)$

$$\nabla\cdot\vec{v} = 0$$

$$\frac{\partial\vec{H}}{\partial t} = \operatorname{curl}(\vec{v}\times\vec{H}') + \eta\nabla^2 H' \qquad (2a)$$

Where ν and η will be considered negligible, the displacement current has been ignored and Ohm's law assumed as well as the normal constitutive relations. The effect of rotation has been lumped in the term $(\hat{f}\times\vec{v})$ and P. In the next section, gravity will drop out; in the later sections it will be neglected since the material will be considered homogeneous and of constant density.

C. The β-plane analogue

A rectangular system of coordinates is adopted with x to the east, y to the north and z vertical to the earth's surface. The following assumptions are made about the variables and the mean fields.

$$\vec{f} = (f_0 + \beta y)\hat{k}$$

$$\vec{v} = \vec{V} e^{i(K_x x + K_y y + \lambda z - \omega t)}$$

where K_x and K_y are specified and V is small, and any mean zonal motion can be eliminated by adjusting f_0

$$\vec{H}' = \vec{H} + \vec{h}\,(x, y, z, t)$$

and $\vec{H} = (H_0 + \gamma y)\,\hat{i}$ is specified

Linearizing equation 2,

$$\frac{\partial \vec{h}}{\partial t} = \vec{H}\cdot\nabla\vec{v} - \vec{v}\cdot\nabla\vec{H} \tag{2b}$$

Then taking the local derivative with respect to time followed by the curl of (1) and substituting (2b) the equations of motion become

$$\frac{\partial^2}{\partial t^2}\,\mathrm{curl}\,\vec{v} + \dot{v}\beta\hat{k} - f_0\frac{\partial}{\partial z}\frac{\partial}{\partial t}\vec{v} = \mathrm{curl}\left(H^2\frac{\partial^2}{\partial x^2}\vec{v}\right) \tag{3}$$

$$\nabla\cdot\vec{v} = 0$$

where H_0 will be assumed much greater than any value of βy.

These equations represent a set of four equations in three unknowns and thus one equation is redundant. Choosing the last three equations and invoking the solubility condition to set the determinant equal to zero it is possible to specify one of the eigenvalues (in this case λ) in terms of the other.

$$\lambda = 0 \quad \text{or}\ \lambda = \pm\frac{a K_x\left[(a K_x^2 - \omega\beta K_x) - a^2 K_y^2 - iK_y b\right]}{a(aK_x - \omega\beta) + f_0\omega(b - f_0\omega K_x)} \tag{4}$$

$$a = H^2 K_x^2 - \omega^2$$

$$b = 2H\gamma K_x^2$$

For each value of λ, U and V can be obtained from

$$(a\lambda^2 + aK_x^2)U + (if_0\omega\lambda^2 + aK_xK_y)V = 0$$
$$(f_0\omega K_x - (b + iaK_y))U + \left[f_0\omega K_y + i(aK_x - \omega\beta)\right]V = 0 \tag{5}$$

and W can be determined from the continuity condition.

$$K_x U + K_y V + \lambda W = 0 \tag{6}$$

It should be noticed that for the two non-zero values of λ that U and V are the same and W differs only in sign.

The other eigenvalue ω should be determined now by the boundary conditions which are those corresponding to a free boundary.

$$W = 0 \qquad \frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = 0 \qquad \text{at } z = 0$$

and all variables finite at $z = -\infty$.

<u>Case $\lambda = 0$</u>

The boundary conditions require for $\lambda = 0$ that all derivatives vanish throughout the fluid and since W vanishes at z zero it must vanish throughout the fluid. Since in this case λ is not functionally dependent on ω the solubility condition on equations (5) determine ω . Thus

$$K_x^2 \left[f_0 \omega K_y + i (a K_x - \omega \beta) \right] - K_x K_y \left[f_0 \omega K_x - (b + i a K_y) \right] = 0 \tag{7}$$

The boundary conditions and the fourth equation, i.e.,

$$(f_0 \omega) \lambda U + i a \lambda V - (b + i a K_y) W = 0 \tag{8}$$

are trivially satisfied in this case.

Case $\lambda_\pm = \pm\lambda$

Any solution should be expressible in terms of u, V, and W determined for the three values of λ :

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A_1 \begin{pmatrix} u_0 \\ V_0 \\ 0 \end{pmatrix} + A_2 \begin{pmatrix} u_\lambda \\ V_\lambda \\ W_\lambda \end{pmatrix} + A_3 \begin{pmatrix} u_\lambda \\ V_\lambda \\ -W_\lambda \end{pmatrix}$$

The boundary conditions give three equations for determining A_1, A_2 and A_3. The solubility condition then should determine ω .

However the solubility condition is trivially satisfied and thus another condition must be used to determine ω . The only remaining condition is the fourth equation which must be satisfied for all acceptable ω . With this condition the problem is formally complete.

In order to obtain an insight into the effect of the magnetic field it is only useful to examine in detail the simplest cases. When λ is not zero the eigenvalue equations are very complicated giving little physical insight. Furthermore, the implications of the boundary conditions and the meaningfulness of the model need further investigation in the case that there is a z - dependence in the solution.

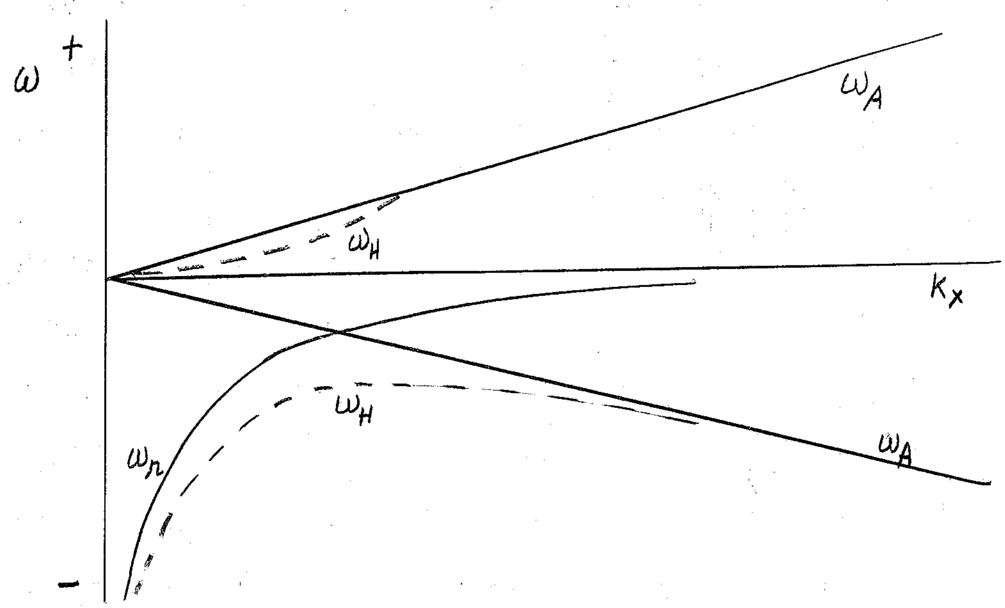
Case $\lambda = 0$ and $K_y = 0$

It should be noticed that $\omega_A = \pm K_x H$ corresponds to unmodified Alfen waves, and that $\omega_R = -\frac{\beta}{K_x}$ corresponds to unmodified Rossby waves.

In the case of no z-dependence and no component of the wave vector in the y-direction there are at low wave numbers both modified Rossby waves in the negative x-direction (West) and modified Alfen waves in the positive x-direction.

$$\omega_H = -\frac{\beta}{2K_x}\left(1 \pm \sqrt{1 + 4\frac{\omega_A^2}{\omega_R^2}}\right) \qquad \text{(a)}$$

These are represented in the figure by dotted curves to be compared with the unmodified waves drawn in solid curves.



It is interesting to note that although the Rossby mode at low wave numbers is altered relatively little in frequency and speed by the presence of a magnetic field, the modified Alfen branch is radically changed in speed dependence on K_x, from a constant value of $\sqrt{\frac{\mu}{4\pi\rho}}\,H$ to a parabolic form which is zero at zero k_x. This is quite a different behaviour about zero to either of the unmodified waves. Furthermore it should be noted, with reference to the earth's core, that these slow-moving modified Alfen waves move with respect to the fluid in a direction opposite to the Rossby wave.

Case $\lambda = 0$ and $K_y \neq 0$

For the case of motion in the y-direction the roots become a little more complicated:

$$\omega_H = +\frac{\omega_r}{2}\left[1 \pm (A^2+B^2)^{\frac{1}{4}}\cos\tfrac{1}{2}\tan^{-1}\left(\tfrac{B}{A}\right) \pm i(A^2+B^2)^{\frac{1}{4}}\sin\tfrac{1}{2}\tan^{-1}\left(\tfrac{B}{A}\right)\right] \qquad (10)$$

$$A = 1 + \frac{4H^2K_x^2}{\omega_r^2} \qquad K^2 = K_x^2 + K_y^2$$

$$B = \frac{8H\gamma K_x^2 K_y}{\omega_r^2 K^2} \qquad \omega_r = -\frac{\beta K_x}{K^2}$$

It is evident that at least two of these four roots are unstable, namely those corresponding to the negative sign of the imaginary part.

In the case that $K_y << K$ which is the only case in which this model is valid:

$$\omega_H = \frac{\omega_r}{2}\left(1 \pm \sqrt{1 + \frac{4\omega_A^2}{\omega_r^2}} \pm i\,\frac{8H\gamma K_y}{2\sqrt{1+\frac{4\omega_A^2}{\omega_r^2}}}\right) \qquad (10b)$$

Thus the real part of the frequency is identical to the case where there was no component of the wave vector in the y-direction, but the amplitude is modified exponentially in time.

It is not possible under the present assumptions to determine when and how the unstable modes might occur, both because the discussion has been limited to strictly local regions of the fluid, and because it appears likely from a discussion of a similar problem by M. Stern that the stability criteria will depend on the over-all profile of the mean magnetic field at least to the extent of determining the second derivative. However the wave modes discussed in this β-plane analogue could provide a detailed mechanism for the transfer of energy between the mean flow and the mean magnetic field.

D. A Perturbation Approach

When the amplitude of an inertial oscillation is not small, it is no longer possible to attempt a local solution such as the previous one. The full geometry of the sphere must be employed. As a first approach to this problem it is useful to find out how far the analysis can be taken by mere scaling of the non-magnetic case, already solved by Greenspan. To this end the field equations can be combined in terms of new variables:

$$\vec{B} = \vec{H} - \vec{v} \qquad P = -p'$$

satisfying the following equation (after Malkus)

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)\vec{B} - (\eta-\nu)\nabla^2\vec{v} = -\frac{1}{\rho}\nabla p - (\vec{B}\cdot\nabla)\vec{B} + 2(\vec{v}\cdot\nabla)\vec{B} + 2\vec{\Omega}\times\vec{v} \qquad (11a)$$

assuming:
$$|\vec{B}| >> |\vec{v}|$$
$$|\nabla^2\vec{B}| >> (\eta-\nu)|\nabla^2\vec{v}|$$
$$|(\vec{B}\cdot\nabla)\vec{B}| >> |(\vec{v}\cdot\nabla)\vec{B}|$$

This becomes

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2\right)\vec{B} = -\frac{1}{\rho}\nabla p - (\vec{B}\cdot\nabla)\vec{B} - 2\vec{\Omega}\times\vec{v}. \qquad (11b)$$

Assuming the mean field $\vec{B} = c\vec{V}$ is the same as assuming it is toroidal since the mean velocity is one associated with solid rotation. (c is approximately 700 for the earth's core, according to Malkus).

The perpendicular component of the velocity is related to the mean and fluctuating part of the field by

$$\frac{\partial \vec{h}}{\partial t} = \operatorname{curl}(\vec{v}_\perp \times \vec{H}) \qquad \text{(assuming } \eta \text{ negligible).}$$

We assume that $(\vec{B}\cdot\nabla)\vec{B}$ can be eliminated by making it a second-order quantity through changing Ω . Furthermore it is assumed, following Malkus, that this term accounts for approximately half of $2\frac{\vec{\Omega}}{c}\times\vec{B}_{\parallel}$.

Thus:

$$\frac{\partial}{\partial t}\vec{H} + \frac{1}{\rho}\nabla p - \frac{\vec{\Omega}}{c}\times\vec{H} = \vec{\Omega}\times\vec{v}_\perp \qquad (12)$$

This equation is similar to the inviscid non-magnetic solution with the addition of the small perturbation term $\vec{\Omega}\times\vec{v}_\perp$.

The magnitude of $\vec{v}_\perp$ relative to $\frac{\vec{H}}{c}$ can be estimated from equation (2a) by assuming $\vec{v}_\perp = \epsilon\frac{\vec{H}}{c}$ and that $\vec{H}$ has a wave form of angular velocity ω and wave number k

Then

$$\epsilon \sim \frac{c\left(\frac{\omega}{k}\right)}{H}$$

where $\left(\frac{\omega}{k}\right)$ is the propagation speed of the wave and H can be characterized by the speed of an Alfen wave. Since the ratio of these two terms is normally small and is here further decreased by c this term can safely be considered a perturbation and as such will not affect the speed or period of an inertial wave to the first order. Thus in the range for the parameters in which the assumptions made are correct, the variable $\vec{H} \propto \vec{v}$ satisfies an equation identical in form to the normal inviscid equation with only a difference in scaling of the frequency (i.e., $c\,\Omega^{-1}$ rather than Ω^{-1}).

In order to understand better the nature of these solutions the inviscid solutions due to Greenspan will be briefly reviewed.

<u>A review of the non-magnetic solutions for an inviscid contained rotating fluid.</u>

The equations are scaled to a unit sphere and the time by Ω^{-1} , cylindrical polar coordinates (r , ω , z) are adopted and inertial oscillations of the form:

$$\left.\begin{matrix} p \\ u \\ v \\ w \end{matrix}\right\} = \left.\begin{matrix} \phi \\ U \\ V \\ W \end{matrix}\right\} e^{\,i(k\omega+\lambda t)}$$

are tried in the equations of motion.

An equation

$$\nabla^2\phi - \frac{4}{\lambda^2}\frac{\partial^2}{\partial z^2}\phi \qquad (13a)$$

is obtained which when the z -coordinate is scaled by $\left(1-\frac{4}{\lambda^2}\right)$ is Laplace's equation in cylindrical polar coordinates. The boundary condition of no normal motion at the spherical boundary becomes:

$$r\frac{\partial\phi}{\partial r}+\frac{2K}{\lambda}\phi+\left(1-\frac{4}{\lambda^2}\right)z\ \frac{\partial\phi}{\partial z} \qquad (13b)$$

To fit the boundary condition on a constant coordinate surface, oblate spheriodal coordinates are introduced:

$$r=\left(C_{nK}^{2}-\eta^2\right)^{\frac{1}{2}}\left(1-\mu^2\right)^{\frac{1}{2}}$$

$$z=\frac{\eta\mu}{C_{nK}X} \text{ where } \quad C_{nK}=\frac{1}{(1-X^2)^{\frac{1}{2}}} \qquad X=\frac{\lambda_{nK}}{2} \qquad (14)$$

On the surface of the sphere

$$\eta=C_{nK}X \qquad \begin{array}{l} r=(1-\mu^2)^{\frac{1}{2}} \\ z=\mu=\cos\theta \end{array}$$

The solution for each mode K and corresponding eigenvalue λ_{nK} is:

$$\phi=P_n^K\left(\frac{\eta}{C_{nK}}\right)P_n^K(\mu) \qquad (15a)$$

with an eigenvalue equation:

$$K P_n^K(X)=(1-X^2)\frac{d}{dX}P_n^K(X) \qquad (15b)$$

The velocities can be recovered from the following relations:

$$\begin{aligned} u &= -\frac{i}{2(1-\lambda^2)}\left(\frac{K}{r}\phi+\frac{\lambda}{2}\frac{\partial\phi}{\partial r}\right) \\ V &= \frac{1}{2\left(1-\frac{\lambda^2}{4}\right)}\left(\frac{\partial\phi}{\partial r}+\frac{\lambda K}{r}\phi\right) \qquad (16) \\ W &= -\frac{1}{i\lambda}\frac{\partial\phi}{\partial z} \end{aligned}$$

The solutions are products of associated Legendre polynomials. These functions are characterized by a close correspondence between functions of the same class i, where the polynomial is written P^{k}_{k+i}. When the first few eigenvalues are inspected, a similarity is also apparent among eigenvalues corresponding to each class. Furthermore the values suggest that the smallest eigenvalue corresponding to symmetric functions P^{k}_{k+2m} are an order of magnitude smaller than any corresponding to odd functions.

These facts together with the supposition that the modes in the core that might cause a perturbation of the mean field at the surface of the earth, some 3000 km above, will be of low order in n (so as to produce a low enough order magnetic pole to be observed at this distance) enables one to pick out by inspection the most likely modes from the first few sets.

It is also possible to set, more or less, an upper limit on the period of such modes. The observed period of the non-dipole field is of the order of 60,000 Ω. The smallest of the first few roots is of the order of one-tenth. When this is scaled by c, the corresponding period is approximately 7,000 Ω or within an order of magnitude of the observed value.

E. <u>A Toriodal Magnetic Field in a Contained Rotating Fluid</u>

In order to check the range of validity of the perturbation approach, it is interesting to compare it to the exact solution of a contained rotating fluid in the presence of a toroidal field. $\vec{H} = K r \hat{\omega}$.

The following scaling is assumed (with stars on the unscaled variables):

$$\vec{r}_* = L\vec{r} \quad t_* = \Omega^{-1} t \quad \vec{q}_* = \epsilon L \Omega \vec{q} \quad \vec{H}_* = KL\vec{H}$$
$$\frac{\vec{P}_*}{\rho_*} = \frac{1}{2}\Omega L^2 (x^2+y^2) + \frac{B^2}{2\mu} + \epsilon L^2 r P \tag{17a}$$

where the characteristic length scale is the radius of the core and K is the uniform value of $\operatorname{curl} \vec{H}$. With this scaling the equations of motion now become:

$$\frac{\partial \vec{v}}{\partial t} + \epsilon \vec{v}\cdot\nabla\vec{v} + \nabla P + 2\hat{k}\times\vec{v} = \frac{C_A^2}{\epsilon \Omega^2 L^2} \vec{H}\cdot\nabla\vec{H} \tag{17b}$$

where $C_A = \sqrt{\frac{\mu}{4\pi\rho}}\, H$ is the Alfen speed.

$$\frac{\partial \vec{h}}{\partial t} + \epsilon \vec{v}\cdot\nabla\vec{H} = \epsilon \vec{H}\cdot\nabla\vec{v} \tag{17c}$$

If the substantial derivative $\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)$ of equation (17b) is taken, a substitution made from equation (17c), and all terms with ϵ as coefficients neglected (this involves neglecting only terms in $\vec{v}\cdot\nabla\vec{v}$ and $\vec{v}\cdot\nabla(\nabla p)$) then the following equation is arrived at:

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial \omega^2}\right)\vec{v} + \nabla\dot{p} + 2\hat{k}\times\dot{v} = 0 \tag{18}$$

where dots indicate local derivatives and

$$c = \frac{C_A}{\Omega R} \sim 10^{-3}$$ for the core of the earth.

By a method similar to the inviscid approach the following are derived:

Pressure Equation

$$\nabla^2\phi - \left(\frac{c^2}{a} + \frac{4b^2}{da}\right)\frac{\partial^2}{\partial z^2}\phi \qquad \begin{array}{l} a = (\lambda^2 - c^2K^2) \\ b = (\lambda - c^2K) \\ d = (\lambda^2 - c^2/K^2H) \end{array} \tag{19a}$$

Boundary Condition

$$r\frac{\partial\phi}{\partial r} + z\frac{\partial\phi}{\partial z} - \left(\frac{c^2}{a} + \frac{4b^2}{d}\right) z\frac{\partial\phi}{\partial z} + \frac{2bK}{d}\phi \tag{19b}$$

with the same transformation of coordinates, but with

$$x^2 = \frac{1}{\left(\frac{c^2}{d} + \frac{4b^2}{da}\right)} \qquad C_{nK} = \frac{1}{1-y^2} \tag{20a}$$

the solution becomes:

$$\phi = P_n^k\left(\frac{\eta}{C_{nx}}\right)P_n^k(\mu) \tag{20b}$$

with

$$K + f(y)\,P_n^K(y) - \frac{1-y^2}{y}\,\frac{\partial P_n^k(y)}{\partial y} = 0 \tag{20c}$$

where

$$f(y) = \frac{2b}{a} = \frac{2(\lambda - c^2K)}{(\lambda^2 - c^2(K^2+1))}$$

and λ is substituted for in terms of y .

It follows that for the perturbation approach to be valid the eigenvalue equation must be functionally of the same form as the non-magnetic case and λ must be related to y by a constant. This is true in the range $1 > 2\lambda > C$ and $K^2 >> 1$.

In the case that $C >> \frac{\lambda}{K} >> C^2$ the equation takes a different functional form with

$$y^2 \approx \frac{c^4}{\lambda^2}$$

$$f(y) \sim \frac{2y}{c^2}$$

F. Conclusion

Examination of the modification of inertial waves due to a magnetic field indicates, in the case of simple β-plane analogue, that where as one branch of the solution smoothly approaches non-magnetic Rossby wave velocities at low wave number, the Alfen wave at high wave number never approaches zero. The other branch, although it approaches the Alfen wave speeds at high wave number, has quite a distinct behaviour at low wave number, approaching a zero speed as the wave number approaches zero.

Although the complexity of the solution for large amplitude modes is such that no easy division can be made between the different types of waves, it is apparent that the similarity argument of section D is only valid for modes that approach the non-magnetic ones. The very slow modes behave in a systematically different manner and it is in this range that the observed period of the earth's non-dipole field is to be found, if it is indeed at all related to an inertial oscillation.

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A Study of the Equatorial Undercurrent

Michael Devine

I Introduction

The equatorial undercurrent is a high-velocity subsurface current, moving eastward under a wind-driven westward surface flow. This eastward flow, observed to some extent in all three oceans beginning at an average depth of about 50 m, is in the form of a thin ribbon-shaped current, approximately centered about the equator. It is about 200 m thick and 300 km wide in the Pacific. It's core speed is often in excess of 100 cm/sec, considerably higher than the speed of the surface flow. It's volume transport has been estimated to be as much as half that of the Gulf Stream.

The governing dynamics of the current structure at and very near the equator are extremely difficult to elucidate. There is evidence of upwelling and a horizontal temperature minimum at the equator, so that thermal effects are probably important. The theoretical breakdown of the Ekman layer assumption and the observed breakdown of the thermocline structure at the equator indicate that friction must be taken into account through most of the oceanic troposphere. Examination of the magnitude of the non-linear inertial term in the observed equatorial undercurrent forces one to the conclusion that inertial effects may be fully as important as viscous ones.

Thermal effects are not taken explicitly into account in the model presented below, since inclusion of a realistic heat equation would make the entire problem intractable. The simplest "thermal" assumption, that of a homogeneous ocean, also appears to be unsatisfactory, requiring a sea surface sloping downward from west to east in order to sustain the current. This condition is not fulfilled in the Indian Ocean, where a considerable undercurrent has been observed below a sea surface sloping upward from west to east. This unfavorable situation, however, exists only near the surface. The horizontal pressure gradient is observed to reverse itself with depth, becoming favorable to the undercurrent at a depth of about 100 m. Thus a two-layer model, allowing the entire upper layer to adjust instead of only the sea surface, should be considerably more realistic than a homogeneous model. The two-layer assumption has been made in this study.

The fitting of a boundary layer in which both viscous and inertial effects are important into an interior Sverdrup solution is a problem which has not been generally solved, and we will therefore restrict our attention to the immediate vicinity of the equator. We will impose zonal symmetry on the current, with the intention of studying the mean mid-ocean current structure, rather than zonal changes in that strucutre. It will be seen that strict zonal symmetry cannot exist due to zonal changes in layer depth, but this assumption will be retained as a necessary first approximation. It will be assumed that the sea surface is acted on by a constant east wind, this again being regarded as a first approach to reality.

II. The Layer Depth at the Equator

Our general model consists of a hydrostatic upper layer of density ρ, and a motionless lower layer of density ρ^*. Let the interface be at depth h at (x_1, y_1) and at depth $h+h^*$ at (x_2, y_2). At depth $h+h^*$ we have, at $(x_1\ y_1)$

$$p^* = g\rho h + g\rho^* h^* + \text{const.} \tag{1}$$

and at $(x_2\ y_2)$

$$p = g\rho(h+h^*) + \text{const.} \tag{2}$$

No motion in the lower layer requires that there be no horizontal pressure gradients there, so that

$$h^* + \text{const.} = -\frac{\rho}{\rho^*} h \tag{3}$$

The horizontal pressure gradients in the upper layer, which, by the hydrostatic assumption persist throughout the entire layer, are then given by

$$\frac{p_x}{\rho} = g\left(\frac{\rho^* - \rho}{\rho^*}\right) h_x = g^* h_x \tag{4}$$

$$\frac{p_y}{\rho} = g\left(\frac{\rho^* - \rho}{\rho^*}\right) h_y = g^* h_y \tag{5}$$

where $$g^* = g\left(\frac{\rho^* - \rho}{\rho^*}\right)$$

To get the behavior of the layer depth at the equator, we turn now to the Sverdrup transport equation, assumed to hold within at most a degree or two of the equator. These give generally

$$\tau(x) - \beta y V + \int_h^0 \frac{p_x}{\rho}\, dz - \frac{p(h)}{\rho} h_x = 0 \tag{6}$$

$$T(y) + \beta y\ U + \int_h^0 \frac{p_y}{\rho}\, dz - \frac{p(h)}{\rho}\, h_y = 0 \tag{7}$$

$$U_x + V_y = 0 \tag{8}$$

or, with (4) and (5) assuming $p(0) = 0$

$$T(x) - \beta y V - 2g^* h h_x = 0 \tag{9}$$

$$T(y) + \beta y U - 2g^* h h_y = 0 \tag{10}$$

In our β-plane coordinate system T is the impressed wind stress while U and V are components of the horizontal mass transport. We assume that the equatorial wind regime $T(x) = T$ constant $T(y) = 0$ holds. $\partial/\partial y$ (9) - $\partial/\partial x$ (10) then gives, with (8)

$$\beta y V = 0 \tag{11}$$

The meridional component of the deep geostrophic transport balances exactly that of the Ekman transport. Equation (9) now gives the zonal slope in the layer depth as

$$g^* h_x = \frac{T}{2h} \tag{12}$$

By an argument similar to that in boundary layer theory, although not susceptible of exact justification, we continue (12) through the equatorial boundary layer and assume that it holds also at the equator. However, the relation (12) still leaves our problem in an intractably complex form. Variation in layer depth governed by the general form of (12) will destroy the zonal symmetry we require. In addition, the boundary condition at the layer interface, $U(-h) = V(-h) = W(-h) = 0$ contains a layer depth h which must be determined as part of the solution. To eliminate these difficulties we select a reference depth h_0 , from which h is assumed to undergo only small variations, and set

$$g^* h_x = \frac{\tau}{2h_0} \tag{13}$$

This assumption loses the greatest benefits of the two-layer model, and constrains the analysis to follow the lines indicated by Dr. A.R.Robinson in his analysis of a homogeneous model.

The question remains of the accuracy of the approximation (13). A test of the validity of (13) and a first correction to the results obtained could be gotten by repeating the analysis below at another longitude x_1 .

III. The Equatorial Flow Field

With the previous discussion in mind, the horizontal β-plane equations may be written

$$VU_y + WU_z - \nu U_{zz} - \beta y V + \frac{\nu}{2h_0} = 0 \tag{1}$$

$$VV_y + WV_z - \nu V_{zz} + \beta y U + g^* h_y = 0 \tag{2}$$

$$V_y + W_z = 0 \tag{3}$$

Restricting our attention to the immediate vicinity of the equator, we expand each velocity component in a power series about $y = 0$. We regard the equator as a free boundary and do not distinguish between flows north and south of the equator. This requires that U and W be symmetric, and V antisymmetric about $y = 0$. The expanded flow components are then

$$U(y,z) = U(z) + O(y^2) \tag{4}$$

$$V(y,z) = y\left[V(z) + O(y^2)\right]$$

$$W(y,z) = W(z) + O(y^2)$$

The equations of motion then become, to lowest order in y and with (2) differentiated with respect to z to eliminate h_y

$$WU' - \nu U'' + \frac{T}{2h_0} = 0 \tag{5}$$

$$VV' + WV'' - \nu V''' + \beta U' = 0 \tag{6}$$

$$V + W' = 0 \tag{7}$$

where $\dfrac{dF}{dz} = F'$.

The boundary conditions, letting $z = 0$ be the layer interface are

$$\nu U'(h_0) = T \qquad \nu V'(h_0) = 0 \qquad W(h_0) = 0 \tag{8}$$

$$U(0) = 0 \qquad V(0) = 0 \qquad W(0) = 0$$

For further study, a non-dimensional form of the equations is required. This non-dimensionalization will be carried out with the goal of obtaining a tractable form containing both viscous and inertial terms. In particular, we will seek to exhibit inertial effects as a first-order correction to a zero-order viscous flow. With this in mind, introduce the non-dimensional quantities, which appear on the left-hand side of the equations, as

$$x = \frac{x}{L} \qquad y = \frac{y}{L} \qquad z = \frac{z}{h_0}$$

$$u = \frac{u}{V_0} \qquad v = \frac{v}{V_0}\left(\frac{A_{0v}}{(\beta L) h_0^2}\right) \qquad w = \frac{w}{V_0}\,\frac{L}{h_0}\left(\frac{A_{0v}}{(\beta L) h_0^2}\right)$$

$$A_v = \frac{\gamma}{A_{0v}} \qquad \tau = \frac{\tau}{\tau_0}$$

For convenience, we impose $\gamma = A_{0v}\,\text{cm}^2/\text{sec}$ $\tau = -2$ dyne/cm^2 $\tau_0 = -1$ dyne/cm^2, the latter two scalings corresponding to a constant easterly wind stress, and scale the velocity, with the surface boundary condition in mind, by $V_0 \; 2 \, \frac{h_0 \tau_0}{A_{0v}}$.

The complete set of non-dimensional equations is then

$$\frac{\varepsilon}{\gamma^2} w u' - u'' + 1 = 0 \tag{9}$$

$$\frac{\varepsilon}{\gamma^2}\left(v v' + w v''\right) - v''' + u' = 0 \tag{10}$$

$$v + w' = 0 \tag{11}$$

with boundary conditions

$$\begin{array}{lll} u'(1) = 1 & v'(1) = 0 & w(1) = 0 \\ u(0) = 0 & v(0) = 0 & w(0) = 0 \end{array} \tag{12}$$

where $\varepsilon = \dfrac{V_0}{(\beta L)L}$ $\qquad \gamma = \dfrac{A_{0V}}{(\beta L)h_0^2}$

The system (9) - (12) may be expanded in the perturbation parameter $R = \frac{\varepsilon}{\gamma^2}$. Define

$$U = U_0 + R\,U_1 + \dots \tag{13}$$

$$V = V_0 + R\,V_1 + \dots$$

$$W = W_0 + R\,W_1 + \dots$$

The zero order system is

$$-U_0'' + 1 = 0 \tag{14}$$

$$-V_0''' + U_0' = 0 \tag{15}$$

$$V_0 + W_0' = 0 \tag{16}$$

$$U_0'(1) = 1 \qquad V_0'(1) = W(1) = U(0) = V(0) = W(0) = 0 \tag{17}$$

with solution

$$U_0 = z^2/2 \tag{18}$$

$$V_0 = z^4/24 - \frac{9z^2}{80} + \frac{7z}{120} \tag{19}$$

$$W_0 = -\frac{z^5}{120} + \frac{3z^3}{80} - \frac{7z^2}{240} \tag{20}$$

(18) - (20) may now be substituted into the first-order system which is

$$W_0 U_0' - U_1''' = 0 \tag{21}$$

$$V_0 V_0' + W_0 V_0''' - V_1''' + U_1' = 0 \tag{22}$$

$$V_1 + W_1' = 0 \tag{23}$$

$$U_1'(1) = V_1'(1) = W_1(1) = U_1(0) = V_1(0) = W_1(0) = 0 \tag{24}$$

The complete non-diemnsional velocity field to first order which emerges is

$$U(z) = z^2/2 + \frac{R}{960}\left(-\frac{z^8}{7} + \frac{6z^6}{5} - \frac{7z^5}{5} + \frac{33}{35}z\right) \tag{25}$$

$$\begin{aligned} V(z) = {} & \frac{z^4}{24} - \frac{9z^2}{80} + \frac{7z}{120} + \\ & + \frac{R}{960}\left(\frac{4z^{10}}{1890} - \frac{4z^7}{90} + \frac{27z^6}{200} - \frac{21z^5}{100} + \frac{49z^4}{360} + \frac{11z^3}{70} - \frac{3z^2}{10} + \frac{27z}{250}\right) \end{aligned} \tag{26}$$

$$\begin{aligned} W(z) = {} & -\frac{z^5}{120} + \frac{3z^3}{80} - \frac{7z^2}{240} - \\ & - \frac{R}{960}\left(\frac{4z^{11}}{20790} - \frac{z^8}{180} + \frac{27z^7}{1400} - \frac{7z^6}{200} + \frac{49z^5}{1800} + \frac{11z^4}{280} - \frac{z^3}{10} + \frac{27z^2}{500}\right) \end{aligned} \tag{27}$$

It must be kept in mind that the scale velocity V_0 is negative, due to the negative τ_0 , and that R is negative for the same reason. When we restore dimensional velocities in terms of non-dimensional z we therefore get

$$U(z) = -|V_0|\frac{z^2}{2} + \frac{|R||V_0|}{960}\left(-\frac{z^8}{7} + \frac{6z^6}{5} - \frac{7z^5}{5} + \frac{33}{35}z\right) \tag{28}$$

$$\begin{aligned} V(z) = {} & -\frac{|V_0|}{\gamma}\left(\frac{z^4}{24} - \frac{9z^2}{80} + \frac{7z}{120}\right) + \\ & + \frac{|R||V_0|}{960\gamma}\left(\frac{4z^{10}}{1890} - \frac{4z^7}{90} + \frac{27z^6}{200} - \frac{21z^5}{100} + \frac{49z^4}{360} + \frac{11z^3}{70} - \frac{3z^2}{10} + \frac{27}{250}z\right) \end{aligned} \tag{29}$$

$$\begin{aligned} W(z) = {} & -\frac{V_0}{\gamma}\frac{h_0}{L}\left(-\frac{z^5}{120} + \frac{3z^3}{80} - \frac{7}{240}z^2\right) + \\ & + \frac{|R||V_0|}{960\gamma}\frac{h_0}{L}\left(\frac{4z^{11}}{20790} - \frac{z^8}{180} + \frac{27z^7}{1400} - \frac{7z^6}{200} + \frac{49z^5}{1800} + \frac{11z^4}{280} - \frac{z^3}{10} + \frac{27}{500}z^2\right) \end{aligned} \tag{30}$$

IV. Discussion

The dependence of the velocity field upon the chosen values of h_0 and A_{0V} is critical enough to make any direct comparison of the results with observation unwarrented. It is instructive, however, to examine $U(z)$ for imposed values of h_0 and A_{0V} to see if the model is capable of producing a realistic vertical profile of the zonal velocity component. It is not a priori clear that it can do so, since it is conceivable that the obtaining of a realistic V_0 might necessitate a totally unreasonable R. We observe that, due to the presence of the linear term $\left(\frac{R}{960}\right)\left(\frac{33}{35}\right) z$, an Eastward flow counter to the wind-driven surface flow will appear at some $z > 0$. For purposes of comparison with reality we observe that $R >$ 830 gives an eastward flow throughout the entire depth $0 < z \leq 1$, while $R <$ 500 or so gives the result that the magnitude of the maximum undercurrent is considerably less than the magnitude of the surface velocity. With this in mind we select $h_0 = 2 \cdot 10^4$ cm and $A_{0V} = 1.33 \cdot 10^2$ cm^2/sec which give V_0 = 300 cm/sec R = 588. The resultant behavior of $U(z)$ is shown below, where the z-coordinate represents depth below the surface in meters.

The result is qualitatively comparable to observation, but is not particularly good quantitatively giving, for example, a considerably smaller maximum undercurrent velocity than has been generally observed. Perhaps better agreement could be obtained by

z (m)	$U(z)$ (cm/sec)
0	-40
20	- 5
40	+11
60	26
80	38
100	46
120	43
140	37
160	29
180	15
200	0

a more judicious choice of h_0 and A_{0V}. It should be emphasized that this sample velocity profile is intended only to illustrate the ability of the model to reproduce the observed strong reversal with depth of the zonal velocity. The study as a whole is intended only as a qualitative indication of the controlling dynamics of the equatorial undercurrent.

The first-order velocity field may be substituted into the second equation of motion III-(2) to determine the meridional variation of layer depth h in the immediate vicinity of the equator. The result of attempting this has not been conclusive, the slope changing from very slightly negative (downward) to very slightly positive as R increases through the range $500 < R < 800$.

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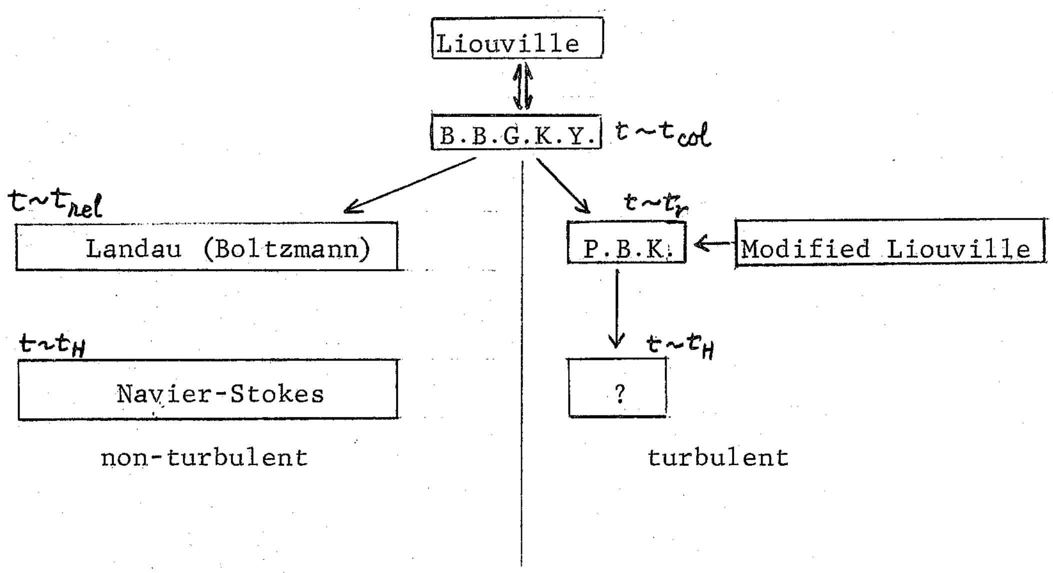
Microscopic Descriptions of Long-range Correlations in a Gas

Uriel Frisch

Abstract

An attempt is made to describe situations with long-range correlations (such as turbulence) using as a starting point a Gibbsian phase-space ensemble, instead of a collection of laminar flows. Hydrodynamic velocity correlations are expressed in terms of particle distribution functions. This requires the introduction of a space smoothing operator. The equations governing the dynamics of long-range correlations on a time scale much longer than the collision time form a hierarchy describing a cascade process for particle correlations. It is shown that the closure assumption made by Prigogine, Balescu and Krieger has an unphysical consequence.



