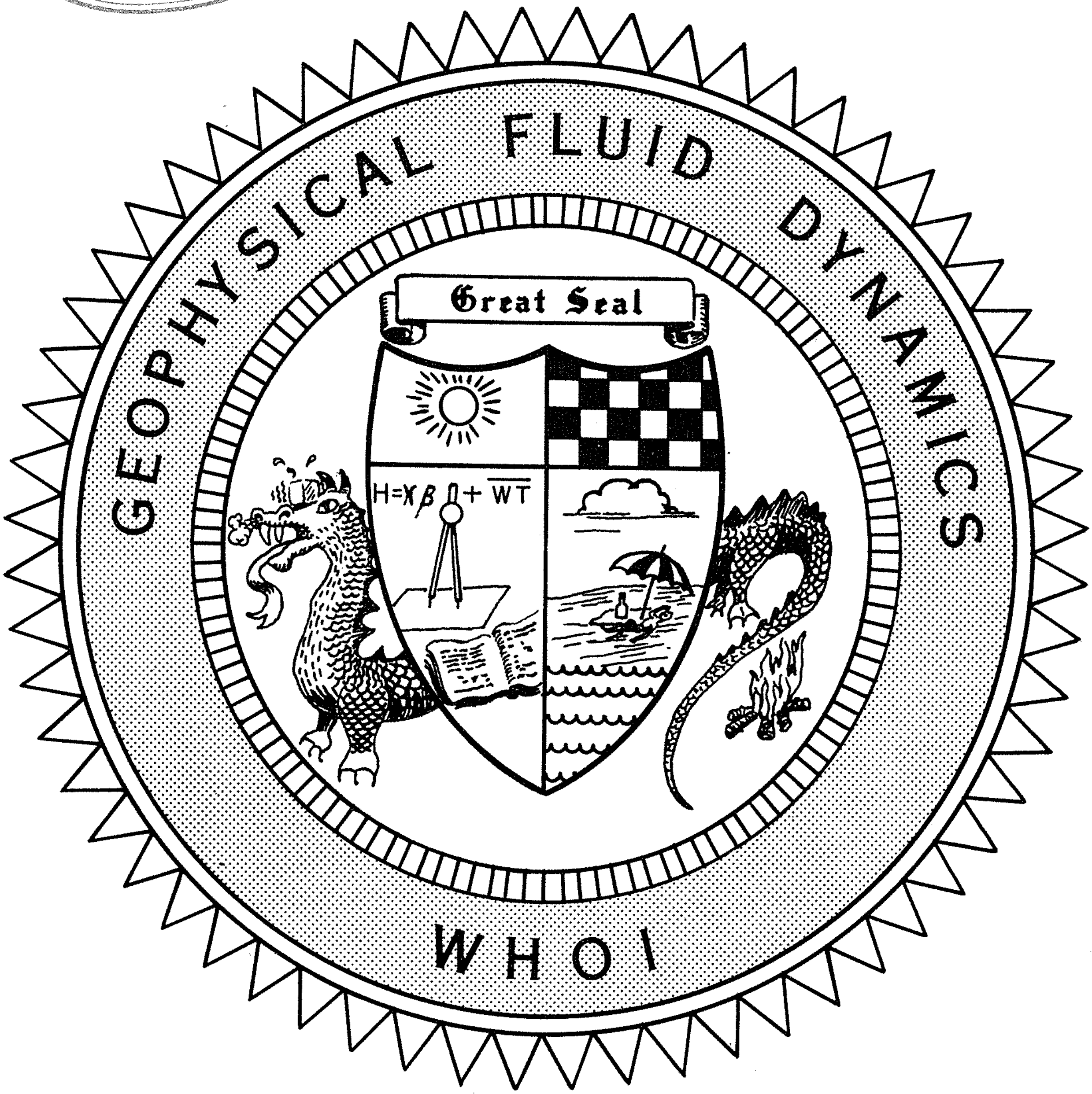
1962



STUDENT LECTURES

Notes on the 1962

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

Reference No. 62-33

LIST OF PARTICIPANTS

Prof. Arnt Eliassen, University of Oslo, Blindern, Norway

Dr. George B. Field, Princeton Observatory, Princeton, New Jersey

Dr. Raymond Hide, Massachusetts Institute of Technology

Dr. R. H. Kraichnan, Institute for Mathematical Sciences, New York

Dr. Willem V.R.Malkus, University of Southern California at Los Angeles

Dr. Leon Mestel, Trinity College, Cambridge University, England

Dr. Eric Mollo-Christensen, Massachusetts Institute of Technology

Dr. Derek Moore, Bristol University, Bristol, England

Dr. Alan Robinson, Harvard University

Dr. Claus Rooth, Woods Hole Oceanographic Institution

Dr. Edward A. Spiegel, Institute for Mathematical Sciences, New York

Dr. Melvin Stern, Woods Hole Oceanographic Institution

Prof. Henry Stommel, Harvard University

Dr. Alar Toomre, Massachusetts Institute of Technology

Dr. George Veronis, Woods Hole Oceanographic Institution

STUDENT FELLOWS

Peter Bryant, Emmanuel College, Cambridge, England

James Holton, Massachusetts Institute of Technology

Joseph Pedlosky, Massachusetts Institute of Technology

Hans Thomas Rossby, Royal Technical University, Stockholm, Sweden
(now M.I.T.)

Pierre Souffrin, Institut d'Astrophysique, Paris, France

Roger T. Williams, University of Southern California at Los Angeles

Acknowledgment

This volume contains the manuscripts of the student research lectures. We should like to thank the National Science Foundation for the financial support which made this year's program of study and research possible.

In addition we owe our gratitude to Dr. Leon Mestel of Cambridge University for his lecture series "Topics in Astrophysical Hydrodynamics", and to Dr. Robert Kraichnan of New York University for his series "Topics in Turbulence Theory".

Mrs. Mary Thayer is to be thanked again for her conscientious preparation of this volume from the manuscripts. Although each author is responsible for his contribution, the time factor has prevented them from checking the printed version.

Melvin Stern.

Table of Contents

Wind Flow over Water Waves

by

Peter J. Bryant

Wind Flow over Water Waves

by

Peter J. Bryant

1. Introduction

