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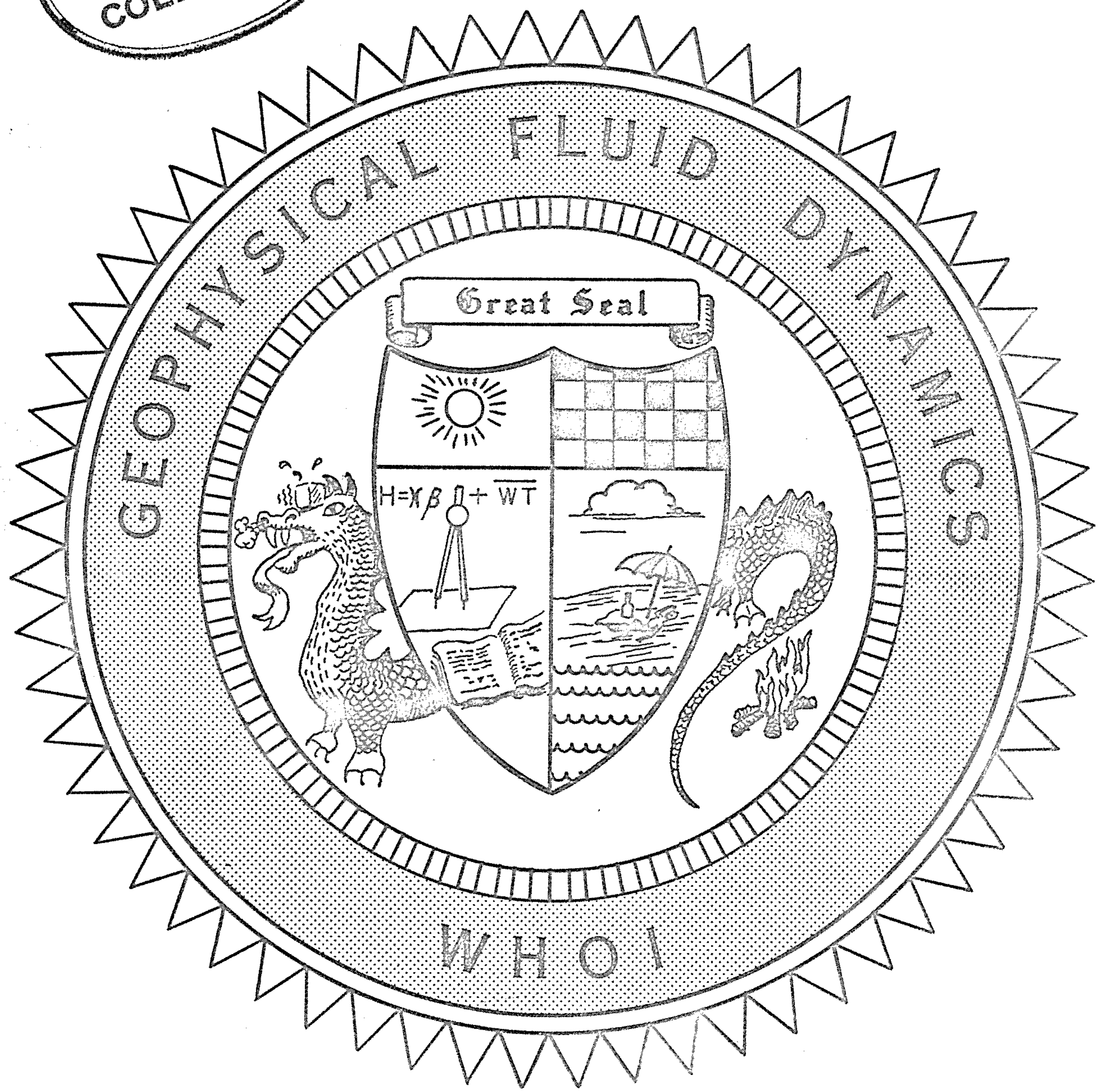
61 - 39

Vol. III



Participant Lectures

Contents of the Volumes

Reference No. 61-~~33~~ 39

LIST OF PARTICIPANTS

Regular WHOI Staff Members

C. Rooth

G. Veronis

Visiting Staff Members and Post-doctoral Participants

A. Arking (NASA)

A. P. Burger (NPRL, Pretoria, South Africa)

G. B. Field (Princeton Univ.)

N. P. Fofonoff (Nanaimo, British Columbia)

S. Fritz (U.S.Weather Bureau)

A. S. Furumoto (Univ. Hawaii)

J. Herring (NASA)

L. N. Howard (MIT)

R. H. Kraichnan (NYU)

F. B. Lipps (Johns Hopkins Univ.)

W.V.R. Malkus (UCLA)

D. W. Moore (Bristol Univ., England)

E. A. Spiegel (NYU)

H. Stommel (Harvard Univ.)

P. Welander (Nat'l.Research Council, Stockholm, Sweden)

Student Fellows

I. Dugstad (Met.-Oceanog.)(Univ. of Oslo, Norway)

L. W. MacMillan (Physics)(Univ.British Columbia, Vancouver, Canada)

M. T. Mork (Met.-Oceanog.)(Univ. of Oslo, Norway)

S. Nagarajan (Physics)(NYU)

P. P. Niiler (Applied Math.)(Brown Univ.)

D. H. Peregrine (Applied Math.)(Cambridge Univ.,England)

S. I. Rosencrans (Math.)(MIT)

F. C. Shure (Physics)(Univ. of Michigan)

Notes on the 1961

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

Volume III

PARTICIPANTS' LECTURES

Third Volume Edited by

Mary C. Thayer and George Veronis

Editors' Preface

This volume contains the manuscripts of the student research lectures as well as research contributions by senior participants in the summer program. The staff guided the selection of the students' topics with several goals in mind. One goal was to isolate that part of a problem which might prove to be tractable in an effort of eight weeks or so. The more important goal was to find "open-ended" problems which would continue to challenge the student after his return to the university.

The degree of direction by the sponsor varied a great deal. In a few cases, there were frequent conferences and discussions about fruitful avenues of approach. In other cases, there was essentially no contact except one of encouragement and interest. The efforts cover a wide spectrum in originality also. Some of the reports represent a more extended study of material presented in the course of lectures - others are original contributions which are being prepared for publication.

Because of time limitations it was not possible for the notes to be edited and reworked. The reports may contain errors the responsibility for which must rest on the shoulders of the participant-author. It must be emphasized that this volume in no way represents a collection of reports of completed and polished work.

In cases of joint-authorship the names have been listed alphabetically with no attempt to distinguish between senior and junior contributor.