1. Direct and double average statistical description of a turbulent gas.

Let us recall some definitions and results for the description of a classical gas.

A set of N particles without external forces interacting through two body central forces of short range deriving from a potential V has the Hamiltonian

$$H = \sum_{n=1}^{N} \frac{P_n^2}{2m} + \sum_{1\le j<n\le N} V_{jn} \qquad (1.1)$$

where

$$V_{jn} = V\left(\left|\vec{x}_j - \vec{x}_n\right|\right) \qquad (1.2)$$

$\vec{X}_n$ is the position vector of the n^{th} particle and $\vec{P}_n$ its momentum. It is convenient to introduce the phase space Ω, the coordinates of which are $\vec{X}_1, \vec{P}_1, \vec{X}_2, \vec{P}_2; \dots; \vec{X}_N, \vec{P}_N$.*

A point in phase space moves according to the equations

$$\frac{\partial H}{\partial X_j} = -\dot{P}_j \qquad (1.3)$$

$$\frac{\partial H}{\partial \vec{P}_j} = \dot{x}_j \qquad (1.4)$$

where the dot denotes time differentiation.

If instead of a single system we consider an ensemble of systems described by a density $f_N(\vec{X}_1, \text{—}, \vec{P}_N, t)$ in

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*Instead of momentum we shall often use velocities $\vec{U}_n$.

$\{\vec{X}\}$ stands for $\vec{X}_1, \text{—}, \vec{X}_N; \{\vec{P}\}$ for $\vec{P}_1$ — $\vec{P}_n$

$(dX)^N (dP)^N$ = volume element in phase space.

phase space, this density varies according to Liouville's equation.

$$\frac{\partial f_N}{\partial t} + \left[f_N , H \right] = 0 \qquad (1.5)$$

where the second term is the Piosson bracket.

Mathematically the problem of solving the Hamiltonian equations or the Liouville equation are identical. However the Liouville equation gives us the possibility of considering the collection of all the systems with some initially prescribed macroscopic constraints (such as local hydrodynamic velocity) The observable value of any macroscopic quantity A is the average value, weighted with f_N , of the corresponding microscopic quantity.

$$A_{macro} = \int (dx)^N (dv)^N f_N \left(\{\vec{x}\}, \{\vec{v}\}\right) A\left(\{\vec{x}\}, \{\vec{v}\}\right) \qquad (1.6)$$

We now note that the information contained in f_N is actually redundant. For all quantities of macroscopic interest, such as the density, the hydrodynamic velocity etc., the quantity $A\left(\{\vec{x}\}, \{\vec{v}\}\right)$ is a function of the positions and velocities of a very small number of particles (generally one!). We therefore introduce the reduced distribution functions of s particles. They will be defined as follows:

$$f_s\left(\vec{x}_1, -, \vec{x}_s; \vec{v}_1, -, \vec{v}_s; t\right) = \frac{N!}{(N-s)!} \int (dx)^{N-s} (dv)^{N-s} f_N \qquad (1.7)$$

where f_N is integrated over all the particles which are not in f_s . In terms of these functions the most important macroscopic quantities are

Local density

$$n(x;t) = \int dv_1 f_1(\vec{x}_1 \vec{v}_1 ; t) \tag{1.8}$$

Local hydrodynamic velocity

$$\vec{U}(\vec{x};t) = \int dv_1 \vec{v}_1 f_1(\vec{n}, \vec{v}_1 ; t) \tag{1.9}$$

Other quantities such as the hydrodynamic velocity covariance will be given later.

It is also useful to introduce the two particles correlation function.

$$G_2(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2 ; t) = f_2(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2 ; t) - \\ - f_1(\vec{x}_1, \vec{v}_1 ; t) f_1(\vec{x}_2, \vec{v}_2 ; t) \tag{1.10}$$

which vanishes when the two particles are uncorrelated.

We see that the macroscopic constraints are not sufficient to determine the initial f_N , furthermore they do not determine f_s but only some of its low-order moments. We have too little initial information: mathematically the problem is not well posed. However we usually hope that the system will more or less rapidly <u>forget</u> the <u>detailed</u> initial conditions and that macroscopic quantities will (asymptotically) only depend upon initial macroscopic quantities.

What are these macroscopic quantities?

For <u>non-turbulent flow</u> we are given initially local density, momentum and temperature, all expressible in terms of f_1 . It is then convenient to derive an equation for the evolution of f_1 (Boltzmann's equation, for instance) and by taking velocity moments, to get the hydrodynamic equations. This will be discussed in Paragraph 3.

In the turbulent incompressible* case the hydrodynamic velocity is not a given function of $\vec{x}$, but a <u>random function</u> of $\vec{x}$ which is characterized by its initial moments (velocity covariance, etc.). We shall show in Paragraph 2 that in contrast to the previous situation the set of these hydrodynamic moments depends upon reduced particle distribution functions of <u>any order</u> (even for moments involving <u>only two</u> points but <u>many</u> velocities).

Let us call this the <u>direct statistical description</u>.

In order to avoid the introduction of particle distribution functions one usually uses the <u>double average description</u> of turbulence: the turbulent flow is replaced by a collection of non-turbulent flows (called the <u>realizations</u> of the turbulent flow) each of which obeys the Navier-Stokes equation; the expectation value of any hydrodynamic quantity is then its mean value over this collection. Obviously this involves two steps:

1. An average over the phase space ensemble describing the same non-turbulent flow.

2. An average over those hydrodynamic flows.

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*We are then mainly interested in the hydrodynamic velocity.

Let us stress some important differences between the direct and double average description, restricting ourselves to homogeneous turbulence.

In the direct description we have a homogeneous f_N , i.e.

$$f_N(\vec{x}_1+\vec{h},\ldots,\vec{x}_N+\vec{h},\{\vec{p}\};t)=f_N(\{\vec{x}\};\{\vec{p}\};t) \qquad (1.11)$$

for arbitrary $\vec{h}$ and long range correlations, i.e. particle correlation functions have a range which is much longer than in the equilibrium case, being of the order of the hydrodynamic correlation range.

In the double average description we have a non-homogeneous f_N for each realization but without long range particle correlations.

Nevertheless if we assume small hydrodynamic velocities compared with r.m.s. particle velocities the non-turbulent flow and the turbulent flow are hardly to be distinguished from a gas in equilibrium, on any microscopic time or space scale (Maxwell's demon could not distinguish between them!). We shall therefore approximate sometimes purely microscopic quantities (such as particle correlations in non-turbulent flow) by their equilibrium value.

A possible link between the two statistical descriptions.

Instead of a collection of realizations let us introduce a collection of one-particle distribution functions each of which obeys the Boltzmann equation*. The hydrodynamic velocity moments** are then expressed in terms of f_1 moments assuming unit particle density, as for instance

$$\langle \vec{U}(\vec{x}) \cdot \vec{U}(\vec{x}') \rangle = \int \vec{v} \cdot \vec{v}' \langle f_1(\vec{x}, \vec{v}) f_1(\vec{x}', \vec{v}') \rangle \, dv \, dv' \qquad (1.12)$$

This is seen to be another double average description and will be called the Boltzmann double average. We shall now expend the s-particle distribution function of a non-turbulent flow in terms of one particle distribution functions and 1, 2, ... s particle correlation functions.

We have already introduced the two particle correlation function G_2

$$f_2(\alpha, \beta) = f_1(\alpha) f_1(\beta) + G_2(\alpha, \beta) \qquad (1.13)$$

The Greek letters are shorthand notations of position and velocity vector of particles. When particles α and β are not correlated f_2 factorizes.

Let us introduce a third particle γ, which we shall first assume uncorrelated to α and β, the three particle distribution function is then

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*To the first order Chapman-Enskog approximation there is a one-to-one correspondence with realizations because

$$f_1(\vec{x}, \vec{v}; t) = n(x) \left(\frac{m}{2\pi k T(x)} \right)^{\frac{3}{2}} \exp \left[- \frac{m(\vec{v} - \vec{U}(x))^2}{2kT(x)} \right]$$

**See also Paragraph 2b.

$$f_3(\alpha,\beta,\gamma) = f_2(\alpha,\beta) f_1(\gamma)$$
$$= f_1(\alpha) f_2(\beta) f_1(\gamma) + G_2(\alpha,\beta) f_1(\gamma) \tag{1.14}$$

Two other terms similar to $G_2(\alpha,\beta) f_1(\gamma)$ have to be added (α and γ correlated, β and γ correlated) and finally a term where all the particles are correlated $G_3(\alpha,\beta,\gamma)$. This correlation function G_3 is then <u>defined</u> by

$$f_3(\alpha,\beta,\gamma) = f_1(\alpha) f_1(\beta) f_1(\gamma) + \Sigma\, G_2(\alpha,\beta) f_1(\gamma) + G_3(\alpha,\beta,\gamma) \tag{1.15}$$

Higher order correlation functions are defined similarly. The correlation functions are here of purely microscopic origin (collisions) and will be approximated by their equilibrium value i.e., will be taken the same for all the realizations of a turbulent flow. This approximation gives very simple expressions for s-particles distribution functions which become functionals of one particle distribution functions only.* Averaging now over the ensemble of one particle distribution functions we get

$$\langle f_2(\alpha,\beta)\rangle = \langle f_1(\alpha) f_1(\beta)\rangle + G_2(\alpha,\beta) \tag{1.16}$$

$$\langle f_3(\alpha,\beta,\gamma)\rangle = \langle f_1(\alpha) f_1(\beta) f_1(\gamma)\rangle + \Sigma\, G_2(\alpha,\beta)\varphi_1(\gamma) + G_3(\alpha,\beta,\gamma) \tag{1.17}$$

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*The correct approach would be to express even the correlation functions as functionals of one particle distribution functions (see Andrews (1960)) but these are seen to differ only slightly from the equilibrium values.

where

$$\varphi_1(\gamma) = \langle f_1(\gamma) \rangle \tag{1.18}$$

The only contributions to long range correlations are averages of products of f_1 such as $\langle f_1(\vec{x}_1, \vec{v}_1) f_1(\vec{x}_2, \vec{v}_2) \rangle$

2. Hydrodynamic velocity moments in terms of particle distribution functions.

a) Direct statistical description.

Assuming an incompressible homogeneous turbulent gas without mean flow, we want to derive the macroscopic (hydrodynamic) information from the microscopic one, i.e. from the particle distribution functions.

What are the generalizations to turbulence velocity moments* of the well-known formula giving the mean velocity?

$$\vec{U}(\vec{x}) = \int \vec{v} f_1(\vec{x}, \vec{v})\, dv \tag{2.1}$$

Batchelor's (1956) notations for moments will be used, writing capital for hydrodynamic velocities. We shall make the assumption of continuity of hydrodynamic moments with respect to the space variables (Batchelor 1956, Chap. 2, Parag. 4). This implies that moments involving several times the same point are limits of the corresponding moments, where all the points are distinct; for

*We shall be interested in simultaneous moments only.

instance

$$\lim_{\vec{x}_2 \to \vec{x}_1} \langle U_i(\vec{x}_1)\, U_j(\vec{x}_2) \rangle = \langle U_i(\vec{x}_1)\, U_j(\vec{x}_1) \rangle \qquad (2.2)$$

This means that when point $\vec{x}_2$ is approaching point $\vec{x}_1$ the random velocities become totally correlated (or identical in every realization). It is a typical hydrodynamic situation and is not true for the corresponding microscopic velocity moments.

Actually the hydrodynamic velocity moments do not refer to velocities at well-defined points (in the mathematical sense) but to <u>mean velocities</u> in a <u>small volume</u> (we can not make infinitesimal hot wires). We shall therefore introduce a characteristic length δ (called the smoothing length) which will be large compared to the mean free path but small compared to any characteristic length of the turbulence. We could then divide the space occupied by the gas into cells of dimension δ and define the hydrodynamic moments as mean values over cells of microscopic velocity moments. As this would, however, give rise to discrete hydrodynamic moments (functions of cell indices) we replace the averaging process by a <u>smoothing</u> process.

The function

$$S(x,y,z) = \frac{1}{\pi^3 x y z} \sin\frac{x}{\delta} \sin\frac{y}{\delta} \sin\frac{z}{\delta} \qquad (2.3)$$

is the Fourier transform of the function which is equal to 1 inside the cube

$$-\frac{1}{\delta} \le k_x \le \frac{1}{\delta}$$
$$-\frac{1}{\delta} \le k_y \le \frac{1}{\delta} \qquad (2.4)$$
$$-\frac{1}{\delta} \le k_z \le \frac{1}{\delta}$$

and 0 outside.

For any function $f(\vec{x})$ its <u>smoothed</u> function $\tilde{f}(\vec{x})$ is

$$\tilde{f}(\vec{x}) = S * f = \int S(\vec{y}) f(\vec{x}-\vec{y})\, dy \qquad (2.5)$$

If f depends upon several arguments $\vec{x}_1, \vec{x}_2, \vec{x}_3$

$$\tilde{f}(\vec{x}_1,\vec{x}_2,\vec{x}_3) = S(\vec{x}_1) S(\vec{x}_2) S(\vec{x}_3) * f(\vec{x}_1,\vec{x}_2,\vec{x}_3) \qquad (2.6)$$

The smoothing operation is linear and commutes with differentiation. It does not affect any function having a characteristic scale of variation much larger than δ ("macrofunction"); it "kills" any function which goes to zero within a distance much smaller than δ (microscopic correlation functions, for instance). The hydrodynamic velocity moments are then derived from the <u>smoothed</u> particle distribution functions

$$\langle V_{i_1}(\vec{x}_1) V_{i_2}(\vec{x}_2) \cdots V_{i_p}(\vec{x}_p) \rangle = \int dv_1 \cdots dv_p\, v_{i_1} \cdots v_{i_p} \tilde{f}_p(\vec{x}_1 \cdots \vec{x}_p, \vec{v}_1 - \vec{v}_p) \qquad (2.7)$$

What happens when two (or more) of the $\vec{x}$'s coincide? Let us consider the velocity covariance for instance. By the continuity assumption

$$\langle V_i(\vec{x}_1) V_j(\vec{x}_1) \rangle = \lim_{\vec{x}_2 \to \vec{x}_1} \int dv\, dv'\, v_i v_j' \tilde{f}_2(\vec{x}_1, \vec{x}_2, \vec{v}, \vec{v}') \qquad (2.8)$$

The smoothing must be performed before taking the limit and with respect to both $\vec{x}_1$ and $\vec{x}_2$. This would not matter if f_2 were not a very rapidly varying function of $\vec{x}_2 - \vec{x}_1$ when this vector has a length less than the mean free path. We shall therefore write

$$\langle V_i(\vec{x}_1) V_j(\vec{x}_1) \rangle = \int dv\, dv'\, dx_2\, v_i v_j' \tilde{f}_2(\vec{x}_1, \vec{x}_2, \vec{v}, \vec{v}')\, \delta(\vec{x}_1 - \vec{x}_2) \qquad (2.9)$$

where $\delta(\vec{x}_1 - \vec{x}_2)$ is the Dirac distribution not to be confused with the smoothing length.

We conclude that hydrodynamic velocity moments of order p are expressible in terms of smoothed p-particle distribution functions, <u>even when they involve less than</u> p points.

<u>Observation</u>: It would of course be <u>incorrect</u> to write

$$\sum_i \langle U^2(\vec{x}_1) \rangle = \int v^2 f_1(\vec{x}_1, \vec{v})\, dv \qquad (2.10)$$

the l.h.s. being twice the turbulent kinetic energy density and the r.h.s. twice the molecular kinetic energy density.

<u>Boltzmann double average description</u>

Using the fact that for each realization we have

$$\vec{U}(\vec{x}) = \int \vec{v} f_1(\vec{x}, \vec{v})\, dv \qquad (2.11)$$

we get immediately

$$\langle U_{i_1}(\vec{x}_1) \cdots U_{i_p}(\vec{x}_p) \rangle = \int dv_1 \cdots dv_p\, v_{i_1} \cdots v_{i_p} \langle f_1(\vec{x}_1, \vec{v}_1) \cdots f_1(\vec{x}_p, \vec{v}_p) \rangle \qquad (2.12)$$

This remains true even when some points coincide.

This result is consistent with the double average result (eqn. 2.7) and eqn. (1.16; 1.17) relating p-particle distribution functions to one particle distribution functions, because the smoothing operation makes all short range molecular correlations vanish.

In the Boltzmann double average description we can do more than relate hydrodynamic velocity moments to particle distribution functions. We shall show that to the first order Chapman-Enskog approximation* $\langle f_1(\vec{x}_1,\vec{v}_1) - f_1(\vec{x}_p,\vec{v}_p)\rangle$ is expressible in terms of the hydrodynamic random velocity and pressure.

To the first order Chapman-Enskog approximation we have for a given realization

$$f_1(\vec{x},\vec{v}) = n\left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left[-\tfrac{1}{2}m\beta(\vec{v}-\vec{U})^2\right] \tag{2.13}$$

where $\vec{U}$ and $\beta = \frac{1}{kT}$ are functions of $\vec{x}$. The pressure is related to the temperature by

$$p = nkT \tag{2.14}$$

where the particle density n is constant.

We thus write f_1 as a local functional of p and $\vec{U}$

$$f_1(\vec{x},\vec{v}\,;\,p,\vec{U}) \tag{2.15}$$

Introducing the joint probability distribution

$$\Pi(\vec{x}_1-\vec{x}_2,\ p_1-p_2,\ \vec{U}_1-\vec{U}_2) \tag{2.16}$$

*See for instance Chapman and Cowling (1939) of p and $\vec{U}$ at $\vec{x}_1$ —— $\vec{x}_s$ we get

$$\langle f_1(\vec{x}_1,\vec{v}_1) - f_1(\vec{x}_s,\vec{v}_s)\rangle = \int f_1(\vec{x}_1,\vec{v}_1,p_1,\vec{U}_1) - f_1(\vec{x}_s,\vec{v}_s;p_s,\vec{U}_s)\prod dp_1\ldots dp_s\, dU_1\ldots dU_s. \tag{2.17}$$

In a more explicit manner we shall express velocity moments of the $\langle f\cdots f\rangle$ in terms of hydrodynamic velocity and pressure moments. Let us take $\int v_2 v'^2 \langle f_1(\vec{x},\vec{v}) f_1(\vec{x}',\vec{v}')\rangle\, dv\, dv'$ for instance.

For each realization we split the velocity into a mean part and a thermal part

$$\vec{v} = \vec{U} + \vec{c} \qquad v^2 = U^2 + 2\vec{u}\cdot\vec{c} + c^2 \tag{2.18}$$

We know that when the velocity distribution is locally Maxwellian

$$\int \vec{c} f(\vec{x},\vec{c})\, dc = 0 \quad \text{and} \quad \int c^2 f(\vec{x},\vec{c})\, dc = 3p^* \tag{2.19}$$

obtaining thus

$$\int v^2 v'^2 \langle f_1(\vec{x},\vec{v}) f_1(\vec{x}',\vec{v}')\rangle\, dv\, dv' = \langle U^2 U'^2\rangle + 3\langle U^2 p'\rangle + 3\langle p U'^2\rangle + 9\langle pp'\rangle \tag{2.20}$$

Observation

Second-order Chapman-Enskog corrections (viscosity corrections) would introduce moments involving space derivatives of velocities.

3. Dynamics of long range correlations.

Let us first recall some results of non-equilibrium

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*assuming unit density.

statistical mechanics (see for instance Prigogine 1962 or Balescu 1963).

From the mathematical point of view the time evolution of a classical N-body system is determined by the first order partial differential equation of Liouville

$$\frac{\partial f_N}{\partial t} + [f_N, H] = 0 \tag{1.5}$$

and given initial conditions.

This is a reversible equation, but we know that a many-body system actually has an irreversible behaviour. One possible solution of this paradox is that the equations governing the dynamics depend dramatically upon the time scale we are interested in. In a dilute gas there are essentially three characteristic times:

the duration of a single collision t_{col} $(\sim 10^{-12} - 10^{-13}$ sec$)$

the relaxation time t_{rel} (time it takes the <u>local</u> velocity distribution to become Maxwellian $(\sim 10^{-8} - 10^{-9}$ sec$)$

the hydrodynamic time t_H (ratio of characteristic length of macroscopic inhomogeneities to mean hydrodynamic velocity $(\sim 1 - 10^{-2}$ sec$)$

We expect reversible behaviour for $t \sim t_{col}$ and irreversible for

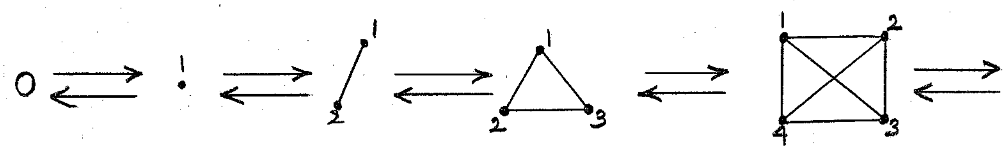
$$t \sim t_{rel} \text{ or } t \sim t_H$$

Integrating Liouville's equation over all velocities and positions except those belonging to the s first particle and recalling the definition (eqn. 1.7) of s particle distribution functions we get the so-called BBGKY equations (Born, Bogolioubov, Green, Kirkwood, Yvon) first introduced by Yvon (1935)

$$\frac{\partial f_s}{\partial t} + \sum_{j=1}^{s} \vec{v}_j \cdot \frac{\partial}{\partial \vec{x}_j} f_s = \frac{1}{m} \sum_{j=1}^{s} \int dx_{s+1} dv_{s+1} \left(\frac{\partial}{\partial \vec{x}_j} V_{j,s+1} \right) \cdot \frac{\partial}{\partial \vec{v}_j} f_{s+1} \tag{3.1}$$

where $V_{j,s+1}$ is the interaction potential between particles j and $s+1$. Each of these equations describes the change in the s particle distribution function due to collisions of these particles with an arbitrary particle of the gas. It is obvious that the characteristic time for these equations is t_{col} and, as expected they are reversible (change $\vec{v}$ and t into $-\vec{v}$ and $-t$). This set of equations is not closed; each f_s being related to f_{s+1}. They form a hierarchy equivalent to Liouville's equation. Some attempts have been made to solve them by making a closure assumption relating higher order distribution functions to lower ones. The justification of such assumptions is always very difficult and generally not very convincing.

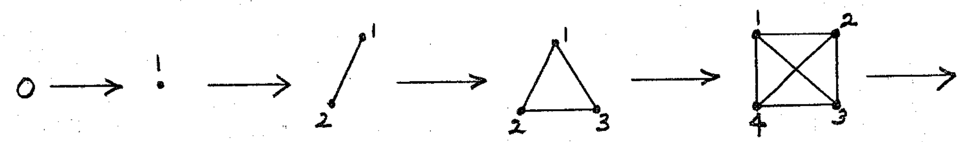
It can be shown (Prigogine, Balescu) that if the BBGKY equations are modified to get equations for the s particle correlation functions (or rather their Fourier transforms) each equation then relates the time derivative of an s particle correlation function to s-1, s and s+1 particle correlation functions, corresponding respectively to creation, propagation and destruction of correlations (see Prigogine (1962) for correct definitions). If we represent an s particles correlation by lines connecting the correlated particles* we have the following diagram representation of the dynamics of correlation for $t \sim t_{col}$

 (3.2)

O is the space average of the one particle distribution function, which is actually the zeroth order correlation function.

Let us go a step further. If one wants to find the behaviour of the f_s for times much longer than t_{col}, one has to get the resultant effect of a large number of collisions or rather of a large number of creations, propagations and destructions, in a language of correlation dynamics. It can be shown that after a time of the order of t_{col} the s particle distribution functions become functionals of the one particle distribution function and that the latter obeys a closed equation (Bogolioubov, 1946; Andrews,1960; Prigogine, 1962).

The dynamics of correlation for $t \gg t_{col}$ is shown to be



In order to get these results one has to make an assumption about the order of magnitude and the range of initial correlations which is not true when long range correlations are present. (See Appendix A for the evaluation of a destruction diagram.)

*In equilibrium situations this is a Mayer diagram.

Let us first discuss the short range case. In a dilute gas (density sufficiently low to neglect simultaneous three bodies collisions) the resultant equation for f_1 is Boltzmann's equation. We shall only write it for "weak interactions" when the deflections of particles due to a single collision are small (Landau or Fokker-Planck approximation).

$$\frac{\partial f_1(\vec{x}_1,\vec{v}_1,t)}{\partial t} + \vec{v}_1 \frac{\partial}{\partial \vec{x}_1} f_1 = \frac{8\pi^4}{m^2} \int dv_2\, dx_2\, d\ell\, V_\ell^2\, \vec{\ell}\cdot\vec{\partial}_1\, \delta(\vec{\ell}\cdot\vec{g}_{12})\, \vec{\ell}\cdot\vec{\partial}_{12}\, f_1(\vec{x}_1,\vec{v}_1,t)\, f_1(\vec{x}_2,\vec{v}_2;t)\, \delta(\vec{x}_1-\vec{x}_2) \quad (3.4)$$

$$V_{\vec{\ell}} = \frac{1}{(2\pi)^3} \int dx\, V(\vec{x})\, e^{-i\vec{\ell}\cdot\vec{x}} \quad (3.5)$$

is the Fourier transform of the interaction potential

$$\vec{\partial}_1 = \frac{\partial}{\partial \vec{v}_1}\,;\quad \vec{\partial}_{12} = \frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2}\,;\quad \vec{g}_{12} = \vec{v}_1 - \vec{v}_2 \quad (3.6)$$

δ is the Dirac distribution.

It is a non-linear, closed, irreversible equation. This irreversibility is expressed by the H-theorem. Boltzmann's equation has two characteristic times: t_{rel} and t_H. The relaxation time is characteristic for the r.h.s. collision term which tends to Maxwellianize the local velocity distribution function, but there is the flow term $\vec{v}\cdot\frac{\partial f}{\partial \vec{x}}$ having the characteristic time t_H which prevents the local velocity distribution from being exactly Maxwellian before a time much longer than t_{rel} (in the non-homogeneous case only).

If we take the mass, velocity and energy moments of Boltzmann's equation we get the usual hydrodynamic equations having the characteristic time t_H because the collision term gives vanishing contribution (due to the conservation laws and the local character of the interaction, expressed by $\delta(\vec{x}_1 - \vec{x}_2)$).

After this somewhat lengthy introduction let us study the dynamics of long range correlations in the frame of the direct statistical description. We shall use Prigogine's and Balescu's notations, particularly the concept of pseudo-cycles (see for instance chapter 7 of Balescu's book). The reader not familiar with this theory should assume equation (3.11).

We assume a homogeneous system with initial long range correlations. λ is the perturbation parameter in the Hamiltonian.

$$H = H_0 + \lambda V \tag{3.7}$$

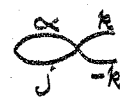
The usual factorization conditions for correlations having all their wave vectors in the macroscopic (hydrodynamic) range does not hold. In contrast to short-range correlations the initial long-range correlations are independent of λ , being of external origin.

We want to derive equations for the long-range correlations valid for times $t >> t_{col}$. In the <u>non-turbulent</u> case the equation for the velocity distribution function is simply

$$\frac{\partial \varphi(\alpha)}{\partial t} = \overset{\alpha}{\underset{\gamma}{\bigcirc}} \varphi(\alpha)\, \varphi(j) \tag{3.8}$$

Where the cycle operator is the velocity-dependent part of the operator on the r.h.s. of eqn. (3.4), there is no contribution from higher correlations which would give rise to uncompensated powers of λ . In the turbulent case we find that the most general contribution to φ is a succession of semi-connected homogeneous cycles and of connected pseudo-cycles, the whole diagram having no external line to the left. We recall that a pseudo-cycle may have external lines but these are confined to the macro-range.

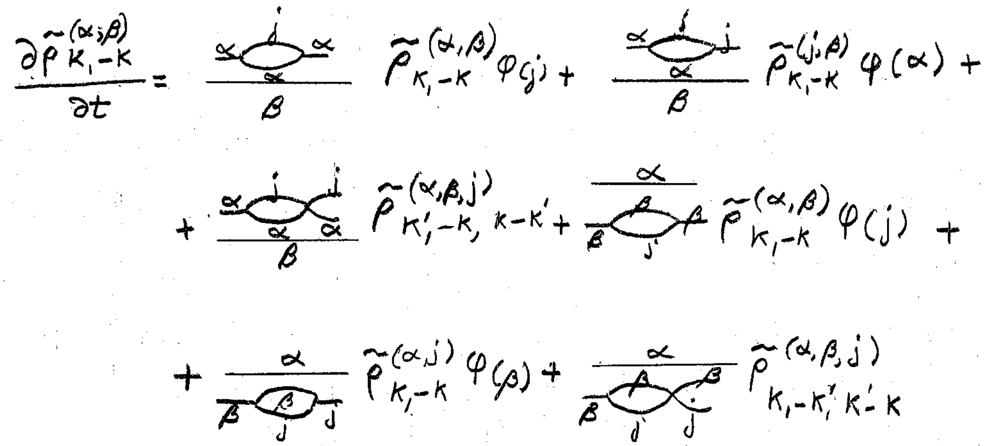
Taking into account this new class of diagrams we obtain the following equation for

$$\frac{\partial \varphi(\alpha)}{\partial t} =$$  $$\varphi(\alpha)\,\varphi(j) +$$  $$\tilde{\rho}_{k_1,-k}^{(\alpha,j)} \qquad (3.9)$$

where k has to be in the macro-range or, in other words, $\tilde{\rho}$ has to be the Fourier transform of a smoothed function (if we define the macro-range by $|k| < \frac{1}{\delta}$).

The diagram  acting on a non-factorized correlation involves a destruction of long-range correlations. In Appendix A it is shown that destructions of long-range correlations have a much longer "life" than destructions of short-range correlations (see also Prigogine, 1962, chapter 8, paragraph 7).

A straightforward generalization to higher order correlations gives for two-particle correlations



(3.10)

It should be mentioned that because of the external origin of long-range correlations there are no creation contributions to these. Writing the higher order equations and transforming them back to ordinary space we obtain the following <u>hierarchy</u> of equations for the <u>smoothed</u> distribution functions

$$\frac{\partial f_s}{\partial t} + \sum_{j=1}^{s} \vec{v}_j \cdot \frac{\partial f_s}{\partial \vec{x}_j} = \sum_{j=1}^{m} \int dx_{s+1}\, dv_{s+1}\, \mathcal{O}_{j,s+1}\, \tilde{f}_{s+1}\, \delta(\vec{x}_j - \vec{x}_{s+1}) \tag{3.11}$$

where $\mathcal{O}_{\alpha\beta}$ is the collision operator

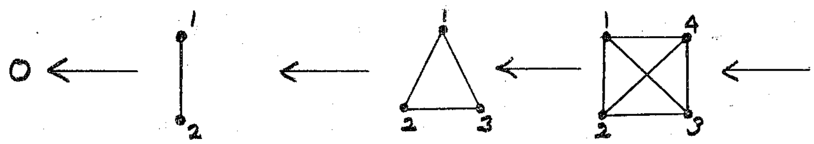
$$\mathcal{O}_{\alpha\beta} = \frac{8\pi^4\lambda^2}{m^2} \int dl\, V_e^2\, \vec{l}\cdot\partial_{\alpha\beta}\, \delta(\vec{l}\cdot g_{\alpha\beta})\, \vec{l}\cdot\partial_{\alpha\beta} \tag{3.12}$$

where the notations are the same as for eqn. (3.4). These equations have been introduced by Prigogine, Balescu and Krieger (1960)(see also Krieger, 1961).*

These equations describe the evolution of the smoothed distribution functions which are the only ones contributing to hydrodynamic velocity moments. They do not relate them to micro-

*We shall call them the P.B.K. hierarchy.

scopic functions and could therefore be an interesting starting point for the study of long-range correlations in the direct statistical approach. This hierarchy is similar to the BBGKY but has very different properties: it is irreversible and it describes correlation dynamics involving no creations



(3.13)

We shall now show that this hierarchy is consistent with the Boltzmann double average description and the identification of smoothed distribution functions with averages of products of one particle non-homogeneous distribution functions. Starting from Landau's equation for a given realization which can be written

$$\frac{\partial f_1}{\partial t} + \vec{v}_1 \frac{\partial f}{\partial x_1} = \int dv_2\, dx_2\, O_{12}\, f_1 f_2\, \delta(\vec{x}_1 - \vec{x}_2) \tag{3.14}$$

We form the time derivative of $f_1(\vec{x}_1,\vec{v}_1) - - f_1(\vec{x}_s,\vec{v}_s)$

$$\frac{\partial}{\partial t}\left(f_1(\vec{x}_1,\vec{v}_1)\cdots f_1(\vec{x}_s,\vec{v}_s)\right) + \sum_{j=1}^{s} v_j \frac{\partial}{\partial x_j}\left(f_1(\vec{x}_1,\vec{v}_1) — f_1(\vec{x}_s,\vec{v}_s)\right) = \tag{3.15}$$
$$= \sum_{j=1}^{s} \int dv_{s+1}\, dx_{s+1}\, O_{j,s+1}\, f_1(\vec{x}_1,\vec{v}_1) - f_1(\vec{x}_s\vec{v}_s)\, f_1(\vec{x}_{s+1},\vec{v}_{s+1})\, \delta(\vec{x}_j - \vec{x}_{s+1})$$

Taking the average over-all realization we obtain a hierarchy which is identical with the former (eqn. 3.11).

An attempt to close the hierarchy has been made by Prigogine, Balescu and Krieger. This is discussed in Appendix B.

Additional observations

1) A modified Liouville equation

Let us indicate that the P.B.K. like the B.B.G.K.Y. hierarchy is derivable from a Liouville equation of a modified type.

This can be written

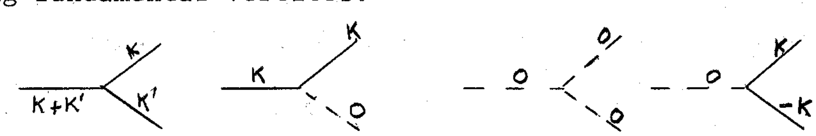
$$\frac{\partial f_N}{\partial t} + \sum_1^N \vec{v}_j \cdot \frac{\partial f_N}{\partial \vec{x}_j} = \sum_{i,j} \delta(\vec{x}_i - \vec{x}_j)\, O_{ij} \cdot \tilde{f}_N \qquad (3.16)$$

where O_{ij} is defined by equation (3.12).

The "force term" of this equation is local and velocity dependent. It does not correspond to particle interactions but rather to the interaction of two small fluid elements.

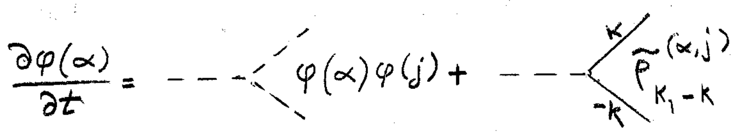
2) A modified diagram technique

We have seen that the fundamental diagram is the cycle in Prigogine's and Balescu's formalism. A modified diagram is then obtained, replacing the cycle by a single interaction vertex. If we represent zero wave vectors by dashed lines we obtain the following fundamental vertices:

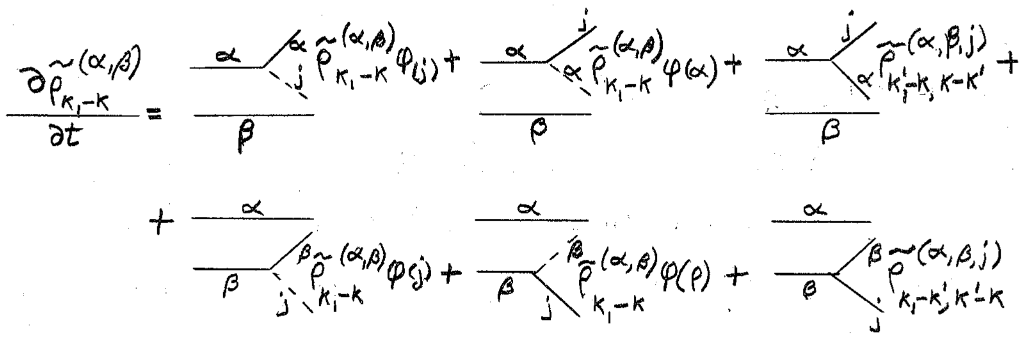


(3.17)

Equations (3.9) and (3.10) are rewritten

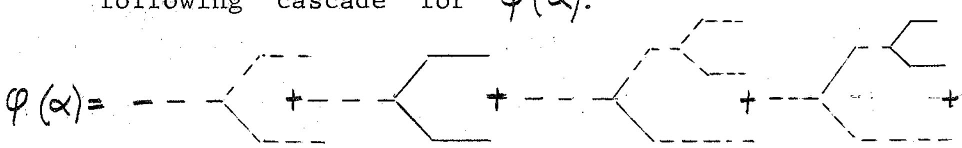


(3.18)



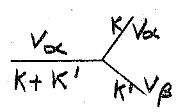
(3.19)

In terms of these diagrams the resolvent of the modified Liouville equation can be expanded, using horizontal lines for free propagation. This gives for instance the following "cascade" for $\varphi(\alpha)$:



(3.20)

It is possible that a restriction of diagram



(3.21)

to velocities satisfying the "phase conservation" condition

$$K' \cdot V_\alpha = K' \cdot V_\beta \tag{3.22}$$

might give some "leading contribution" because they give multiple poles in the resolvent formalism. Is it possible to derive from the B.P.K. hierarchy having the characteristic time t_{rel}, a set of equations having the characteristic time t_H?