When a turbulent wind begins to blow over an infinite horizontal water surface, the initial disturbances in the water are due to the turbulent stress fluctuations at the water surface. However, as soon as a wave pattern is generated, there will be a coupling between the motion of the water surface and the air motion, modifying the surface stress pattern. This investigation is into the nature of this coupling. A complete review of the subject of wind generation of water waves has been made by Phillips (1962).

It has been the custom to assume that the air motion is laminar and that the turbulent fluctuations contribute only to the shape of the mean velocity profile. This assumption is examined, and a form of justification attempted.

When this assumption has been made, the air motion is described by the Orr-Sommerfeld equation. In order to derive a qualitative solution, the inviscid motion of separate parcels of fluid is examined, and deductions are made from the solutions obtained concerning the real state, in the manner suggested by Lighthill (1962).

2. Equations for the air motion

The water surface is assumed to be of the form

$$\eta = a \sin k(x - ct)$$

where a is a slowly varying function of time. This can be interpreted as being one Fourier component of the real surface, because interactions between these Fourier components are non-linear, and in this investigation non-linear interactions are being neglected. The y-axis is chosen to be vertically upwards, with origin in the mean free surface, and the motion is assumed to be two dimensional. This is not strictly true for the turbulent fluctuations, but the fluctuations appearing here may be considered as the z-means of the actual fluctuations.

The equations to be considered are therefore

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned} \tag{2.1}$$

The mean of these equations will be taken in two ways. If f(x,y,t) represents one of the functions appearing, define

$$\bar{f}^{x} = \lim_{X \to \infty} \frac{1}{2X} \int_{-X}^{+X} f \, dx \tag{2.2}$$

and $$\bar{f}^{t} = \frac{1}{T}\int_0^T f \, dt$$

where $\frac{1}{a}\frac{da}{dt} \ll \frac{1}{T} \ll n,$

n being the wave frequency. To be more precise, T can be chosen to be an integral number of periods, satisfying the relation. It has been assumed here that the development time of the amplitude is very much greater than the wave period, and physically this can be seen to be true, provided that the wind is not too strong.