All those who took part in the summer program are grateful to the National Science Foundation for its encouragement and financial support of the program.

Mary Thayer

George Veronis

Table of Contents

1. List of Seminars

Alewyn P. Burger Baroclinic Instability

George B. Field Thermal Instability in the Interstellar Medium

Nicholas P. Fofonoff Inertial Circulation I
Inertial Circulation II

Louis N. Howard and Willem V.R. Malkus A Statistical Theory of Turbulent Convection

Robert H. Kraichnan General Considerations in Isotropic Homogeneous Turbulence
Heuristic Models of Isotropic Homogeneous Turbulence
The Closure Problem in Isotropic Homogeneous Turbulence

Robert R. Long The Turbulent Jet

Eric Mollø-Christensen Large-scale Circulation in an Oscillating Tank

Edward A. Spiegel The Effect of Radiative Transfer on Convective Instability
Convection in a Polytropic Atmosphere
A Mechanistic Theory of Turbulent Convection

Melvin Stern Instability of Jets in a Stratified Rotating System

Henry Stommel A Model of the Ocean Thermocline

George Veronis Penetrative Convection

Pierre Welander A Model of the Ocean Thermocline

A Study of Ekman Layers

by

Ivar Dugstad

I Introduction

The way ocean and atmosphere interact with each other is of great interest both to the meteorologist and the oceanographer. Here one aspect of this interaction is considered, namely how the ocean modifies the wind-field in the atmosphere and how the wind-stress at the surface of the ocean sets up currents. We consider the following idealized situation:

Let both the ocean and the atmosphere be in neutral stability conditions. The geostrophic wind is constant. All mean quantities except the pressure are functions of z only. In the ocean we assume no horizontal pressure gradients, so that the motion there is entirely due to the wind-stress on the surface. Conditions are steady state.

II The equations

For the type of motion we want to consider, atmosphere and ocean can be treated as being incompressible. When referred to a coordinate system fixed to the earth, the equations of motion are then

$$\frac{D\bar{u}}{dt} - f\bar{v} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{u}}{\partial x} - \rho\overline{u'u'}\right) + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right) + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'}\right) \tag{1}$$

$$\frac{D\bar{v}}{dt} + f\bar{u} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{v}}{\partial x} - \rho\overline{v'u'}\right) + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{v}}{\partial y} - \rho\overline{v'v'}\right) + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{v}}{\partial z} - \rho\overline{v'w'}\right) \tag{2}$$

The notation is standard. In our case the equations reduce to:

$$-f\bar{v} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'}\right) \tag{3}$$

$$f\bar{u} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{v}}{\partial z} - \rho\overline{v'w'}\right) \tag{4}$$

We now assume that the turbulence stresses (or Reynolds stresses) may be written in terms of an eddy transfer coefficient of momentum K defined by:

$$-\rho\overline{u'w'} = \rho K\frac{\partial \bar{u}}{\partial z} \tag{5}$$

$$-\rho\overline{v'w'} = \rho K\frac{\partial \bar{v}}{\partial z} \tag{6}$$

Introducing the kinematic viscosity $\nu = \frac{\mu}{\rho}$ and using the fact that in turbulent flow $K \gg \nu$ we may write the equations of motion: (the bars are dropped from now on)

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right) \tag{7}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right) \tag{8}$$

Ekman and Taylor independently solved these equations making the assumption that K is a constant. The result is the well-known Ekman spiral. It is now known that in the layer closest to the surface K is proportional to z, both over the ocean and the ground. Ellison (1) therefore solved the equations with K proportional to z throughout the atmosphere. His results compare favorably with observations. Ellison finds that the angle between the geostrophic wind and the surface wind is of the order of 7=8 degrees at 55° lat., whereas it is 45° according to Ekman's Theory.

III Solution for atmosphere-ocean

We shall solve the equations (7), (8) with K being proportional to height in the atmosphere and proportional to depth in the ocean. The surface velocity of the ocean will be taken into account. The expected velocity profile in the plane is shown on figure 1.

It appears to be convenient to write the velocity $\underset{\sim}{v}$ as:

$$\underset{\sim}{v} = \underset{\sim}{v_s} + \underset{\sim}{v_r} \tag{9}$$

where $\underset{\sim}{v_s}$ is the surface velocity of the ocean, and $\underset{\sim}{v_r}$ is the velocity relative to a coordinate system moving with the velocity

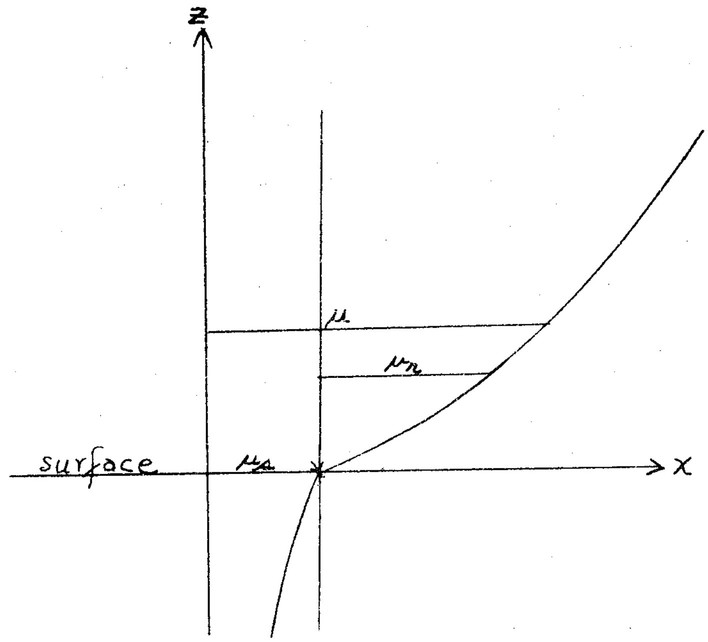


Fig. 1

$\underset{\sim}{v_r}$. Let the orientation of this coordinate system as well as the earth-bound system be so that the relative wind $\underset{\sim}{v_r}$ at the surface is in the positive x-direction.