Appendix A.

Time dependence of a destruction diagram

Let us study the time dependence of the simplest diagram corresponding to a destruction of correlations.

$$\left\langle\begin{matrix}\alpha\\\beta\end{matrix}\right. \rho^{(\alpha,\beta)}_{k,-k} = \frac{1}{2\pi i}\oint_c dz\, e^{-izt} \frac{1}{z}\int dk\, V_k\, \vec{k}\cdot\vec{\partial}_{\alpha\beta} \frac{1}{\vec{k}\cdot\vec{g}_{\alpha\beta}-z}\, \rho^{(\alpha,\beta)}_{k,-k} \qquad \text{(A.1)}$$

Carrying out the z integration we obtain

$$\vec{\partial}_{\alpha\beta}\cdot\int_0^t dt_1 \int dx\, \vec{F}(\vec{x})\, G_2\left(\vec{x}-\vec{g}_{\alpha\beta}t_1, \vec{v}_\alpha, \vec{v}_\beta; t_1\right) \qquad \text{(A.2)}$$

where $\vec{F}(\vec{x})$ is the interaction force and G_2 the two particle correlation function and

$$\vec{x} = \vec{x}_\alpha - \vec{x}_\beta \qquad \text{(A.3)}$$

Let us assume that $\vec{F}(\vec{x})$ has the range a and the correlation function the range A.

If a and A are of the same order of magnitude we see immediately that the destruction vanishes after a time of the order of $t_{col} = \frac{a}{|\vec{g}|}$. If, on the contrary, A is of the hydrodynamic order of magnitude the destruction vanishes only after a time of the order of t_H and becomes proportional to t for

$$t_{col} << t << t_H. \qquad \text{(A.4)}$$

Appendix B.

A closure assumption by Prigogine, Balescu and Krieger

In order to obtain a closed equation for $\tilde{f}_2(\alpha,\beta)$ they expressed $\tilde{f}_3(\alpha\beta\gamma)$ in terms of $\tilde{f}_2$ making the following assumption

$$\tilde{f}_3(\alpha,\beta,\gamma)\delta(\vec{x}_\alpha-\vec{x}_\beta)=\frac{1}{\varphi(\vec{v}_\beta)}\tilde{f}_2(\alpha,\gamma)\tilde{f}_2(\beta,\gamma)\delta(\vec{x}_\alpha-\vec{x}_\beta) \tag{B.1}$$

This should be true after a time of the order of the relaxation time. We shall show that this leads to a contradiction. Consider an initial situation where there are long-range correlations but extending only over a distance small compared to the dimension of the system. We could for instance imagine a great number of stirring devices acting independently upon the gas and remove them suddenly. Initially two points of the fluid are correlated only when they are acted upon by the same stirring device (let us say, if their distance is less than a). We know that such a situation will not last forever, correlations of range longer than a being built up in hydrodynamic times. However, after a time of the order of t_{rel} the initial situation remains practically unchanged.

Let us choose $\vec{x}_\alpha$ and $\vec{x}_\gamma$ such that

$$|\vec{x}_\alpha-\vec{x}_\gamma| >> a. \tag{B.2}$$

α and γ being not correlated:

$\tilde{f}_2(\alpha,\gamma)$ factorizes into $\varphi(\alpha)\,\varphi(\gamma)$

$\tilde{f}_2(\beta,\gamma)$ factorizes into $\varphi(\beta)\,\varphi(\gamma)$

$\tilde{f}_3(\alpha,\beta,\gamma)\,\delta(\vec{x}_\alpha-\vec{x}_\beta)$ factorizes into $\varphi(\gamma)\,\tilde{f}_2(\alpha,\beta)\,\delta(\vec{x}_\alpha-\vec{x}_\beta)$

Prigogine's closure assumption implies then

$$\tilde{f}_2(\alpha,\beta)\,\delta(\vec{x}_\alpha-\vec{x}_\beta)=\varphi(\alpha)\,\varphi(\beta) \tag{B.3}$$

We have shown in Paragraph 2 eqn.(2.9) that

$$\langle\vec{U}(\vec{x}_\alpha)\cdot\vec{U}(\vec{x}_\alpha)\rangle=\int\vec{V}_\alpha\cdot V_\beta\, f_2(\vec{x}_\alpha,\vec{x}_\beta,\vec{V}_\alpha,\vec{V}_\beta)\,\delta(\vec{x}_\alpha-\vec{x}_\beta)\,dv_\alpha\, dv_\beta \tag{B.4}$$

but as

$$\int\vec{V}_\alpha\,\varphi(\vec{V}_\alpha)\,dv_\alpha=\vec{0}\quad\text{(No mean flow)} \tag{B.5}$$

This would lead to

$$\langle\vec{U}\cdot\vec{U}\rangle=0 \tag{B.6}$$

The turbulent kinetic energy density would thus be zero.

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Upper Bound on Heat Transport by Turbulent Convection

Benjamin Halpern

Abstract

Upper bounds for the heat flux through a horizontally infinite layer of fluid heated from below are obtained by maximizing the heat flux subject to an integral constraint derived from the heat equation and differential constraint derived from the momentum equation. This variational problem is solved only for infinite Prandtl number σ, and large Rayleigh number R. Under these conditions the Nusselt number N is found to be bounded by $CR^{\frac{1}{3}}$ where C is some constant.

Introduction

Consider a horizontally infinite layer of fluid heated from below. In order to set an upper bound on the heat flux Howard (1963) found the maximum heat flux among a larger class of fields than those which satisfy the Boussinesq equations. He sought a maximum among those fields which satisfy the continuity equation, the boundary conditions, the requirements of homogeneity and the "power integrals". This paper approaches a very similar problem. A maximum heat flux is sought among all fields which satisfy one of the power integrals, a differential equation derived from the momentum equation, the boundary conditions, and the requirements of homogeneity. The problem is simplified by considering only the case of infinite Prandtl number and large Rayleigh number.

Mathematical formulation

The Boussinesq equations may be written as follows:

$$u_t + u\cdot\nabla u + \rho^{-1}\nabla p - \alpha g \theta \hat{k} = \nu \nabla^2 u \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

$$T^*_t + u\cdot\nabla T^* = k \nabla^2 T^* \tag{3}$$

where $u = (u, v, w)$ is the velocity vector, T^* the temperature field, θ the deviation of T^* from its horizontal average, ρ the mean density, α the coefficient of thermal expansion of the acceleration of gravity, ν and k the coefficients of kinematic viscosity and thermometric conductivity, p the deviation of the pressure from the hydrostatic pressure field corresponding to the horizontally averaged temperature, and $\hat{k}$ the vertical unit vector. The layer of fluid is taken to be $0 \leq z \leq d$, and the boundary conditions are that

$$T^*(0) = T_0, \quad T^*(d) = T_0 - \Delta T,$$

$$w(0) = w(d) = \frac{\partial^2 w}{\partial z^2}\Big/_0 = \frac{\partial^2 w}{\partial z^2}\Big/_d = 0$$

(slippery boundary)

Besides these boundary conditions the fluid is assumed to be statistically steady in time and statistically homogeneous in the horizontal planes z = constant. In particular we require the existence and constancy in time of horizontal averages of the various functions describing the flow, and of the vanishing of the horizontal averages of the horizontal velocity components; the horizontal average of the vertical component is then zero also as a consequence of the continuity equation. We shall use a horizontal bar to denote the horizontal average, and brackets $\langle\ \rangle$ to denote the average over the layer.

We will now derive the equations which will be the constraints in the variational problem which follows.

If (3) is averaged horizontally one obtains

$$\frac{d\overline{wT^*}}{dz} = \kappa \frac{d^2 \overline{T^*}}{dz^2}$$

Since $\overline{w} = 0$ $\overline{wT^*} = \overline{w\theta}$ so $\overline{w\theta} - \kappa \frac{d\overline{T}^*}{dz}$ = const.

At the boundaries $w = 0$ so $\overline{wT} = 0$ and also

$$-\kappa \frac{d\overline{T}^*}{dz} = -\overline{\kappa \frac{dT^*}{dz}} = H$$

$H \equiv$ the heat flux

Setting $\beta = -\frac{d\overline{T}^*}{dz}$ we have

$$H = \kappa\beta + \overline{w\theta} \tag{4}$$

Averaging over the layer we get

$$H = \kappa\beta_m + \langle w\theta \rangle \tag{5}$$

and $$\beta_m \equiv \langle \beta \rangle = -\frac{1}{d}\int_0^d \frac{dT^*}{dz}dz = +\frac{\Delta T}{d} \tag{6}$$

Now if (3) is multiplied by θ and averaged we obtain

$$-\langle \beta w\theta \rangle = \kappa\langle \theta \nabla^2 \theta \rangle \tag{7}$$

Next take the z-component of the double curl of (1) and use (2) to arrive at

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 W = -g\alpha\,\nabla_1^2\theta + L \qquad (8)$$

where $\nabla_1^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

and

$$L \equiv \frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}(u\cdot\nabla u) + \frac{\partial}{\partial y}(u\cdot\nabla v) - \nabla_1^2(u\cdot\nabla w)\right]$$

We now introduce dimensionless (primed) parameters:

$$\vec{r} = d\,\vec{r}\,'$$

$$u = \left(\frac{k}{d}\right)u'$$

$$\theta = (\Delta T)\theta'$$

$$\beta = \left(\frac{\Delta T}{d}\right)\beta'$$

$$H = \left(\frac{k\Delta T}{d}\right)N$$

and define the Rayleigh and Prandtl numbers by

$$R = \frac{\alpha g \Delta T d^3}{k\nu}$$

$$\sigma = \frac{\nu}{k}$$

Now equations (4) - (8) in non-dimentional form (dropping primes)

$$N = \beta + \overline{W\theta} \qquad (9)$$

$$N = 1 + \langle W\theta\rangle \qquad (10)$$

$$\beta_m = 1 \qquad (11)$$

$$\langle\theta\nabla^2\theta\rangle = -\langle\beta W\theta\rangle \qquad (12)$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W = -R\,\nabla_1^2\theta + \frac{1}{\sigma}L \qquad (13)$$

Now in the limit where $\sigma \to \infty$ (13) becomes

$$\nabla^4 W = R\nabla_1^2\theta \tag{14}$$

The boundary conditions are now

$$\theta = W = W_{zz} = 0 \quad at \quad z = 0,1 \tag{15}$$

From (9) and (10) we have

$$\beta = 1 + \langle W\theta\rangle - \overline{W\theta} \tag{16}$$

Statement of the Problem

Find the maximum of $N = 1 + \langle W\theta\rangle$ among all fields W, θ and β which satisfy

$$\nabla^4 W = R\nabla_1^2\theta \tag{17}$$

$$\langle\theta\nabla^2\theta\rangle = -\langle\beta W\theta\rangle \tag{18}$$

$$\beta = 1 + \langle W\theta\rangle - \overline{W\theta} \tag{19}$$

$$\theta = W = W_{zz} = 0 \quad at \; z = 0,1 \tag{20}$$

We will discuss only the case where $R >> 1$. Introducing Lagrange multipliers $p = p(\vec{x})$, $\lambda_1 = \lambda_1(z)$, λ = const. we form the functional

$$I = 1 + \langle W\theta\rangle + \langle p(\nabla^4 W - R\nabla_1^2\theta)\rangle + \lambda\left(\langle\theta\nabla^2\theta\rangle + \langle\beta W\theta\rangle\right) +$$
$$+ \langle\lambda_1\left(\beta - 1 - \langle W\theta\rangle + \overline{W\theta}\right)\rangle$$

and we will require $\delta I = 0$ under independent variations of β, θ and W. p is assumed to satisfy the boundary conditions $p = \nabla^2 p = 0$ at $z = 0,1$.

We thus obtain

$$\lambda\,\overline{W\theta} + \lambda_1 = 0 \tag{21}$$

$$\theta + \nabla^4 p + \lambda\beta\theta - \langle\lambda_1\rangle\,\theta + \lambda_1\theta = 0 \tag{22}$$

$$W - R\nabla_1^2 p + 2\lambda\nabla^2\theta + \lambda\beta W - \langle\lambda_1\rangle W + \lambda_1 W = 0 \tag{23}$$

Regrouping terms we have

$$\nabla^4 p + \left(\lambda\beta + 1 - \langle\lambda_1\rangle + \lambda_1\right)\theta = 0 \tag{24}$$

$$2\lambda\nabla^2\theta + \left(\lambda\beta + 1 - \langle\lambda_1\rangle + \lambda_1\right)W - R\nabla_1^2 p = 0 \tag{25}$$

Averaging (21) we find

$$\langle\lambda_1\rangle = -\lambda\langle W\theta\rangle \tag{26}$$

Therefore

$$\begin{aligned}\lambda\beta + 1 - \langle\lambda\rangle + \lambda_1 &= \lambda\left(1 + \langle W\theta\rangle - \overline{W\theta}\right) + 1 + \lambda\langle W\theta\rangle - \lambda\,\overline{W\theta} \\ &= \lambda\left(1 + 2\left(\langle W\theta\rangle - \overline{W\theta}\right)\right) + 1\end{aligned} \tag{27}$$

So (24) and (25) become

$$\nabla^4 p + \left[\lambda\left(1 + 2\left(\langle W\theta\rangle - \overline{W\theta}\right)\right) + 1\right]\theta = 0 \tag{28}$$

$$2\lambda\nabla^2\theta + \left[\lambda\left(1 + 2\left(\langle W\theta\rangle - \overline{W\theta}\right)\right) + 1\right]W - R\nabla_1^2 p = 0 \tag{29}$$

<u>Now in order to express λ in terms of integrals</u>, multiply (28) by W and (29) by θ and average each. Then note that

$$\langle W\nabla^4 p\rangle = \langle(\nabla^4 W)p\rangle = \langle R\nabla_1^2\theta p\rangle = R\langle\theta\nabla_1^2 p\rangle \tag{30}$$

Therefore by adding equations we get

$$2\lambda\langle\theta\nabla^2\theta\rangle+\langle 2\left[\lambda(1+2(\langle W\theta\rangle-\overline{W\theta}))+1\right]W\theta\rangle=0 \tag{31}$$

Using (18) and (19) we have

$$\langle\theta\nabla^2\theta\rangle=-\langle\beta W\theta\rangle=-\langle W\theta\rangle-\langle W\theta\rangle^2+\langle\overline{W\theta}^2\rangle \tag{32}$$

Therefore

$$2\lambda\left[\langle W\theta\rangle+\langle W\theta\rangle^2-\langle\overline{W\theta}^2\rangle\right]=2\lambda\langle W\theta\rangle+4\lambda\langle W\theta\rangle^2-4\lambda\langle\overline{W\theta}^2\rangle+2\langle W\theta\rangle$$

$$\lambda\left(\langle\overline{W\theta}^2\rangle-\langle W\theta\rangle^2\right)=\langle W\theta\rangle$$

$$\lambda=\frac{\langle W\theta\rangle}{\langle(W\theta-\langle W\theta\rangle)^2\rangle} \tag{33}$$

Formula (33) allows us to estimate the magnitude of λ for large R . It is assumed that β is nearly equal to N in a boundary layer of thickness h and equal to zero in the interior. Since

$$\langle\beta\rangle=1$$

$$2NR=1$$

$$\therefore \quad R=\frac{1}{2N}$$

because there is a boundary layer at $z=0$ and 1.

Equation (19) implies

$$N=\beta+\overline{W\theta} \quad \text{and} \quad N-1=\langle\overline{W\theta}\rangle$$

$$\therefore \quad \overline{W\theta}=\begin{cases}0 \text{ for} & 0\le z\le h \\ & 1-h\le z\le 1 \\ N & h<z<1-h\end{cases}$$

$$\therefore \langle(\overline{W\theta}-\langle\overline{W\theta}\rangle)^2\rangle=(N-1)^2\cdot 2R+1\cdot(1-2R)$$

Since we assume $N>>1$ we have approximately

$$\lambda = \frac{\langle W\theta \rangle}{\langle (\overline{W\theta} - \langle \overline{W\theta} \rangle)^2 \rangle} = \frac{N-1}{(N-1)^2 2R} \approx \frac{1}{(N-1)2R}$$

$$\approx \frac{1}{2NR-2R} \approx 1$$

That is $\lambda \sim 1$ for large R (34)

Separation into Eigenfunctions of ∇_1^2

Since equations (17), (21), (22) and (23) do not have any coefficients depending on x or y, we may "separate" them by eigenfunctions of ∇_1^2. We will make the assumption that the maximum of N will be found among solutions of (17) - (19), (21) - (23) which have a single α. More precisely we assume that the solutions which maximize N have the form

$$W = W(z) \sin \pi\alpha_1 x \sin \pi\alpha_2 y$$

$$\theta = \theta(z) \sin \pi\alpha_1 x \sin \pi\alpha_2 y$$

$$p = P(z) \sin \pi\alpha_1 x \sin \pi\alpha_2 y$$

and $\alpha^2 = \alpha_1^2 + \alpha_2^2$

(For a fuller discussion of this step, see the paper by Howard listed at the end.)

Note that $\overline{W\theta} = \frac{1}{4} W(z)\theta(z)$

From now on W, θ, and P will denote $W(z)$, $\theta(z)$ and $P(z)$, respectively.

Also note that $\nabla_1^2 \rightarrow -\alpha^2\pi^2$

$$\nabla^4 = \left(\frac{\partial^2}{\partial z^2} + \nabla_1^2\right)^2 \rightarrow \left(\frac{\partial^2}{\partial z^2} - \alpha^2\pi^2\right)^2$$

<u>Assumption about $W(z)$ and $P(z)$</u>

We now make some assumptions as to the behaviour of W and P near the boundary which can only be justified <u>a posteriori</u>.

These assumptions are

$$\begin{aligned} W(z) &= Az \\ P(z) &= 0 \end{aligned} \tag{35}$$

(The assumption $P(z) = \text{const} \neq 0$ and $P(z) = B(z)$ lead to strong inconsistancies in the heat equation, $N = 1 + \langle W\Theta\rangle$.)

<u>Integration of the Θ equation</u>

Equation (29) now yields

$$\frac{d^2\Theta}{dz^2} = \left(\pi^2\alpha^2 + \tfrac{1}{4}A^2z^2\right)\Theta - \frac{1}{2\lambda}\left(\lambda + 1 + \tfrac{1}{2}\lambda\langle W\Theta\rangle\right)A$$

This equation has the solution

$$\Theta(z) = \frac{Q}{D_n(0)}\int_0^\infty D_n(y)\sin\frac{\sqrt{A}}{2}zy\,dy \tag{36}$$

where

$$Q = \frac{-\left(\lambda + 1 + \frac{1}{2}\lambda\langle W\Theta\rangle\right)}{\lambda\sqrt{A}} = \frac{1 - \frac{1}{\lambda} - 2N}{\sqrt{A}} \tag{37}$$

$$n = -\frac{1}{2} - \frac{\pi^2\alpha^2}{A} \tag{38}$$

and $D_n(y)$ is the parabolic cylinder function satisfying

$$\frac{d^2}{dy^2}D_n(y) + \left(n + \tfrac{1}{2} - \tfrac{1}{4}y^2\right)D_n(y) = 0$$

The requirement of the heat equation

We must have

$$N = 1 + \frac{1}{4}\langle W\Theta\rangle = 1 + \frac{1}{2}\frac{QA}{D_n(0)}\int_0^{\frac{1}{2}}\int_0^{\infty} D_n(y)\, z \sin\frac{\sqrt{A}}{2} zy \, dy\, dz \tag{39}$$

The asymptotic behaviour of the integrals can be found through an integration by parts and switching the order of integration and assuming that $A \to \infty$ as $R \to \infty$. The result is

$$N = 1 - \frac{\sqrt{A}\, Q}{2} + \frac{Q \quad D_n'(0)\pi}{D_n(0)} \tag{40}$$

Using (37) we have

$$N = 1 + N + \frac{1}{2\lambda} - \frac{1}{2} - \frac{2\pi D_n'(0)}{D_n(0)}\frac{N}{\sqrt{A}} + \frac{2\pi D_n'(0)}{D_n(0)}\frac{(1-\frac{1}{\lambda})}{\sqrt{A}}$$

Thus

$$A \sim CN^2 \quad \text{for large } R \tag{41}$$

where $\quad C = \left(\dfrac{4\pi D_n'(0)}{D_n(0)(1-\frac{1}{\lambda})}\right)^2$

Next we consider the "w equation"

$$\nabla^4 W = -R\alpha^2\pi^2\Theta \tag{42}$$

Using (36) it is readily found that one solution of (42) is

$$W(z) = K\int_0^{\infty}\frac{D_n(y)\sin\frac{\sqrt{A}}{2}yz\, dy}{\left(\frac{Ay^2}{4}+\pi^2\alpha^2\right)^2} \tag{43}$$

where $K = -R\alpha^2\pi^2 \dfrac{1-\frac{1}{\lambda}-2N}{\sqrt{A}\, D_n(0)}$

This solution satisfies the boundary conditions

$w = w'' = w'''' = w^6 = \ldots = 0$ at $z = 0$ and also the order of magnitude of $W(\frac{1}{2})$ is A .

If we require that the initial slope of $W(z)$ as given by (43) is the same as the slope assumed through (35) we obtain

$$A = \frac{\sqrt{A}\,K}{2}\int_0^\infty \frac{D_n(y)\,y\,dy}{\left(\frac{Ay^2}{4} + \pi^2\alpha^2\right)^2} \tag{44}$$

Let $\frac{\sqrt{A}}{2}y = x$

Then

$$A = \frac{2K}{\sqrt{A}}\int_0^\infty \frac{D_n\left(\frac{2x}{\sqrt{A}}\right)x\,dx}{(x^2+\pi^2\alpha^2)^2}$$

Now if $A \to \infty$ as $R \to \infty$ we get for large R

$$A = \frac{2K}{\sqrt{A}}\int_0^\infty \frac{D_n(0)\,x\,dx}{(x^2+\pi^2\alpha^2)^2}$$

$$A = +\left[\frac{4RN}{A} + \frac{R(\frac{1}{\lambda}-1)}{A}\right]\alpha^2\pi^2\int_0^\infty \frac{x\,dx}{(x^2+\pi^2\alpha^2)^2}$$

$$A = \frac{2RN}{A} + \frac{R(\frac{1}{\lambda}-1)}{2A}$$

<u>Final results</u>

So for large R , H and A we have

$$A^2 \approx 2RN \tag{45}$$

Using (41) we have

$$C^2N^4 \approx 2RN$$

$$N \approx \left(\frac{2}{C^2}\right)^{\frac{1}{3}} R^{\frac{1}{3}} \tag{46}$$

$$A \approx 2^{\frac{2}{3}} C^{-\frac{1}{3}} R^{\frac{2}{3}} \tag{47}$$

It should be noted that equation (46) does not involve α, so that maximizing N with respect to α is not possible through (46). The one-third power in equation (46) is the essential result of this paper.

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Non-linear Energy Transfer in a Rossby-wave Spectrum

Kern Kenyon

Abstract: A perturbation method is used to evaluate the energy flux in a Rossby-wave spectrum due to a weak non-linear coupling between the spectral components. It is found that at the second order in the perturbation scheme resonance occurs which represents a continuous energy transfer between groups of three waves satisfying certain selection rules. It is found that the energy flux vanishes for an initially white, isotropic spectrum.

Introduction: Non-linear energy transfer in a gravity-wave spectrum has been studied in some detail by Hasselmann (1961, 1962a, 1962b). Phillips (1960) discovered the existence of unsteady third order perturbations representing a continuous energy transfer between discrete wave components, and Longuet-Higgins (1962) explicitly evaluated the rate of growth of the tertiary wave for two intersecting wave trains. Pedlosky (1962) has studied some aspects of non-linear Rossby-waves.

What follows is the application of the general theory developed by Hasselmann (1961) to a much simpler physical system than the gravity wave model he considered but one which is still geophysically interesting. We will discuss the non-linear energy transfer between different wave components in a two-dimensional, incompressible ocean on a β-plane. Interest will be focussed entirely on the non-linear process; the important generating and dissipating mechanisms will be left out.

Simply stated the problem is an initial-value one: given the energy spectrum of Rossby-waves at some initial time, calculate it at a later time, or equivalently calculate its time rate of change. In general we expect a time change in the initial spectrum due to the non-linearities alone. It turns out that for an initially white spectrum there is no non-linear energy redistribution, and the spectrum remains white.

The analysis proceeds as follows: First an amplitude expansion of the dependent variable is made in the governing equation. Next a separation of variables in the form of a Fourier expansion in wave number $\underline{k}$ is introduced, the interaction coefficients are calculated, and resonance is investigated. Since resonance occurs already at the second order, the perturbation expansion needs to be carried out to only the third order (instead of to the fifth order as in Hasselmann (1961)) in order to completely describe the energy transfer process. The kinetic energy is expanded in a perturbation series, and certain statistical assumptions are made which greatly simplify the calculations. The expression for the energy transfer is then obtained.

Basic Equations: We approximate the real ocean by a model ocean which is unbounded, two-dimensional, incompressible, and rotating with an angular velocity which is a linear function of the north-south coordinate. The equations appropriate to describe the motion of such an ocean are

$$\rho \frac{\partial u}{\partial t} + \rho L u - \rho \Omega v + \frac{\partial p}{\partial x} = 0 \tag{1}$$

$$\rho \frac{\partial v}{\partial t} + \rho L v + \rho \Omega u + \frac{\partial p}{\partial y} = 0 \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where $L = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$, $\Omega = \Omega(y) = f_0 + \beta y$, ρ, f_0, β are constants, y is positive to the north, and x is positive to the east. By using the stream function defined as $u = \Psi_y$, $v = -\Psi_x$ we satisfy 3) exactly, and 1) and 2) can be combined by eliminating the pressure to give the governing equation for Ψ.

$$\frac{D}{Dt} \nabla^2 \Psi + \beta \Psi_x = 0 \tag{4}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + L, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \beta = \frac{\partial \Omega}{\partial y}$$

<u>Perturbation Equations:</u> We assume a zero order state ${}_0u = {}_0v = 0$, ${}_0p$ = constant. Then we expand Ψ in a perturbation series.

$$\Psi = {}_0\Psi + \epsilon\, {}_1\Psi + \epsilon^2\, {}_2\Psi + \epsilon^3\, {}_3\Psi + \ldots \tag{5}$$

where ${}_0\Psi$ = constant, and ϵ is assumed to be small and can be considered a wave slope. If 5) is inserted in 4) and different powers of ϵ are equated to zero, we get the following three perturbation equations:

$$O\, {}_1\Psi = 0 \tag{6}$$

$$O\,{}_2\psi = -L_1\nabla^2{}_1\psi \qquad 7)$$

$$O\,{}_3\psi = -L_1\nabla^2{}_2\psi - L_2\nabla^2{}_1\psi \qquad 8)$$

where $O = \frac{\partial}{\partial t}\nabla^2 + \beta\frac{\partial}{\partial x}$, $L_1 = {}_1u\frac{\partial}{\partial x} + {}_1v\frac{\partial}{\partial y} = {}_1\psi_y\frac{\partial}{\partial x} - {}_1\psi_x\frac{\partial}{\partial y}$, $L_2 = {}_2\psi_y\frac{\partial}{\partial x} - {}_2\psi_x\frac{\partial}{\partial x}$.

Separation of Variables: For an arbitrary perturbation order n we assume the formal expansion

$${}_n\psi = \sum_{\underline{k}=-\infty}^{\infty} {}_n\psi_{\underline{k}}(t)\,e^{i\underline{k}\cdot\underline{x}} \qquad 9)$$

If ${}_n\psi$ is to be real, we must have ${}_n\psi_{\underline{k}} = {}_n\psi^*_{-\underline{k}}$, where the star denotes complex conjugate. The components of $\underline{k}$, and $\underline{x}$ are $\underline{k} = \ell\hat{x} + m\hat{y}$, $\underline{x} = x\hat{x} + y\hat{y}$, where $\hat{x}(\hat{y})$ is a unit vector along the $x(y)$ axis. We assume a linear solution of the form

$${}_1\psi = \sum_{\underline{k}=-\infty}^{\infty} {}_1\psi_{\underline{k}}\, e^{i(\underline{k}\cdot\underline{x} + \omega t)} \qquad 10)$$

which is a sum of waves traveling westward. If we put 10) into 6) we find the dispersion relation

$$\omega = \frac{\beta\ell}{k^2} = \frac{\beta\ell}{\ell^2 + m^2} \qquad 11)$$

Interaction Coefficient: If 9) and 10) are put into 7) we get an equation for the second order Fourier amplitude

$$P\,{}_2\psi_{\underline{k}} = \sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}} D_{\underline{k}_1,\underline{k}_2}\,{}_1\psi_{\underline{k}_1}\,{}_1\psi_{\underline{k}_2}\, e^{i(\omega_1+\omega_2)t} \qquad 12)$$

where $P = \frac{d}{dt} - i\omega$, $\omega_1 = \frac{\beta \ell_1}{k_1^2}$, etc., and the sum is over all $\underline{k}_1$ and $\underline{k}_2$ subject to the selection rule written below the summation sign. The interaction coefficient $D_{\underline{k}_1, \underline{k}_2}$ can be written as

$$D_{\underline{k}_1, \underline{k}_2} = \frac{\hat{z} \cdot (\underline{k}_1 \times \underline{k}_2)(k_1^2 - k_2^2)}{2k^2} \qquad 13)$$

where $\hat{z}$ is a unit vector perpendicular to the xy-plane and is only used to make the expression a scalar. As discussed by Pedlosky (1962) the coefficient has the properties of being symmetric in the indices $\underline{k}_1$, $\underline{k}_2$ and vanishing if $k_1 = k_2$ or $\underline{k}_1$ and $\underline{k}_2$ are parallel. We can say at this point that there will be no non-linear energy transfer in a unidirectional spectrum.

Quadratic Resonance: What we mean by quadratic resonance is the the existence of a solution for ${}_2\psi_{\underline{k}}$ in equation 12) which grows linearly with time. Quadratic resonance is possible if the following two selection rules are satisfied simultaneously

$$\underline{k}_1 + \underline{k}_2 = \underline{k} \qquad 14)$$

$$\omega_1 + \omega_2 = \omega \qquad 15)$$

To show that 14) and 15) can be satisfied together we consider the special case $\omega_1 = \omega_2 = \omega/2$. For fixed ω 11) is the equation for a circle in the ℓ, m plane

$$\left(\ell - \frac{\beta}{2\omega}\right)^2 + m^2 = \left(\frac{\beta}{2\omega}\right)^2 \qquad 16)$$

The circle corresponding to ω_1 and ω_2 has twice the radius of the one corresponding to ω (see figure 1). Figure 1 shows an obvious example which satisfies 14) and 15) for the special case $\omega_1 = \omega_2$. It happens that for this example the interaction coefficient vanishes because $\underline{k}_1 = \underline{k}_2$, but it is true that for any $\underline{k}$ on the smaller circle one can find a $\underline{k}_1$ and a $\underline{k}_2$ on the larger circle which satisfy 14). In this special case all possible $\underline{k}_1$ and $\underline{k}_2$ satisfying 14) lie on the portion of the large circle defined by its intersection with the line $\ell = 2/3$ if the radius of the large circle is taken as unity.

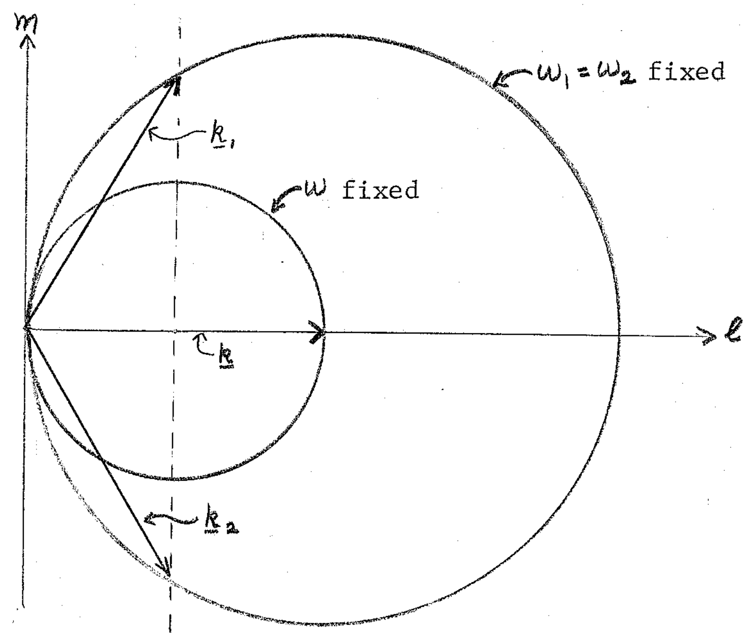

Figure 1

In the general case for fixed $\omega_1 \neq \omega_2 \neq \omega$ one can satisfy the two selection rules for a continuous but limited range of $\underline{k}_1$ and $\underline{k}_2$.

<u>Asymptotic Response</u>: In order to solve 12) for ${}_2\psi_{\underline{k}}$ and the corresponding equation for ${}_3\psi_{\underline{k}}$ and to evaluate the energy transfer we need to know the asymptotic response of the P operator to stationary and non-stationary forcing functions.

The solution of the equation

$$P\psi = \left[\frac{d}{dt} - i\omega\right]\psi = e^{i\omega' t} \qquad 17)$$

for the initial value $\psi = 0$ at $t = 0$ is $I_1(\omega, \omega'; t)$ where

$$I_1(\omega, \omega'; t) = \frac{i}{\omega - \omega'}\left[e^{i\omega' t} - e^{i\omega t}\right] \text{ for } \omega \neq \omega' \qquad 18a)$$

$$I_1(\omega, \omega'; t) = t\, e^{i\omega t} \text{ for } \omega = \omega' \qquad 18b)$$

By contour integration in the complex ω' plane we can compute the quantity

$$\frac{d}{dt}\int_{-\infty}^{\infty} I_1(\omega, \omega'; t) I_1^*(\omega, \omega'; t)\, g(\omega')\, d\omega' = 2\pi g(\omega) \qquad 19)$$

where $g(\omega)$ is an arbitrary continuous real function. In other words we have

$$\frac{d}{dt} I_1 I_1^* = 2\pi\,\delta(\omega - \omega') \qquad 20)$$

The solution $I_2(\omega, \omega_1, \omega_2, \omega - \omega_1; t)$ of the equation

$$P\psi = I_1(\omega_2, \omega - \omega_1; t)\, e^{i\omega_1 t} \qquad 21)$$

for $\psi = 0$ at $t = 0$ is

$$I_2 = \frac{i t e^{i\omega t}}{\omega_1 + \omega_2 - \omega} - \frac{1}{(\omega_1 + \omega_2 - \omega)^2}\left[e^{i(\omega_1 + \omega_2)t} - e^{i\omega t}\right] \qquad 22)$$

for $\omega_1 + \omega_2 \neq \omega$. Another quantity which we will use later is

$$\frac{d}{dt} Re\left[I_2 e^{-i\omega t}\right] = \pi\,\delta(\omega_1+\omega_2-\omega) \qquad 23)$$

which is computed in the same way as 20). Equations 17) through 23) are direct analogies to the equations for a forced, undamped harmonic oscillator in section 3 of Hasselmann (1961).

<u>Perturbation Amplitudes</u>: We can write down the second order perturbation amplitude ${}_2\psi$ from 9), 12), and 18) in the form

$$ {}_2\psi = \sum_{\underline{k}} e^{i(\underline{k}\cdot\underline{x}+\omega t)}\, {}_2\psi'_{\underline{k}} \qquad 24)$$

where

$$ {}_2\psi'_{\underline{k}} = \sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}} D_{\underline{k}_1,\underline{k}_2}\, {}_1\psi_{\underline{k}_1}\, {}_1\psi_{\underline{k}_2}\, I_1(\omega,\ \omega_1+\omega_2;t)\, e^{-i\omega t} \qquad 25)$$

Now by using 8), 9), and 24) we can write down the equation for the third order Fourier amplitude ${}_3\psi_{\underline{k}}$ analogous to 12).

$$\begin{aligned} P\,{}_3\psi_{\underline{k}} &= \sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}} D_{\underline{k}_1,\underline{k}_2}\left({}_1\psi_{\underline{k}_1}\,{}_2\psi'_{\underline{k}_2} + {}_1\psi_{\underline{k}_2}\,{}_2\psi'_{\underline{k}_1}\right) e^{i(\omega_1+\omega_2)t} \qquad 26)\\ &= 2\sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}} D_{\underline{k}_1,\underline{k}_2}\,{}_1\psi_{\underline{k}_1}\,{}_2\psi'_{\underline{k}_2}\, e^{i(\omega_1+\omega_2)t}\\ &= 2\sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}}\ \sum_{\substack{\underline{k}'+\underline{k}''\\=\underline{k}_2}} D_{\underline{k}_1,\underline{k}_2}\, D_{\underline{k}',\underline{k}''}\,{}_1\psi_{\underline{k}'}\,{}_1\psi_{\underline{k}''}\,{}_1\psi_{\underline{k}_1}\, I_1(\omega_2,\omega'+\omega'';t)\, e^{i\omega_1 t}\end{aligned}$$

The first step uses the symmetry of the interaction coefficients and the second step uses 25). The solution of 26) is written symbolically as

$$ {}_3\psi_{\underline{k}} = 2\sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}}\ \sum_{\substack{\underline{k}'+\underline{k}''\\=\underline{k}_2}} D_{\underline{k}_1,\underline{k}_2}\, D_{\underline{k}',\underline{k}''}\,{}_1\psi_{\underline{k}'}\,{}_1\psi_{\underline{k}''}\,{}_1\psi_{\underline{k}'}\, I_2(\omega,\omega_1,\omega_2,\omega'+\omega'';t) \qquad 27)$$

<u>Statistical Assumptions</u>: Before discussing the energy we will introduce two simplifying statistical assumptions. The first statistical assumption which will be used is that in the linear approximation the sea state is described by Fourier amplitudes $\psi_{\underline{k}}$ which are initially statistically independent for different wave numbers $\underline{k}$. Because of the Central Limit Theorem this amounts to assuming that the sea is initially homogeneous, and Gaussian. (The second assumption is that the sea is stationary to the first order.) What we mean by statistical independence is that the ensemble average of the product of two or more Fourier amplitudes $\psi_{\underline{k}_1} \psi_{\underline{k}_2} \psi_{\underline{k}_3} \psi_{\underline{k}_4}$, for example, can be written as the product of the individual ensemble averages if the $|\underline{k}_i|$ are all different (i = 1,2,3,4). However if $\underline{k}_2 = -\underline{k}_1$ and $\underline{k}_4 = -\underline{k}_3$, then we have $|\overline{\psi_{\underline{k}_1}}|^2 \; |\overline{\psi_{\underline{k}_3}}|^2$ for $|\underline{k}_3| \neq |\underline{k}_1|$.