The origin in x is chosen to be travelling with the wave velocity c, so as to reduce the wave motion to a near steady state.

Let

$$\begin{aligned} u(x,y,t) &= U(y) - c + u_1(x,y) + u_2(x,y,t) \\ v(x,y,t) &= v_1(x,y) + v_2(x,y,t) \\ p(x,y,t) &= P(y) + p_1(x,y) + p_2(x,y,t) \end{aligned} \tag{2.3}$$

where, using f as above,

$$\begin{aligned} &\overline{f_2}^x = \overline{f_2}^t = 0 \qquad \text{ie} \quad f_2 = f - \overline{f}^t \\ &\overline{f_1}^x = 0 \qquad\qquad\quad \text{ie} \quad f_1 = \overline{f}^t - \overline{f}^x \end{aligned} \tag{2.4}$$

f_1 is that part of the air motion which is coupled with the water motion, and f_2 is the random air motion.

If f, g are any two such functions, not necessarily different, then

$$\begin{aligned} \overline{f_1 g_2}^t &= 0 \\ \overline{f_2^2}^t &= \overline{f_2^2}^x \end{aligned} \tag{2.5}$$

the latter equality arising from the properties of homogeneity in the x direction and near stationarity for the random air motion.

Also $\overline{f_1^2}^x$, $\overline{f_2^2}^x$, $\overline{f_2^2}^t$ are independent of x, and vary only slowly with t. (When there is no confusion, the meaned variable will not be indicated.)

The boundary conditions are taken to be

$u_1, v_1, p_1 \rightarrow 0$ as $y \rightarrow \infty$ and $u_2, v_2 = 0$ on $y = 0$.

Taking the means of equations (2.1) in the various ways, and carrying out integration, obtain

$$\begin{gathered} \overline{u_1 v_2}^x + \overline{u_2 v_1}^x = 0 \\ \overline{v_1 v_2}^x = 0 \end{gathered} \tag{2.6}$$

$$\overline{v_1^2} + \overline{v_2^2} = \frac{P_\infty}{\rho} - \frac{P(y)}{\rho} + \overline{v_2^2}_\infty \tag{2.7}$$

$$\frac{\partial}{\partial y}\left(\overline{u_1 v_1} + \overline{u_2 v_2}\right) = \nu \frac{\partial^2 U}{\partial y^2} \tag{2.8}$$

$$(U-c)\frac{\partial u_1}{\partial x} + v_1 \frac{\partial U}{\partial y} + \frac{\partial}{\partial x}(u_1^2) + \frac{\partial}{\partial y}(u_1 v_1 - \overline{u_1 v_1}) = -\frac{1}{\rho}\frac{\partial p_1}{\partial x} + \nu \nabla^2 u_1 \tag{2.9}$$

$$(U-c)\frac{\partial v_1}{\partial x} + \frac{\partial}{\partial x}(u_1 v_1) + \frac{\partial}{\partial y}(v_1^2 - \overline{v_1^2}) = -\frac{1}{\rho}\frac{\partial p_1}{\partial y} + \nu \nabla^2 v_1 \tag{2.10}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{2.11}$$

Equations (2.6) could probably have been assumed initially, but were not since they follow from the assumptions already made. Equation (2.7) gives the variation of the mean pressure, which here is the driving force of the system. Equation (2.8) will be integrated below, since it contains a singularity at the critical layer. Equations (2.9) (2.10) (2.11) with the non-linear terms excluded, are the Orr-Sommerfeld equations, and are the justification for the assumption of laminar air motion, but with the mean velocity profile dependent on the total fluctuations.

3. Description of the air motion

The motion represented by the inviscid Orr-Sommerfeld equation (the Rayleigh equation) is now examined. The subscript 1 is omitted, and the origin of co-ordinates is fixed in space.