The geostrophic value of $\underset{\sim}{v_r}$ is denoted by v_{rg}, it is defined by:

$$-f(v_s + v_{rg}) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{10}$$

$$f(u_s + u_{rg}) = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{11}$$

By use of (9), (10) and (11) the equations of motion become:

$$-f(v_r - v_{rg}) = \frac{\partial}{\partial z}\left(K\frac{\partial u_r}{\partial z}\right) \tag{12}$$

$$f(u_r - u_{rg}) = \frac{\partial}{\partial z}\left(K\frac{\partial v_r}{\partial z}\right) \tag{13}$$

With $W = u_{rg} - u_r - i(v_{rg} - v_r)$ the two last equations may be written in the form:

$$f W = i \frac{\partial}{\partial z}\left(K \frac{\partial W}{\partial z} \right) \tag{14}$$

At this point it seems appropriate to consider the semi-empirical law for the velocity profile near the surface. In neutral stability conditions the flow near the surface can be compared with flow in pipes and channels (1), even though the driving mechanisms are different. The dominant dynamical features are the turbulent shear stresses and the frictional drag at the surface. The pressure force and the Coriolis force can be neglected. The profile is given by:

$$u = \frac{u^*}{k}\left[\ln \frac{z}{z_o} + B_1\left(\frac{u^* z_o}{\nu} \right) \right] \quad \text{(Nikuradse)} \tag{15}$$

Here $u^* = \sqrt{\frac{\tau}{\rho}}$, the friction velocity

τ is the shear stress in the surface layer

k is von Karman's constant 0.4

z_o is the roughness parameter

B_1 is a function which has the following properties:

If $\frac{u^* z_o}{\nu} < 0.13$ then $B_1 = 2.2 + \ln \frac{u^* z_o}{\nu}$ and the flow is said to be aerodynamically smooth.

If $\frac{u^* z_o}{\nu} > 3$ then $B_1 = 0$ and the flow is said to be aerodynamically rough.

If $0.13 < \frac{u^* z_o}{\nu} < 3$ the flow is called intermediate.

For aerodynamically rough flow the profile is then given by:

$$u = \frac{u^*}{k} \ln \frac{z}{z_0} \tag{16}$$

and for aerodynamically smooth flow by:

$$u = \frac{u^*}{k} \ln \frac{9u^*z}{\nu} \tag{17}$$

Consider the shear stress in the surface layer,

Using (15) we find that

$$K = \frac{\tau}{\rho \frac{\partial u}{\partial z}} = k u^* z \tag{18}$$

With this value of K equation (14) becomes

$$fW = i k u^* \frac{\partial}{\partial z}\left(z \frac{\partial W}{\partial z}\right). \tag{19}$$

Since $W = W(z)$ we can write (19) as

$$\frac{d^2 W}{dx^2} + \frac{1}{x}\frac{dW}{dx} + iW = 0 \tag{20}$$

where the new variable $x = \left(\frac{4fz}{ku^*}\right)^{1/2}$

The solution which is finite for large x is

$$W = A H_0^{(1)}(i^{1/2}x) \tag{21}$$

The Hankel function $H_0^{(1)}(i^{1/2}x)$ is tabulated in e.g. Yahnke and Emde, 1945. It has a logarithmic form for small x. Using this we can write:

$$W = u_{rg} - u_r - i(v_{rg} - v_r) = \frac{1}{2}A + \frac{2iA}{\pi}(\ln x - \ln 2 + \gamma) \tag{22}$$

where A is a constant to be determined and γ is Euler's constant.

We now distinguish between two cases.

a) Aerodynamically rough flow

The profile near the surface is given by (16) which in terms of χ becomes

$$u_r = \frac{2u^*}{k} \ln x + \frac{u^*}{k} \ln \frac{ku^*}{4f z_0} \tag{23}$$

From (22) and (23) we then find:

$$A = \frac{i\pi u^*}{k}$$

$$u_{rg} = \frac{u^*}{k}\left(2(\ln 2 - \gamma) + \ln \frac{ku^*}{4f z_0}\right) \tag{24}$$

$$v_{rg} = -\frac{u^*}{k} \cdot \frac{\pi}{2} \tag{25}$$

b) Aerodynamically smooth flow

The profile near the surface is given by (17), which in terms of χ becomes:

$$u_r = \frac{2u^*}{k} \ln x + \ln \frac{9ku^{*2}}{4\nu f} \tag{26}$$

Using (22) and (26) we find:

$$u_{rg} = \frac{u^*}{k}\left[2(\ln 2 - \gamma) + \ln \frac{9ku^{*2}}{4\nu f}\right] \tag{27}$$

$$v_{rg} = -\frac{u^*}{k} \cdot \frac{\pi}{2} \tag{28}$$

Note that the y-component is the same in both cases.

Before we consider the ocean a few words about the roughness parameter seems appropriate. Over ground z_0 is found to be closely related to the roughnesses on the surface, and these are in most instances not affected greatly by the wind. Plotting observations of u on a $u, \log z$ diagram and drawing a straight line through the points one finds z_0 where the line intersects the z-axis.

The interface between ocean and atmosphere is a self-roughening surface. The wind itself is causing the roughnesses. Exactly how z_0 is related to waves is not fully understood, but observations indicate that only the shortest waves and ripples are responsible for the drag (2). According to Hay (3) there is a simple relation between z_0 and u^* given by:

$$z_0 = 8 \cdot 10^{-5} u^{*2} \tag{29}$$

His measurements were made in aerodynamically rough flow.

The available information on the velocity profile due to the windstress in the ocean is very limited. As a working hypothesis it will therefore be assumed that the profile under the surface follows a logarithmic law, and that the motion can be described in terms of a roughness parameter and a friction velocity as in the atmosphere.

Let the parameters in the ocean be marked with the subscript w, and change the direction of the z-axis by putting $z = -z'$. The equations (12) and (13) then become:

$$-f(v_s + v_n) = \frac{\partial}{\partial z'}\left(K_w \frac{\partial u_n}{\partial z}\right) \tag{30}$$

$$f(u_s + u_n) = \frac{\partial}{\partial z'}\left(K_w \frac{\partial v_n}{\partial z'}\right) \tag{31}$$

With $K_w = k u_w^* z'$ we obtain: (dropping the primes)

$$i f W_w + k u_w^* \frac{d}{dz}\left(z \frac{dW_w}{dz}\right) = 0 \tag{32}$$

with $$W_w = u_r + u_s - i(v_r + v_s).$$

Putting $\chi_w = \left(\frac{4fz}{k u^*_w}\right)^{1/2}$ the solution of (32) is:

$$W_w = A_w H_0^{(1)}(i^{1/2}\chi_w). \tag{33}$$

Again we distinguish between two cases.