The assumption of the statistical independence of the Fourier amplitudes is a critical point because we know that the non-linear resonant interactions will build up correlations. The correlation which is developed between three interacting wave groups will be lost in a sense because after interacting the three wave groups will propagate in different directions with different speeds due to the dispersive nature of the medium. Each of the three wave groups then interacts with other wave groups, and at each successive interaction it is hoped that the assumption of statistical independence will hold. This is analogous to the assumption made by Boltzmann in deriving his collision formula that each binary collision is independent of all the others. Hasselmann (1962) has pointed out the usefulness of the wave particle analogy in the case of weak non-linearities.

Mean Energy: In this model the total energy is kinetic energy. The kinetic energy density E is

$$E = \frac{1}{2}\rho\left[|U|^2 + |V|^2\right] = \frac{1}{2}\rho\left[|\Psi_x|^2 + |\Psi_y|^2\right] \tag{28}$$

If the perturbation series 5) is inserted in 28) we get

$$E = {}_2E + {}_3E + {}_4E + \dots \tag{29}$$

The odd orders give rise to odd statistical moments which vanish on account of the Gaussian assumption. The mean second order energy is

$${}_2\overline{E} = \frac{1}{2}\rho\sum_{\underline{k}} k^2\, \overline{|{}_1\Psi_{\underline{k}}|^2} \tag{30}$$

where the bar denotes ensemble average. The next non-vanishing term in 29) is ${}_4E$.

$${}_4E = \frac{1}{2}\rho\left[|{}_2U|^2 + |{}_2V|^2 + {}_1U\,{}_3U^* + {}_1U^*\,{}_3U + {}_1V\,{}_3V^* + {}_1V^*\,{}_3V\right] \tag{31}$$

$$= \frac{1}{2}\rho\left[|{}_2\Psi_x|^2 + |{}_2\Psi_y|^2 + 2\,\mathrm{Re}\left({}_1\Psi_x^*\,{}_3\Psi_x + {}_1\Psi_y^*\,{}_3\Psi_y\right)\right]$$

By using 24) and 25) we compute the first two terms of ${}_4E$ as

$$\frac{1}{2}\rho\sum_{\underline{k}}\ \sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}}\ \sum_{\substack{\underline{k}_3+\underline{k}_4\\=\underline{k}}} k^2 D_{\underline{k}_1,\underline{k}_2} D_{\underline{k}_3,\underline{k}_4}\ \overline{{}_1\Psi_{\underline{k}_1}\,{}_1\Psi_{\underline{k}_2}\,{}_1\Psi^*_{\underline{k}_3}\,{}_1\Psi^*_{\underline{k}_4}}\ \times \tag{32a}$$

$$\times\ I_1(\omega, \omega_1+\omega_2; t)\, I_1^*(\omega, \omega_3+\omega_4; t)$$

$$= \rho\sum_{\underline{k}}\ \sum_{\substack{\underline{k}_1+\underline{k}_2\\=\underline{k}}} k^2 D^2_{\underline{k}_1,\underline{k}_2}\ \overline{|{}_1\Psi_{\underline{k}_1}|^2}\ \overline{|{}_1\Psi_{\underline{k}_2}|^2}\ I_1 I_1^*\ ,\ I_1 = I_1(\omega, \omega_1+\omega_2; t)$$

due to the assumption of statistical independence of the ${}_1\psi_{\underline{k}}$. We compute the second two terms of ${}_4\bar{E}$ in a similar manner from 10) and 27)

$$4\rho \sum_{\underline{k}} \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} k^2 D_{\underline{k}_1,\underline{k}_2} D_{\underline{k},-\underline{k}_1} |{}_1\psi_{\underline{k}}|^2 |{}_1\psi_{\underline{k}_1}|^2 Re\,(I_2 e^{-i\omega t}) \qquad 32b)$$

where $I_2 = I_2(\omega, \omega_1, \omega_2, \omega-\omega_1 ; t)$. We now compute the time rate of change of the mean fourth order energy from 20), 23), 32a), 32b)

$$\frac{\partial_4 \bar{E}}{\partial t} = 2\pi\rho \sum_{\underline{k}} \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} k^2 D_{\underline{k}_1,\underline{k}_2} \Big\{ D_{\underline{k}_1,\underline{k}_2} \overline{|{}_1\psi_{\underline{k}_1}|^2}\, \overline{|{}_1\psi_{\underline{k}_2}|^2} + \qquad 33)$$

$$+ 2\, D_{\underline{k},-\underline{k}_1} \overline{|{}_1\psi_{\underline{k}}|^2}\, \overline{|{}_1\psi_{\underline{k}_1}|^2} \Big\} \delta(\omega_1 + \omega_2 - \omega)$$

Energy Spectrum: We introduce the instantaneous space spectrum of the sea as follows:

$$\bar{E} = \iint_{-\infty}^{\infty} F(\underline{k})\, d\underline{k} \qquad 34)$$

where $F(\underline{k}) = {}_2F(\underline{k}) + {}_4F(\underline{k}) + \ldots$ because of 29) and the Gaussian assumption. Equation 33) becomes in terms of the spectrum

$$\frac{\partial_4 F(\underline{k})}{\partial t} = \frac{8\pi}{\rho} \int_{-\infty}^{\infty}\!\!\cdots\!\!\int k^2 D_{\underline{k}_1,\underline{k}_2} \Big\{ \frac{D_{\underline{k}_1,\underline{k}_2}}{k_1^2 k_2^2}\, {}_2F(\underline{k}_1)\, {}_2F(\underline{k}_2) + \qquad 35)$$

$$+ 2\frac{D_{\underline{k},-\underline{k}_1}}{k^2 k_1^2}\, {}_2F(\underline{k})\, {}_2F(\underline{k}_1) \Big\} \delta(\underline{k}_1 + \underline{k}_2 - \underline{k})\, \delta(\omega_1 + \omega_2 - \omega)\, d\underline{k}_1\, d\underline{k}_2$$

To the order of approximation considered we can drop the perturbation indices on the spectra, and by using a slightly more condensed notation we rewrite 35) as

$$\frac{\partial F_3}{\partial t} = \frac{8\pi}{\rho} \int_{-\infty}^{\infty}\!\!\cdots\!\int \frac{D_3}{k_1^2 k_2^2} \Big\{ D_3 k_3^2 F_1 F_2 - D_2 k_2^2 F_1 F_3 - \qquad 36)$$

$$- D_1 k_1^2 F_2 F_3 \Big\} \delta(\omega_1 + \omega_2 - \omega_3)\, \delta(\underline{k}_1 + \underline{k}_2 - \underline{k}_3)\, d\underline{k}_1\, d\underline{k}_2$$

where $D_3 = D_{\underline{k}_1, \underline{k}_2}$, $D_2 = D_{\underline{k}_1, \underline{k}_3}$, $D_1 = D_{\underline{k}_2, \underline{k}_3}$, $F_3 = F(\underline{k}_3)$, etc. Equation 36) is valid for all time provided the assumption of the initial statistical independence of the first order Fourier amplitudes is not violated.

Energy Conservation: Energy is conserved at each perturbation order in this model because we have included no generating or dissipating mechanisms. We see from 30) and the assumption of first order stationarity that energy is conserved to the second order. The mean third order energy vanishes. Conservation of energy at the fourth order is expressed as

$$\iint_{-\infty}^{\infty} \frac{\partial F_3}{\partial t}\, d\underline{k}_3 = 0 \qquad 37)$$

For arbitrary $F(\underline{k})$ this imposes the following relationship among the three coefficients

$$D_1 k_1^2 + D_2 k_2^2 = D_3 k_3^2 \qquad 38)$$

subject to the two selection rules for quadratic resonance 14) and 15). Written in detail 38) is

$$\underline{k}_2 \times \underline{k}_3 (k_2^2 - k_3^2) + \underline{k}_1 \times \underline{k}_3 (k_1^2 - k_3^2) = \underline{k}_1 \times \underline{k}_2 (k_1^2 - k_2^2) \qquad 39)$$

By using 14) $(\underline{k}_1 + \underline{k}_2 = \underline{k}_3)$ we see that 39) is identically satisfied.

Momentum Conservation: If we use the form of conservation of momentum in Hasselmann (1962), then the fourth order momentum conservation

$$\iint_{-\infty}^{\infty} \frac{\partial F_3}{\partial t} \frac{\underline{k}_3}{\omega_3} d\underline{k}_3 = 0 \qquad 40)$$

imposes the following condition on the interaction coefficients

$$\frac{D_1 k_1^2 \underline{k}_1}{\omega_1} + \frac{D_2 k_2^2 \underline{k}_2}{\omega_2} = \frac{D_3 k_3^2 \underline{k}_3}{\omega_3} \qquad 41)$$

subject to the two selection rules 14) and 15).

Now 38) and 41) are three homogeneous equations for the three interaction coefficients, and for a non-trivial solution a certain determinant must vanish, where

$$det = \frac{(k_1 k_2 k_3)^2}{\omega_1 \omega_2 \omega_3} \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \underline{k}_1 & \underline{k}_2 & \underline{k}_3 \end{vmatrix} = 0 \qquad 42)$$

The determinant vanishes because of the selection rules. Thus the interaction coefficients are consistent with conservation of energy and momentum to the fourth order.

Square Vorticity Conservation: Another quantity which is conserved to the fourth order is mean square vorticity $\overline{|\nabla^2 \psi|^2}$. The expression for this analogous to 40) is

$$\iint_{-\infty}^{\infty} \frac{\partial F_3}{\partial t} k_3^2 \, d\underline{k}_3 = 0 \qquad 43)$$

which is identical with the east-west component of the momentum conservation equation 40).

Number Density: First we write the energy equation 36) in a simple form by noticing that if we define a new set of coefficients A_1, A_2, A_3 by $D_1 = \omega_1 k_2^2 k_3^2 A_1$, $D_2 = \omega_2 k_1^2 k_3^2 A_2$, $D_3 = \omega_3 k_1^2 k_2^2 A_3$, then we can show from the selection rules 14) and 15) that $A_1 = A_2 = A_3 \equiv a$.

$$\frac{\partial F_3}{\partial t} = b \int_{-\infty}^{\infty}\!\!\cdots\!\int (a k_1 k_2 k_3)^2 \omega_3 \{\omega_3 F_1 F_2 - \omega_2 F_1 F_3 - \omega_1 F_2 F_3\} \delta(\omega_1 + \omega_2 - \omega_3)\, \delta(\underline{k}_1 + \underline{k}_2 - \underline{k}_3)\, d\underline{k}_1\, d\underline{k}_2 \qquad 44)$$

where $b = 8\pi/\rho$. We now define the number density $n(\underline{k})$ (see Hasselmann 1962) as $n(\underline{k}) = F(\underline{k})/\alpha\omega$, where α is an arbitrary constant. In terms of the number density equation 44) is

$$\frac{\partial n_3}{\partial t} = b \int_{-\infty}^{\infty}\!\!\cdots\!\int \sigma \{n_1 n_2 - n_1 n_3 - n_2 n_3\} \delta(\omega_1 + \omega_2 - \omega_3)\, \delta(\underline{k}_1 + \underline{k}_2 - \underline{k}_3)\, d\underline{k}_1\, d\underline{k}_2 \qquad 45)$$

where $\sigma = (a k_1 k_2 k_3)^2 (\alpha \omega_1 \omega_2 \omega_3)$. The number density is not a conservative quantity as Hasselmann found it was for gravity waves.

Concluding Remarks: As pointed out by Hasselmann equation 45) has a formal similarity with the Boltzmann equation for the rate of change of the number density in phase space of a spatially homogeneous distribution of mass particles. The basic collision process here is the collision and annihilation of two particles to produce a third particle, and corresponding to this there are two inverse collisions.

Some interesting properties of the energy transfer process can be inferred from equation 44). (1) If all three values of the spectrum are equal for an interacting group then the net energy transfer for that group vanishes. In the special case of a white spectrum, $F(\underline{k})$ = constant, the total energy transfer vanishes.

(2) If the spectral values are equal at two points, then the spectral value at the third point changes in the direction of the other two; i.e. if $F_1 = F_2$ then $\frac{\partial F_3}{\partial t} \propto (F_1 - F_3)$.

(3) If $\underline{k}_1, \underline{k}_2, \underline{k}_3$ are colinear for an interacting group, then the interaction vanishes irrespective of the spectral values. This follows from the form of the interaction coefficient 13). Thus there is no non-linear energy transfer within a unidirectional spectrum.

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Finite Amplitude Instability in Quasi-steady Convection

Ruby Krishnamurti

INTRODUCTION

Suppose a collimated beam of light is passed horizontally through a layer of fluid in steady cellular convection. This originally uniform beam, after emerging from the fluid, displays uniformly spaced bright vertical lines. These lines are produced by columns of cold descending fluid acting as cylindrical lenses. By rotating the beam about a vertical axis, thus observing the fluid from different directions, the horizontal plan form is determined from the number of angular positions (two for rectangles, three for hexagons) at which bright lines are observed, and from the spacing of these lines.

Linear stability theory does not allow us to predict the plan form that will be realized in an experiment. In an attempt to remove this degeneracy some work (Palm 1960; Segel and Stuart 1961) has been directed towards the non-Boussinesq effect of variation of kinematic viscosity with temperature. Although it is stated in the literature as an experimental result that convective motion tends towards a regular hexagonal pattern, this is not always the case. Hexagonal cells, the only form whose sign of the vertical motion is a relevant quantity, are observed with asymmetric boundary conditions (rigid bottom, free top), but with symmetric boundary conditions squares are observed. However, even with symmetric boundaries, hexagons can be obtained by another asymmetry: suppose the layer is only heated from below or only cooled from above. In the first case, the mean temperature of the fluid is rising $\left(\frac{\partial \overline{T}}{\partial t} = \eta > 0\right)$ and the conduction temperature profile is curved as in figure 1(a). In the second case $\eta < 0$ and the profile is as in figure 1(b).

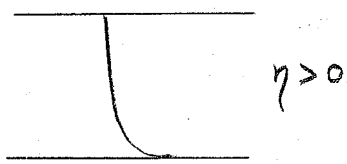

Fig. 1(a)

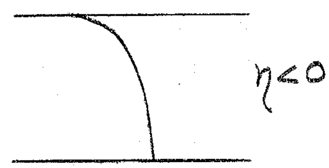

Fig. 1(b)

This type of asymmetry has been observed to give rise to hexagons even with symmetric boundary conditions. Furthermore, by reversing the asymmetry $(\eta < 0$ to $\eta > 0)$ the motion at the center of the hexagon has been observed (by the optical method described above) to change its sign. However, since squares are in fact observed under laboratory conditions, it would appear that there must be a finite amount of asymmetry before hexagons can be realized.

GOVERNING EQUATIONS AND THE METHOD OF SOLUTION

The governing equations in the Boussinesq approximation are written as follows:

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T = -\underset{\sim}{v} \cdot \nabla T \tag{1}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \underset{\sim}{v} = -\nabla \tilde{P}/\rho_0 - \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} + \gamma (T - T_0) \underset{\sim}{k} \tag{2}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{3}$$

where

κ = thermometric conductivity

$\gamma = g\alpha$, α = coefficient of thermal expansion

$\tilde{p} = p - gz$

The horizontal average (denoted by a bar) of equation (1) gives us the mean field equation:

$$\frac{\partial \overline{T}}{\partial t} - \kappa \frac{\partial^2}{\partial z^2} \overline{T} = -\frac{\partial}{\partial z}(\overline{WT}) \qquad (4)$$

where $T = T - \overline{T}$.

Equations (1) and (4) give the equation governing the fluctuating temperature:

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T = -\frac{\partial \overline{T}}{\partial z} W - h \qquad (5)$$

where $h = \underset{\sim}{v} \cdot \nabla T - \frac{\partial}{\partial z} \overline{WT}$

By cross differentiating and eliminating other variables the following equation is obtained for the vertical velocity (see Malkus and Veronis, 1958):

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 W + \gamma \frac{\partial \overline{T}}{\partial z} \nabla_1^2 W = \\ = -\gamma \nabla_1^2 h + \left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) L \qquad (6)$$

where L is the z-component of the curl curl of the velocity advection.

The equations are non-dimensionalized by setting

$$t = \frac{d^2}{\kappa} t'$$

$$\underset{\sim}{x} = d\, \underset{\sim}{x}'$$

$$\underset{\sim}{V} = \frac{\kappa}{d} V'$$

$$T = \frac{\kappa \nu}{\gamma d^3} T'$$

$$\overline{T} = \Delta T\, \overline{T}'$$

where ΔT is the difference in temperature between bottom and top boundary. The resulting equations (where primes are omitted) are

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W + R\frac{\partial \overline{T}}{\partial z}\nabla_1^2 W = -\nabla_1^2 h + \frac{1}{\sigma}\left(\frac{\partial}{\partial t} - \nabla^2\right) L \tag{7}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W = \nabla_1^2 T + \frac{1}{\sigma} L \tag{8}$$

$$R\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right)\overline{T} = -\frac{\partial}{\partial z}\overline{WT} \tag{9}$$

where $R = \frac{g\alpha}{\kappa\nu}\Delta T d^3$, the Rayleigh number

$\sigma = \frac{\nu}{\kappa}$, the Prandtl number

In the absence of convection, equation (9) is the heat conduction equation to be solved for the following boundary conditions. Suppose the temperature difference between top and bottom remains constantly at ΔT , but that both boundaries are uniformly changing temperature by an amount ηt . In dimensionless form these conditions are

$$\bar{T} = \eta t \frac{d^2}{\kappa \Delta T} \quad \text{at} \quad z = 0$$

$$\bar{T} = -1 + \eta t \frac{d^2}{\kappa \Delta T} \quad \text{at} \quad z = 1$$

Defining the dimensionless parameter

$$\eta' = \eta \frac{d^2}{\kappa \Delta T}$$

and making the transformation

$$\bar{\mathcal{J}} = \bar{T} - \eta' t$$

the equation governing $\bar{\mathcal{J}}$ is

$$\frac{\partial}{\partial t} \bar{\mathcal{J}} - \frac{\partial^2}{\partial z^2} \bar{\mathcal{J}} = -\eta'$$

Steady state solutions are parabolic functions of z , giving us

$$\bar{T} = \eta/_2 \left(z^2 - z\right) - z + \eta t \tag{10}$$

(where the prime has been omitted from η and will be understood to mean the dimensionless quantity). If η is a small quantity the parabolic part of $\bar{T}$ can be regarded as a small perturbation on the linear part. However, since we wish to apply the amplitude expansion method of Malkus and Veronis, we require exact solutions of the linear stability problem. Therefore we order the solutions not only in ϵ but also in η , as follows:

$$\begin{aligned}
W &= \sum_{n,m=0}^{\infty} \epsilon^{n+1} \eta^m W_{nm} \\
T &= \sum_{n,m=0}^{\infty} \epsilon^{n+1} \eta^m T_{nm} \\
\bar{T} &= \sum_{n,m=0}^{\infty} \epsilon^{n} \eta^m \bar{T}_{nm} \\
R &= \sum_{n,m=0}^{\infty} \epsilon^{n} \eta^m R_{nm}
\end{aligned} \tag{11}$$

Substituting these expansions into the governing non-linear equations produces the following infinite set of linear inhomogeneous equations:

To order ϵ^1, η^0,

$$\overline{T}_{00} = -z$$
$$\mathcal{L} W_{00} = 0 \qquad (12a)$$

where $\mathcal{L} = \left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 - R_{00}\nabla_1^2$

To order ϵ^1, η^1,

$$\mathcal{L} W_{01} = R_{01}\nabla_1^2 W_{00} - R_{00}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{00}$$
$$\overline{T}_{01} = \frac{1}{2}(z^2 - z) \qquad (12b)$$

To order ϵ^1, η^2,

$$\mathcal{L} W_{02} = R_{02}\nabla_1^2 W_{00} - R_{00}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{01} - R_{01}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{00}$$
$$\overline{T}_{02} = 0 \qquad (12c)$$
$$\overline{T}_{0m} = 0, \quad m > 1$$

To order ϵ^2, η^0

$$\mathcal{L} W_{10} = R_{10}\nabla_1^2 W_{00} - \nabla_1^2 h_{\binom{0}{0}\binom{0}{0}} + \frac{1}{\sigma}\left(\frac{\partial}{\partial t} - \nabla^2\right) L_{\binom{0}{0}\binom{0}{0}}$$
$$\overline{T}_{1m} = 0, \; all\, m \qquad (12d)$$

To order ϵ^2, η^1

$$\begin{aligned} L W_{11} = & - R_{00}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{10} \\ & - R_{01}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{10} \\ & - R_{01}\frac{\partial \overline{T}_{10}}{\partial z}\nabla_1^2 W_{00} \\ & - R_{11}\frac{\partial \overline{T}_{00}}{\partial z}\nabla_1^2 W_{00} \\ & - R_{10}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{00} \\ & - R_{10}\frac{\partial \overline{T}_{00}}{\partial z}\nabla_1^2 W_{01} \end{aligned} \qquad (12e)$$

To order ϵ^3, η^0,

$$\mathcal{L} W_{20} = -R_{10} \frac{\partial \overline{T}_{00}}{\partial z} \nabla_1^2 W_{10} - R_{20} \frac{\partial \overline{T}_{00}}{\partial z} \nabla_1^2 W_{00} - R_{00} \frac{\partial \overline{T}_{20}}{\partial z} \nabla_1^2 W_{00} -$$

$$- \nabla_1^2 \left\{ h_{\binom{0}{0}\binom{1}{0}} + h_{\binom{1}{0}\binom{0}{0}} \right\} + \frac{1}{\sigma}\left(\frac{\partial}{\partial t} - \nabla^2\right) \left\{ L_{\binom{0}{0}\binom{1}{0}} + L_{\binom{1}{0}\binom{0}{0}} \right\} \quad (12f)$$

$$R_{00} \left(\frac{\partial^2}{\partial z^2} \overline{T}_{20} \right) = \frac{\partial}{\partial z} \overline{W_{00} T_{00}}$$

To order ϵ^3, η^1,

$$\begin{aligned} \mathcal{L} W_{21} = & -R_{00} \frac{\partial \overline{T}_{01}}{\partial z} \nabla_1^2 W_{20} - R_{01} \frac{\partial \overline{T}_{00}}{\partial z} \nabla_1^2 W_{20} \\ & -R_{11} \frac{\partial \overline{T}_{00}}{\partial z} \nabla_1^2 W_{10} - R_{10} \frac{\partial \overline{T}_{01}}{\partial z} \nabla_1^2 W_{10} \\ & -R_{10} \frac{\partial \overline{T}_{00}}{\partial z} \nabla_1^2 W_{11} - R_{21} \frac{\partial \overline{T}_{00}}{\partial z} \nabla_1^2 W_{00} \\ & -R_{20} \frac{\partial \overline{T}_{01}}{\partial z} \nabla_1^2 W_{00} - R_{20} \frac{\partial \overline{T}_{00}}{\partial z} \nabla_1^2 W_{01} \\ & -R_{01} \frac{\partial \overline{T}_{20}}{\partial z} \nabla_1^2 W_{00} - R_{00} \frac{\partial \overline{T}_{21}}{\partial z} \nabla_1^2 W_{00} \\ & -R_{00} \frac{\partial \overline{T}_{20}}{\partial z} \nabla_1^2 W_{01} \\ & - \nabla_1^2 \left\{ h_{\binom{1}{1}\binom{0}{0}} + h_{\binom{1}{0}\binom{0}{1}} + h_{\binom{0}{1}\binom{1}{0}} + h_{\binom{0}{0}\binom{1}{1}} \right\} + \\ & + \frac{1}{\sigma}\left(\frac{\partial}{\partial t} - \nabla^2\right) \left\{ L_{\binom{1}{1}\binom{0}{0}} + L_{\binom{1}{0}\binom{0}{1}} + L_{\binom{0}{1}\binom{1}{0}} + L_{\binom{0}{0}\binom{1}{1}} \right\} \end{aligned}$$

$$R_{00} \frac{\partial^2 \overline{T}_{21}}{\partial z^2} + R_{20} \frac{\partial^2 \overline{T}_{01}}{\partial z^2} = \frac{\partial}{\partial z} \left\{ \overline{W_{01} T_{00}} + \overline{W_{00} T_{01}} \right\}$$

The R_{nm} are to be evaluated from the solubility condition

$$\langle \widetilde{W}_{00} \mathcal{L} W_{nm} \rangle = 0$$

and ϵ is to be interpreted as the amplitude of the W_{00} part of W :

$$\langle W_{00} W \rangle = \epsilon$$

To order ϵ^1, η^0, we have $R_{00} = 657$ for free boundaries and for hexagonal cells,

$$W_{00} = \frac{2}{\sqrt{3}}\left\{2\cos\frac{2\pi x}{\sqrt{3}L}\cos\frac{2\pi}{3L}y + \cos\frac{4\pi}{3L}y\right\}\sin\pi z$$

$$\equiv \frac{2}{\sqrt{3}}\phi_1 \sin\pi z$$

where $L = \frac{4}{3\alpha}$ is the length of one side of the hexagon.

$$T_{00} = \frac{2}{\sqrt{3}} N_0 \phi_1 \sin\pi z$$

$$U_{00} = -\frac{2}{\alpha}\sin\frac{2\pi x}{\sqrt{3}L}\cos\frac{2\pi y}{3L}\cos\pi z$$

$$V_{00} = -\frac{2}{\sqrt{3}\alpha}\left\{\cos\frac{2\pi x}{\sqrt{3}L}\sin\frac{2\pi y}{3L} + \sin\frac{4\pi y}{3L}\right\}\cos\pi z$$

$$N_0 = \pi^2\frac{(1+\alpha^2)^2}{\alpha^2}$$

To order ϵ^1, η^1,

$$R_{01}\langle W_{00}\nabla_1^2 W_{00}\rangle = R_{00}\left\langle W_{00}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{00}\right\rangle$$

$$= 0$$

$$W_{01} = \phi(x,y)\sum_{n=2}^{\infty} a_n \sin n\pi z$$

where $a_n = \frac{-2}{\sqrt{3}}\,\frac{\alpha^2 R_{00}}{\pi^4}\,\frac{I_n}{\left[(n^2+\alpha^2)^3 - R_{00}\frac{\alpha^2}{\pi^4}\right]}$

$$I_n = 2\int_0^1 dz\; z\sin\pi z\sin n\pi z \;-\frac{1}{2}\delta_{n1}$$

$$= 0 \text{ unless } n \text{ is even}$$

The other fields T_{01}, U_{01}, V_{01} are similarly expressed.

To order ϵ^1, η^2

$$R_{02}\langle W_{00}\nabla_1^2 W_{00}\rangle = R_{00}\langle W_{00}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{01}\rangle$$

For $\alpha^2 = \frac{1}{2}$,

$$R_{02} = 1.97$$

Thus we have, to order ϵ^1, η^1 that $R_{01} = 0$, expressing the fact that R is a minimum.

To order ϵ^1, η^2

$$R = R_{00} + 1.97\,\eta^2$$

For a mean temperature change of one degree per hour in fluid of depth $d = 1cm$, $\Delta T = 1°C$, $\kappa = 10^{-3}$ c.g.s. units, ν = 0.1 stokes, η^2 = 0.78 and the change in Rayleigh number is negligible.

To order ϵ^2, η^0,

$$R_{10} = 0$$

$$W_{10} = -\frac{9\sqrt{3}}{4\pi}(C_1\phi_1 + C_2\phi_2)\sin 2\pi z$$

$$C_1 = \frac{1 + 3/\sigma}{729/4 - \frac{R_{00}}{\pi^4}}, \qquad C_2 = \frac{1 + 11/3\sigma}{1331/4 - 3R_{00}/\pi^4}$$

as computed by Malkus and Veronis (1958).

To order ϵ^2, η^1,

$$R_{11}\langle W_{00}\nabla_1^2 W_{00}\rangle = R_{00}\langle W_{00}\frac{\partial \overline{T}_{01}}{\partial z}\nabla_1^2 W_{10}\rangle$$

Note that for rolls, $W_{10} = 0$. For rectangles, W_{10} has a horizontal plan form with horizontal wave number twice that in W_{00}. Thus for rolls and rectangles the horizontal integration gives $R_{11} = 0$. Only in case of hexagons does W_{10} have a horizontal form which overlaps with that of W_{00} . For hexagons, with $\alpha^2 = \frac{1}{2}$,

$$R_{11} = +1.4$$

and

$$W_{11} = \phi_1 \sum_{2}^{\infty} a_n' \sin n\pi z + $$
$$+ \phi_2 \sum_{2}^{\infty} a_n'' \sin n\pi z$$

where $a_n' = \frac{27}{8\pi} C_1 a_n$

$$a_n'' = -\frac{2}{\sqrt{3}} \alpha^2 \pi^2 R_{00} \left(\frac{81}{8\pi} C_2\right) \frac{I_n}{\pi^6 \left[(n^2 + 3\alpha^2)^3 - \frac{3\alpha^2}{\pi^4} R_{00}\right]}$$

Then T_{11} is determined from

$$\nabla_1^2 T_{11} = -\nabla^4 W_{11} - \frac{1}{\sigma} \left[L_{\binom{0}{1}\binom{0}{0}} + L_{\binom{0}{0}\binom{0}{1}} \right]$$

We have, so far, to order ϵ^2, η^2,

$$\epsilon = \frac{R - R_{00}}{\eta R_{11}}$$

For $R > R_{00}$, we see that the motion in the center of the hexagon is upward or downward according as $\eta > 0$ or $\eta < 0$, since $R_{11} > 0$. Note that we can have finite amplitude instability for $R < R_{00}$ and that in this case the hexagonal plan form is the only possible solution and is therefore the stable solution.

To order ϵ^3, η^0,

$$R_{20} = +30.6$$

for $\alpha^2 = \frac{1}{2}$ and for infinite Prandtl number.

To order ϵ^3, η^1 , we find for $\alpha^2 = \frac{1}{2}$ and for infinite Prandtl number,

$$\mathcal{R}_{21} = -4.05 \times 10^{-2}$$

and

$$\epsilon = \frac{-\eta \mathcal{R}_{11} \pm \sqrt{\eta^2 \mathcal{R}_{11}^2 - 4(\mathcal{R}_{20}+\eta \mathcal{R}_{21})(\mathcal{R}_{00}-\mathcal{R})}}{2(\mathcal{R}_{20}+\eta \mathcal{R}_{21})}$$

The heat flux curve will appear as in figure 2.

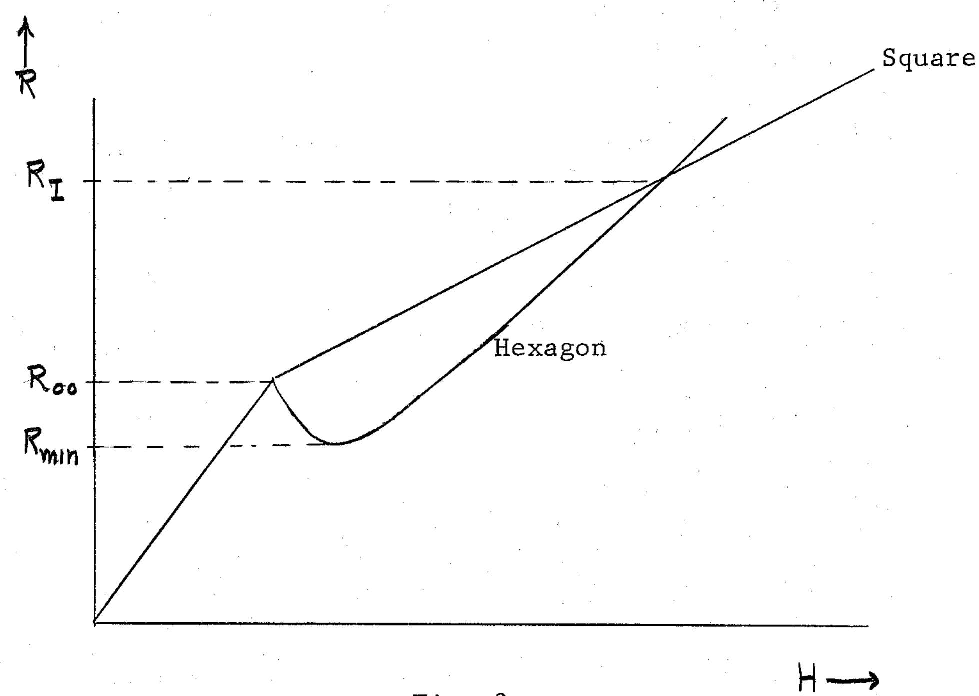

Fig. 2

Heat transport vs. Rayleigh number.

The minimum of the hexagon curve (figure 2) occurs at

$$\mathcal{R}_{00} - \mathcal{R}_{min} = \frac{\eta^2 \mathcal{R}_{11}^2}{4(\mathcal{R}_{20}+\eta \mathcal{R}_{21})}$$

$$\simeq 0.016\, \eta^2$$

The heat flux curve for hexagons intersects that for squares at

$$\mathcal{R}_I = \frac{\eta^2 \mathcal{R}_{11}^2 \mathcal{R}_{2s}}{(\mathcal{R}_{20} + \eta \mathcal{R}_{21} - \mathcal{R}_{2s})^2}$$

where $1/\mathcal{R}_{2s}$ is the slope of the straight line representing the heat flux for squares.

$$\mathcal{R}_{2s} \equiv \mathcal{R}_{20}^{SQUARES} + \eta \mathcal{R}_{21}^{SQUARES}$$

Had there been no asymmetry in heating, the heat flux curve for hexagons would be a straight line with slope (on $\mathcal{R}-H$ plot) greater than the slope for squares. With the asymmetric heating, there is a range of $\mathcal{R}$ for which hexagons transport more heat than do squares. The area of the loop between the two heat flux curves is a measure of the asymmetry required in order that hexagons be realized. This area is

$$A = \frac{1}{2} \frac{\eta^4 \mathcal{R}_{11}^4 \mathcal{R}_2}{(\mathcal{R}_{20} + \eta \mathcal{R}_{21} - \mathcal{R}_{2s})^4} - \frac{\mathcal{R}_{20} + \eta \mathcal{R}_{21}}{2} \left\{ \frac{\eta^2 \mathcal{R}_{11}^2}{(\mathcal{R}_{20} + \eta \mathcal{R}_{21} - \mathcal{R}_{2s})^2} - \frac{2\eta^2 \mathcal{R}_{11}^2}{(\mathcal{R}_{20} + \eta \mathcal{R}_{21})^2} \right\} +$$
$$+ \frac{2}{3} \eta \mathcal{R}_{11} \left\{ \left[\frac{\eta \mathcal{R}_{11}}{\mathcal{R}_{20} + \eta \mathcal{R}_{21} - \mathcal{R}_{2s}} \right]^{3/4} - 2 \left[\frac{\eta \mathcal{R}_{11}}{\mathcal{R}_{20} + \eta \mathcal{R}_{21}} \right]^{3/4} \right\}$$

Experimental data of quasi-steady heat flux shows scatter near the critical Rayleigh number. This is probably related to the hysteresis effect in which the heat flux is different depending upon whether the Rayleigh number is being increased or decreased.

SUMMARY

The calculations are summarized in table 1.

TABLE 1

	ϵ^1	ϵ^2	ϵ^3
η^0	$R_{00} = 657$	$R_{10} = 0$	$R_{20} = 30.6$ $\sigma = \infty$
η^1	$R_{01} = 0$	$R_{11} = 1.40$	$R_{21} = -4.05 \times 10^{-2}$ $\sigma = \infty$
η^2	$R_{02} = 1.97$		

For $R > R_{00}$, the motion in the center of the hexagon is upwards or downwards according as the mean temperature is increasing or decreasing.

For $R < R_{00}$, finite amplitude instability is possible in which case hexagons are the only possible solution and is therefore the stable solution. This is an example of finite amplitude instability which is easily accessible to observation.

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An Analytic Approach to Some Problems in Turbulence Theory

Steven A. Orszag

I. Introduction

Recently it has been found that many of the properties of thermal turbulence in a thin layer of fluid heated from below (Bénard problem) are retained when certain selected "fluctuating self-interaction" terms are omitted from the Boussinesq approximation to the Navier-Stokes equation. It was found by numerical calculation[1] of these "mean field equations" that few physically unrealistic features are introduced. From the results of these calculations, Herring observed that the vertical velocity was largely composed of a single Fourier component. Then making the approximation of considering only a single Fourier component, he was able to derive[2] analytic results which agreed with the numerical results to within one per cent, provided the horizontal wave number is close to that which maximizes the total heat transport.

The success of this analytic approximation to the mean field equations encourages one to apply the same technique to other problems. In particular, there is the problem of penetrative convection in a 4^{o}C water layer. Water, as is well known, is densest at 3.98^{o}C and hence if we have a parallel layer of water, whose bottom temperature is say, 0^{o}C and whose top temperature is greater than 4^{o}C, the layer from 0^{o}C to 3.98^{o}C is convectively unstable while the upper layer is stable. The problem, then, is to investigate the nature of this situation.

However, before one can apply the technique to other problems, particularly problems which lack the symmetry of the Bénard problem, it is necessary to understand exactly how the technique works. Unfortunately, Herring's original procedure makes it difficult to do this, because his results and method of approximation are intimately wound up in each other in the space of the Fourier coefficients of the functions. Therefore, an alternative approach to the analytic calculation is suggested which clearly presents the hypotheses, the approximations, and the techniques used in the problem. The results obtained differ from Herring's in some essential aspects, but, again, for horizontal wave numbers close to that which maximizes the total heat transport the results agree within a few per cent.