The frequency of oscillation of a parcel of air seen by an observer moving with the mean wind is $k(U(y) - c)$. Hence the vertical velocity, v, and the displacement, h, of this parcel are given by

$$v = v_o(y) \cos k(U-c)t \tag{3.1}$$

$$h = v_o(y) \frac{\sin k(U-c)t}{k(U-c)} \tag{3.2}$$

The equation of continuity shows that

$$u = U(y) - \frac{dv_o(y)}{dy}\frac{1}{k}\sin k(U-c)t - v_o(y)\frac{d\epsilon(y)}{dy}\frac{1}{k}\cos k(U-c)t \tag{3.3}$$

where $\epsilon(y)$ is the phase and is defined by

$$k(U-c)t = k(x-ct) + \epsilon(y) \tag{3.4}$$

The vorticity ω is

$$\begin{aligned}\omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= -\frac{\partial U(y-h)}{\partial y}\end{aligned}$$

by the Kelvin Helmholtz conservation of vorticity,

ie $$\omega = -\frac{\partial u}{\partial y} + h\,\frac{\partial^2 u}{\partial y^2} \tag{3.5}$$

to the first order.

Taking the mean of the horizontal equation of motion, and using the equation of continuity

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial y}\left(\overline{uv}\right) = \overline{\omega v} \tag{3.6}$$

all these terms being of the second order.

$$\begin{aligned} \overline{\omega v} &= U''(y)\,\overline{hv} \\ &= U''(y)\,v_0^2(y)\;\overline{\frac{\sin 2k(U-c)t}{2k(U-c)}} \\ &= U''(y)\,v_0^2(y)\;\frac{\pi\,\delta(y-y_c)}{2k\,U'(y_c)} \quad \begin{pmatrix} y = y_c \\ \text{at } U = c \end{pmatrix} \\ &= \frac{\pi}{2k}\,\frac{U''(y_c)}{U'(y_c)}\,v_0^2(y_c)\,\delta(y-y_c) \end{aligned} \tag{3.7}$$

where the relation used is

$$\overline{\frac{\sin zt}{z}} = \lim_{t\to\infty} \frac{\sin zt}{z} = \pi\,\delta(z) \tag{3.8}$$

and $\delta(z)$ is the Dirac δ-function, defined as a generalised function.

Integrating (3.6)

$$\begin{aligned} \overline{uv} &= \frac{\pi}{2k}\,\frac{U''(y_c)}{U'(y_c)}\,v_0^2(y_c) & y < y_c \\ &= 0 & y > y_c \end{aligned} \tag{3.9}$$

where the boundary condition $u \to U_\infty$, $v \to 0$ as $y \to \infty$ is used.

Also, direct from (3.1), (3.3)

$$\overline{uv} = -\frac{1}{2k}\frac{d\epsilon(y)}{dy} v_0^2(y)$$

Hence

$$\begin{aligned}\frac{d\epsilon(y)}{dy} v_0^2(y) &= -\pi \frac{U''(yc)}{U'(yc)} v_0^2(yc) \quad & y < yc \\ &= 0 & y > yc\end{aligned} \qquad (3.10)$$