a) <u>Aerodynamically rough flow</u>

For small χ_w the profile is given by:

$$u_r = -\frac{2u^*_w}{k}\ln\chi - \frac{u^*_w}{k}\ln\frac{k u^*_w}{4f z_{0w}} \tag{34}$$

and, as in the case of the atmosphere we find:

$$A_w = \frac{i\pi u^*_w}{k}$$

$$u_s = \frac{u^*_w}{k}\left[2(\ln 2-\gamma) + \ln\frac{k u^*_w}{4f z_{0w}}\right] \tag{35}$$

$$v_s = -\frac{u^*_w}{k}\,\frac{\pi}{2} \tag{36}$$

b) <u>Aerodynamically smooth flow</u>

For small χ_w the profile is given by:

$$u_r = \frac{2u^*_w}{k}\ln\chi_w - \frac{u^*_w}{k}\ln\frac{9k{u^*_w}^2}{4\nu_w f} \tag{37}$$

and in this case we find:

$$u_s = \frac{u^*_w}{k}\left[2(\ln 2-\gamma) + \ln\frac{9k{u^*_w}^2}{4f\nu_w}\right] \tag{38}$$

$$v_s = -\frac{u^*_w}{k}\cdot\frac{\pi}{2} \tag{39}$$

At the interface between ocean and atmosphere the stress must be continuous:

$$\rho {u^*}^2 = \rho_w {u^*_w}^2 \tag{40}$$

IV Discussion

From our equations we can now see that if the flow is aerodynamically smooth in both atmosphere and ocean the whole flow is determined by one quantity such as the gradient wind or the friction velocity u^*. Assuming Hay's relation (29) to be true the above holds also for the case of rough flow in the atmosphere - smooth flow in the ocean.

When the flow in the ocean is rough we also need information about z_{ow}. Perhaps a relation analogous to (29) exists in the ocean also, i.e.

$$z_{ow} = \text{const} \cdot u_w^{*2} = \text{const} \cdot \alpha^2 u^{*2} \qquad (41)$$

and the whole flow would again be determined by u^*.

Let $\alpha \equiv \sqrt{\frac{\rho}{\rho_w}}$ $\left(\alpha \approx \frac{1}{28}\right)$, and consider rough flow in both media. From (9), (24), (25), (35), (36) and (40) we then find:

$$u_s = \alpha\left(u_{rg} + \frac{u^*}{k} \ln \frac{\alpha z_o}{z_{ow}}\right) = \frac{\alpha}{1+\alpha}\left(u_g + \frac{u^*}{k} \ln \frac{\alpha z_o}{z_w}\right) \qquad (42)$$

$$v_s = \alpha\, v_{rg} = \frac{\alpha}{1+\alpha}\, v_g \qquad (43)$$

If $z_{ow} = \alpha z_o$ we see that there is a simple relation between the geostrophic wind and the surface-velocity . The surface-velocity is along the isobars and equal to about $\frac{1}{28}$ times the geostrophic wind.

From the definitions of χ and χ_w it is seen that if $\chi = \chi_w$ then $z_w = \alpha z$, i.e. the depth of the Ekman layer in the ocean is $\frac{1}{28}$ times the depth of the Ekman layer in the atmosphere.

Tables 1 and 2 display some of the flow characteristics assuming Hay's relation to be true. The latitude, which enters through the Coriolis parameter, is set equal to 55°. ψ_1 is the angle between the surface wind $\underset{\sim}{v_r}$ and the geostrophic wind $\underset{\sim}{v_{rg}}$.

Table 1.

Atmospheric flow

Smooth				Rough			
u^*	ψ_1	u_{rg}	v_{rg}	ψ_1	u_{rg}	v_{rg}	z_n
174	$4^{\circ}12'$	9313	683	$7^{\circ}58'$	4885	683	$3.97 \cdot 10^5$
64	$4^{\circ}38'$	3106	251	$7^{\circ}19'$	1957	251	$1 \cdot 46 \cdot 10^5$
24	$5^{\circ}09'$	1047	94	$6^{\circ}46'$	793	94	$5 \cdot 48 \cdot 10^4$
9.5	$5^{\circ}45'$	370	37	$6^{\circ}20'$	333	37	$2 \cdot 17 \cdot 10^4$
3.2	$6^{\circ}41'$	107	12.5	$5^{\circ}57'$	120	12.5	$7 \cdot 31 \cdot 10^3$

z_n is the lowest height at which $u_r = u_{rg}$.

All quantities are in c.g.s. units.

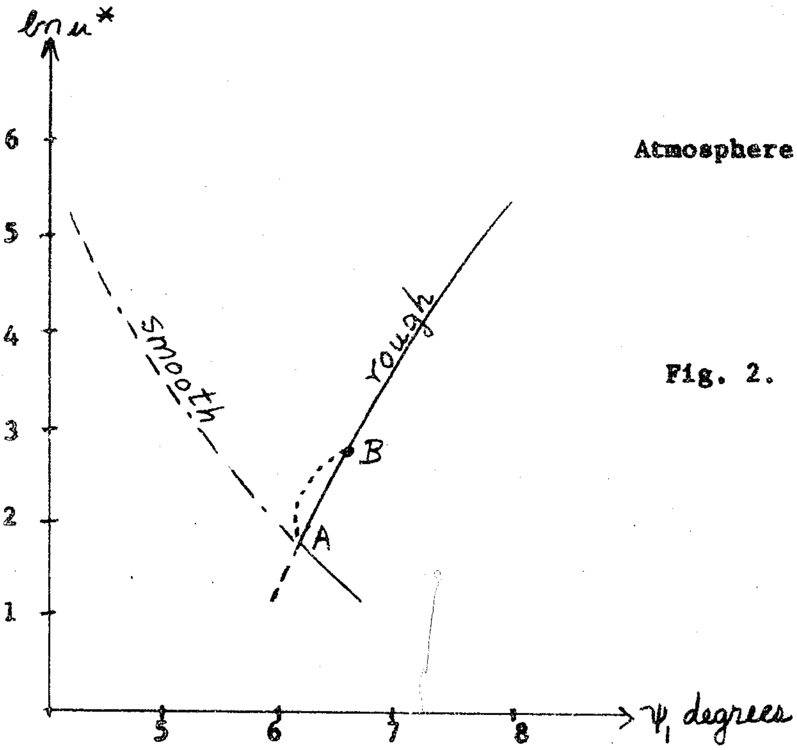
These tables show that the surface velocity is not very sensitive to the value of the roughness parameter z_{0w}. The extreme values of the angle between the x-axis and v_a, ψ_2 are $4^{\circ}50'$ and $13^{\circ}14'$. Both are probably very unrealistic, the first because it is based upon the assumption of rough flow at very small velocities, the last because of the large value of z_{0w}. Note that ψ_1 and ψ_2 decrease with increasing stress when the flow is smooth whereas ψ_1 and ψ_2 increase with increasing stress for rough flow.