II. Mean Field Equations

We consider a plane parallel layer of fluid between two plates at temperatures T_0, T_1 (Fig. 1). We use the Boussinesq

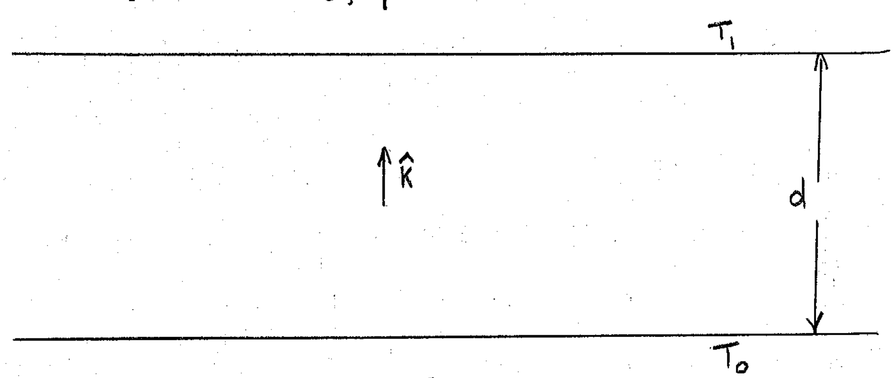

Fig. 1

set[3] of equations which approximates the Navier-Stokes set by assuming the density to obey the law

$$\rho = \rho_0 \left(1 - \alpha \left(T - T_0\right)\right) \tag{1}$$

in the buoyancy term of the momentum equation and to be constant in the other terms. Thus the other equations of the Boussinesq set are the momentum equation

$$\rho_0 \frac{\partial \underset{\sim}{v}}{\partial t} + \rho_0 (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{v} = -\rho_0 g \left(1 - \alpha (T - T_0)\right) \hat{k} - \nabla P + \rho_0 \nu \nabla^2 \underset{\sim}{v} \tag{2}$$

the heat equation

$$\frac{\partial T}{\partial t} + \left(\underset{\sim}{v} \cdot \nabla\right) T = \varkappa \nabla^2 T \tag{3}$$

and the continuity equation,

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{4}$$

where $\hat{k}$ is a unit vector in the positive z-direction and $\varkappa$, ν, g are thermometric conductivity, viscosity, and gravity, respectively.

Using (4) we may take horizontal average of equation (3) obtaining

$$\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial z} \langle w \theta \rangle = \varkappa \frac{\partial^2 \overline{T}}{\partial z^2} \tag{5}$$

where $\langle \ \rangle$ denotes horizontal average, $T = \overline{T} + \theta$ where $\langle \theta \rangle = 0$ and w is the vertical component of $\underset{\sim}{v}$. Equation (5) does presuppose some sort of boundedness or homogeneity condition at infinity. Subtracting (5) from (3) we obtain

$$\frac{\partial \theta}{\partial t} - \kappa \nabla^2 \theta = - w \frac{\partial \overline{T}}{\partial z} - h \tag{6}$$

where

$$h \equiv \underset{\sim}{v} \cdot \nabla \theta - \frac{\partial}{\partial z} \langle w\theta \rangle \tag{7}$$

By taking the z-component of the curl of the curl of (2), we obtain

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \nabla^2 w = g\alpha \nabla_1^2 \theta + L \tag{8}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{9}$$

is the horizontal Laplacian and

$$L \equiv \frac{\partial}{\partial z} \left[\frac{\partial}{\partial x} \left((\underset{\sim}{v} \cdot \nabla) u \right) + \frac{\partial}{\partial y} \left((\underset{\sim}{v} \cdot \nabla) v \right) \right] - \nabla_1^2 \left((\underset{\sim}{v} \cdot \nabla) w \right) \tag{10}$$

where

$$\underset{\sim}{v} = (u, v, w).$$

In the case of a statistically steady temperature field, $\frac{\partial \overline{T}}{\partial t} = 0$, and we may obtain a first integral of (5),

$$\kappa \beta + \langle w\theta \rangle = H \tag{11}$$

where H = constant and $\beta = - \frac{d\overline{T}}{dz}$.

We now introduce dimensionless (primed) parameters:

$$\underset{\sim}{r} = d\, \underset{\sim}{r}' \tag{12}$$

$$\underset{\sim}{v} = \frac{\kappa}{d} \underset{\sim}{v}' \tag{13}$$

$$\theta = (\Delta T) \theta' \tag{14}$$

$$\overline{T} = (\Delta T)\overline{T}' \tag{15}$$

$$t = \frac{d^2}{\kappa} t' \tag{16}$$

where d is the total depth of the fluid layer, and $\Delta T = T_0 - T_1$ is the temperature excess of the bottom plate over the top.

We further define a Rayleigh number

$$R = \frac{g\alpha}{\kappa\nu} \Delta T d^3 \tag{17}$$

a Prandtl number $$\sigma = \frac{\nu}{\kappa} \tag{18}$$

and in the case of a statistically steady flow a Nusselt number

$$N = \frac{H}{\kappa \frac{\Delta T}{d}} \tag{19}$$

In the following all our variables will be non-dimensional and we shall drop the primes. Our equations (5), (6), (8), (11) in dimensionless form become

$$\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial z} \langle W\theta \rangle = \frac{\partial^2 \overline{T}}{\partial z^2} \tag{20}$$

$$\frac{\partial \theta}{\partial t} - \nabla^2 \theta = -W \frac{\partial \overline{T}}{\partial z} - h \tag{21}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W = R\,\nabla_1^2 \theta + \frac{1}{\sigma} L \tag{22}$$

and for statistically steady flows

$$\beta + \langle W\theta \rangle = N \tag{23}$$

The terms h of (21) and $\frac{1}{\sigma} L$ of (22) have the form of a deviation of a bilinear fluctuating quantity from its horizontal mean (fluctuating self-interaction). By dropping these terms we obtain the system of "mean field equations" which consists of (20) and two additional equations:

$$\frac{\partial \theta}{\partial t} - \nabla^2 \theta = \beta W \tag{24}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W = R\,\nabla_1^2 \theta \tag{25}$$

where

$$\beta(z) \equiv -\frac{\partial \overline{T}}{\partial z}$$

As noted by Herring, the procedure of deleting fluctuating self-interactions gives equations which obey the same conservation laws associated with the complete set of convection equations (conservation of kinetic energy and entropy). Furthermore, the significance of discarding the fluctuating self-interactions may be seen as corresponding to a closure scheme for closing the hierarchy of moment equations obtained from (2), (3), and (4). The mean field equations then correspond to discarding the third order cumulants in the second order moment equation.

Next, we may note that the horizontal dependence of W, θ may be separated out of the mean field equations, there being coupling between different horizontal wave numbers only through the mean temperature field. In this case, the simplest situation is to have a single horizontal wave number. Making this assumption the equations must be interpreted for suitably chosen

α as giving the r.m.s. properties of the field. Herring's numerical calculations show that the single α calculation is not a limitation, in that it is sufficient to study a single α, in particular, the α which supports the convection.

We shall assume that the mean field is statistically steady, and hence according to a theorem of Spiegel[4] W and θ are independent of time. Instead of the assumptions of a steady state and neglect of h and $\frac{1}{\sigma} L$ in (21) in (22) we could equally well study the case of infinite Prandtl number σ, neglect h, and assume a steady state and again arrive at the mean field equations.

Finally we shall use free boundary conditions:

$$\frac{\partial^m}{\partial t^m} W(0,t) = \frac{\partial^m}{\partial z^m} W(1,t) = 0 \quad m = 0, 2, 4, \cdots \tag{26}$$

$$\bar{T}(0,t) = 1 \qquad \bar{T}(1,t) = 0 \tag{27}$$

$$\theta(0,t) = 0 \qquad \theta(1,t) = 0 \tag{28}$$

III Analytic Approximation

We remarked above that we shall consider a single horizontal wave number. Therefore the fields W, and θ depend on x, y through a function f_α where

$$\nabla_1^2 f_\alpha = -\alpha^2 \pi^2 f_\alpha \tag{29}$$

and

$$\langle f_\alpha^2 \rangle = 1 \tag{30}$$

Then set
$$W = W(z)\, f_\alpha(x,y) \tag{31}$$

$$\Theta = \Theta(z)\, f_\alpha(x,y) \tag{32}$$

Then equations (23), (24), (25) take the form

$$N = \beta(z) + W(z)\,\Theta(z) \tag{33}$$

$$\left(\frac{d^2}{dz^2} - \alpha^2\pi^2\right)\Theta(z) = -\beta(z)\,W(z) \tag{34}$$

$$\left(\frac{d^2}{dz^2} - \alpha^2\pi^2\right)^2 W(z) = R\alpha^2\pi^2\,\Theta(z) \tag{35}$$

We now make the additional assumption that to zeroth order (Fig. 2)

$$W(z) = \frac{A}{\pi} \sin \pi z \tag{36}$$

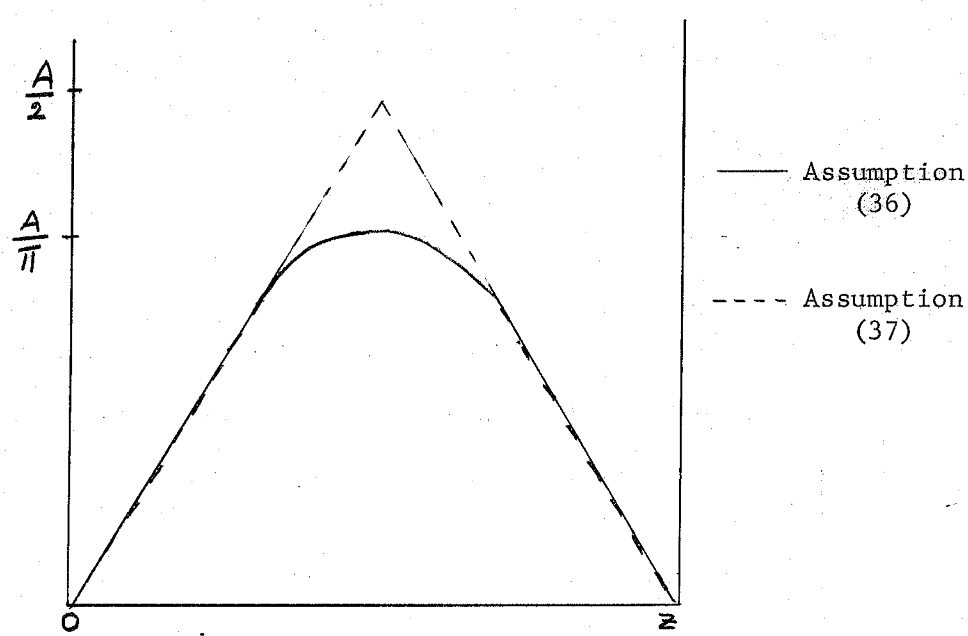

Fig. 2

It can be shown that this assumption is equivalent in the limit $R \to \infty$, $\alpha = O(1)$ to the assumption that (Fig. 2)

$$W(z) = \begin{cases} Az & 0 \le z \le \frac{1}{2} \\ -A(z-1) & \frac{1}{2} < z \le 1 \end{cases} \tag{37}$$

Equivalence here means that, using the method described below, both fields for fixed R, α give the same amplitude A and Nusselt number N. This equivalence may be proved using the remark below that the amplitude A is determined in boundary layers at $z = 0, 1$ and that Mathieu functions and parabolic cylinder functions are asymptotically identical as $z \to 0$ in their definition (see below).

Assumption (37) introduces considerable simplicity into the approach, in that it allows solutions to the equations to be obtained in terms of Fourier transforms of known functions rather than as integrals over Green's functions.

Using (33) in (34) and assumption (37) for $W(z)$ (there is symmetry about $z = \frac{1}{2}$) we obtain:

$$\frac{d^2}{dz^2}\theta = \left(\alpha^2\pi^2 + A^2 z^2\right)\theta - NAz \tag{38}$$

If, in (38), we make the change of variable $z = \sqrt{\frac{1}{2A}}\, y$ we obtain the differential equation

$$\frac{d^2\theta}{dy^2} = \left(\frac{\alpha^2\pi^2}{2A} + \frac{1}{4}y^2\right)\theta - \frac{N}{2\sqrt{2A}}\, y \tag{39}$$

with the boundary condition $\theta(0) = 0$.

The homogeneous part of equation (39)

$$\frac{d^2\psi}{dy^2} = \left(\frac{1}{4}y^2 + \epsilon\right)\psi, \quad \epsilon = \frac{\alpha^2\pi^2}{2A} > 0 \tag{40}$$

defines the parabolic cylinder function $D_{-\frac{1}{2}-\epsilon}(y)$ which, for ϵ a negative half-integer is the Harmite function which arises in the quantum mechanics of harmonic oscillators. (If we had made assumption (36) instead of (37) we would arrive at functions of the elliptic cylinder which have as eigenfunctions the Mathieu functions).

The function $D_{-\frac{1}{2}-\epsilon}(y)$ is that non-trivial solution of (40) which vanishes exponentially at $y = +\infty$. Obviously for $\epsilon > 0$, $D_{-\frac{1}{2}-\epsilon}(y) > 0$ for $y \geq 0$. If the other linearly independent solution of (40) is assumed to be of the form $\varphi_\epsilon = \chi D_{-\frac{1}{2}-\epsilon}$ it is easily seen that

$$\varphi_\epsilon(y) = D_{-\frac{1}{2}-\epsilon}(y)\int_0^y \frac{1}{\left(D_{-\frac{1}{2}-\epsilon}(\xi)\right)^2}\, d\xi \tag{41}$$

$\varphi_\epsilon(y)$ has the properties: $\varphi_\epsilon(0) = 0$, $\varphi_\epsilon'(0) = \frac{1}{D_{-\frac{1}{2}-\epsilon}(0)}$ and φ_ϵ increases exponentially to ∞ at ∞. It is well known that an integral representation for $D_{-\frac{1}{2}-\epsilon}$ exists[5]:

$$D_{-\frac{1}{2}-\epsilon}(y) = \frac{1}{\Gamma(\frac{1}{2}+\epsilon)}\, e^{-y^2/4}\int_0^\infty e^{-yt-\frac{1}{2}t^2} t^{\epsilon-\frac{1}{2}}\, dt \tag{42}$$

From this, it easily follows that

$$\frac{D'_{-\frac{1}{2}-\epsilon}(0)}{D_{-\frac{1}{2}-\epsilon}(0)} = -\sqrt{2}\,\frac{\Gamma(\frac{3}{4}+\frac{\epsilon}{2})}{\Gamma(\frac{1}{4}+\frac{\epsilon}{2})} \tag{43}$$

This integral representation has the immediate consequence that $D_{-\frac{1}{2}-\epsilon}(y)$ is a continuous function of ϵ for $\epsilon > -\frac{1}{2}$ and that it tends to zero uniformly as $y \to \infty$ with $-\frac{1}{2}+\delta \le \epsilon \le M$ where $\delta > 0$, $M < \infty$.

We now use the properties of the parabolic cylinder functions to study equation (39). If we take the Fourier transform of $D_{-\frac{1}{2}-\epsilon}$ we obtain the functions

$$\Phi_\epsilon(y) = \int_0^\infty D_{-\frac{1}{2}-\epsilon}(\xi) \sin \tfrac{1}{2}\xi y \, d\xi \qquad (44)$$

$$\Psi_\epsilon(y) = \int_0^\infty D_{-\frac{1}{2}-\epsilon}(\xi) \cos \tfrac{1}{2}\xi y \, d\xi \qquad (45)$$

These functions satisfy the equations

$$\Phi_\epsilon''(y) = \left(\tfrac{1}{4}y^2+\epsilon\right)\Phi_\epsilon - \tfrac{1}{2} D_{-\frac{1}{2}-\epsilon}(0) y \qquad (46)$$

$$\Psi_\epsilon''(y) = \left(\tfrac{1}{4}y^2+\epsilon\right)\Psi_\epsilon + D'_{-\frac{1}{2}-\epsilon}(0) \qquad (47)$$

Since $\Phi_\epsilon(0) = 0$,

$$\Theta(z) = \frac{N}{\sqrt{2A}\, D_{-\frac{1}{2}-\epsilon}(0)} \int_0^\infty D_{-\frac{1}{2}-\epsilon}(\xi) \sin\left(\sqrt{\tfrac{A}{2}}\,\xi z\right) d\xi + C\varphi_\epsilon\left(z\sqrt{2A}\right) \qquad (48)$$

is the general solution of (39) which satisfies the condition $\Theta(0) = 0$. It can easily be seen that $C \to 0$ as $A \to \infty$ exponentially, and, in this case, we make the further assumption that the term $C\varphi_\epsilon$ can be completely neglected. This is true because when (33) is integrated from $z = 0$ to $z = 1$, as we shall see below, the term

$$2\int_0^{\frac{1}{2}} A\,C\,z\,\varphi_\epsilon(z\sqrt{2A})\,dz$$

must be $O(1)$ or smaller. This implies since φ_ϵ increases exponentially as $z\sqrt{2A} \to \infty$ that C must vanish exponentially, at least, as $A \to \infty$. This suggests taking $C \equiv 0$. This assumption is actually too strong. The assumption needed is that

$$2\int_0^{\frac{1}{2}} A\,C\,z\,\varphi_\epsilon\left(z\sqrt{2A}\right)dz = O(1) \text{ as } A \to \infty$$

In fact, the assumption that A is determined in the boundary layers relieves us from making any assumption at all about C.

We now consider how well (33) is fulfilled by (48). We shall see that the requirement that (33) is satisfied to $O(1)$ determines A as a function of N, α. Integrating (33) from $z=0$ to $z=1$ and using the boundary condition on $\overline{T}$:

$$N = 1 + 2\int_0^{\frac{1}{2}} W(z)\,\theta(z)\,dz \tag{49}$$

Using (48) and integrating by parts we see that

$$\theta(z) = \frac{N}{Az} + \frac{N}{AD_{-\frac{1}{2}-\epsilon}(0)\,z}\int_0^{\infty} D'_{-\frac{1}{2}-\epsilon}(\xi)\cos\sqrt{\frac{A}{2}}\,z\,\xi\,d\xi \tag{50}$$

Using this expression in (49) we obtain

$$N = 1 + 2\int_0^{\frac{1}{2}}\left\{N + \frac{N}{D_{-\frac{1}{2}-\epsilon}(0)}\int_0^{\infty} D'_{-\frac{1}{2}-\epsilon}(\xi)\cos\sqrt{\frac{A}{2}}\,z\,\xi\,d\xi\right\}dz \tag{51}$$

Interchanging the order of integration in (51) we obtain

$$N = 1 + N + \frac{2\sqrt{2}N}{\sqrt{A}\,D_{-\frac{1}{2}-\epsilon}(0)} \int_0^\infty D'_{-\frac{1}{2}-\epsilon}(\xi)\,\frac{\sin\left(\frac{1}{2}\sqrt{\frac{A}{2}}\,\xi\right)}{\xi}\,d\xi \tag{52}$$

Now as $A \to \infty$ the integral on the right-hand side of (52) may be evaluated by the Dirichlet integral formula to give

$$\frac{\pi}{2} D'_{-\frac{1}{2}}(0) + O\left(\frac{1}{\sqrt{A}}\right) \tag{52a}$$

since $\epsilon \to 0$ as $A \to \infty$.

We have, in applying the Dirichlet integral formula, used the property of continuity of $D_{-\frac{1}{2}-\epsilon}(y)$ as a function of ϵ mentioned above. We may also note that the value of the integral on the right is determined in the boundary layer as $A \to \infty$. (We shall presently identify the limit $A \to \infty$ as $N \to \infty$). This is so because all that is needed in the Dirichlet integral on the right side of (52) to assure (52a) is the argument $\sqrt{\frac{A}{2}}\,\delta\,\xi$ in the sine function where $\delta > 0$.

Therefore (52) becomes

$$N = 1 + N + \frac{\pi\sqrt{2}\,N D'_{-\frac{1}{2}}(0)}{\sqrt{A}\,D_{-\frac{1}{2}}(0)} + O\left(\frac{N}{A}\right) \tag{53}$$

Therefore to $O(1)$ if we assume $A \to \infty$ as $N \to \infty$ we obtain

$$-1 = \frac{\pi\sqrt{2}\,N}{\sqrt{A}} \; \frac{D'_{-\frac{1}{2}}(0)}{D_{-\frac{1}{2}}(0)} \tag{54}$$

and thus

$$A = 2\pi^2 N^2 \left(\frac{D'_{-\frac{1}{2}}(0)}{D_{-\frac{1}{2}}(0)}\right)^2 + O(N) \tag{55}$$

Using (43) this becomes

$$A = 4.53\, N^2 + O(N) \tag{56}$$

Thus we see that we can obtain an amplitude A if we satisfy the heat equation (33) to $O(1)$ as $N \to \infty$. The degree to which this relation seems to be satisfied by the numerical calculations will be discussed below.

We now use equation (48) in the momentum equation (35). If we rewrite (35) using the identity

$$\left(\frac{d}{dz} + A\right)\varphi \equiv e^{-Az}\frac{d}{dz}\left(e^{Az}\varphi\right) \tag{57}$$

in the form

$$e^{-\alpha\pi z}\frac{d^2}{dz^2}e^{2\alpha\pi z}\frac{d^2}{dz^2}e^{-\alpha\pi z}w(z) = R\alpha^2\pi^2\theta(z) \tag{58}$$

we may solve for $W(z)$ using (48).

Doing this we obtain a particular solution to (58) which also satisfies the boundary conditions (26):

$$w(y) = \frac{RN\alpha^2\pi^2}{AD_{\frac{1}{2}-\epsilon}(0)}\int_0^\infty \frac{D_{-\frac{1}{2}-\epsilon}\left(\sqrt{\frac{2}{A}}\,x\right)\sin xy}{(x^2+\alpha^2\pi^2)^2}\,dx \tag{59}$$

We now have several choices as to what course to follow. We can assume, for example, that the first derivative of $w(y)$ given by (59) must be given by A and thus get a relationship between A, N, R, α which with (56) determines N as a function of R, α. A word of caution is appropriate at this point. The following paradox seems to arise. Setting

$$W(z) = \frac{A}{\pi} \sin \pi z \quad \text{in}$$

equation (35) we obtain

$$\frac{A}{\pi}(\alpha^2\pi^2 + \pi^2)^2 \sin \pi z = R\alpha^2\pi^2\theta(z)$$

and going through with our plan to evaluate derivatives at 0 we obtain

$$A(\alpha^2+1)^2 = \frac{R\alpha^2}{\pi^2} \left.\frac{d\theta}{dz}\right|_{z=0} \qquad (59a)$$

But as is seen from (48)

$$\left.\frac{d\theta}{dz}\right|_{z=0} \propto \sqrt{A}$$

so that (59a) implies

$$R \propto \sqrt{A} \propto N$$

This is clearly in contradiction with $N \propto R^{\frac{1}{3}}$. The difficulty arises in an incorrect passage to the limit $z \to 0$ in the derivation of (59a) from (35).

The solution to the differential equation is given by an integral over a Green's function in which the inhomogeneous term enters from $0, \infty$ not just from 0 to δ .

We see from (59) and the remark concerning the continuity of $D_{-\frac{1}{2}-\epsilon}(x)$ that as $A \to \infty$

$$W'(0) = \frac{RN\alpha^2\pi^2}{A}\int_0^\infty \frac{x\,dx}{(\alpha^2\pi^2+x^2)^2} + O\left(\frac{RN}{A^{3/2}}\right)$$

so

$$W'(0) = \frac{RN}{2A} + O\left(\frac{RN}{A^{3/2}}\right) \qquad (60)$$

Setting $W'(0) = A$ we obtain

$$A^2 = \frac{RN}{2} + O(R) \qquad (61)$$

and therefore using (56)

$$20.5 N^3 = \frac{R}{2} + O(R^{2/3})$$

or

$$N = 0.29\, R^{1/3} + O(1) \tag{62}$$

Another assumption that can be made to get from (59) to a relationship between R, N, α is that the first sine Fourier component, A_1, of W given by (59) should equal $\frac{A}{\pi}$ as expressed in (36). This gives

$$A_1 = 4 \int_0^{\frac{1}{2}} w(y) \sin \pi y \, dy \tag{63}$$

Thus

$$A_1 = \frac{4RN\alpha^2\pi^2}{AD_{-\frac{1}{2}-\epsilon}(0)} \int_0^{\frac{1}{2}} \sin \pi y \int_0^{\infty} \frac{D_{-\frac{1}{2}-\epsilon}\left(\sqrt{\frac{2}{A}}\, x\right) \sin xy}{(x^2+\alpha^2\pi^2)^2} \, dx \tag{64}$$

Interchanging order of integration we find

$$A_1 = -\frac{4RN\alpha^2\pi^2}{AD_{-\frac{1}{2}-\epsilon}(0)} \int_0^{\infty} D_{-\frac{1}{2}-\epsilon}\left(\sqrt{\frac{2}{A}}\, x\right) \frac{x \cos\frac{x}{2}\, dx}{(x^2-\pi^2)(\alpha^2\pi^2+x^2)^2} \tag{65}$$

and, as before, this becomes

$$A_1 = -\frac{4RN\alpha^2\pi^2}{A} \left\{ \int_0^{\infty} \frac{x \cos\frac{x}{2}\, dx}{(x^2-\pi^2)(\alpha^2\pi^2+x^2)^2} + O\left(\frac{1}{\sqrt{A}}\right) \right\} \tag{66}$$

Setting $\frac{x}{2} = y$ we obtain

$$A_1 = -\frac{RN\alpha^2\pi^2}{4A} \left\{ F(\alpha) + O\left(\frac{1}{\sqrt{A}}\right) \right\} \tag{67}$$

where

$$F(\alpha) = \int_0^{\infty} \frac{y \cos y \, dy}{\left(y^2 - \frac{\pi^2}{4}\right)\left(y^2 + \frac{\alpha^2\pi^2}{4}\right)^2} \tag{68}$$

We may rewrite $F(\alpha)$ in the form

$$F(\alpha) = G(\alpha) + H(\alpha) \tag{69}$$

where

$$G(\alpha) = \frac{\pi}{2}\int_0^\infty \frac{\cos y\, dy}{\left(y - \frac{\pi^2}{4}\right)\left(y^2 + \alpha^2 \frac{\pi^2}{4}\right)^2} \tag{70}$$

$$H(\alpha) = \int_0^\infty \frac{\cos y\, dy}{\left(y + \frac{\pi}{2}\right)\left(y^2 + \alpha^2 \frac{\pi^2}{4}\right)^2} \tag{71}$$

$G(\alpha)$ may be evaluated by residues to give

$$G(\alpha) = \frac{-16}{\pi^3 (1+\alpha^2)^2} \tag{72}$$

$H(\alpha)$ may be estimated for $\alpha \simeq 1$ as follows:

$H(\alpha)$ is a convergent alternating series:

$$H(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\cos y\, dy}{\left(y + \frac{\pi}{2}\right)\left(y^2 + \alpha^2 \frac{\pi^2}{4}\right)^2} + \sum_{n=0}^{\infty} \int_{n\pi + \frac{\pi}{2}}^{(n+1)\pi + \frac{\pi}{2}} \frac{\cos y\, dy}{\left(y + \frac{\pi}{2}\right)\left(y^2 + \alpha^2 \frac{\pi^2}{4}\right)^2} \tag{73}$$

and so

$$H(\alpha) \leq \int_0^{\frac{\pi}{2}} \frac{\cos y\, dy}{\left(y + \frac{\pi}{2}\right)\left(y^2 + \alpha^2 \frac{\pi^2}{4}\right)^2} \leq \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dy}{\left(y^2 + \alpha^2 \frac{\pi^2}{4}\right)^2}$$

$$= \frac{2^4}{\alpha^3 \pi^4}\left\{\frac{1}{2}\arctan\frac{1}{\alpha} + \frac{\alpha}{\alpha(1+\alpha^2)}\right.$$

$$= \frac{4}{\pi^4}\left(\frac{\pi}{2} + 1\right) \qquad \alpha \simeq 1 \tag{74}$$

In general, we may expect $H(\alpha) \simeq 0$. In (78) below, we will assume $H(\alpha) = 0$.

Therefore for $\alpha \simeq 1$ from (72) and (74) we have

$$F(\alpha) \leq -\frac{2}{\pi^4}(\pi - 2) \tag{75}$$

From (67) setting $A_1 = \frac{A}{\pi}$ we obtain

$$\frac{A^2}{N} = -\frac{R\alpha^2\pi^3}{4}F(\alpha) + O\left(\frac{R}{\sqrt{A}}\right) \tag{76}$$

so that

$$N = .72\,\alpha^{2/3}\,F(\alpha)^{\frac{1}{3}}\,R^{\frac{1}{3}} + O(1) \tag{77}$$

Using (56) and (72) taking $H(\alpha) = 0$ so $F(\alpha) = G(\alpha)$ we obtain

$$20.5N^3 \cong \frac{4R\alpha^2}{(1+\alpha^2)^2}$$

Therefore

$$N \simeq .58\ R^{1/3}\frac{\alpha^{2/3}}{(1+\alpha^2)^{2/3}} \tag{78}$$

For $\alpha = 1$ this becomes $N \simeq .36\,R^{\frac{1}{3}}$

Finally, it follows from (50) and the Riemann-Lebesgue lemma that since $D'_{-\frac{1}{2}-\epsilon}(\xi)$ is integrable

$$\Theta(z) = \frac{N}{Az}\left(1 + O\left(\frac{1}{\sqrt{A}}\right)\right) \quad A \to \infty \tag{79}$$

Furthermore, from (59) it follows that in the limit $A \to \infty$

$$w(y) = \frac{RN\alpha^2\pi^2}{A}\int_0^\infty \frac{\sin xy}{(\alpha^2\pi^2 + x^2)^2}\,dx + O\left(\frac{RN}{A^{3/2}}\right) \tag{79a}$$

This integral may be evaluated by Fourier techniques. It further follows that

$$\beta(z) = -\frac{N}{D_{-\frac{1}{2}-\epsilon}(0)}\int_0^\infty D'_{-\frac{1}{2}-\epsilon}(\xi)\cos\left(\sqrt{\frac{A}{2}}\,z\,\xi\right)d\,\xi \tag{79b}$$

Typical $w, \theta, \overline{T}$ fields are shown in Figs. 2, 3, 4.

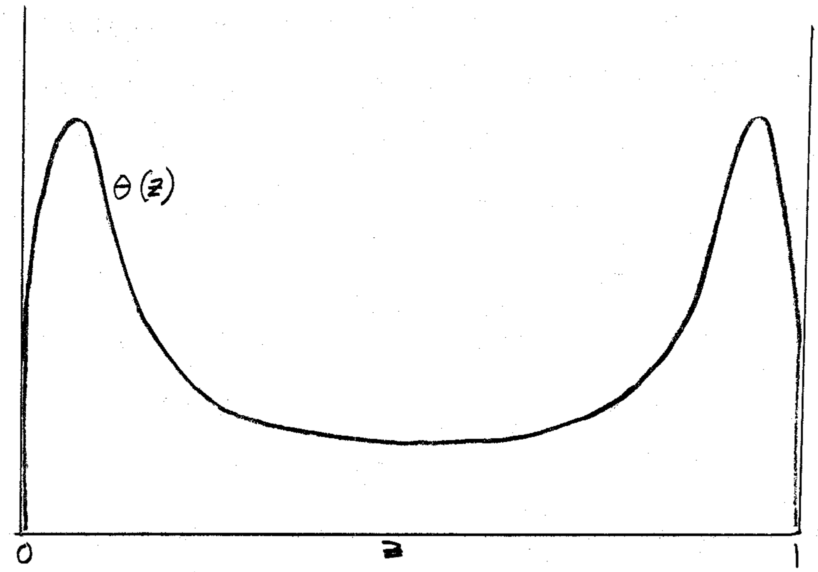

Fig. 3

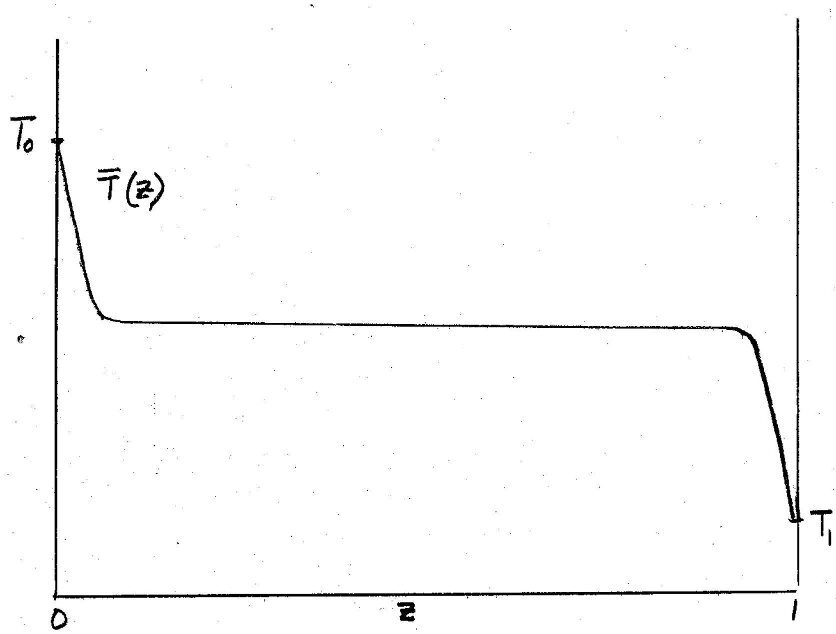

Fig. 4

IV. Discussion of the Method

We shall first restate the approach. Assume an explicit form for the vertical velocity field (single horizontal α)W. Use this in the mean field equations to get an ordinary differential equation for $\theta(z)$, which can be solved. Using $\theta(z)$ and the assumed $W(z)$ in the integrated conservation of heat flux equation, determine the amplitude of $W(z)$ so as to satisfy this equation to $O(1)$ as the Nusselt number N tends to ∞. Then use $\theta(z)$ in the momentum equation of the mean field set to recalculate $W(z)$. Determine the relation between the Rayleigh number R and N by assuming either that the derivative of the iterated W field at the origin is the same as that initially assumed or that the first Fourier component of iterated W field is the same as that initially (if the initial W field is a sine function).

We shall now compare (56) with the numerical and analytic results obtained by Herring. For $R = 10^5, 10^6$ $\alpha = 1.5$ (56) compares very favorably with Herring's numerical results. (These are the only published results of Herring which are applicable.) Herring's analytic result, suitably normalized is

$$A = 4.53\, X(\alpha) N^2 \tag{80}$$

where $X(\alpha)$ is a function some typical values of which are:

$$
\begin{aligned}
X(1) &= 0.9 \\
X(1.5) &= 0.82 \\
X(2.0) &= 0.73 \\
X(3.0) &= 0.55 \\
X(4.0) &= 0.45 \\
X(5.0) &= 0.37
\end{aligned}
\tag{81}
$$

Thus we see that (80) and (56) are within 20% of each other for $\alpha \simeq 1$ but that for larger values of α the two expressions diverge. Since the amplitude A is determined in the boundary layers at $z=0,1$ $(N \to \infty)$ by the condition of maximum dissipation, there does not seem to be any reason why there should be such a strong α dependence of A as that given by (80).

However, there does not seem to be any a priori reason for excluding an α dependence of the R, N relation. In any case the R, N relation found numerically by Herring for maximum heat flux is

$$N = 0.31 R^{1/3} \tag{82}$$

and analytically

$$N = 0.27 R^{1/3} \quad \alpha \simeq 1 \tag{83}$$

with a definite α dependence on both these quantities.

In this expression, α is the scale of the horizontal field that supports the convection. These expressions are to be compared to (78) and (62). The comparison is seen to be reasonable.

V. Penetrative Convection

In the case of the penetrative convection of a layer of water with 4°C region, equation (1) is replaced by

$$\rho = \rho_0 \left(1 - \alpha (T - T_0)^2\right) \tag{84}$$

where $T_0 = 4°C$, and equation (2) by

$$\rho_0 \frac{\partial \underset{\sim}{v}}{\partial t} + \rho_0 (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{v} = -\rho_0 g \left(1 - \alpha (T - T_0)^2\right) \hat{k} - \nabla p + \rho_0 \nu \nabla^2 \underset{\sim}{v} \tag{85}$$

Proceeding as in Section II we obtain the equations

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial z} \langle w\theta \rangle = \kappa \frac{\partial^2 \bar{T}}{\partial z^2} \tag{5}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 w = 2 g \alpha (\bar{T} - T_0) \nabla_1^2 \theta + g \alpha \nabla_1^2 \theta^2 + L \tag{86}$$

$$\frac{\partial \theta}{\partial t} - \kappa \nabla^2 \theta = -w \frac{\partial \bar{T}}{\partial z} - h \tag{6}$$

where h is given by (7), and L is given by (10).

We now introduce dimensionless (primed) parameters. Let ΔT be the temperature difference (positive) between 4°C

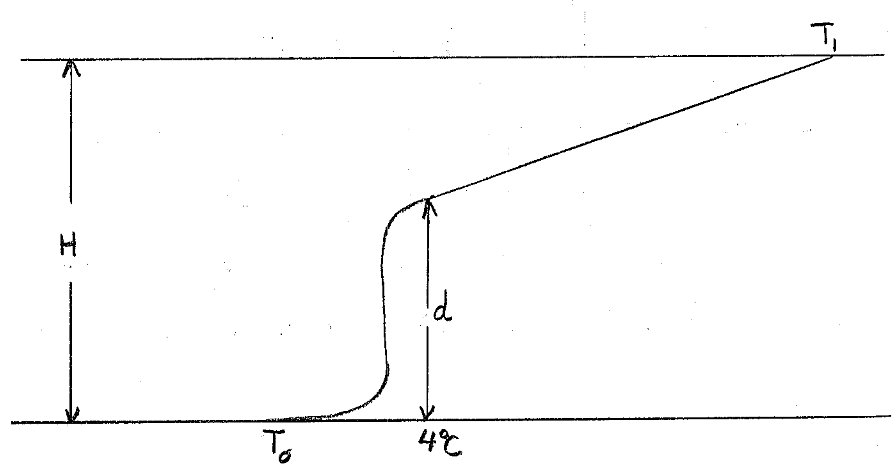

Fig. 5 Penetrative Convection

and the bottom plate. Let d = height of the 4^{o}C layer (Fig. 5). Set $T_0 = 0$.