ie $\frac{d\epsilon(y)}{dy} > 0$ for $y < yc$

and $\epsilon(y) = \text{constant}$ $y > yc$

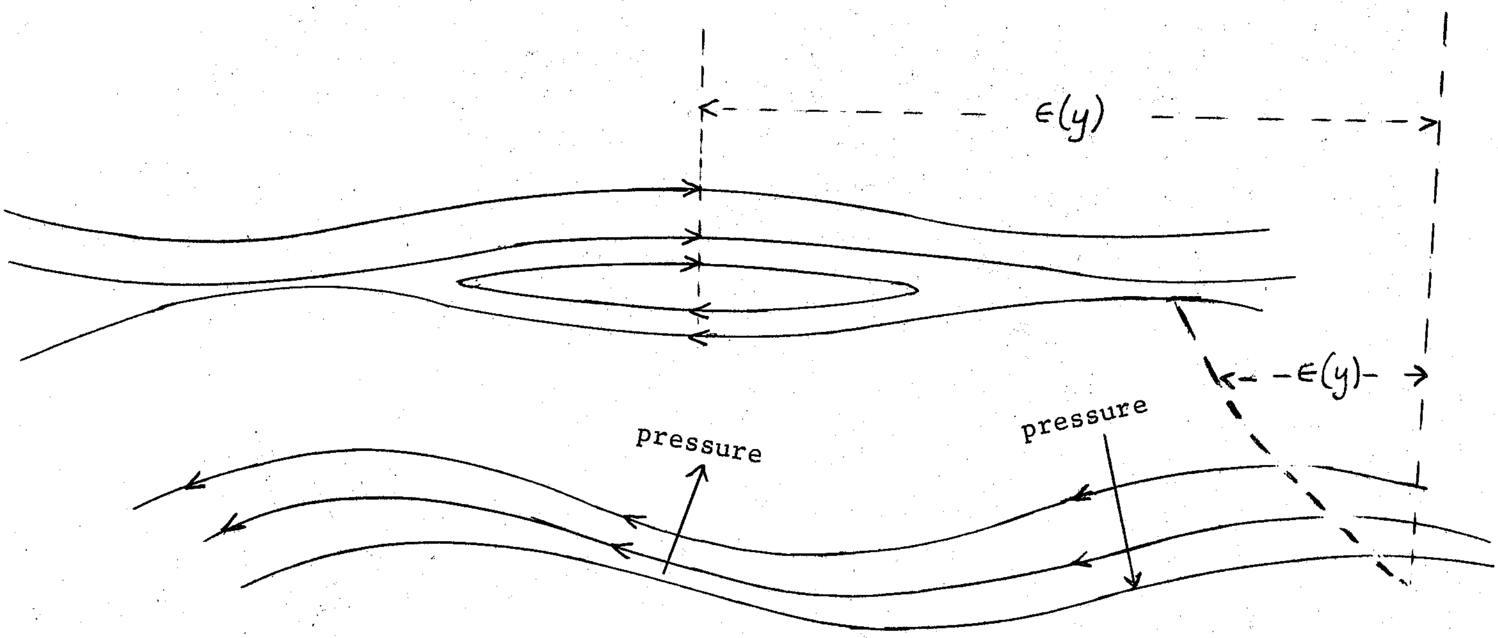
At $y = yc$, $\epsilon(y)$ takes two values differing by π, depending on whether yc is approached from above or below. This discontinuity is dictated by physical considerations, having the consequence that equation (3.1) must have the two solutions

$$v(y) = \pm v_0(y)$$

at the critical layer.

The streamline pattern, seen by an observer riding with the surface wave train, is shown in the figure.

The streamlines in the figure indicate the pressure distribution on the wave, and it is seen to be such as to cause a positive transfer of horizontal momentum from the air to the water waves. The critical layer has the 'catseye' structure, which also appears at the critical layer in stability problems, and it shows how this layer is one of concentrated vorticity.



We can now integrate equation (2.8) to give

$$\overline{u_1 v_1} + \overline{u_2 v_2} = \nu \frac{dU}{dy} + \overline{u_2 v_2}_{\infty} \qquad y > y_c$$

$$= \nu \frac{dU}{dy} + \overline{u_2 v_2}_{\infty} + \frac{\pi}{k} \frac{U''(y_c)}{U'(y_c)} v_0^2(y_c) \qquad y < y_c$$

The total stress, $\tau(x, y)$ is defined as

$$\tau(x,y) = \nu \frac{dU}{dy} - \overline{uv}^{t}$$

Hence

$$\bar{\tau}(x,y) = -\overline{u_2 v_2}_{\infty} \qquad y > y_c$$

$$= -\overline{u_2 v_2}_{\infty} - \frac{\pi}{2k} \frac{U''(y_c)}{U'(y_c)} v_0^2(y_c) \qquad y < y_c \qquad (3.11)$$

and all terms on the right of equations (3.11) are positive.