Table 2. Ocean flow

	Smooth			Rough												
				$z_{ow} = \alpha^2 z_o$			$z_{ow} = \alpha \cdot z_o$			$z_{ow} = z_o$			$z_{ow} = \frac{1}{\alpha} z_o$			
μ^*	ψ_2	μ_s	v_s	ψ_2	μ_s	v_s	ψ_2	μ_s	v_s	ψ_2	μ_s	v_s	ψ_2	μ_s	v_s	z_{hw}
174	5°20'	262	24	6°07'	227	24	7°58'	174	24	9°52'	138	24	13°14'	102	24	$1.42 \cdot 10^4$
64	6°03'	85	9	5°45'	89	9	7°19'	70	9	8°58'	57	9	11°33'	44	9	$5.21 \cdot 10^3$
24	7°07'	27.6	3.4	5°25'	35.5	3.4	6°46'	28.3	3.4	8°18'	23.3	3.4	10°32'	18.3	3.4	$1.96 \cdot 10^3$
9.5	8°07'	9.34	1.3	5°08'	14.9	1.3	6°20'	11.9	1.3	7°29'	9.9	1.3	9°21'	7.9	1.3	$7.75 \cdot 10^2$
3.2	10°05'	2.53	0.4	4°50'	5.31	0.4	5°57'	4.3	0.4	6°20'	3.6	0.4	7°51'	2.9	0.4	$2.61 \cdot 10^2$

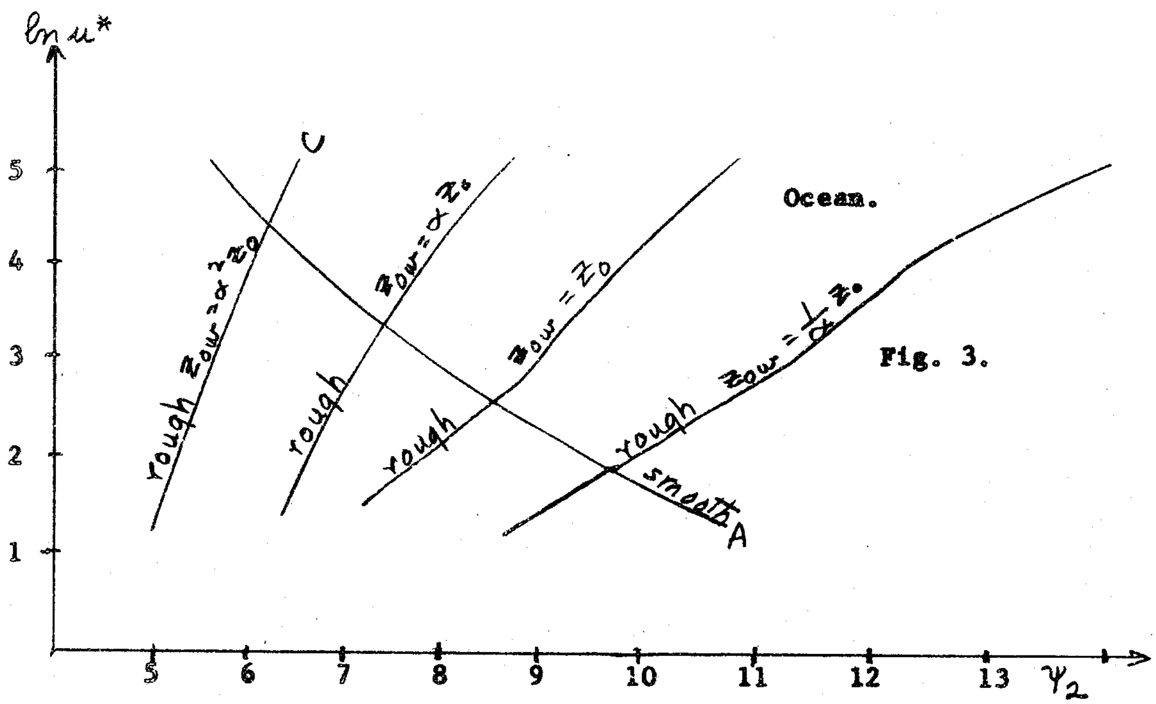
ψ_2 = angle between $\underset{\sim}{v}_2^g$ and v_s^g .

This is also shown on fig. 2 and 3 where Ψ_1 and Ψ_2 are plotted against $\ln u^*$.



Atmosphere.

Fig. 2.



Ocean.

Fig. 3.

This is also shown on Figs. 2 and 3 where Ψ_1 and Ψ_2 are plotted against $\ln u^*$.

Looking at Figure 2, it is interesting to speculate on which flow form actually is chosen. If the stress is to increase continuously with the wind the change of flow regime obviously must take place where the two lines intersect. The flow changes from smooth to rough at the point A, the actual flow is given by the full-drawn part of the curves. This is equivalent to saying that the preferred flow for a given geostrophic wind is the one that gives the maximum wind stress at the surface, i.e. the maximum transfer of momentum. One point that supports this theory is that at the intersection point

$$\frac{u^* z_0}{\nu} = \frac{6 \cdot 8 \cdot 10^{-5} \cdot 36}{0.13} \approx 0.13$$

i.e. it has the value at which the flow enters the intermediate regime. $\frac{u^* z_0}{\nu} = 3$ at $\ln u^* = 2.83$, that is at point B on Figure 2. The flow in the intermediate range is then indicated by the dotted line from A to B.

By analogous reasoning the preferred flow form in the ocean should be the one that gives a minimum $\underset{\sim}{v_s}$ for a given value of the surface stress. Since v_s is the same for rough and smooth flow, this means that u_s has a minimum. Now since the vertically integrated transport in the x-direction vanishes it is also equivalent so that the preferred motion is the one in which

$$\int_0^\infty |u| dz \quad \text{has a minimum.}$$

V. Final remarks

This brief study points out the need of careful observations of velocity in and above the ocean surface. When making observations in the air the surface velocity of the ocean should be taken into account, z_0 for example should be defined as the height at which $u = u_s$ if observations are made from a point fixed relative to earth.

The assumption of neutral stability in the ocean is probably not a serious restriction since the conditions are close to neutral in the Ekman layer there. In the atmosphere the conditions are often stable and one should look at this case. The time-dependent problem should also be considered. The geostrophic flow in the ocean can be included and, as shown by Ellison, one can also easily handle the case where the geostrophic wind varies linearly with height. A closer examination of the roughening process of the interface might give information about ocean waves.