Let:

$$\underset{\sim}{r} = d\underset{\sim}{r}' \tag{87}$$

$$\underset{\sim}{v} = \frac{\kappa}{d}\underset{\sim}{v}' \tag{88}$$

$$\theta = (\Delta T)\theta' \tag{89}$$

$$\bar{\bar{T}} = (\Delta T)\bar{\bar{T}}' \tag{89a}$$

$$t = \frac{d^2}{\kappa}t' \tag{90}$$

$$R = \frac{g\alpha(\Delta T)^2}{\kappa\nu}d^3 \tag{91}$$

$$\sigma = \frac{\nu}{\kappa} \tag{92}$$

The non-dimensionalized equations (dropping primes) are:

$$\frac{\partial\bar{\bar{T}}}{\partial t} + \frac{\partial}{\partial z}\langle w\theta\rangle = \kappa\frac{\partial^2\bar{\bar{T}}}{\partial z^2} \tag{93}$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = 2R\bar{T}\nabla_1^2\theta + R\nabla_1^2\theta^2 + \frac{1}{\sigma}L \tag{94}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = -w\frac{\partial\bar{\bar{T}}}{\partial z} - h \tag{95}$$

In this case, we take the mean field equations to be

$$N = \beta(z) + \langle w\theta\rangle \tag{96}$$

where $\beta(z) = -\frac{d\bar{\bar{T}}}{dz}$, N is the Nusselt number, $N = \frac{H}{\kappa\frac{\Delta T}{d}}$

$$\nabla^4 w = -2R\bar{\bar{T}}\nabla_1^2\theta \tag{97}$$

$$\nabla^2\theta = -\beta w \tag{98}$$

Note, however, that in this case the conservation laws are no longer fulfilled since we have also dropped a term which is not a fluctuating self-interaction, $R\nabla_1^2\theta^2$. The set still has the property that different horizontal wave numbers interact only through the heat flux equation (96). Again it is convenient to consider only a single horizontal α in which case the equations become:

$$N = \beta(z) + W(z)\,\theta(z) \tag{99}$$

$$\left(\frac{d^2}{dz^2} - \alpha^2\pi^2\right)^2 W = 2R\alpha^2\pi^2\,\overline{T}\,\theta(z) \tag{100}$$

$$\left(\frac{d^2}{dz^2} - \alpha^2\pi^2\right)\theta = -\beta W \tag{101}$$

The only difference between equations (99) - (101) and (33) - (35) is the appearance of the factor $2\overline{T}$ in the momentum equation (100). The procedure described in III above disregards the character of the momentum equation until the last step of determining N as a function of R. Of course, the momentum equation entered into our choice of initial function (36) or (37) but only through the characteristic of having a high order differential operator on the left-hand side of (35), not through the nature of the right-hand side.

The difficulty then arises as to how we can hope to get information about penetrative convection when our technique disregards the differences between it and ordinary Bénard convection until the final steps. The only avenue of approach that seems to be open in such a case is to use the momentum equation implicitly in our choice of initial velocity field $W(z)$ and to emphasize the similarities between the two types of convection. Such a procedure was tried without success. It was found that a well-ordered scheme was not generated and that the corrections from iteration could not be made small in the region of interest. It is at this point that numerical computations would be very useful in order to provide some basis for guessing an initial approximation. The difficulty is of a similar nature to the paradox pointed out in III above. With an assumed initial velocity field as shown in Fig. 6, the difficulty arises in

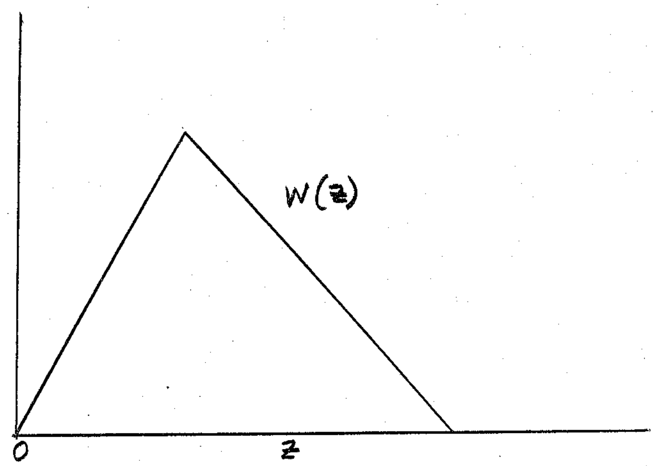

Fig. 6

the part of the field where $W=0$. It arises because the only scale available for the field in this region is determined by α and does not boundary layer. What is needed is a more careful passage to the limit $W \longrightarrow 0$ in this region. This has not been done successfully, however.

The author wishes to thank Dr. W. V.R. Malkus for introducing him to Dr. Herring's work and suggesting that it might be applicable to penetrative convection. He also wishes to acknowledge the many useful discussions with Dr. Malkus and the other members of the Geophysical Fluid Dynamics Course.

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A Problem in Mountain Wave Theory

John G. Pierce

A. Introduction

Observations[1] of the basic airstream velocity over the Sierras show that in summer the basic structure consists of lower altitude westerlies and higher altitude easterlies. Such a situation in which the basic velocity changes direction provides a very difficult problem in mountain wave theory. This is because of a singularity which appears in the governing differential equations when the basic velocity $U(z)=0$. To date this problem has not been solved analytically. Numerical studies[2] have been carried out by arranging the integration grid so that U changes sign between grid points. The results of these studies suggest that energy from the mountain perturbation can be propagated through the zero point of the basic velocity, and can, in some cases, excite strong "lee" waves on the upstream side of the mountain at high altitudes.

The present report deals with some of the difficulties which are inherent in any model containing a reversing wind. The specific model under consideration here is shown in Fig. I. It consists of two layers, each having uniform wind shear. It is also assumed that each layer is isothermal, although both layers need not be at the same temperature. This assumption guarantees that the Väisälä-Brunt frequency is a constant in each

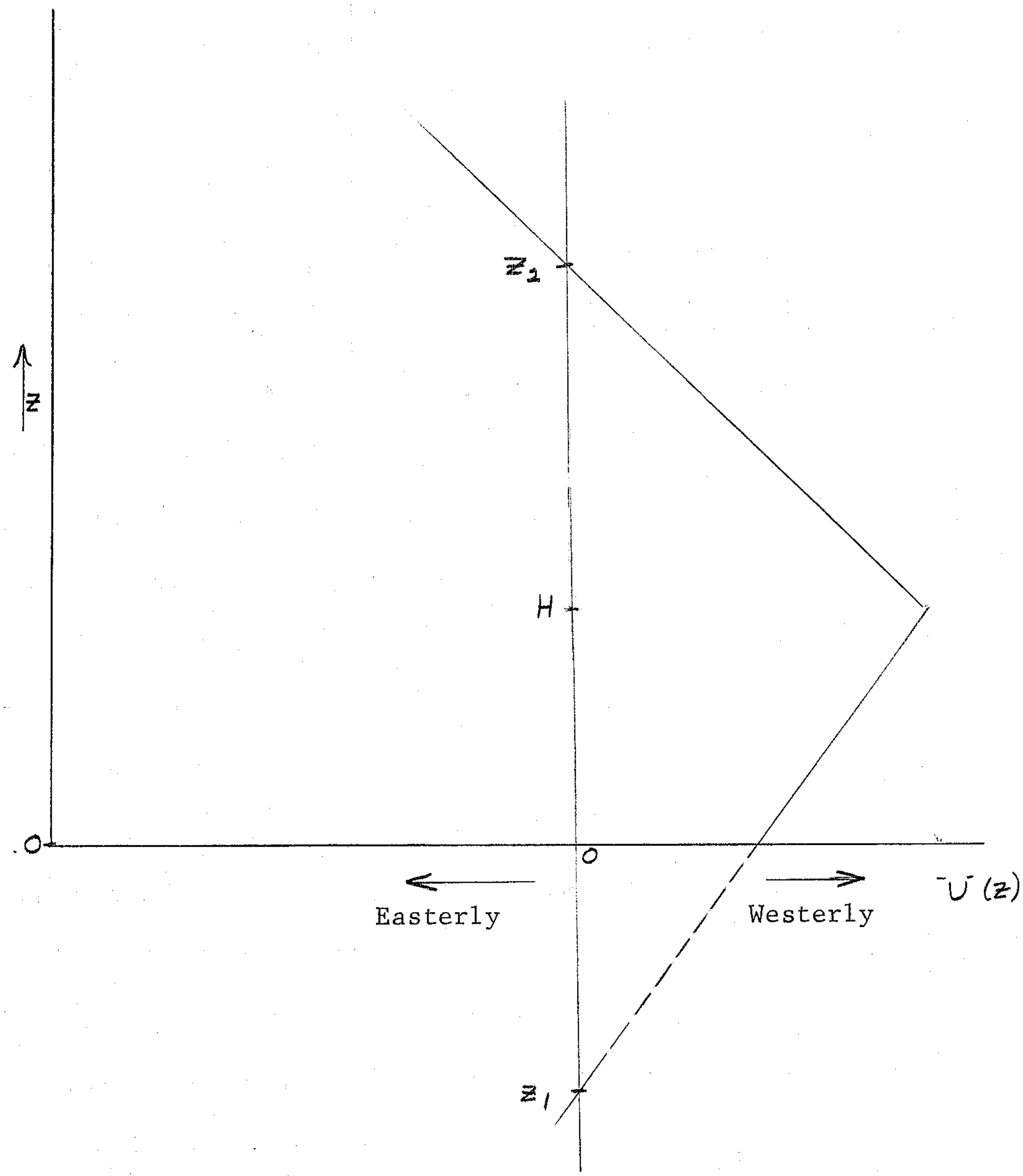

Fig. 1.

layer, and that the only variation in the coefficients of the differential equations is due to the variable wind velocity.

The problem will be formulated as an initial value problem, so that certain aspects of the stability of such an arrangement can be investigated in the course of solving the general problem.

Specifically we consider that the flow is exactly that shown in Fig. I for $t < 0$. At $t = 0$ some perturbation is introduced, and the solution is required to satisfy the condition that the total velocity be tangential to the surface of the mountain. The situation as $t \to \infty$ is examined by looking at the asymptotic behavior of the inverse Laplace transform.

B. Formulation of the Differential Equation

The physical problem is governed by the following basic equations:

Conservation of momentum in two dimensions:

$$\rho \frac{Du}{Dt} = - \frac{\partial p}{\partial x} ; \tag{1}$$

$$\rho \frac{Dw}{Dt} = - \frac{\partial p}{\partial z} - \rho g ; \tag{2}$$

Continuity:

$$\frac{D\rho}{Dt} = - \rho \left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right] ; \tag{3}$$

The equation of adiabatic motion:

$$\frac{D}{Dt} \left(p \rho^{-\gamma} \right) = 0 \tag{4}$$

where

$$\gamma = c_p / c_v . \tag{5}$$

In addition to these differential equations, we need an equation of state, for which we use the perfect gas law:

$$p/\rho = RT = \text{const. in each layer} \tag{6}$$

A more convenient form of the adiabatic equation is

$$\frac{Dp}{Dt} = -\gamma p \left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right] . \tag{7}$$

In each layer the basic flow can be represented by an expression of the form

$$u = \bar{U} = \alpha \left(z + z_0 \right) \tag{8}$$

where α and z_0 are different for the two layers.

From Eq. (2), this implies, for the basic state,

$$\frac{\partial p}{\partial z} = - \rho g , \tag{9}$$

which, combined with Eq. (6), can be integrated to give the pressure and density distributions of the basic state.

$$p_0 = p_s e^{-\beta z} ; \qquad \rho_0 = \rho_s e^{-\beta z} = \frac{\beta p_s}{g} e^{-\beta z} , \tag{10}$$

where

$$\beta = g/RT . \tag{11}$$

The next step is to linearize the equations of motion for the perturbation quantities, which are introduced by assuming variables of the form

$$\rho = \rho_0 (z) + \rho (x, z, t) ; \tag{12}$$

$$p = p_0 (z) + p (x, z, t) ; \tag{13}$$

$$u = U (z) + u (x, z, t) ; \tag{14}$$

$$w = 0 \quad + w (x, z, t) . \tag{15}$$

With these assumptions, the basic equations become

$$(\rho_0+\rho)\left[\frac{\partial u}{\partial t}+(U+u)\frac{\partial u}{\partial x}+w\frac{\partial}{\partial z}(U+u)\right]=-\frac{\partial p}{\partial x}\ ; \tag{16}$$

$$(\rho_0+\rho)\left[\frac{\partial w}{\partial t}+(U+w)\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}-\rho g\ ; \tag{17}$$

$$\frac{\partial \rho}{\partial t}+(U+u)\frac{\partial \rho}{\partial x}+w\frac{\partial}{\partial z}(\rho_0+\rho)=-(\rho_0+\rho)\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]\ ; \tag{18}$$

$$\frac{\partial p}{\partial t}+(U+u)\frac{\partial p}{\partial x}+w\frac{\partial}{\partial z}(p_0+p)=-\gamma(p_0+p)\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]. \tag{19}$$

When these equations are linearized, we obtain

$$\rho_0\left[\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+w\frac{\partial U}{\partial z}\right]=-\frac{\partial p}{\partial x}\ ; \tag{20}$$

$$\rho_0\left[\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right]=-\frac{\partial p}{\partial z}-\rho g\ ; \tag{21}$$

$$\frac{\partial \rho}{\partial t}+U\frac{\partial \rho}{\partial x}+w\frac{\partial \rho_0}{\partial z}=-\rho_0\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]\ ; \tag{22}$$

$$\frac{\partial p}{\partial t}+U\frac{\partial p}{\partial x}+w\frac{\partial p_0}{\partial z}=-\gamma p_0\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]. \tag{23}$$

We now recall from the properties of the basic state that

$$\frac{\partial U}{\partial z}=\alpha\ ;\ \frac{\partial p_0}{\partial z}=-\beta p_0\ ;\ \frac{\partial \rho_0}{\partial z}=-\beta\rho_0\ ; \tag{24}$$

and that α and β may be different in the two layers. In addition we perform integral transforms over x (Fourier) and t (Laplace), so that only the z-dependence remains in the differential equations. The results of these operations are

$$\ell\hat{u}=u(0)-\frac{ik}{\rho_0}\hat{p}-\alpha\hat{w}\ ; \tag{25}$$

$$\ell\hat{w}=w(0)-\frac{\hat{p}'}{\rho_0}-\frac{g}{\rho_0}\hat{\rho}\ ; \tag{26}$$

$$\ell \hat{\rho} = \rho(0) - \rho_0 \left[ik\hat{u} + \hat{w}' - \beta \hat{w} \right] \tag{27}$$

$$\ell \hat{p} = p(0) - p_0 \left[\gamma ik\hat{u} + \gamma \hat{w}' - \beta \hat{w} \right] \tag{28}$$

The notation used here is

$$\hat{f} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \int_0^{\infty} e^{-\bar{p}t} f(x,z,t) dt \ ; \tag{29}$$

$$f(0) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \, f(x,z,0) \ ; \tag{30}$$

$$\ell = \ell(z) = \bar{p} + ikV(z). \tag{31}$$

The prime dentoes differentiation with respect to z .

Next there follow some lengthy manipulations to obtain an ordinary differential equation for the vertical velocity component $\hat{w}$. The result of this is

$$\begin{aligned} & W''\left[1 + \frac{\ell^2}{c^2k^2}\right] - \beta W'\left[1 + i\ell\left(\frac{2\alpha}{\beta k c^2}\right) + \frac{\ell^2}{k^2c^2}\right] + \\ & + \beta^2 W\left[\left(-\frac{k^2}{\beta^2} + \frac{i\alpha k}{\beta \ell} - \frac{k^2 g}{\beta \ell^2}\right) + \frac{1}{c^2}\left(-\frac{2\ell^2}{\beta^2} + \frac{i\alpha \ell}{\beta k} - g/\beta - \right.\right. \\ & \left.\left. -2\frac{\alpha^2}{\beta^2} - \frac{i\alpha k g}{\ell \beta^2} + \frac{g^2k^2}{\ell^2\beta^2}\right) + \frac{1}{c^4}\left(-\frac{\ell^4}{\beta^2 k^2} + \frac{g^2}{\beta^2}\right)\right] = S' \end{aligned} \tag{32}$$

Here S' is a source term depending on the initial values. We shall return to consider this term later. We have also introduced the velocity of sound c , by the relation

$$c^2 = \frac{\gamma g}{\beta}. \tag{33}$$

Since Eq. (32) is far too difficult to attack analytically, we seek to reduce it to tractable form by throwing away terms which may be very small for the real atmosphere. To accomplish this we first introduce scaling factors as follows:

1) a horizontal length scale determined by the width of the mountain:

$$L \sim 1/k \; ; \tag{34}$$

2) a vertical scale height for density:

$$H_1 \sim \frac{1}{\beta} \; ; \tag{35}$$

3) a characteristic velocity and a scale height for shear:

$$\alpha \sim \frac{V}{H_2} \; . \tag{36}$$

Dimensionless combinations of these which turn out to be important are the Mach number

$$M = \frac{V}{c} \; , \tag{37}$$

and the inverse Froude number

$$\frac{1}{F_1} = \frac{g H_1}{V^2} \tag{38}$$

In terms of these parameters, the ordering of the terms in Eq.(32) is as follows:

$$\begin{aligned} & W''\left[1+O(M^2)\right]-\beta W'\left[1+O\left(\frac{H_1}{H_2}M^2\right)+O(M^2)\right]+ \\ & +\beta^2 W\left\{\left[O\left(\frac{H_1^2}{L^2}\right)+O\left(\frac{H_1}{H_2}\right)+O\left(\frac{1}{F_1}\right)\right]+\right. \\ & +O(M^2)\times\left[O\left(\frac{H_1^2}{L^2}\right)+O\left(\frac{H_1}{H_2}\right)+O\left(\frac{1}{F_1}\right)+O\left(\frac{H_1^2}{H_2^2}\right)+\right. \\ & \left.\left.+O\left(\frac{H_1}{H_2}\frac{1}{F_1}\right)+O\left(\frac{1}{F_1^2}\right)\right]+O(M^4)\times\left[O\left(\frac{H_1^2}{L^2}\right)+O\left(\frac{1}{F_1^2}\right)\right]\right\} \end{aligned} \tag{39}$$

We must now ask what values these parameters assume for a reasonable atmospheric model. The following set[3] seems appropriate to the summer observations near the Sierras

$$\left[\begin{array}{ll} g \sim 10\ m/sec^2 ; & c \sim 350\ m/sec \\ V \sim 25\ m/sec ; & L \sim 10^4 - 10^5 m \\ H_1 \sim H_2 \sim 10^4 m & \end{array}\right] \tag{40}$$

Actually it will be more useful to ignore the horizontal scaling, and solve for k as an eigenvalue in terms of the other parameters of the problem. For the set (40) we find

$$M^2 \sim \frac{1}{100} ; \quad \frac{1}{F_1} \sim 100 ; \quad \frac{H_1}{H_2} \sim 1. \tag{41}$$

This sort of scaling shows that most of the terms in Eq. (32) are $\leq O(10^{-2})$. Throwing these away and keeping only those of $O(1)$ or greater, we have

$$W'' - \beta W' + \beta^2 W \Big[-\frac{k^2}{\beta^2} - \frac{g}{\beta c^2} + \frac{g^2}{\beta^2 c^4} + \frac{i\alpha k}{\beta^2 \ell}\left(\beta - 2g/c^2\right) - \frac{k^2 g}{\beta^2 \ell^2}\left(\beta - g/c^2\right) \Big] = S'' \tag{42}$$

This equation can be reduced to canonical form by the substitution

$$f = e^{-\beta z/2} W \tag{43}$$

We then find

$$f'' + f\left[-k^2 - \frac{\beta^2}{4} - \frac{N^2}{c^2} + \frac{i\alpha k}{g\ell}\left(N^2 - \delta^2\right) - \frac{k^2 N^2}{\ell^2} \right] = e^{-\beta z/2} S'. \tag{44}$$

Here we have introduced the Värsälä-Brunt frequency

$$N^2 = -g\left(-\beta + g/c^2\right). \tag{45}$$

Also

$$\delta^2 = g^2/c^2 \tag{46}$$

We turn next to the scaling of the source term S'. In its raw form this term is

$$-\Big\{W(0)\Big[\frac{k^2}{l}+\frac{2l}{c^2}+\frac{l^3}{c^4k^2}\Big]+u(0)\Big[-\frac{ik\beta}{l}+\frac{1}{c^2}\Big(\frac{igk}{l}+2\alpha-\frac{i\beta l}{k}\Big)+$$
$$+\frac{1}{c^4}\Big(\frac{igl}{k}\Big)\Big]+u'(0)\Big[\frac{ik}{l}+\frac{il}{c^2k}\Big]+\frac{p(0)}{\rho_0}\Big[\frac{1}{c^2}\Big(\frac{gk^2}{l^2}+\frac{i\alpha k}{l}\Big)+ \tag{47}$$
$$+\frac{1}{c^4}\Big(g+\frac{i\alpha l}{k}\Big)\Big]+\frac{p'(0)}{\rho_0}\Big[-\frac{1}{c^2}-\frac{l^2}{c^4k^2}\Big]+\frac{\rho(0)}{\rho_0}\Big[-\frac{gk^2}{l^2}-\frac{2g}{c^2}-\frac{gl^2}{c^4k^2}\Big].$$

We introduce the scalings of Eqs.(34) - (36). In addition we assume

$$W(0)\sim\frac{H_1}{L}\quad U(0)\sim\frac{p(0)}{\rho} \tag{48}$$

We then use the same criterion for retention of terms as in the operator part of the equation. As a result the terms we keep are

$$S' = -\frac{\rho(0)}{\rho_0}\frac{gk^2}{l^2}+\Big[u(0)\Big(-\frac{ik\beta}{l}+\frac{ikg}{c^2l}\Big)+u'(0)\frac{ik}{l}\Big]+\frac{p(0)}{\rho_0}\Big(-2g/c^2\Big) \tag{49}$$

Two features deserve comment here. In the actual scaling the term involving $\rho(0)$ is approximately an order of magnitude larger than the other terms. In addition, it appears as the coefficient of $1/l^2$. We shall see subsequently that $1/l^2$ is the strongest singularity appearing in the evaluation of the inverse Laplace transform, so its coefficient is the major term in the asymptotic behavior of the solution. Thus, for two separate reasons, we see that initial density perturbations can be expected to dominate over initial perturbations in velocity and pressure.

Before the formulation of the differential equation is complete, it is convenient to introduce some further transformations to achieve a more compact form. Let

$$\lambda^2 = \beta^2 + 4k^2 + \frac{4N^2}{c^2} \tag{50}$$

and

$$Z = \lambda z . \tag{51}$$

The Equation (44) and the source term (49) can then be written

$$\frac{d^2 f}{dZ^2} + f\left[-\frac{1}{4} + \frac{N^2 - \delta^2}{\lambda g}\frac{1}{Z + Z_0 - i\sigma} + \frac{N^2}{\alpha^2 (Z + Z_0 - i\sigma)^2}\right] =$$

$$= S_0(Z) + \frac{S_1(Z)}{Z + Z_0 - i\sigma} + \frac{S_2(Z)}{(Z + Z_0 - i\sigma)^2} \tag{52}$$

where

$$S_0(Z) = -\frac{e^{\beta z/2}\, 2g}{\rho_0 c^2 \lambda^2}\, \rho(z,0) ; \tag{53}$$

$$S_1(Z) = \frac{e^{-\beta z/2}}{\alpha}\left[\frac{d}{dZ} u(0) - \frac{N^2}{g\lambda} u(0)\right] ; \tag{54}$$

$$S_2(Z) = \frac{e^{-\beta z/2} g}{\alpha^2 \rho_0}\, \rho(z,0) ; \tag{55}$$

and

$$\sigma = \frac{p\lambda}{k\alpha} . \tag{56}$$

C. The Boundary Conditions

The basic boundary condition to be satisfied is that the velocity be tangential to the surface of the mountain. If this is rigorously applied it results in the introduction of unpleasant non-linearities which we have tried to eliminate in the formulation of the differential equations. For reasons of simplicity and tradition we will also linearize the boundary conditions. Previous analytical and numerical work[4] indicates that this is not a severe approximation when the mountain slope is not steep and when there are no large shears near the surface.

Formally, let the contour of the mountain be represented by a function

$$z = \zeta(x) \tag{57}$$

The slope is then $\dfrac{d\zeta}{dx}$

and the rigorous boundary condition is

$$\frac{w(x, \zeta(x))}{U(\zeta(x)) + u(x, \zeta(x))} = \frac{d\zeta}{dx} \tag{58}$$

The linearization consists of setting

$$w(x,0) = \frac{d\zeta}{dx}\, U(0). \tag{59}$$

When we look at this condition in the Fourier transform representation, we have

$$w_k(0) = ik\,\zeta_k\, U(0) \equiv b_k \tag{60}$$

This will suffice as the lower boundary condition.

In addition we need a condition at infinity. We will simply take this to be that the solution is bounded at infinity. This insures that the amount of energy radiated away to infinity is also bounded. A more detailed discussion of this condition and its physical consequences is found in reference 4.

We finally have a well-posed problem. The differential equation is Eq. (52) with the boundary conditions

$$\left.\begin{array}{l} f\,(z=0) = b_K \\ f\,(z=\infty) = 0 \end{array}\right] \qquad (61)$$

D. The Formal Solution

We can obtain a formal solution to this system by the application of well-known Green's function techniques.

We have an equation of the form

$$\mathcal{L}_z(f) = S(z) \qquad (62)$$

The solution is then

$$f(z) \quad \int G(z/z')\, S'(z')\, dz' \qquad (63)$$

where $G(z\,|\,z')$ satisfies

$$\mathcal{L}_z\left(G(z\,|\,z')\right) = \delta(z - z') \qquad (64)$$

and the boundary conditions on f as well.

To construct the Green's function it is necessary to know the basic solutions to the equation

$$\mathcal{L}_{\mathcal{Z}}(f) = 0 \tag{65}$$

With a slight change of notation, the homogeneous equation for the present operator is

$$\frac{d^2 f}{d\mathcal{Z}^2} + f\left[-\frac{1}{4} + \frac{s}{\mathcal{Z}+\mathcal{Z}_0 - i\sigma} + \frac{1/4 - m^2}{(\mathcal{Z}+\mathcal{Z}_0 - i\sigma)^2}\right] = 0 \tag{66}$$

where

$$s = \frac{N^2 - \delta^2}{\lambda g} \tag{67}$$

and

$$\frac{1}{4} - m^2 = \frac{N^2}{\alpha^2} \tag{68}$$

The basic solutions of Eq. (66) are the confluent hypergeometric functions discussed by Whittaker and Watson[5], and frequently referred to simply as Whittaker functions. The two independent solutions are

$$f_1(\mathcal{Z}) = W_{s,m}(\mathcal{Z}) \tag{69}$$

$$f_2(\mathcal{Z}) = W_{-s,m}(-\mathcal{Z}) \tag{70}$$

These functions are analytic functions of the complex variable $\mathcal{Z}$, possessing a branch point at the origin. They are rendered unique by making a branch cut along the negative real axis. In our applications we shall need to use these functions for negative real argument. In such cases we shall consistently use

$$f(-x) \equiv \lim_{G \to 0} f\left(\rho e^{i(\pi-\epsilon)}\right). \tag{71}$$

Useful properties of the Whittaker functions are their asymptotic expansions,

$$f_1(z) \sim z^{s} e^{-z/2} \quad ; \tag{72}$$

$$f_2(z) \sim (-z)^{-s} e^{z/2} \quad ; \tag{73}$$

and their expansions about the origin

$$f_1 \simeq \frac{\Gamma(-2m)}{\Gamma(1/2-m-s)} z^{\frac{1}{2}+m} + \frac{\Gamma(2m)}{\Gamma(1/2+m-s)} z^{\frac{1}{2}-m} \tag{74}$$

$$f_2 \simeq \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m+s)} (-z)^{\frac{1}{2}+m} + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m+s)} (-z)^{\frac{1}{2}-m} \tag{75}$$

In all realistic atmospheric models $N^2/\alpha^2 > \frac{1}{4}$, so examination of Eq. (68) shows that m is pure imaginary. Under these circumstances, we may wonder about the behavior of the Whittaker functions near the origin. That is, does the following limit exist?

$$\lim_{z \to 0} \quad z^{\frac{1}{2}+i\mu}$$

To investigate this question, we write

$$z^{i\mu} = e^{i\mu \ln z} = \cos \mu \ln z + i \sin \mu \ln z \tag{76}$$

and then look separately at

$$\lim_{z \to 0} z^{\frac{1}{2}} \cos \mu \ln z \text{, and } \lim_{z \to 0} z^{\frac{1}{2}} \sin \mu \ln z \tag{77}$$

The sine and cosine functions can be expanded in their Taylor series, e.g.,

$$\cos\mu \ln z = \sum a_n(\mu \ln z)^{2n} \tag{78}$$

$$\sin\mu \ln z = \sum a_n(\mu \ln z)^{2n+1} \tag{79}$$

The limit of a typical term of one of these series is

$$\lim_{z\to 0} z^{\frac{1}{2}}(\ln z)^n = \lim_{z\to 0} \frac{(\ln z)^n}{\frac{1}{z^{\frac{1}{2}}}} \tag{80}$$

In the latter form, the limit appears as ∞/∞ , and we may use L'Hospital's rule to evaluate it.

$$\lim_{z\to 0} \frac{(\ln z)^n}{1/z^{\frac{1}{2}}} = \lim_{z\to 0} \frac{\frac{d}{dz}(\ln z)^n}{\frac{d}{dz}\,\frac{1}{z^{\frac{1}{2}}}} = -2n \lim_{z\to 0} \frac{(\ln z)^{n-1}}{\frac{1}{\sqrt{z}}} \tag{81}$$

An n-fold application of this rule will finally give

$$\lim_{z\to 0} (-2)^n n!\,\sqrt{z} \to 0 \tag{82}$$

Since this is true for any n , the limits in Eqs.(77) exist and are equal to zero. Hence both solutions f_1 and f_2 are zero at the origin for m pure imaginary.

We now use the basic solutions to construct the Green's function by noting the following: For $z \neq z'$, $G(z/z')$ satisfies the homogeneous equation. In addition, the derivative of G must satisfy a jump condition at $z = z'$. That is

$$\lim_{\epsilon\to 0} \frac{dG(z'+\epsilon|z')}{dz} - \frac{d}{dz}\, G(z'-\epsilon\,(z') = 1 \tag{83}$$

This condition is obtained by integrating Eq.(64) from $z'-\epsilon$ to $z'+\epsilon$.

We shall be interested in cases for which the source function is confined to the lower layer $(z' < H)$. With this understanding we can write down the Green's function in the form

$$G = A f_1(z + z_1 - i\sigma) + B f_2(z + z_1 - i\sigma) \tag{84}$$

for $z < z' < H$

$$G = C f_1(z + z_1 - i\sigma) + D f_2(z + z_1 - i\sigma) \tag{85}$$

for $z' < z < H$

$$G = E f_1(z_2 - z - i\sigma) + F f_2(z_2 - z - i\sigma) \tag{86}$$

for $z > H > z'$.

The values of the constants $A - F$ are determined by the boundary conditions Eq.(61), the jump condition Eq.(83), and the additional requirements that G and $\frac{dG}{dz}$ be continuous at $z = H$. After some lengthy eliminations we find

$$\begin{aligned} AD = b_k W\Big[& f_2(z_1 + H - i\sigma) f_2'(z_2 - H - i\sigma) - f_2'(z_1 + H - i\sigma) f_2(z_2 - H - i\sigma) + \\ & + f_2(z_1 - i\sigma)\Big[f_2'(z_2 - H - i\sigma)\{f_1(z_1 + H - i\sigma) f_2(z_1 + z' - i\sigma) - f_1(z_1 + z' - i\sigma) \\ & f_2(z_1 + H - i\sigma)\} + f_2(z_2 - H - i\sigma)\{f_2(z_1 + z' - i\sigma) f_2'(z_1 + H - i\sigma) - \\ & - f_2(z_1 + z' - i\sigma) f_1'(z_1 + H - i\sigma)\}\Big] \end{aligned} \tag{87}$$

$$\begin{aligned} BD = b_k W\Big[& f_1'(z_1 + H - i\sigma) f_2'(z_2 - H - i\sigma) - f_2(z_1 + H - i\sigma) f_2(z_2 - H - i\sigma) + \\ & + f_1(z_1 - i\sigma)\Big[f_2'(z_2 - H - i\sigma)\{f_1(z_1 + z' - i\sigma) f_2(z_1 + H - i\sigma) - f_2(z_1 + z' - i\sigma) \\ & f_1(z_1 + H - i\sigma)\} + f_2(z_2 - H - i\sigma)\{f_2(z_1 + z' - i\sigma) f_1'(z_1 - H - i\sigma) - \\ & - f_1(z_1 + z' - i\sigma) f_2'(z_1 + H - i\sigma)\}\Big] \end{aligned} \tag{88}$$

$$CD = \left\{ f_2(z_1+H-i\sigma) f_2(z_2-H-i\sigma) - f_1'(z_1+H-i\sigma) f_2(z_2-H-i\sigma) \right\} \times$$
$$\times \left\{ -b_K W + \left[f_1(z_1-i\sigma) f_2(z_1+z'-i\sigma) - f_2(z_1-i\sigma) f_1(z_1+z'-i\sigma) \right] \right\} \tag{89}$$

$$DD = \left\{ f_1(z_1+H-i\sigma) f_2'(z_2-H-i\sigma) - f_1'(z_1+H-i\sigma) f_2(z_2-H-i\sigma) \right\} \times$$
$$\times \left\{ -b_K W + \left[f_1(z_1-i\sigma) f_2(z_1+z'-i\sigma) - f_2(z_1-i\sigma) f_1(z_1+z'-i\sigma) \right] \right\} \tag{90}$$

$$ED = 0 \tag{91}$$

$$FD = b_K W^2 - W\left(f_1(z_1-i\sigma) f_2(z_1+z'-i\sigma) - f_2(z_1-i\sigma) f_1(z_1+z'-i\sigma) \right) \tag{92}$$

where

$$D = W \Big[f_1(z_1-i\sigma) \Big\{ f_2'(z_1+H-i\sigma) f_2(z_2-H-i\sigma) -$$
$$- f_2'(z_2-H-i\sigma) f_2(z_1+H-i\sigma) \Big\} + f_2(z_1-i\sigma)$$
$$\Big\{ f_1(z_1+H-i\sigma) f_2'(z_2-H-i\sigma) - f_1'(z_1+H-i\sigma) f_2(z_2-H-i\sigma) \Big\} \Big] \tag{93}$$

and W is the Wronskian of the two solutions f_1 and f_2,

$$W = f_1 f_2' - f_2 f_1' \tag{94}$$

To evaluate the Wronskian we note that a simple manipulation of the differential equations shows that

$$\frac{dW}{dz} = 0. \tag{95}$$

Hence W can be evaluated for any value of z. It proves simplest to use the asymptotic expansions (72) and (73) and evaluate W as $z \longrightarrow \infty$. The result is

$$W = (-1)^{n} \tag{96}$$

This completes the specification of the Green's function.

E. The inverse transform.

We now ask what information can be obtained about the behavior of the solutions for large t by an asymptotic evaluation of the inverse Laplace transform. Formally we have

$$f_k(z,t) = \frac{1}{2\pi i}\int_C e^{pt}\,dp \int dz'\, G(z/z')\, S(z'). \tag{97}$$

To discuss the asymptotic behavior it is necessary to have a knowledge of the singularities of the integrand in the $p(\sigma)$ plane. We have already seen that the S term has poles of orders one and two. In addition we must worry about the zeros of D defined by Eq. (93).

Let us consider only the term which provides the strongest singularity in the p- plane. This is the S_2 term. For utmost simplicity, let us consider a point source of density perturbation at some height $z_s < H$, and write

$$S_2(z) = S_k\, \delta(z - z_s) \tag{98}$$

The z' integration can be performed immediately, with the result

$$f_{k,p}(z) = \frac{S_k}{D(i\sigma)} f_2(z_2 - z - i\sigma)\Big[b_k W^2 - W\big(f_1(z_1 - i\sigma) f_2(z_1 + z_s - i\sigma) - f_2(z_1 - i\sigma) f_1(z_1 + z_s - i\sigma)\big)\Big] \Big/ (z_1 + z_s - i\sigma)^2 \tag{99}$$

For the purposes of the p-integration, the singularities of

(99) are at $i\sigma = z_1 + z_s$ and at the zeros of $D(i\sigma)$. If we can assume, with Case[6] that D has only simple zeros, then $i\sigma = z_1 + z_s$ is a pole of second order for the term multiplying $b_k W^2$, but it is a branch point for the terms multiplying W, since the f functions of the argument $z_1 + z_s - i\sigma$ have branch points there themselves. Generally

$$\frac{1}{2\pi i}\int_c f_{p,k}(z) e^{pt} dp = \frac{S_k b_k W^2}{2\pi i}\int_c \frac{f_2(z_2 - z_1 - i\sigma)}{(z_1 + z_s - i\sigma)^2 D} e^{pt} dp - \tag{100}$$

$$- \frac{S_k W}{2\pi i}\int_c \frac{\left[f_1(z_1 - i\sigma) f_2(z_1 + z_s - i\sigma) - f_2(z_1 - i\sigma) f_1(z_1 + z_s - i\sigma)\right]}{(z_1 + z_s - i\sigma)^2 D} e^{pt} dp$$

The first integral can be evaluated by the residue theorem, giving

$$S_k b_k W^2 \left[\frac{f_2(z_2 - z_1 - z_1 - z_s)}{D\big|_{i\sigma = z_1 + z_s}} \right] t e^{-i(z_1 + z_s)\alpha k t} \tag{101}$$

$+ \sum$ oscillatory terms.