The stream function for the flow is

$$\psi = \frac{1}{k} v_0(y) \sin k(U-c)t \tag{3.12}$$

where
$$v = \frac{\partial \psi}{\partial x}, \quad u = -\frac{\partial \psi}{\partial y} + U(y)$$

Substitution in equation (2.9) gives

$$\frac{p}{\rho} = (U-c)\frac{\partial \psi}{\partial y} - U'\psi \tag{3.13}$$

and then replacing ψ

$$\frac{p}{\rho} = \left(\frac{U-c}{k} \frac{dv_0(y)}{dy} - \frac{1}{k} U'(y) v_0(y)\right) \sin k(U-c)t + \frac{U-c}{k} v_0(y) \frac{d\epsilon(y)}{dy} \cos k(U-c)t \tag{3.14}$$

The part of the pressure containing $\sin k(U-c)t$ is in phase with the displacement, and is the part which can lead to Kelvin-Helmholtz instability at the water surface. The term containing $\cos k(U-c)t$ is in quadrature with the displacement, and describes the transfer of horizontal momentum from the air wave motion to the water wave motion. To obtain a measure of this transfer, consider the mean horizontal pressure component, which is

$$\overline{\frac{p}{\rho} \frac{\partial h}{\partial x}} = -\overline{uv} - \frac{\pi}{2k} v_0^2(y_c)\, \delta(y-y_c) \tag{3.15}$$

after some calculation. Thus, except at the critical layer, the horizontal component of the pressure is equal to the Reynolds stress.

The Rayleigh equation is

$$\frac{\partial^2 \psi}{\partial y^2} - k^2 \psi - \frac{U''}{U-c} \psi = 0 \tag{3.16}$$

Substituting for ψ , and reducing, obtain

$$\left(\frac{1}{k} \frac{d^2 v_0(y)}{dy^2} - \frac{1}{k} v_0(y) \left(\frac{d\epsilon(y)}{dy} \right)^2 - k v_0(y) - \frac{U''}{U-c} \frac{1}{k} v_0(y) \right) \sin k(U-c)t$$
$$+ \frac{\pi}{k} \frac{U''(y_c)}{U'(y_c)} v_0(y_c)\, \delta\left(y - y_c\right) = 0 \tag{3.17}$$

This curious equation becomes

for $y > y_c$

$$\frac{d^2 v_0(y)}{dy^2} - k^2 v_0(y) - \frac{U''}{U-c} v_0(y) = 0 \tag{3.18}$$

for $0 < y < y_c$

$$\frac{d^2 v_0(y)}{dy^2} - k^2 v_0(y) - \frac{U''}{U-c} v_0(y) - \frac{A^2}{(v_0(y))^3} = 0 \tag{3.19}$$

where $\quad A = \pi \dfrac{U''(y_c)}{U'(y_c)} v_0^2(y_c)$

The mean of the left hand side of (3.17) is zero for all y, including the critical layer, and the apparent time dependence disappears. The effect of the phase change appears explicitly in equation (3.19).

4. Conclusions

The mean value in equations (3.7) is taken over t from $t = 0$ to ∞, and presupposes that $\frac{da}{dt} = 0$. If instead, the mean value is taken from $t = 0$ to T, as in equation (2.2), a broadened and slightly distorted δ-function is obtained, leading to a distortion of the step function in equations (3.9), and a small difference in the algebraic results. However, the overall description of the flow remains unchanged, and it is a matter only of algebraic manipulation to make the correction.

The general effect of viscosity is to reduce velocity gradients. So it may be expected that the infinite gradients appearing at the critical layer would be reduced symmetrically, leaving a broadened δ -function as a description of the loss of momentum from the mean air flow to the air wave flow. Dissipation would also lead to an increase in the area under this function, compared with the true δ -function. The effect of viscosity on the normal stresses at the air-water interface is negligible, and viscosity becomes important only in the calculation of the tangential stresses.