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Investigation of Cell Size in Thermal Convection

by

S.Fritz, F.B.Lipps and D.W.Moore

Introduction (S.Fritz)

In studies of Bernard cell thermal convection, a typical ratio of L/d is about 3 to 1, where "L" is the horizontal dimension and "d" is the vertical dimension. This cell "size" or ratio of about 3:1 is found both experimentally and theoretically.

However there are geophysical phenomena which may be described by a typical ratio L/d of 20 or 30 to 1. One such phenomenon has been observed from the satellite TIROS I [1]. In that case cumulus and stratocumulus clouds were arranged in patterns with horizontal dimensions of about 30 miles. The clouds were located below a temperature inversion; and the height of the inversion was about one mile. Thus, if we assume that the entire depth of atmosphere, below the cloud, comprised the thickness of the layer in which convection was occurring, the rate of L/D was about 30:1.

What was the basic cause which produced such cells? Many possible reasons, dealing with thermal convection and other phenomena might be involved. But in order to investigate some models which might at least suggest the trend of the cell size, and to apply the theory studied in this course, we decided to study some simple quasi-physical models.

First, the cloud arrays are characterized by some known phenomena. The clouds are produced in those portions of the cells where the motion is upward. In those portions of the cells where the air is cloudless, the motion is probably downward, at least near the upper boundary (at the inversion). Moreover, at least two phenomena occur when the cloud is present. First, stratocumulus clouds are very good radiators of thermal energy to space; they are approximately black body radiators at the temperature of the cloud "top". Evaporation from the cloud "top" may also contribute to cooling there. On the other hand, as the clouds condense they release the heat of condensation, and so may act as heat sources.

Thus the clouds act as heat sinks near their upper boundaries. They also act as heat sources throughout the part of the cloud where condensation is occurring.

We may also imagine that <u>prior</u> to the formation of the actual cloud, there may be somewhat more water <u>vapor</u> in the upward moving part of the cell than in the downward part. This water vapor may then contribute to energy changes by emitting energy to space from each unit volume of air throughout the fluid. D.M.Moore investigated this problem, with the results shown in Part II below.

Finally, we note that with a superadiabatic temperature gradient $\Delta T_o/\Delta z$ of $1^{o}K$ in a layer 1 km deep, the Rayleigh no., R, is about 10^{18} if we use molecular values for κ and ν. In the case of the cellular pattern under discussion, the difference of temperatures between the ocean water near the surface and the temperature on shipboard was about 3 to 5°C. The temperature gradient in the layer of air between the ship and the inversion was approximately the adiabatic gradient. Thus there is a question as to the proper value of ΔT_o to choose. However, the presence of cumulus clouds also suggests "fully developed" turbulence with a high Rayleigh number.

Therefore, because of the very high Rayleigh number (assumed, 10^{18}) Lipps and Fritz also examined the linear theory for unstable vertical velocity distributions. Specifically if $W = e^{st}\ g(x,y)\ h(z)$, we derived the variation of s with wave number when $s > 0$.

It turns out that in most cases examined, the cell size becomes <u>smaller</u> than in the Jeffrey's free-free solution, rather than larger. The main exception to this was the situation when the air was <u>heated</u> near the top boundary in the upward flowing part of the cell. This suggests that condensation may be a factor in producing large cells. Even then the variation in wave number may be small.

Thus, to find the explanation for the "large" cells, seen in TIROS I photographs, still other factors must be investigated, although condensation may play a role.

The solutions to the three types of problem are given below. These problems are:

(I) Non-uniform heating or cooling at the boundary (linear).

(II) Non-uniform heating or cooling in the fluid; boundaries held at constant temperature (non-linear).

(III) Variation of wave number for very large Rayleigh number (linear).

References:

[1] A.F.Krueger and S. Fritz, Cellular Cloud Patterns revealed by TIROS I. Tellus, 13:1-7: Feb.1961.

I. Marginal stability with non-uniform cooling or heating at the upper boundary

by S. Fritz, D.M.Moore, and F.B.Lipps.

The problem:

Given: a layer of fluid in which heat leaves the upper boundary at those places where "w" is upward.

To find: what will be the cell size?

The cell size is defined by the ratio of the horizontal to the vertical dimension of the cells.

To avoid non-linearities, it was finally decided to assume heat flow away from the fluid (for the perturbation quantities) at the boundary in the upward flowing part of the cell, and heat flow into the fluid in the downward flowing part of the cell. Ideally, it would have been desirable to allow heat loss as given but to allow neither heat loss nor heat gain in the downward part of the cell.

The condition just described approximates the condition for radiative heat loss from the cloud at the upper boundary. Reversing the signs would correspond to the condition of heat release by condensation at the cloud "top".

In this problem we assume:

(a) free boundaries

(b) at $z = 0$,

$$T = 0, \quad w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

(c) at z = d (the depth of the fluid)

$$w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

$$\text{and } k\frac{\partial T}{\partial z} = c_2 \frac{\partial w}{\partial z}, \quad \text{or} \quad \frac{\partial T}{\partial z} = c_1 \frac{\partial w}{\partial z}$$

This last condition assumes that the heat conduction will equal the radiative heat loss; and further the radiative heat loss is proportional to $\left(\frac{\partial w}{\partial z}\right)$. This was done to assure that the heat loss would be a maximum where "w" was a maximum in the interior of the fluid. This was also done as a compromise for ease in mathematical evaluation.

We accept the equations for marginal stability in the Jeffreys theory as presented in class lectures. Using equations (63V) and (65V)* we get

$$\nu(-M^2 + D^2)^2 w = \gamma M^2 T \qquad (1)$$

where $M = \frac{\pi\alpha}{d}$, and $D^n = \frac{\partial^n}{\partial z^n}$

On differentiating eq.(1) with respect to "z", substituting for $\frac{\partial T}{\partial z}$ from the "radiative" boundary condition from assumption "c", and expanding the quantity in brackets on the left hand side, we get

$$M^4 D(w) - 2M^2 D^3(w) + D^5(w) = \frac{c_1 \gamma M^2}{\nu} D(w) \qquad (2)$$

Now let

$$w = g(x,y)h(z)$$

*Equation nos. followed by "V" refer to equation nos. in notes of the lectures by Dr. Veronis.

and substitute in eq.(2). In the resulting equation collect terms and set

$$\xi = \pi \frac{z}{d} . \tag{4}$$

From this we get the boundary condition at $z = d$ or $\xi = \pi$; namely,

$$h^{V} - 2\alpha^2 h''' + \alpha^2(\alpha^2 - p)h' = 0 \tag{5}$$

where the "h's" are derivatives with respect to " ξ " and

$$p = c_1 \frac{\gamma}{\nu} \frac{d^2}{\pi^2}$$

Now assume

$$h(z) = \sum_1^{\infty} A_n \sin n \xi , \quad \text{(see 73V)} \tag{6}$$

The three derivatives shown in eq.(5) are easily written down following the procedure shown in eqs.(74-76V). For example

$$h^{V} = \frac{1}{\pi} h^{IV}(\pi-) + \sum_1^{\infty} \left[n^5 A_n + (-1)^n \frac{2}{\pi} h^{IV}(\pi-) \right] \cos n \xi . \tag{7}$$

In eq.(7) $h^{IV}(0+) = 0$ because $D^4(h) = 0$ at $z = 0$ from assumption (b).