The behavior of the second integral of (100) can be estimated by replacing the integrand by one which has the same behavior near the singularity, i.e.

$$\sim -\frac{S W}{2\pi i} \frac{f_1(-z_s)}{D\big|_{i\sigma = z_1 + z_s}} \int_c \frac{f_2(z_1 + z_s - i\sigma) e^{pt}}{(z_1 + z_s + i\sigma)^2} +$$

$$+ \frac{S_k W}{2\pi i} \frac{f_2(-z_s)}{D\big|_{i\sigma = z_1 + z_s}} \int_c \frac{f_1(z_1 + z_s - i\sigma) e^{pt}}{(z_1 + z_s - i\sigma)^2} \tag{102}$$

Near the singularity we use the expansions (74) and (75) for

f_1 and f_2. This yields, disregarding the various coefficients, two integrals of the form

$$I_1 = \int_{C'} (z_1 + z_s - i\sigma)^{-3/2+m} e^{pt} dp \tag{103}$$

$$I_2 = \int_{C'} (z_1 + z_s - i\sigma)^{-3/2-m} e^{pt} dp \tag{104}$$

where C' is a contour around the branch point, starting and ending at infinity in the left-half plane. A bit of manipulation puts these integrals in a form such that they correspond to Hankel's contour integral as function of the reciprocal Γ function (Whittaker and Watson, chap. 12). One can then write

$$I_1 = \left(\frac{ik\alpha}{\lambda}\right)^{\frac{3}{2}-m} \frac{e^{-i(z_1+z_s)k\alpha t} t^{1/2-m}}{\Gamma(3/2-m)} \tag{105}$$

and

$$I_2 = \left(\frac{ik\alpha}{\lambda}\right)^{\frac{3}{2}+m} \frac{e^{-i(z_1+z_s)k\alpha t} t^{1/2+m}}{\Gamma(3/2+m)} \tag{106}$$

In addition there will be oscillatory terms due to the zeros of D in the second term of (100). The time dependence of the Fourier components is clearly exhibited by Eqs. (101), (105), and (106).

The analysis from page   on has all been based on the presumption used by Case, that D has only simple zeros. This is, however, not true. D has a branch point where $z_1 - i\sigma$ is zero, since both factors f_1 and f_2 have branch points there. A close analysis of the behavior of $\mathcal{D}$ near the branch point has led the author to the view that it is not even possible to treat this point by the usual method of excluding it from the Bromwich contour by a branch cut. This is because in any finite distance from the branch point along some ray, there is a denumerable infinity of poles. Hence any branch cut which we might wish to put around the branch point cannot be deformed without crossing some poles and thus changing the value of the integral.

The truth of this statement can be seen by looking at the behavior of $\mathcal{D}$ for $\tau \equiv z_1 - i\sigma \sim 0$. Use of the expansions (74) and (75) leads us to

$$D \sim \tau^{\frac{1}{2}} e^{-i\mu \ln\tau} \left[e^{2i\mu \ln\tau} + C \right], \tag{107}$$

where C is a complex constant depending on the expansion coefficients and on the other functions in the full expression for $\mathcal{D}$, evaluated at $\tau = 0$. Writing $\tau = re^{i\theta}$ and $C = \rho e^{i\varphi}$, we see the $\mathcal{D}$ has zeros for

$$e^{-2\mu\theta} = \rho \quad \text{and} \quad 2\ln r = \varphi + 2n\pi \tag{108}$$

where n is any positive or negative integer. From this we see that as $r \longrightarrow 0$ along the ray defined by the first equation (108), $\mathcal{D}$ has an infinity of zeros. No method of treating such a situation is known to the author at present.

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Ocean Circulation Models for Regions of High Latitude

Stephen Pond

Introduction

In considering regions of high latitude, the region of particular interest is that of the Antarctic. This region is a very complicated one. There are both wind and thermal driving effects. There are undoubtedly important topographic effects, at least in some parts of the region. In addition, the Antarctic current is unique, in that part of it is not blocked by meridional barriers at all. While this region is small, we shall see from considering models both with and without this break in the meridional barrier (the Drake Passage), that this feature appears to play a major role in determining the gross features of the circulation.

Because the region is so complicated, very simplified models are considered in the hope that they will illustrate some of the major features of the flow and that they will lead to a better understanding of the Antarctic region so that a more complete solution may eventually be found.

Because of the relatively high latitudes (40^{o}-70^{o}S), the usual beta-plane approximation is not valid. The first step is to find equations appropriate to the interior region of an ocean at high latitudes (but still somewhat removed from the pole).

The Interior Equations

As in obtaining the equations for the beta-plane, the starting point is the complete equations of motion in spherical coordinates (Landau and Lifshitz, p.52). θ in the equations is the co-latitude rather than the latitude ($\theta = 0$ at the North Pole and increases to π at the South Pole).

Eventually, a transformation to a Cartesian system will be made, using the transformation:

$$
\begin{aligned}
x &= R \sin \theta_0 \varphi & \qquad V_\varphi &= u \\
y &= R(\theta_0 - \theta) & \qquad V_\theta &= -v \\
z &= r - R & \qquad V_r &= w
\end{aligned}
$$

where x is positive to the east, y to the north and z upwards. θ_0 is a reference value of θ and R is the radial distance to the origin of the Cartesian coordinate system (which can be taken at any convenient point between the bottom and surface of the ocean). V_φ , V_θ , V_r are the velocities in the φ , θ , r directions, respectively; u , v, w are the velocities in the x , y , z directions, respectively. Since y decreases when θ increases, the sign of θ-components of vectors will have to be changed. In the equations, geometric factors will be retained without approximation so that we could simply use the transformation $z = r - R$ and the approximation $r \approx R$ but we shall make the transformation to the more familiar Cartesian form which is exactly equivalent. Before making the transformation we shall obtain the vorticity equation for the interior in spherical form. When the transformation is made, we shall then have the equivalent of the Sverdrup relation. In obtaining the spherical coordinate form we shall make the one approximation $H/R << 1$ (H is the total depth of the ocean) since this is always a very good approximation and simplifies the equations considerably.

The continuity equation (assuming the fluid to be incompressible) is

$$\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial \varphi} + \frac{V_\theta \cot\theta}{r} = 0$$

where the term $\frac{2V_r}{r}$ has been neglected since it will be $O(H/R)$ of the term $\frac{\partial V_r}{\partial r}$.

Next we must check to see of the viscous and inertial terms are small compared to the linear terms. The same notation as that of Robinson's lectures is used. A typical velocity is 10 cm/s; the Coriolis parameter, $f \geqslant 10^{-4}\, s^{-1}$; a minimum length scale is 10^8 cm (1000 km). The coefficient of the inertial terms,

$$\varepsilon \sim \frac{V_0^2/L}{f_0 V_0} = \frac{V_0}{f_0 L} \leqslant 10^{-3}$$

The coefficient of horizontal 'viscous' terms,

$$\Gamma \sim \frac{A_{oh}}{f_0 L^2} \leq 10^{-2}$$

These values should be regarded as upper bounds. γ , the coefficient for the vertical 'viscous' terms is usually about the same order as Γ (for $\gamma = 10^{-2}$, the Ekman depth would be 300 m which is larger than is usually found in the oceans).

The interior flow should be geostrophic provided that none of the viscous or inertial terms contains a geometric factor which makes it $O(100)$ of the spherical terms which have the same form as the Cartesian ones. (Our non-dimensional parameters are based on the spherical terms which have the same form as the Cartesian ones.) We find on examining these terms that all the terms which differ from the ones of Cartesian form are of the same order or smaller.

The Coriolis force is given by $2\vec{\Omega} \times \vec{V}$ which in spherical coordinates is

$$2\Omega\left(-V_\varphi \sin\theta,\ -V_\varphi \cos\theta,\ V_\theta \cos\theta + V_r \sin\theta\right)$$

where the order of components is r, θ, φ. Consider the term

$V_\theta \cos\theta + V_r \sin\theta$

$$(2)/(1) \text{ is } \frac{W_0}{V_0}\tan\theta \sim \frac{H}{L}\tan\theta << 1$$

where W_0/V_0 is $O(H/L)$ from the continuity equation. Therefore we neglect this second term. We also neglect the Coriolis term in the vertical equation and assume the fluid to be hydrostatic.

Therefore the Coriolis terms are

$-f V_\varphi$ for the V_θ equation

$f V_\theta$ for the V_φ equation, $f = 2\Omega\cos\theta$

We want to put the driving force of the wind into the equations. Consideration of the stresses acting on an infinitesimal spherical element shows that the stresses have the form

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{2\tau_{\theta r}}{r}$$

where $\tau_{\theta r}$ is the stress in the θ direction across a plane perpendicular to the r-direction. However, the term $\frac{2\tau_{\theta r}}{r}$ is $O(H/R)$ and therefore can be neglected.

The equations of motion for the interior are:

$$2\Omega\cos\theta\, V_\theta = -\frac{1}{\rho_0 r\sin\theta}\frac{\partial p}{\partial \varphi} + \frac{1}{\rho_0}\frac{\partial \tau_{\varphi r}}{\partial r} \tag{1}$$

$$-2\Omega\cos\theta\, V_\varphi = -\frac{1}{\rho_0 r}\frac{\partial p}{\partial \theta} + \frac{1}{\rho_0}\frac{\partial \tau_{\theta r}}{\partial r} \tag{2}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} - \rho g \tag{3}$$

where the Boussinesq approximation has been made ($\rho = \rho_0$ except when multiplied by g).

The continuity equation may be rewritten as

$$\frac{\partial V_\varphi}{\partial \varphi} + \sin\theta\frac{\partial V_\theta}{\partial \theta} = -r\sin\theta\frac{\partial V_r}{\partial r} - V_\theta\cos\theta \tag{4}$$

We take $\frac{\partial}{\partial\theta}\sin\theta \times (1) - \frac{\partial}{\partial\varphi}(2)$ to obtain

$$2\Omega\cos 2\theta\, V_\theta + 2\Omega\cos\theta\left\{\frac{\partial V_\varphi}{\partial\varphi} + \sin\theta\frac{\partial V_\theta}{\partial\theta}\right\} = \tag{5}$$

$$= \frac{1}{\rho_0}\frac{\partial}{\partial r}\left\{\sin\theta\frac{\partial \tau_{\varphi r}}{\partial\theta} + \cos\theta\,\tau_{\varphi r} - \frac{\partial \tau_{\theta r}}{\partial\varphi}\right\}$$

Substituting from the continuity equation for $\frac{\partial V_\varphi}{\partial\varphi} + \sin\theta\frac{\partial V_\theta}{\partial\theta}$ and dividing by $r\sin\theta$ gives:

$$-\frac{2\Omega \sin\theta}{r} V_\theta - f\frac{\partial V_r}{\partial r} = \frac{1}{\rho_0 r}\frac{\partial}{\partial r}\left\{\frac{\partial \tau_{\varphi r}}{\partial \theta} + \cot\theta\, \tau_{\varphi r} - \frac{1}{\sin\theta}\frac{\partial \tau_{\theta r}}{\partial \varphi}\right\} \qquad (6)$$

Equation (6) is the vorticity equation. As a check, we note that near the equator $\sin\theta \cong \sin\theta_0$ and $L/R \cot\theta_0 \ll 1$ so that the equation reduces to the beta-plane form.

If the transformation to Cartesian coordinates is now applied to equation (6), we obtain

$$\beta v - f w_z = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left\{-\tau_y^{(x)} + \frac{\sin\theta_0}{\sin\theta}\tau_x^{(y)} + \frac{\cot\theta}{R}\tau^{(x)}\right\} \qquad (7)$$

where $\beta = \frac{2\Omega \sin\theta}{R}$, $\tau^{(x)}$ and $\tau^{(y)}$ are stress components in the x- and y-directions, and subscripts now denote partial differentiation. Next we integrate over z to obtain the integrated vorticity equation. The stresses at the bottom are neglected but we do not assume a lower stationary layer. The condition that there is no flow through the bottom implies that $w_{z=D} = (\vec{v}_D \cdot \nabla) D(x,y)$ where $\vec{v}_D$ is the velocity near the bottom and $D(x,y)$ is the depth. The surface vertical velocity is neglected since surface slopes are always very small.

The vertically integrated equation is then

$$\beta V + f(\vec{v}_D \cdot \nabla) D = \frac{1}{\rho_0}\left\{\frac{\sin\theta_0}{\sin\theta}\tau_x^{(y)} - \tau_y^{(x)} + \frac{\cot\theta}{R}\tau^{(x)}\right\} \qquad (8)$$

where $V = \int_{-D}^{h} v\, dz$ and the stresses are those at the surface.

We note that this equation is very similar to the one which leads to the Sverdrup relation. The beta-term is unchanged except that beta is no longer a constant and the wind-stress terms are somewhat more complicated.

If we consider an ocean for which $\tau = \hat{i}\,\tau(y)$ as is usually done (east-west winds only) the stress term becomes

$$\frac{\cot\theta}{R}\,\tau^{(x)} - \tau_y^{(x)}$$

If the assumption of a lower stationary layer or an ocean of constant depth is made then the second term of equation (8) vanishes and we can integrate the equation immediately since the stress term is a function of y only.

Often in considering oceanic regions at lower latitudes the form of $\tau(y)$ is taken to be $\tau^* \cos\frac{\pi y}{2L}$ (τ^* is the amplitude of the wind stress). If we choose a somewhat skewed form for $\tau(y) = \tau^* \frac{\cos\frac{\pi y}{2L}}{\sin\theta}$ then we get a very simple form for the driving term, $\tau^* \frac{\pi}{2L} \frac{\sin\frac{\pi y}{2L}}{\sin\theta}$ which is very similar to the beta-plane term $\tau^* \frac{\pi}{2L} \sin\frac{\pi y}{2L}$. In general, if we put $\tau(y) = \frac{f(y)}{\sin\theta}$, the driving term in the vorticity equation becomes $-f'(y)/\sin\theta$.

To be consistent in our approximation we must evaluate $(\vec{V}_D \cdot \nabla) D$ in spherical coordinates and then transform to the x , y plane.

$$(\vec{V}_0 \cdot \nabla D) = \frac{\sin\theta_0}{\sin\theta}\, u_D D_x + v_D D_y$$

The vorticity equation then has the form

$$\beta V + f\left(\frac{\sin\theta_0}{\sin\theta}\, u_D D_x + v_D D_y\right) = T'(y) \tag{9}$$

where $T'(y) = \frac{\cot\theta}{R}\tau^{(x)} - \tau_y^{(x)}$ and we have one simple example of a form which might be used for T' .

We can consider two simple models:

1. A barotropic flow with bottom topography. In this case we can consider the wind driving to be a body force and that the flow is two-dimensional or we can integrate to the bottom of the Ekman layer and drive the flow by the divergence of the Ekman flow instead of T'. The first case will be considered for simplicity. Then $u_D = U/D$, $v_D = V/D$ and equation (9) becomes

$$\beta V + \frac{f}{D}\left\{\alpha(\theta) U D_x + V D_y\right\} = T'(y) \tag{10}$$

where $\alpha(\theta) = \sin\theta_o / \sin\theta$ and $U = \int_{-D}^{h} u\, dz$.

2. An ocean with a constant depth or a lower stationary layer. Then (9) becomes

$$\beta V = T'(y) \tag{11}$$

and the interior solution is of a Sverdrup form. The boundary conditions must then be satisfied by fitting boundary layers where necessary.

BAROTROPIC FLOW WITH BOTTOM TOPOGRAPHY

In general equation (10) has the form:

$$a(x,y)\Psi_x + b(x,y)\Psi_y = c(x,y)$$

where $\alpha(\theta)\Psi_x = V$; $\Psi_y = -U$; $a = \alpha(\theta)(\beta + \frac{f}{D} D_y)$;

$b = \alpha(\theta)\frac{f}{D} D_x$ and $c = T'$.

Ψ_x is defined in this way so that the stream function in both the x, y and θ, φ- planes will be the same.

This equation can be solved by the method of characteristics. Consider lines which are given by

$$\frac{dx}{a} = \frac{dy}{b} = K = const.$$

Then on these lines $\frac{d\psi}{c} = K$ also since $d\psi = \psi_x dx + \psi_y dy$. We solve the equation $\frac{dy}{dx} = \frac{b}{a}$ to obtain the family characteristic lines. Along these lines $y = y(x,\gamma)$ where γ is a parameter (the arbitrary constant of integration). Then we solve the equation

$$\frac{d\psi}{dx} = \frac{c(x, y(x,\gamma))}{a(x, y(x,\gamma))}$$

which has the solution $\psi = A(y) + \int_{x_0}^{x} \frac{c}{a}\, dx$

$A(y)$ is determined from the specified values of ψ along the line $x = x_0$, that is, from the boundary conditions. Finally, we solve the equation $y = y(x,\gamma)$ for γ to obtain $\gamma = \gamma(x,y)$ and substitute for γ in the equation for ψ. If the boundary conditions give ψ along a line $y = y_0$ then we use y instead of x as our independent variable. Formally, this procedure gives the solution although it may be very difficult to find analytical solutions. When this is the case the solution can be obtained numerically.

The method can be illustrated by a very simple case. Take D to be of the form $D = D_0(1 + d \sin \frac{n\pi x}{M})$, $d < 1$, M is the width of the ocean, n is an integer. Consider a region for which beta is constant; then $\alpha(\theta) = 1$ also.

The equation for the characteristics is

$$\frac{dy}{dx} = -\frac{f}{\beta}\,\frac{D_x}{D}$$

$$\frac{\beta}{f_0+\beta y}\,dy = -d\ln D$$

$$d\ln(f_0+\beta y) = -d\ln D$$

and the solution is $f_0+\beta y = \frac{\gamma}{D}$

For the wind stress we take the form $\tau^{(x)} = \tau^*\,\frac{y^2/2L^2}{\sin\theta}$ which leads to a linear driving term $T' = -\frac{\tau^* y}{L^2\sin\theta}$. The boundary condition which we shall use is that $\psi = 0$ on $x = 0$. $x = 0$ will be the eastern boundary of our ocean. The model will not describe the flow over the whole ocean but will show the effects of bottom topography on the interior of the flow. For this model, ψ has the form

$$\psi = -\int_0^x \frac{\tau^*}{\beta^2L^2\sin\theta_0}\left(\frac{\gamma}{D}-f_0\right)dx$$

$$= -\frac{\tau^*}{\beta^2L^2\sin\theta_0}\left\{\frac{-2\gamma}{\frac{n\pi}{M}D_0\sqrt{1-d^2}}\tan^{-1}\left[\sqrt{\frac{1-d}{1+d}}\tan\left(\frac{\pi}{4}-\frac{n\pi x}{2M}\right)\right]-f_0x\right\}\Bigg|_0^x$$

Suppose $d \ll 1$, then

$$\psi \cong -\frac{\tau^*}{\beta^2L^2\sin\theta_0}\left\{\frac{\gamma x}{D_0}-f_0x\right\}$$

Substituting for $\gamma = D(f_0+\beta y)$ gives

$$\psi \cong -\frac{\tau^*}{\beta^2L^2\sin\theta_0}\left\{\left(1+d\sin\frac{n\pi x}{M}\right)(f_0+\beta y)-f_0\right\}x$$

As a check, we note that for $d = 0$ ψ has the form given by the Sverdrup equation. The model without bottom topography is similar to the one used for the interior by Morgan and Charney in their studies of inertial boundary layers.

Next non-dimensional coordinates are introduced to make it simpler to plot the stream function. As a particular example, x and y are normalized using a length scale L which makes the non-dimensional beta equal to 1. The range of x is taken to be 0 to -2; the range of y to be 0 to 1; η is taken to be 4.

$$y = y'L \qquad x = x'L \qquad \beta^* = \frac{\beta L}{f_0} = 1 \tag{10}$$

Then
$$\psi = \frac{\tau^* f_0}{\beta^2 L \sin\theta_0}\,\psi'$$

$$\psi' = -\left[x'y' + d\,x'(1+y')\sin 2\pi x'\right]$$

The equations for the stream lines are given by

$$y' = \frac{-d \sin 2\pi x' - \psi'}{x'(1 + d \sin 2\pi x')}$$

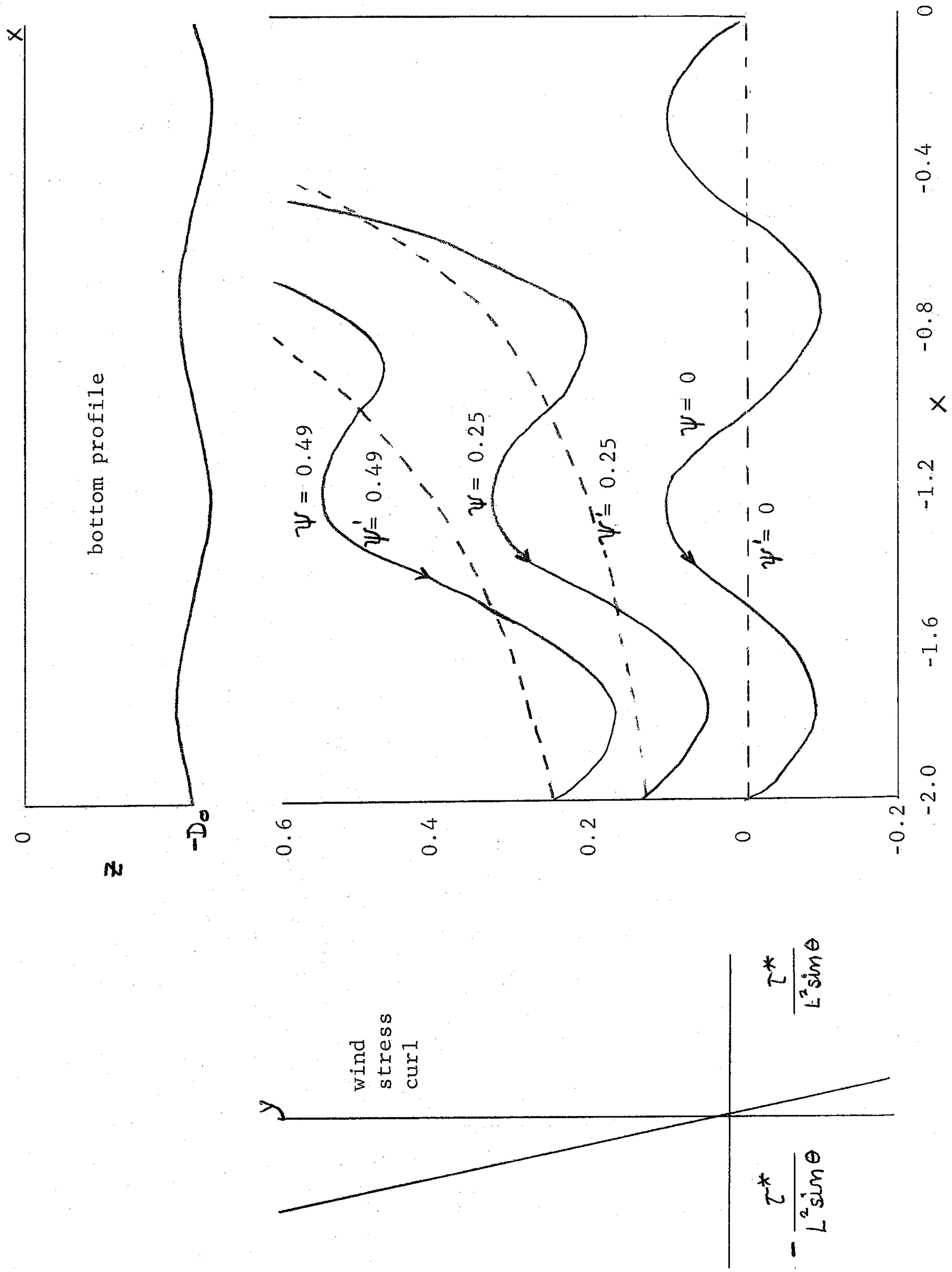


Figure 1. Bottom profile, wind stress curl, and stream lines for a beta-plane ocean. —ψ for $D = D_0(1 + 0.1 \sin 2\pi x)$. ---ψ' for $D = D_0$ for comparison.

Figure 1 shows the bottom profile, the wind stress curl and the stream function over part of this region. The dashed lines show the stream function for the case $D = D_0$ for comparison. d was taken to be 0.1 although this value is somewhat larger than should be used if our approximation for ψ is to be valid.

AN OCEAN WITH CONSTANT DEPTH

The equation which will govern the interior flow for this region is

$$\beta V = T'(y) \tag{11}$$

or introducing ψ by $\alpha(\theta)\,\psi_x = V,\ \psi_y = -U$ we have

$$\alpha(\theta)\beta\,\psi_x = T'(y) \tag{11'}$$

A region similar to the Antarctic will be considered: at first we shall assume that there is a complete meridional barrier and no Drake Passage. The region considered extends from 50°S to approximately 70°S. Actually the latitude of the Antarctic continent varies from 70°S to slightly less than 65°S but such a variable southern boundary would make the model too complicated. As we shall see later in our discussion of the boundary layers, the southern boundary can be put at any latitude so we could move it northward to correspond to the mean value of the latitude of the Antarctic continent if we wished to do so. The wind stress is assumed to be a function of y only and to have only an east-west component, with westerlies over most of the basin but easterlies south of 65^{o}. The mathematical form for $\tau^{(x)}(y)$ is taken to be

$$\tau^{(x)}(y) = \tau^{*} \frac{\cos \frac{\pi y}{2L}}{\sin \theta}$$

where L is the length corresponding to 50^{o} to 65^{o} latitude range and τ^{*} is the amplitude of the wind stress. The driving term T' then has the form

$$T' = \frac{\tau^{*}\pi}{2L} \quad \frac{\sin \frac{\pi y}{2L}}{\sin \theta}$$

Both these forms look very much like the cosine and sine forms used in discussing the circulation in a mid-latitude gyre but are slightly skew.

The equation for the interior of this region then has the form

$$\psi_x = \frac{\tau^{*}\pi}{2L\sin\theta_0} \frac{R}{2\Omega\sin\theta} \sin\frac{\pi y}{2L}$$

where $\beta = \frac{2\Omega\sin\theta}{R}$ has been written out explicitly. Like the Sverdrup equation, this equation can be integrated immediately to give:

$$\psi = \frac{\tau^{*}\pi R}{4\Omega L\sin\theta_0} \quad \frac{\sin\frac{\pi y}{2L}}{\sin\theta} \left[x - x_0(y)\right]$$

where $x_0(y)$ is an arbitrary function of y which must be determined from the boundary conditions. The stream function can be written in non-dimensional form so that it can be easily sketched

$\psi = \psi_0 \psi'$

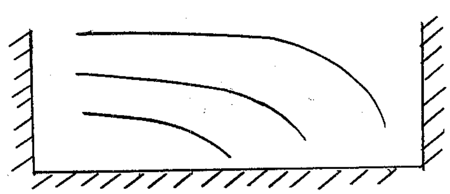
$$\psi_0 = \frac{\tau^{*}\pi R M}{4\Omega L \sin\theta_0} \qquad \psi' = \frac{\sin\frac{\pi y'}{2}}{\sin\theta}\left[x' - x_0'\right]$$

$Ly' = y$; $Mx' = x$; $Mx'_0 = x_0$ (M is the length of the rectangular basin which we obtain when we transform coordinates; the eastern and western boundaries of the rectangular model correspond to the western and eastern coasts of South America and the peninsula of the Antarctic continent which extends towards South America, respectively).

For the complete meridional barrier the boundary condition that $\psi' = 0$ on the eastern boundary is satistied if $x_0 \equiv 1$. ψ' is made equal to zero on the western and southern boundaries by fitting boundary layers to the interior solution. In a later section we shall see that such boundary layers can be constructed and are of limited extent. For the moment we shall examine the gross features of the interior flow and simply use the boundary layers to close the flow without considering the details of these boundary layer regions. The stream lines for the interior can be found from the equation

$$x' = 1 - \frac{\sin\theta}{\sin\frac{\pi y'}{2}}\psi'$$

$\dfrac{\sin\theta}{\sin\frac{\pi y'}{2}}$ is tabulated for fixed intervals of y' (say 0.1 steps) for y' from 0 to -1.3. A value for ψ' is chosen and the value for x' corresponding to each y' is found for that stream line.



The sketch shows the form of the interior solution. This solution looks very similar to the upper half of a mid-latitude

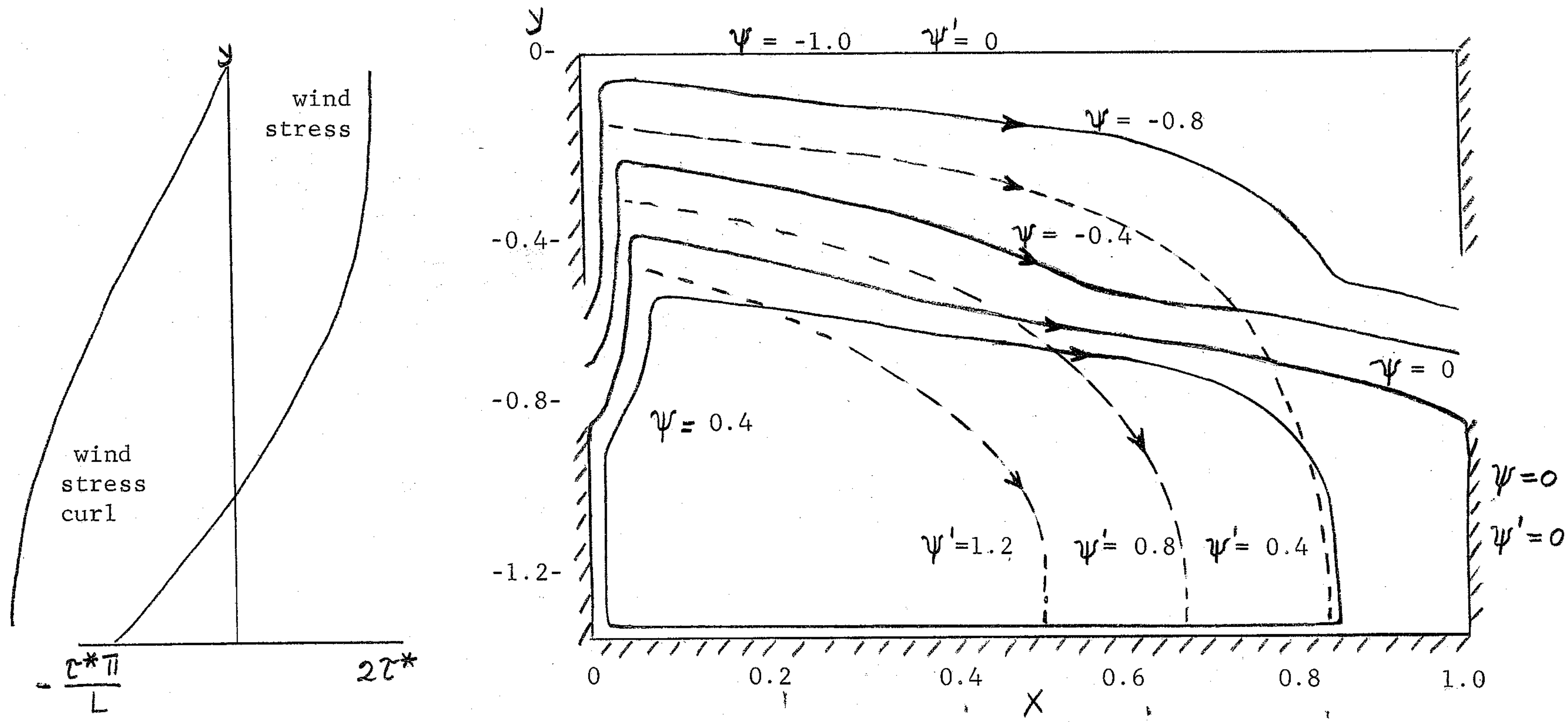


Figure 2. The stream function (ψ), wind stress, and wind stress curl for a model in which a flow through Drake Passage is required by the boundary conditions. The stream function (ψ') for a complete meridional barrier is shown for comparison.

gyre (see for example Stommel, 1960, page 92). It does not look very much like the Antarctic circumpolar current (even when we note that there is approximately a 10 to 1 vertical exaggeration). This result is not very surprizing since we know that there is, in the real case, a hole in the barrier corresponding to the Drake Passage between South America and Antarctica. We know also that there is a flow through this gap. Thus ψ' will be zero on the lower part of the meridional boundary and $-c$ on the upper part. The Drake Passage is taken to be centered on 60°S and to be 5° of latitude wide. This new boundary condition can be satisfied by picking a special form for $x_0'(y)$.

$$
\begin{aligned}
x_0'(y) &= 1 \qquad y' < -\tfrac{5}{6} \\
&= 1 + 3c \frac{\sin\theta}{\sin\frac{\pi y'}{2}} \left(y' + \tfrac{5}{6}\right) \qquad -\tfrac{5}{6} < y' < -\tfrac{1}{2} \\
&= 1 + c \frac{\sin\theta}{\sin\frac{\pi y'}{2}} \qquad -\tfrac{1}{2} < y' < 0
\end{aligned}
$$

The stream lines are obtained in the same way as before. The equation is now

$$
x' = x_0'(y) - \frac{\sin\theta}{\sin\frac{\pi y'}{2}} \psi' \quad \text{for } c = 1
$$

This specification of a more realistic boundary condition changes the flow pattern considerably. The stream line pattern is shown in Figure 2. The wind stress and its curl are also shown. The interior solution for the case of a complete barrier (dashed lines) is shown for comparison. The boundary layer regions are used to join the stream lines without considering the details of these regions. There is now a strong circumpolar flow in the upper part of the basin and a gyre south of the Drake Passage. The model looks somewhat like the real circulation (note that there is approximately a 10 to 1 vertical exaggeration because of the differences in the north-south and east-west scales). In a sense, this result is not surprizing either since, by requiring a flow through the Drake Passage, we have forced the solution to look more like the real one.

There are, however, some very marked differences between this model flow and the real Antarctic circumpolar current. The southern gyre is stronger and larger than the real case but this is probably an artifact of our modeling of the southern boundary. A more realistic southern boundary would probably split the southern gyre into two weaker gyres because near $x' = 0.5$ the southern boundary extends northward to about $y' = -5/6$. However, the boundary conditions would be very difficult to satisfy in such a more realistic model. The differences in the circumpolar current in the model and in the real world are more difficult to rationalize. In the real current the western boundary region is not so well defined and on the west coast of South America (the eastern side of our model) the stream lines are quite crowded and have strong curvature. These effects might be topographic and it might be possible to use the equation including bottom topography to get a more realistic model. If the stream lines were forced northward near the eastern boundary by a ridge then they would necessarily become more crowded near the eastern boundary (such an effect appears to be present in the real ocean). However, in such a region of strong flow and high curvature of stream lines our simple linear interior equation may not include some essential part of the physics of the problem.

THE BOUNDARY LAYERS

The solutions for the western boundary layer are very similar to those obtained for a beta-plane ocean at mid-latitude except that with each χ - differentiation a factor $\frac{\sin\theta_0}{\sin\theta}$ occurs (so that our equations are consistent with a transformation from spherical coordinates). For a model controlled by bottom friction (discussed in Robinson's lectures) the equation is (in non-dimensional form)

$$\left(\frac{\sin\theta_0}{\sin\theta}\right)^2 \psi_{\xi\xi} + \frac{\sin\theta_0}{\sin\theta}\,\frac{\beta}{f^{1/2}}\,\psi_\xi = 0$$

and the solution is

$$C_1 + C_2\,\exp\left(-\frac{\sin\theta}{\sin\theta_0}\,\frac{\beta}{f^{1/2}}\,\xi\right)$$

where $\xi = \gamma^{-\frac{1}{2}}\chi$. The solution for the beta-plane does not contain the factor $\frac{\sin\theta_0}{\sin\theta}$. The non-dimensional boundary layer thickness $\gamma^{\frac{1}{2}}\frac{f^{1/2}}{\beta}\,\frac{\sin\theta_0}{\sin\theta}$ varies with latitude. The boundary layer becomes narrower as one goes northward.

If lateral friction controls the boundary layer, then a solution of the Munk type is obtained (see Stommel, 1960, pages 93-103 for details). Again the boundary layer thickness is slightly modified. The equation has the form

$$\left(\frac{\sin\theta_0}{\sin\theta}\right)^4 \psi_{4\xi} - \frac{\sin\theta_0}{\sin\theta}\beta\psi_{\xi} = 0$$

where $\xi = \Gamma^{-\frac{1}{3}}x$ ($\Gamma = \frac{A_{oh}}{f_0 M^2}$, A_{oh} is the horizontal eddy viscosity, f_0 is the reference value of the Coriolis parameter and M is the width of the ocean). The boundary layer solution has the form

$$e^{-\frac{1}{2}a(y)\xi}\left(\alpha\cos\frac{\sqrt{3}}{2}a(y)\xi + \delta\sin\frac{\sqrt{3}}{2}a(y)\xi\right)$$

where $a(y) = \beta^{\frac{1}{3}}\frac{\sin\theta}{\sin\theta_0}$.

The non-dimensional boundary layer thickness is $\Gamma^{\frac{1}{3}}\beta^{-\frac{1}{3}}\frac{\sin\theta_0}{\sin\theta}$. In the Munk model the $\frac{\sin\theta_0}{\sin\theta}$ factor does not appear. Again the boundary layer becomes narrower as one goes northward.

These boundary layer solutions can be applied to both the model without and the model with the Drake Passage. The bottom friction model satisfies only one boundary condition - no flow normal to solid boundaries; the Munk-type model, because it has two solutions with decaying behavior, can also be made to satisfy a no-slip condition, that is, no flow along the solid boundaries.

The southern boundary layer is more interesting and more complicated since the solutions are infinite series expansions. The scaling is now done in the y-direction. The non-dimensional Munk-type equation has the form $-\Gamma\nabla^4\psi + \frac{\sin\theta_0}{\sin\theta}\beta\psi_x = T'$

Note that in the boundary layer β , $\sin\theta$, and T' are all approximately constants. The interior solution gives the inhomogeneous term T' so we must find a boundary layer solution for the equation

$$-\Gamma \nabla^4 \psi + \frac{\sin \theta_0}{\sin \theta} \beta \psi_x = 0$$

In using stretched coordinates only the highest order derivatives will occur since any terms with lower order differentiations will be of lower order in Γ. We scale y with Γ to obtain a stretched coordinate $\eta = \Gamma^{-\frac{1}{4}} \beta^{\frac{1}{4}} y$ $\left(\beta = \frac{1}{R} \tan \theta_0\right)$ and the boundary layer equation becomes

$$\psi_{4\eta} - \psi_x = 0$$

The boundary conditions are that $\psi \to 0$ as $\eta \to \infty$, that $\psi(0,x)$ is proportional to $(1-x)$ so that the total stream function vanishes at the boundary. A further boundary condition is that $\frac{\partial \psi_{total}}{\partial y} = 0$ at the boundary.