To summarize: the concentrated vorticity at the critical layer causes transfer of horizontal momentum from the mean air motion to the air wave motion. This transfer may be described mathematically by the term $\overline{\rho\omega\wedge v}$ in equation (3.6), which Lighthill (1962) called the vortex force, and showed its usefulness in describing the physical processes at the critical layer. This horizontal momentum is then carried down to the water waves as a Reynolds stress on the coupled wave motion in the air, and the air Reynolds stress is transmitted directly into the water Reynolds stress at the water surface. Equation (3.10) indicates that a necessary condition for the existence of such an air Reynolds stress is that the phase of the air wave motion is changing, ie $\frac{d\epsilon(y)}{dy} \neq 0$. The figure shows the stream lines produced by this type of motion, and makes it clear why an observer riding with the wave experiences a driving force from behind.

An equivalent description is to represent the air pressure at the water surface as being the real part of

$$p = (\alpha + i\beta)\rho U_1^2 k \eta \tag{4.1}$$

Equation (3.14) indicates how this representation is related to the flow characteristics. U_1 here is a scaling velocity, and α, β are functions of n, the wave frequency. This is the description taken by Miles (1957), and section 2 indicates that to obtain the total fluctuating pressure, one need only add to equation (4.1) the turbulent pressure fluctuations.

This investigation is a development of some of the ideas expressed by Dr. M. J. Lighthill in the lecture which is referred to below.

References

Lighthill (1962) Lecture delivered at the Applied Mechanics Conference, Bristol, April 1962.

Miles (1957) J. Fluid. Mech. 3, 185.

Phillips (1962) J. Geophys. Res., 67, 3135.

An Attempt to Simulate the Cromwell Current in the Laboratory

by

James R. Holton

An Attempt to simulate the Cromwell Current in the Laboratory.

James R. Holton

1. Introduction

The Cromwell Current, or equatorial undercurrent, is an intense ribbon-like zonal jet centered about the equator in the Pacific Ocean. The transport of the current is comparable to that of the Gulf Stream, yet its existence was not suspected until about ten years ago.

That the Cromwell Current has only recently been discovered is due, no doubt, to the paradoxical nature of the circulation in the equatorial Pacific. The surface water drifts slowly westward driven by the mean wind stress, but only a few meters below the surface the Cromwell Current flows eastward, with velocities of 50 to 150 cm/sec in its core at about 100 meters depth. The vertical scale of the current is about 200 meters, its latitudinal extent is about 1.5 degrees on either side of the equator, and its length is several thousand kilometers.

Because the core of this easterly current is at the equator its angular momentum exceeds that of the earth. This has led to occasional statements in the literature that a west to east pressure gradient is required to maintain the current against frictional dissipation. However, it would be possible to maintain an easterly current at the equator without a pressure gradient if there existed horizontal eddies which transferred net momentum equatorward through a negative correlation of the velocities u', v' (where u' and v' are departures from the spatially averaged zonal and meridional velocities respectively). Similar eddies are important momentum transfer agents in the atmosphere and occur in quasi-geostrophic circulations when the streamlines are tilted so that u' is larger when v' is negative than when v' is positive. (See Figure 1).

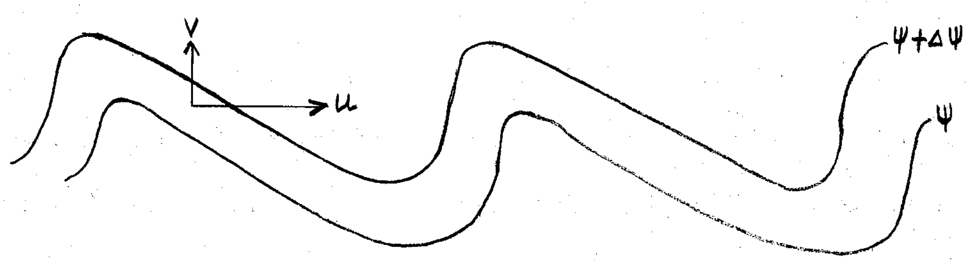


Figure 1.

Streamline field for net equatorward momentum flow.