If we try a solution of the form $\psi = f(x) g(\eta)$, then

$$g'''' f - g f' = 0$$

$$\frac{g''''}{g} = \frac{f'}{f}$$

If we choose the separation constant to be ik where k is real, and $i = \sqrt{-1}$ then

$$f' - ikf = 0$$

$$g'''' - ikg = 0$$

Solutions for f are e^{ikx}. For g we try $g = e^{a\eta}$

$$a^4 - ik = 0$$

$$a^4 = ik$$

For $k > 0$ the two solutions with negative real parts are

$$a_1 = k^{1/4} e^{i\frac{5\pi}{8}}$$

$$a_2 = k^{1/4} e^{i\frac{9\pi}{8}}$$

For $k < 0$ the two solutions with negative real parts are

$$a_1 = |k|^{1/4} e^{i\frac{7\pi}{8}}$$
$$a_2 = |k|^{1/4} e^{i\frac{11\pi}{8}}$$

At $\eta = 0$ we must construct a series expansion for $1 - x$ using the functions e^{ikx}. $1 - x$ is extended over the interval 0 - 2 although the solution is only used over the interval 0 - 1. $\int_0^2 (1-x)\,dx = 0$ so there is no term for $k = 0$ which is not of a boundary layer type. The eigenvalues k are determined by the x-interval; $k = \pm n\pi$, $n = 1, 2, 3 \dots$. The general solution has the form

$$\psi = \sum_{k=-\infty}^{\infty} a_k e^{ikx} (\alpha_k e^{a_{1k}\eta} + \beta_k e^{a_{2k}\eta})$$

where $a_k = \frac{1}{2}\int_0^2 (1-x) e^{-ikx} dx$, $\alpha_k + \beta_k = 1$. The ratio $\frac{\alpha_k}{\beta_k}$ will be determined by the no-slip boundary condition.

A similar solution can be constructed for a boundary layer controlled by bottom friction but this boundary layer will not satisfy the no-slip condition.

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Thermal Circulation in a Deep Rotating Annulus

Sten Gösta Walin

Background

The symmetric flow in a rotating annulus of square cross section with the side walls held at different temperatures has been investigated by Robinson. His solution is obtained by expanding in the thermal Rossby number. This means that the heat equation must be dominated by conduction. The main feature of the solution is an interior with pure zonal motion and thin viscous boundary layers along the walls.

However there are some recent experimental results for a deep rectangular cross section which show an internal structure with, apart from the zonal motion, a relatively strong circulation in the vertical plane, more or less concentrated into a jet. This motion seems to be located where the zonal motion has sharp vertical gradients.

As a first step towards an understanding of this phenomenon, we investigate here some effects of making the height of the annulus large compared to its width.

The fundamental equations

With the notation used by Robinson, the steady symmetrical flow in a rotating annulus of liquid is governed by the following set of equations:

$$-\nu\nabla'^2 \underset{\sim}{V}' + \underset{\sim}{V}'\nabla\underset{\sim}{V}' + 2\Omega\times\underset{\sim}{V}' - \alpha g T' k + \rho_0^{-1}\nabla' p' = 0$$

$$-\kappa\nabla'^2 T' - \underset{\sim}{V}'\nabla' T' = 0 \qquad (1)$$

$$\nabla'\cdot\underset{\sim}{V}' = 0$$

The assumption used to derive these equations can be found in Robinson's paper. Here we only note that all effects of the curvature of the tank are cancelled by making $L/R \ll 1$ and $\Omega^2 R/g \ll 1$ (fig. 1). Thus the equations describe the flow in a rotating infinite straight channel.

The boundary conditions on the system are assumed to be (fig. 1):

$$\left.\begin{array}{l} V' = 0 \quad \text{at} \quad x' = \pm\frac{1}{2}L \text{ and } z = \pm\frac{1}{2}H \\ T' = \pm\frac{1}{2}\Delta T \text{ at } x' = \pm\frac{1}{2}L \\ \dfrac{\partial T'}{\partial z} = 0 \quad \text{at} \quad z' = \pm\frac{1}{2}H \end{array}\right\} \qquad (2)$$

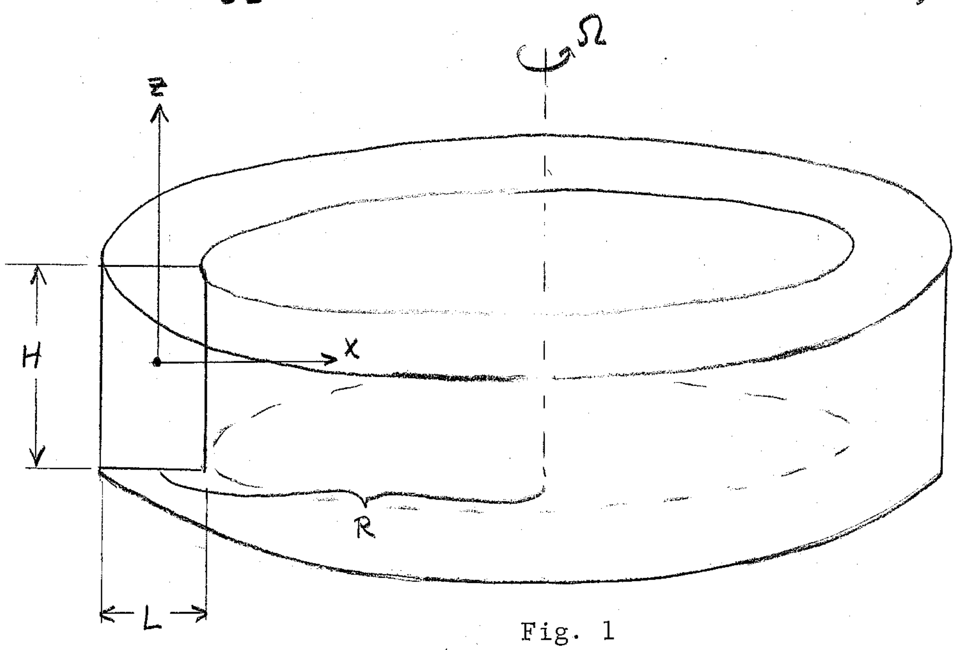

Fig. 1

Equations (1) are non-dimensionalized according to:

$$x' = Lx \;,\; z' = Hz \;,\; T' = \Delta T \cdot T \text{ and } V' = c \cdot V$$

For the velocity scale c we introduce:

$$c = \frac{\alpha g \Delta T}{2\Omega}$$

We shall consider the symmetric state of motion characterized by $\frac{\partial}{\partial y} = 0$ and consequently we can introduce a stream function ψ , defined by $u \equiv \psi_z$, $w \equiv -\psi_x$. Eliminating the pressure, we arrive at the following set of non-dimensional equations:

$$\left.\begin{aligned} &-\varepsilon \nabla_\delta^4 \psi + \beta\delta J(\psi, \nabla_\delta^2 \psi) - \delta v_z + T_x = 0 \qquad &(3a)\\ &-\varepsilon \nabla_\delta^2 V + \beta\delta J(\psi, v) + \delta \psi_z = 0 \qquad &(3b)\\ &-\varepsilon \nabla_\delta^2 T + \beta\sigma\delta J(\psi, T) = 0 \qquad &(3c) \end{aligned}\right\}$$

with the boundary conditions

$$V = \psi = \psi_x = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2} \qquad (4a)$$

$$V = \psi = \psi_z = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \qquad (4b)$$

$$T = \pm\tfrac{1}{2} \quad \text{at} \quad x = \pm\tfrac{1}{2} \qquad (4c)$$

$$T_z = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \qquad (4d)$$

in equations (1) $\varepsilon = \dfrac{\nu}{2\Omega L^2}$; a viscous parameter

$\beta = \dfrac{\alpha g \Delta T}{(2\Omega)^2 L}$; the thermal Rossby number

$\delta = \frac{L}{H}$; a geometrical parameter

$\sigma = \frac{\nu}{\kappa}$; the Prandtl number.

Further in equations (3) we have:

$$\nabla_\delta^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \delta^2 \frac{\partial^2}{\partial z^2}\right) \text{ and } \nabla_\delta^4 \equiv \left(\nabla_\delta^2\right)\left(\nabla_\delta^2\right)$$

$$J(A, B) \equiv A_z B_x - A_x B_z$$

All the parameters except δ are based on the width of the annulus (L). The problem treated by Robinson is obtained by setting $\delta = 1$. In all applications ε is a very small number giving rise to boundary layer type solutions. Here (as well as in Robinson's work) β is assumed small which makes an analytic approach possible. The validity of Robinson's solution further requires $\beta\sigma \cdot \varepsilon^{-\frac{1}{2}}$ to be small which is a still stronger condition. In this case, we have the additional assumption that $\delta \ll 1$. We will find however that the type of solution depends critically on the relative magnitude of ε and δ . Thus we have mainly two cases:

1) $\delta \lesssim \varepsilon \ll 1$

2) $\varepsilon \ll \delta \ll 1$

These two cases will be treated separately in what follows.

Extremely small δ , viscous interior.

For $\delta \to 0$ equation (3a) reduces to $\varepsilon \Psi_{4x} + T_x = 0$. Thus the stream function becomes $O(\varepsilon^{-1})$. We therefore introduce new scales for Ψ and V according to $\Psi = \varepsilon \psi$ and $V = \varepsilon V$ which gives:

$$-\nabla_\delta^4 \Psi + \beta \frac{\delta}{\varepsilon^2} J(\Psi, \nabla_\delta^2 \Psi) - \frac{\delta}{\varepsilon} V_z + T_x = 0 \qquad (5a)$$

$$-\nabla_\delta^2 V + \beta \frac{\delta}{\varepsilon^2} J(\Psi, V) + \frac{\delta}{\varepsilon} \Psi_z = 0 \qquad (5b)$$

$$-\varepsilon \nabla_\delta^2 T + \beta\sigma \frac{\delta}{\varepsilon} J(\Psi, T) = 0 \qquad (5c)$$

From these equations can be seen that for $\delta \ll \varepsilon$, $\frac{\sigma\beta}{\varepsilon}$ and $\beta/\varepsilon \ll 1$ we have an interior solution that is viscously balanced. This solution is characterized by no z-dependence and is given by

$$T_{xx} = 0, \quad \Psi_{4x} = 1, \quad V = 0$$

with boundary conditions

$$T = \pm \tfrac{1}{2}, \quad \Psi = \Psi_x = 0 \quad at \quad x = \pm \tfrac{1}{2}$$

The solution is obtained as

$$\left.\begin{aligned} \Psi &= \psi^{\infty} = C_0 (1 - 8x^2 + 16x^4) \\ T &= x \end{aligned}\right\} \qquad (6)$$

with $C_0 = (24.16)^{-1}$.

Now it remains to be seen if this solution can be joined to the horizontal boundaries at $z = \pm \frac{1}{2}$.

To proceed we expand the involved fields after $\beta \frac{\delta}{\varepsilon^2}$.

We introduce: $\Psi = \sum_{n=0}^{\infty} \left(\beta \frac{\delta}{\varepsilon^2}\right)^n \psi^n, \; V = \sum_{n=0}^{\infty} \left(\beta \frac{\delta}{\varepsilon^2}\right)^n v^n, \; T = \sum_{n=0}^{\infty} \left(\beta \frac{\delta}{\varepsilon^2}\right)^n T^n$

and obtain the lowest order set of equations.

$$\left.\begin{aligned} -\nabla_\delta^4 \psi^0 - \frac{\delta}{\varepsilon} v_z^0 + T_x^0 &= 0 \qquad (7a) \\ -\nabla_\delta^2 v^0 + \frac{\delta}{\varepsilon} \psi_z^0 &= 0 \qquad (7b) \\ \nabla_\delta^2 T^0 &= 0 \qquad (7c) \end{aligned}\right\}$$

Equation (4c) gives $T^0 = X$ which is introduced in equation (4a).

In order to transfer the inhomogeneity to the boundary conditions we introduce $\psi^0 = \varphi^0 + \psi^\infty$

$$-\nabla_\delta^4 \varphi^0 - \frac{\delta}{\varepsilon} v_z^0 = 0 \qquad (8a)$$

$$-\nabla_\delta^2 v^0 + \frac{\delta}{\varepsilon} \varphi_z^0 = 0 \qquad (8b)$$

with boundary conditions:

$$v^0 = \varphi^0 = \varphi_x^0 = 0 \;\; at \;\; x = \pm \tfrac{1}{2} \qquad (9a)$$

$$v^0 = \varphi + \varphi^\infty = \varphi_z = 0 \;\; at \;\; z = \pm \tfrac{1}{2} \qquad (9b)$$

To satisfy the boundary conditions at $z = \pm \frac{1}{2}$ we must have boundary layers where the highest order z-derivatives in equations (8) become important. The equations for the boundary layers are obtained with the transformation

$$\zeta' = \varepsilon^{-\frac{1}{2}} \delta^{-1} (z \pm \tfrac{1}{2}) \;\; \text{near} \;\; z = \mp \tfrac{1}{2}$$

$$\left.\begin{aligned} \varphi^0_{4\zeta'} + \varepsilon^{\frac{1}{2}} v^0_{\zeta'} &= 0 \qquad (10a) \\ v^0_{2\zeta'} - \varepsilon^{-\frac{1}{2}} \varphi^0_{\zeta'} &= 0 \qquad (10b) \end{aligned}\right\}$$

Equations (10) are the Ekman boundary layer equations. The solution is well-known and sets the following requirement on the flow for large ζ' :

$$v^0 \pm \sqrt{2}\,\varepsilon^{-\frac{1}{2}}\left(\varphi^0 + \psi^{\infty}\right) = 0 \qquad z = \mp\frac{1}{2}$$

Clearly this requirement is not fulfilled by our solution ψ^{∞} (that is with $\varphi^0 = 0$ for large ζ') thus another boundary layer will be required between the Ekman layer and the interior.

To obtain the equations in this region we try a new stretching of the z-coordinate:

$\zeta = \frac{\varepsilon}{\delta}\left(z \pm \frac{1}{2}\right)$ near $z = \mp\frac{1}{2}$ which gives:

$$\left.\begin{aligned} -\nabla_\varepsilon^4 \varphi^0 - v^0_\zeta &= 0 \qquad (11a)\\ -\nabla_\varepsilon^2 v^0 + \varphi^0_\zeta &= 0 \qquad (11b)\end{aligned}\right\}$$

with the boundary condition (for the case $z \rightarrow -\frac{1}{2}$)

$$v^0 + \sqrt{2}\,\varepsilon^{-\frac{1}{2}}\left(\varphi^0 + \psi^{\infty}\right) = 0 \quad \text{at} \quad \zeta = 0 \qquad (11c)$$

Equations (8) state that φ^0 and v^0 are of the same order in this region. We now order equations (11) with respect to the small parameter $\varepsilon^{\frac{1}{2}}$ and obtain for the two lowest orders

$$\left.\begin{aligned} \varphi^{00}_{4x} + v^{00}_\zeta &= 0 \qquad (12a)\\ v^{00}_{2x} - \varphi^{00}_\zeta &= 0 \qquad (12b)\end{aligned}\right\}$$

$$\varphi^{00} + \psi^{\infty} = 0 \quad \text{at} \quad \zeta = 0 \qquad (12c)$$

$$\left.\begin{aligned} \varphi^{01}_{4x} + v^{01}_{\zeta} &= 0 \\ v^{01}_{2x} - \varphi^{01}_{\zeta} &= 0 \end{aligned}\right\} \qquad \begin{aligned}(13a)\\(13b)\end{aligned}$$

$$v^{00} + \sqrt{2}\,\varphi^{01} = 0 \quad at \quad \zeta = 0 \qquad (13c)$$

For $\zeta \to \infty$ we require that all fields disappear. The boundary conditions at $x = \pm \frac{1}{2}$ are the same for (φ^{00}, v^{00}) and (φ^{01}, v^{01}) as for φ^{0}, v^{0}.

Equations (13) have the forcing $v^{00}(\zeta = 0)$ in the boundary condition which expresses the fact that the zonal velocity over the Ekman layer introduces a φ-field of order $\varepsilon^{\frac{1}{2}}$.

Equations (12) can be solved by separation of variables. Eliminating v^{00} we obtain:

$$\left.\begin{aligned} &\varphi^{00}_{6x} + \varphi^{00}_{\zeta\zeta} = 0 && (14a)\\ &\varphi^{00} = \varphi^{00}_{x} = \varphi^{00}_{4x} = 0 \quad at \quad x = \pm\tfrac{1}{2} && (14b)\\ &\varphi^{00} + \psi^{00} = 0 \quad at \quad \zeta = 0 && (14c) \end{aligned}\right\}$$

This system has a solution of the form

$$\varphi^{00} = \sum_{n=0}^{\infty} A^{0}_{n}\, \varphi_{n}\, e^{-\alpha_n \zeta}$$

where $\alpha_n > 0$.

The φ_n and the corresponding α_n are the solutions to:

$$\varphi_{n6x} + \alpha_n^2 \varphi_n = 0 \qquad (15a)$$

$$\varphi_n = \varphi_{nx} = \varphi_{n4x} = 0 \quad at \quad x = \pm\tfrac{1}{2} \qquad (15b)$$

which is a well-defined eigenvalue problem. The coefficients

A_n are determined by the boundary condition at $\zeta = 0$.

$$\sum_{n=0}^{\infty} A_n^{\circ} \varphi_n + \psi^{\infty} = 0 \tag{16}$$

To determine A_n we need the orthogonality properties of the function φ_n.

Suppose we have two functions φ_n and φ_n^* that satisfy

$$\left.\begin{aligned} L\varphi_n + \alpha_n^2 \varphi_n &= 0 \\ L\varphi_m^* + \alpha_m^2 \varphi_m^* &= 0 \end{aligned}\right\}$$

where $L \equiv \dfrac{d^6}{dx^6}$

Multiply the first equation with φ_m^* and the second with φ_n, subtract and integrate from $-\frac{1}{2}$ to $+\frac{1}{2}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\varphi_m^* L\varphi_n - \varphi_n L\varphi_m^* + (\alpha_n^2 - \alpha_m^2)\varphi_n \varphi_m^*\right) dx = 0 \tag{17}$$

We now look for φ_m^* such that

$$\int \varphi_m^* L \varphi_n \, dx = \int \varphi_n L \varphi_m^* \, dx$$

Integrating by parts we find that φ_m^* must satisfy the boundary conditions $\varphi_m^* = \varphi_{m2x}^* = \varphi_{m3x}^* = 0$

From equation (15) we find that φ_{m4x} has this property and we define $\varphi_m^* = \varphi_{m4x}$

From equation (17) we now obtain for $n \neq m$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_n \varphi_m^* dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_n \varphi_{m4x} \, dx = 0 \tag{18}$$

which is the required orthogonality relation.

From equation (12a) the v field is obtained as

$$v^{\infty} = \sum \frac{1}{\alpha_n} A_n^{\circ} \varphi_{n4x} e^{-\alpha_n \zeta} = \sum \frac{1}{\alpha_n} A_n^{\circ} \varphi_n^{*} e^{-\alpha_n \zeta}$$

Thus the zonal velocity field is obtained as a series in the adjoint functions φ_n^{*} .

We will now use equation (18) to derive a relation for A_n° . Multiply equation (16) with φ_m^{*} , integrate from $-\frac{1}{2}$ for $\frac{1}{2}$ and use (18):

$$A_m^{\circ} \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_m^{*} \varphi_m \, dz = - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi^{\infty} \varphi_m^{*} \, dx \tag{19}$$

which determines the A_m° .

The right-hand side can be simplified because of the properties of ψ^{∞} and φ_m^{*} . We know that $\psi^{\infty} = \psi_x^{\infty} = 0$ at $x = \pm \frac{1}{2}$ and that $\psi_{4x}^{\infty} = 1$. Integrating by parts we obtain:

$$\int \psi^{\infty} \varphi_m^{*} \, dx = \Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{\alpha_m^2} \varphi_{m5x}$$

Thus we have from (19)

$$A_m^{\circ} = \Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{\alpha_m^2} \varphi_{m5x} \cdot \left(\int \varphi_m^{*} \varphi_m \, dx \right)^{-1} \tag{20}$$

Clearly from equation (20), $A_m^{\circ} = 0$ if φ_{m5x} is an even function of x . This means that we can restrict φ_m to be even (which makes φ_{m5x} odd).

To proceed, we must look at the explicit form of the set of functions φ_m .

Even solutions to equations (15) are of the form:

$$\varphi_n = A\cosh\beta_n\sqrt{3}X\cos\beta_n X + B\sinh\beta_n\sqrt{3}X\sin\beta_n X + C\cos 2\beta_n X \tag{21}$$

where $(2\beta_n)^3 = \alpha_n$ and $\beta_n > 0$.

A, B, C and the eigenvalue α_n are determined from the boundary conditions (15b) and a normalization condition.

The boundary conditions give rise to a homogeneous system of equations in A, B and C. The determinant set equal to zero gives the following equation for β_n after some algebra:

$$\sin(2\beta_n) + \cosh\sqrt{3}\beta_n \sin\beta_n + \sqrt{3}\sinh\sqrt{3}\beta_n\cos\beta_n = 0 \tag{22}$$

For large n we have the approximate solution:

$$\tan\beta_n = -\sqrt{3} \quad \text{or} \quad \beta_n \cong n\pi + \tfrac{2}{3}\pi \tag{23}$$

For $n=0$ we get $\beta_0 \sim \frac{2}{3}\pi \sim 2$ which is not very far from the correct value. Thus $\alpha_0 = (2\beta_n)^3 \sim \frac{1}{2}\cdot 10^2$ is seen to be a rather large number indicating that the maximum penetration depth is only a small fraction of $\frac{1}{\varepsilon}\cdot L$.

Without going into the details of the form of φ_n for large n we can derive the decay properties of A_n^0 for large n. For large n the dominant contribution to φ_n in the interior comes from the term $C\cdot\cos 2\beta_n X$. The other terms are of the same order of magnitude only close to $X = \pm\frac{1}{2}$. For convenience we put $C=1$. This makes $\varphi_{max} = \varphi_m^* \sim (2\beta_n)^4$ and $\varphi_{mxx} \sim (2\beta_n)^5$, both in the interior and close to $X = \pm\frac{1}{2}$. The main contribution to $\int \varphi_m^* \varphi_m \, dx$ comes from the term $\cos 2\beta x$ and we have $\int \varphi_m^* \varphi_m \, dx \sim (2\beta_n)^4$. Inserting in equation (20), we get:

$$A_m^0 \sim \frac{1}{\alpha_m^2}(2\beta_n)^5 \cdot (2\beta_n)^{-4} = (2\beta_n)^{-5} \sim n^{-5}$$

Thus we have a very rapid decay of the coefficients A_m^0.

We will now look briefly at the first order in $\varepsilon^{\frac{1}{2}}$. Equations (13a) and (13b) are identical but ψ^{∞} is replaced by $\frac{1}{\sqrt{2}} v^{00}(\zeta=0)$ in the boundary condition.

We seek a representation of φ^{01} of the form $\varphi^{01} = \sum A_n^1 \varphi_n e^{-\alpha_n \zeta}$ where A_n^1 if given is:

$$\frac{1}{\sqrt{2}} v^{00}_{\zeta=0} + \sum A_n^1 \varphi_n = 0$$

with $v^{00}_{\zeta=0} = \sum \frac{1}{\alpha_n} A_n \varphi_n^*$.

Multiplying with φ_m^* and integrating we obtain:

$$\sum_{n=0}^{\infty} \frac{1}{\alpha_n} A_n^0 \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_m^* \varphi_n^* \, dx = A_m^1 \int \varphi_m^* \varphi_m \, dx$$

The behavior of A_m^1 for large m is thus seen to depend on the behavior of $\int \varphi_m^* \varphi_n^* \, dx$ for large n. By integration by parts using the fact that $\varphi_m^* = \varphi_{m4x}$ we can derive the following relations:

$$\int \varphi_m^* \varphi_n^* \, dx = \frac{1}{\alpha_m^2 - \alpha_n^2} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \left(\alpha_m^2 \varphi_{m2x} \varphi_{n5x} - \alpha_n^2 \varphi_{m5x} \varphi_{n2x} \right)$$

By using an expression for φ_m for large m we obtain:

$$\int \varphi_m^* \varphi_n^* \, dx \sim \frac{\beta_n^5 \cdot \beta_m^5}{(\beta_m^3 + \beta_n^3)}$$

but $\int \varphi_m^* \varphi_n \, dx \sim \beta_n^4$ and $\alpha_n \sim \beta_n^3$

thus $A'_m \sim \sum_{n=1}^{\infty} k_n \frac{\beta_n^2 \beta_m}{\beta_m^3 + \beta_n^3} A_n$ where $k_n \sim O(1)$.

This representation of A'_m is seen to be convergent for all values of m. A rough estimate of the sum gives the result that $A'_m \sim \frac{1}{m^2}$ for large m.

Thus for some value $m = m_K$ $A'_{m_K} \sim \epsilon^{-\frac{1}{2}} A^{\circ}_{m_K}$ at which value of m the representation for $\varphi^{\circ\circ}$ shall be truncated. Since $A^{\circ}_m \sim \frac{1}{m^5}$ we have $m_K^3 \sim \epsilon^{-\frac{1}{2}}$ or $m_K \sim \epsilon^{-\frac{1}{6}}$.

If we go to first order in the expansion after $\beta \frac{\partial}{\epsilon^2}$ we find contributions only in the boundary layer of thickness $\frac{1}{\epsilon}$, all of which are of the magnitude $\frac{\epsilon}{\delta}$ or $\frac{\epsilon}{\delta} \cdot \sigma$ compared to the zero order solutions. Thus we require $\frac{\beta\sigma}{\epsilon} \ll 1$ for the zero order solution to be valid (since $\sigma > 1$).

We now describe briefly the picture we have obtained for the very high rotating annulus (fig. 2). In region III we have a viscously balanced vertical motion without zonal motion. In region II this flow closes and builds up a zonal velocity. At the edge of region I we have balance between the vertical variation of zonal velocity and the temperature gradient (thermal wind relation). Thus at the bottom of region II we have v_z = constant (however v is a function of x). Region I is an Ekman layer required to cancel the zonal velocity and does not otherwise affect the flow field.

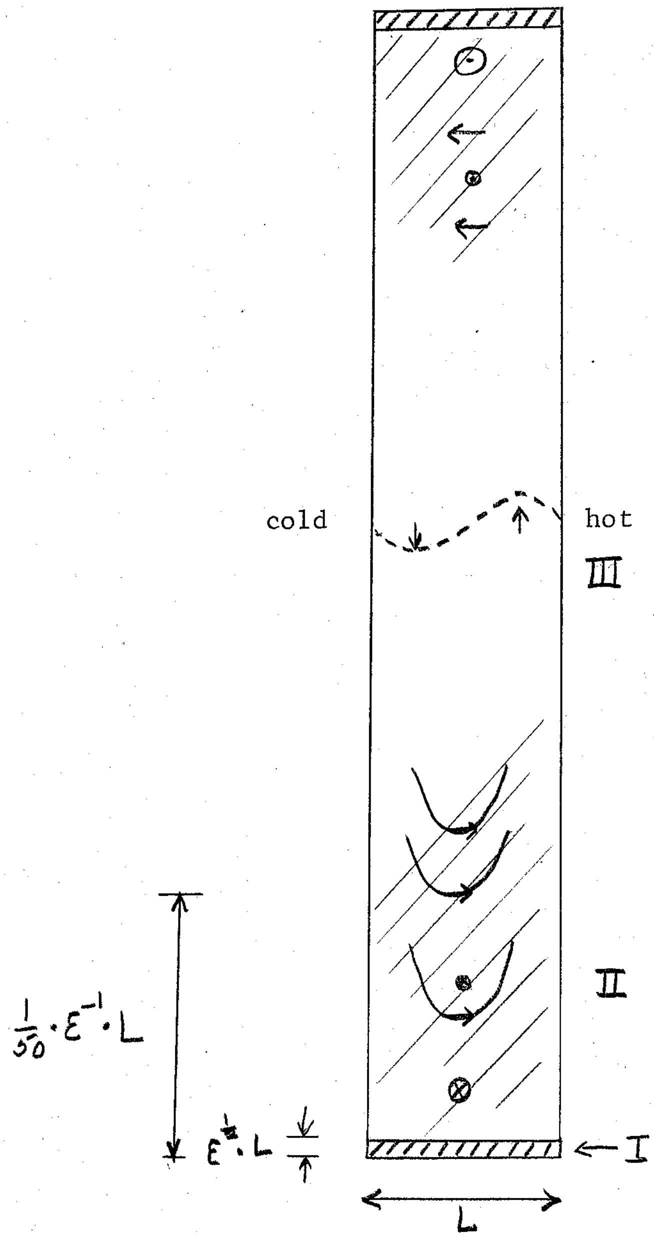

Fig. 2

Moderately small δ

We now assume $\varepsilon \ll \delta \ll 1$. From equation (3a) is found that the terms δv_z and T_x must balance in the interior. Accordingly we introduce $\Psi = \delta\psi$, $V = \delta V$ and obtain:

$$-\varepsilon'\nabla_\delta^4\Psi+\beta' J\left(\Psi,\nabla_\delta^2\Psi\right)-V_z+T_x=0 \quad (24a)$$

$$-\varepsilon'\nabla_\delta^2 V+\beta' J\left(\Psi,V\right)+\Psi_z=0 \quad (24b)$$

$$-\varepsilon'\nabla_\delta^2 T+\beta'\sigma J\left(\Psi,T\right)=0 \quad (24c)$$

where $\varepsilon'=\frac{\varepsilon}{\delta}$ and $\beta'=\frac{\beta}{\delta}$

We now look for modifications introduced by $\delta \ll 1$ in an approach similar to that of Robinson. We expand in β' and obtain the first order set:

$$-\varepsilon'\nabla_\delta^4\Psi^0-V_z^0+1=0 \quad (25a)$$

$$-\varepsilon'\nabla_\delta^2 V^0+\Psi_z^0=0 \quad (25b)$$

The interior is governed by the equations

$$V_z^0=1\ ,\quad \Psi_z^0=0 \quad (26)$$

The boundary layers at the top and bottom are obtained through the transformation

$$\zeta'=\varepsilon^{-\frac{1}{2}}\delta^{-1}z \quad (27)$$

and become ordinary Ekman layers of thickness $\varepsilon^{+\frac{1}{2}}L$ in natural coordinates. From these layers the internal Ψ^0 is determined to be $\Psi^0=\sqrt{2}\,\varepsilon^{\frac{1}{2}}\cdot\frac{1}{2}$.

The sidewall boundary layers are obtained with the transformation:

$$\xi=(\varepsilon')^{-\frac{1}{3}}x \quad (28)$$

We obtain the boundary layer equations

$$-\varepsilon'^{-\frac{1}{3}}\psi^{0}_{4\xi} - v^{0}_{z} = 0 \qquad (29a)$$

$$-\varepsilon'^{\frac{1}{3}}v^{0}_{2\xi} + \psi^{0}_{z} = 0 \qquad (29b)$$

with

$$v^{0}_{z} = -\tfrac{1}{2}z \qquad (29c)$$

$$\psi^{0} = -\sqrt{2}\,\varepsilon^{\frac{1}{2}}\tfrac{1}{2} \qquad (29d)$$

for $\xi = 0$

The inhomogeneities in the boundary condition can be treated separately. The first $\left(v^{0}_{z} = -\frac{1}{2}z\right)$ gives rise to a boundary layer solution with $v^{0} \sim O(1)$ and $\psi^{0} \sim O\left(\varepsilon'^{\frac{1}{3}}\right)$. The second gives a solution with $\psi^{0} \sim O(\varepsilon^{\frac{1}{2}})$ and $v^{0} \sim O(\varepsilon'^{-\frac{1}{3}}\varepsilon^{\frac{1}{2}})$. Since $\varepsilon' = \frac{\varepsilon}{\delta} \gg \varepsilon$ the contribution from the interior $\psi^{0} = -\sqrt{2}\,\varepsilon^{\frac{1}{2}}\frac{1}{2}$ can be neglected. This means that the Ekman layers have no effect on the sidewall boundary layers.

From equation (28) we can see that the thickness of the sidewall boundary layers is $(\varepsilon')^{\frac{1}{3}}L = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{3}}L$. Thus the boundary layer structure disappears when $\varepsilon \sim \delta$ which is consistent with our earlier result. The maximal contribution to ψ in the sidewall boundary layer is $O\left(\varepsilon'^{\frac{1}{3}}\right)$. The vertical velocity ψ_z in these layers is $O(1)$ since $\psi_z \sim (\varepsilon')^{-\frac{1}{3}}\psi$. Thus the vertical velocity along the sidewalls is always of the same order of magnitude as the interior zonal velocity.

The explicit form of the boundary layer solution necessary to cancel the interior v field can be obtained from the case

$\delta = 1$ by replacing ε by ε'.

We now turn our attention to the first order in $\beta' = \frac{\beta}{\delta}$. The equations for the first order fields (ψ', v', T') are:

$$\left.\begin{aligned} -\varepsilon'\nabla_\delta^4\psi' - v_z' + T_x' + J(\psi^\circ, \nabla_\delta^2\psi^\circ) &= 0 \qquad (30a)\\ -\varepsilon'\nabla_\delta^2 v' + \psi_z' + J(\psi^\circ, v^\circ) &= 0 \qquad (30b)\\ -\varepsilon'\nabla_\delta^2 T' + \sigma J(\psi^\circ, T^\circ) &= 0 \qquad (30c)\end{aligned}\right\}$$

Equation (25c) becomes

$$-\varepsilon'\nabla_\delta^2\left(T'/\sigma\right) + \psi_z^\circ = 0$$

Thus T'/σ satisfies the same equation as v° and we have a particular solution

$$T_p' = \sigma v^\circ \qquad (31)$$

T_p' satisfies the boundary condition at $x = \pm\frac{1}{2}$ but not at $z = \pm\frac{1}{2}$. Thus we need a solution (T_h') to the homogeneous equation $\nabla^2 T' = 0$ satisfying the boundary conditions:

$T_{hz}' + T_{pz}' = 0$ at $z = \pm\frac{1}{2}$ and $T_h' = 0$ at $x = \pm\frac{1}{2}$.

T_h' is a solution to Laplace equation and will therefore only affect two square regions near top and bottom of the annulus. This means that the contributions from the two ends will not interact. For $z = \pm\frac{1}{2}$ we have:

$$v_z^\circ = \pm k\cdot\varepsilon^{-\frac{1}{2}}\cdot\delta^{-1} \quad \text{or from (31)}$$

$$T_{pz}' = \pm k\cdot\varepsilon^{-\frac{1}{2}}\cdot\delta^{-1}\cdot\sigma$$

where k is a constant of order 1.

Thus we have for T_h' :

$$\left.\begin{aligned} &\nabla_\delta^2 T_h' = 0 && (32a)\\ &T_{hz}' = \pm k\,\varepsilon^{-\frac{1}{2}}\delta^{-1}\cdot\sigma \quad \text{at } z = \pm\tfrac{1}{2} && (32b)\\ &T_h' = 0 \quad \text{at } x = \pm\tfrac{1}{2} && (32c)\\ &T_h' \to 0 \quad \text{for } \tfrac{z}{\delta} \to \infty && (32d) \end{aligned}\right\}$$

(32 a, c, d) give

$$T_u = \sum A_n \cos\alpha_n e^{-\alpha_n \zeta}$$

where $\alpha_n = (2n+1)\pi$ and $\zeta = \delta^{-1}(\pm z - \frac{1}{2})$

(32b) gives $A_n \sim \varepsilon^{-\frac{1}{2}}\cdot k\cdot\sigma$

Thus $T_h' \sim \varepsilon^{-\frac{1}{2}}\sigma$ which clearly dominates over T_p' .

The interior zonal velocity field forced by T' is obtained from the relation:

$$v_z' = T_x'$$

This relation can be integrated to give a solution with $v = 0$ at $z = \pm\frac{1}{2}$. This solution is independent of z except near $z = \pm\frac{1}{2}$ and is of order $\delta\cdot\varepsilon^{-\frac{1}{2}}\sigma$. V' is found positive near the hot wall and negative near the cold wall. The boundary layer contributions to ψ' induced by v' will be of order $\left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{3}} v' = \delta\varepsilon^{-\frac{1}{2}}\sigma\left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{3}}$. Since we have expanded in $\frac{\beta}{\delta}$ and we require $v' << v^0$ we obtain the requirement

$$\beta\cdot\varepsilon^{-\frac{1}{2}}\cdot\sigma << 1 \qquad (33)$$

Thus we can see that this condition is the same as in the case $\delta = 1$.

To find out if (33) is the strongest condition for the validity of our expansion we now look at the inertial inhomogeneities in equations (30 a,b). We find that the v fields forced by these inhomogeneities are of order $\left(\frac{\varepsilon}{\delta}\right)^{-\frac{1}{3}}$ which gives the condition

$$\beta \cdot \delta^{-1}\left(\frac{\varepsilon}{\delta}\right)^{-\frac{1}{3}} << 1 \tag{34}$$

By comparing (33) and (34) we find that condition (33) is the strongest if δ satisfies:

$$\left(\frac{\delta}{\varepsilon}\right)^{4} >> \sigma^{-6} \tag{35}$$

Since $\sigma \sim 6$ we have a large range of δ where this condition if fulfilled and (33) is the relevant condition.

It is well-known that the boundary layer of thickness $(\varepsilon')^{\frac{1}{3}}$ is not sufficient to take care of an arbitrary internal distribution of zonal velocity. We may also need a layer which is characterized by $\psi \sim O(\varepsilon^{\frac{1}{2}})v$ and $v_z = 0$. The scale of the x-coordinate shall be determined so that the terms $\varepsilon' \nabla_\delta^2 v$ and ψ_z can balance. This gives the boundary layer thickness $\left(\frac{\varepsilon}{\delta^2}\right)^{\frac{1}{4}} \cdot L$. Thus we find that when such a contribution is relevant it will lose its boundary layer character already when $\delta \sim \varepsilon^{\frac{1}{2}}$. In zero order we do not get any boundary layer of this kind. What happens in first order has not yet been established.

Summary

For $\delta \lesssim \varepsilon$ we have found that the side-wall boundary layers reach across the annulus. Thus viscosity is important in the whole region. For $\delta \ll \varepsilon$ it was found that the influence of the horizontal boundaries disappears within a distance of about $(\sigma O \varepsilon)^{-1} \cdot L$. Inside these "boundary layers" the motion consists entirely of vertical motion.

For $\varepsilon \ll \delta \ll 1$ an approach similar to that of Robinson was applied. It was found that the side-wall boundary layers grow thicker when δ is decreased, and become closed circulating cells with no net transportation.

A somewhat surprising result was obtained regarding the range of validity of the zero order solution. It was found that if $\left(\frac{\delta^4}{\varepsilon}\right) \gg \sigma^{-1}$ we require $\beta \varepsilon^{-\frac{1}{2}} \sigma \ll 1$ which condition is not affected by δ.

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