To evaluate the possible importance of this momentum transport mechanism in the oceans would require synoptic observations which are not now available. Thus, in the studies of the Cromwell Current to date the zonal pressure gradient has been the explicit driving force and the eddy flux mechanism has not been considered. In fact, it is known from observations that a west to east pressure gradient does exist along the equator in the region of the Cromwell Current, and further, this pressure gradient disappears east of the Galapagos Islands, as does the Cromwell Current.

In view of the above discussion, it would be inappropriate to use an eddy viscosity coefficient to represent the effects of horizontal mixing, because not only the value, but even the sign of such a coefficient is in doubt for the larger eddy scales, and for the smaller scales the large vertical shear assures the dominance of vertical eddy dissipation.

2. A Physical Model of the Cromwell Current

The equation of motion for the steady state zonal component of the Cromwell Current may be written as,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

where ν is the eddy coefficient of viscosity for vertical mixing and the other symbols have their usual meanings. At the equator $f = 0$ and the above equation indicates that a balance will exist between the pressure force and the inertial and viscous terms. However, the extreme smallness of the Rossby number for large scale ocean circulations indicates that the circulation is quasigeostrophic even very close to the equator, and that the Cromwell Current may be an essentially geostrophic jet with an inerto-viscous boundary layer at the equator.

A possible model for the maintenance of this geostrophic current is the following. As shown by Stommel (Deep Sea Research 6) the mean easterly windstress creates a divergent surface Ekman layer at the equator, because the surface current is to the right of the wind in the Northern Hemisphere and to the left in the Southern Hemisphere. This divergence forces an upwelling to preserve continuity. The upwelling and enhanced vertical mixing raise the thermocline at the equator and a core of cold dense water is maintained at 100 meters depth (see Figure 2).

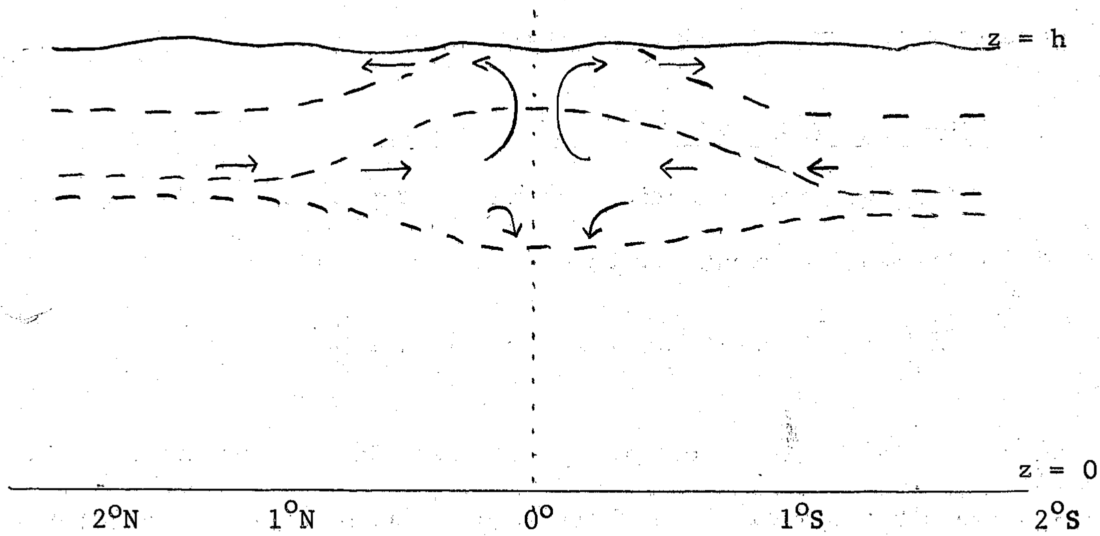


Figure 2.

Dashed lines represent isotherms. Arrows show the pattern of meridional circulation.

The result of this isotherm spread is a pressure gradient in the y direction pointed away from the equator in both hemispheres, which is balanced by an eastward moving geostrophic current.

In summary, it appears that three physical effects are necessary for the formation and maintenance of the equatorial undercurrent (1) the sign reversal of the Coriolis force, (2) a west to east pressure gradient, (3) divergence of the surface Ekman layer.