On substituting for the derivatives in eq. 5, and setting $\xi = \pi$ or $\cos n\pi = (-1)^n$, we get

$$\frac{h^{IV}(\pi-)}{\pi} \left[1+2 \sum_1^{\infty} (+1)^n \right] + \sum_1^{\infty} (-1)^n \left[n^5+2\alpha^2 n^3+\alpha^2(\alpha^2-p)n \right] A_n=0 \tag{8}$$

In the lecture notes, substitution for h(z) from eq.(6)

into the sixth order differential equation for w(see eq.66V), led to the equivalent of (see eq.77-79V)

$$-(n^2+\alpha^2)^3 A_n + R'\alpha^2 A_n = (-1)^n\, n\, \frac{2}{\pi}\, h^{IV}(\pi-) \tag{9}$$

After solving eq(9) for A_n, substitute this result for A_n into eq.(8). We divide by $h^{IV}(\pi-)$ assuming it to be different from zero. The result is

$$-\sum_1^\infty \frac{(n^2+\alpha^2)^2}{F} + \sum \frac{R'}{F} - \sum \frac{pn^2}{F} = -\frac{1}{2\alpha^2} \tag{10}$$

where $F = R'\alpha^2 - (n^2+\alpha^2)^3$

Consider the first term in eq.(10), and add and subtract

$$\frac{1}{n^2+\alpha^2} \tag{11}$$

The first term in eq.(10) then becomes

$$-\sum \frac{R'\alpha^2}{(n^2+\alpha^2)\left[R'\alpha^2-(n^2+\alpha^2)^3\right]} + \sum \frac{1}{n^2+\alpha^2}$$

But

$$\sum_1^\infty \frac{1}{n^2+\alpha^2} = -\frac{1}{2\alpha^2} + \frac{\pi}{2\alpha}\coth \pi\alpha$$

(see appendix for proof)

Therefore eq.(10) becomes

$$\sum_1^\infty \frac{R'}{F}\left(\frac{n^2}{n^2+\alpha^2}\right) - \sum_1^\infty \frac{pn^2}{F} = -\frac{\pi}{2\alpha}\coth \pi\alpha \tag{12}$$

or

$$= \sum_1^\infty \frac{R'n^2}{F}\left[\frac{1}{(n^2+\alpha^2)} - \frac{p}{R'}\right] \tag{13}$$

Since "F" contains a term in n^6, eq.(12) will converge approximately as $\frac{1}{n^4}$. It is therefore useful to examine the variation of the minimum value of R', or R'_c , as a function of "p", even if we use only n = 1. We can then see whether the corresponding value of α, or α_c, increases or decreases with p. The value of p depends on C_1. Thus when p is positive, heat is being lost at the upper boundary in that part of the cell where "w" is upward.

Figure 1 is a plot R' vs α for several values of "p" from -5 to +10. The figure shows that the value of α, at R_c, <u>increases</u> somewhat with <u>increasing</u> "p". That is, as the "radiative" heat loss, at the boundary in the upward moving part of the cell, increases, the horizontal dimension of the cell tends to <u>decrease</u>. On the other hand, as "p" decreases towards negative values, "α" tends to decrease, indicating an <u>increase</u> in the horizontal dimension. A negative value for p corresponds to heating near the boundary, in the upward moving air; this might correspond to heating in the cloud by condensation.

We cannot assign a numerical value to "p" because the magnitude of C_1 is uncertain. When $|p|$ is small, Figure 1 shows that the change in α is only about 10% for a change from p = -5 to p = +10. However if $|p|$ could attain large values, the change in α would be appreciable.

Figure 1 also shows that when p = 0, the value of R' is smaller than in the Jeffreys free boundary solution. Jeffreys found $R'_c = \frac{27}{4}$; in Fig. 1, $R'_c \approx 4$ for $p = 0$. However, it should be noted that only $n = 1$ was used to compute Figure 1.

Regarding cell size, the main result indicates that heating, rather than cooling, is required in the upward moving air, in order to increase the cell size.

Appendix

Proof that

$$\sum_1^{\infty} \frac{1}{n^2+\alpha^2} = -\frac{1}{2\alpha^2} + \frac{\pi}{2\alpha}\coth \pi\alpha \quad .$$

K. Knopp [2] states that

$$\pi x \cot \pi(x) = 1 + 2x^2 \sum_{n=1}^{\infty} \frac{1}{x^2-n^2} \qquad \text{(A)}$$

valid for x , an arbitrary real number distinct from 0, $\pm$ 1, $\pm$ 2, ...

This will also be valid for imaginary arguments.

Therefore let

$$x = i\alpha$$

and substitute into eq.(A), so that

$$\pi(i\alpha) \cot \pi(i\alpha) = 1 + 2\alpha^2 \sum_1^{\infty} \frac{1}{\alpha^2+n^2}$$

$$= \pi\alpha \coth \pi\alpha$$

Therefore

$$\sum_1^{\infty} \frac{1}{\alpha^2+n^2} = -\frac{1}{2\alpha^2} + \frac{\pi}{2\alpha}\coth \pi\alpha \, .$$

References:

2 K.Knopp, Theory and Application of Infinite Series, 1928, Blackie and Son, see p. 207.

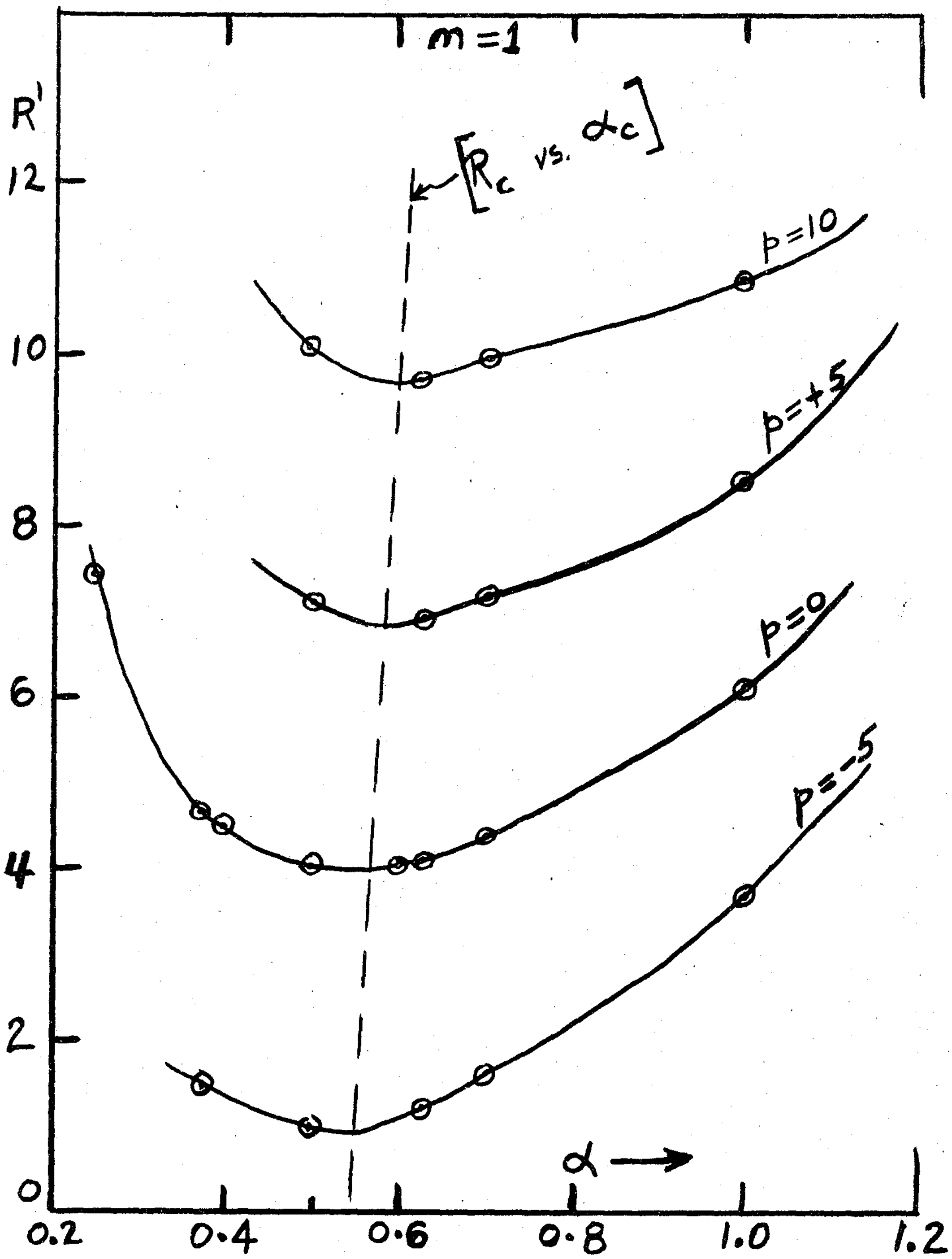

Fig. 1 - Variation of R' with α for several values of p.
Note that α_c increases with increasing .

II. Convective cell size with internal heat sources

by D. W. Moore.

In the previous analysis the radiative heat transfer was confined to the boundaries. The purpose of this section is to examine the case where each volume element of the fluid can lose heat by radiation. The physical picture is that of a diffuse tenuous cloud which fills the entire depth of the fluid where the upward velocity, w, of the fluid is positive. In general, a volume element in the cloud will lose heat by radiation from the condensed vapour and will gain heat by the release of latent heat as the water vapour condenses. These effects are both dependent on the existence in the volume element of condensing water vapour. Thus in regions where the air is falling these effects will be absent, since there will be no water droplets in these regions. This means that the heating is a non-linear function of the vertical velocity w, so that one must examine non-linear equations in order to discuss the stability problem. For this reason, we have had resort to an approximate treatment in terms of the power integrals. It is known that these give reliable information for the usual stability problems so that we may hope that, in the present case, at least the general trend of the results will be revealed.

We shall first derive the modified power integrals appropriate to the situation described. Then a trial function, which is in fact the linear solution for rolls, will be substituted and the amplitudes as functions of the Rayleigh number determined.

We suppose that $-F(w)$ ergs sec^{-1} cm^{-3} are being radiated from each volume element of the fluid. Then the heat equation (8V, p.38, or eqn. 1.1 in ref.(4)) for the total temperature $\mathcal{T}$ may be written

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right)\mathcal{T} - \frac{F(w)}{\rho_m c_p} = - \underline{V} \cdot \nabla \mathcal{T} . \tag{1}$$

Following the analysis of Malkus and Veronis referred to, we define

$$\mathcal{T} = \bar{\mathcal{T}} + T \tag{2}$$

where $\bar{\mathcal{T}}$ is the mean temperature at height z (an overbar denotes a horizontal average) and T the perturbation temperature. On taking the horizontal average of (1) we find

$$\frac{\partial \bar{\mathcal{T}}}{\partial t} + \frac{\partial}{\partial z}(K\beta) = \frac{\bar{F}}{\rho_m c_p} - \frac{\partial}{\partial z}(\overline{wT}), \tag{3}$$

where, as usual, $\beta \equiv -\frac{\partial \bar{\mathcal{T}}}{\partial z}$: note that even for steady motion the mean heat flux $K\beta + \overline{wT}$ is no longer independent of z. If we substitute (3) into (1) we have

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right)T = \beta w - h_1 , \tag{4}$$

where

$$h_1 = \underline{V} \cdot \nabla T - \frac{\partial}{\partial z}(\overline{wT}) - \frac{1}{\rho_m c_p}(F - \bar{F}), \tag{5}$$

which may be compared with eq. 15V, p.36.

If we multiply h_1 by T and take the vertical average, denoted by $(\ )_m$, we find that

$$(\overline{h_1 T})_m = -\frac{1}{\rho_m c_p} (\overline{FT})_m . \qquad (6)$$

Using (6) we now easily find for the temperature power integral

$$\frac{1}{2}\frac{\partial}{\partial t'} (\overline{T'^2})_m = R(\overline{w'T'})_m + \left[(\overline{w'T'})_m^2 - (\overline{w'T'^2})_m + (\overline{w'T'G})_m \right]$$

$$- (\overline{\nabla T' \cdot \nabla T'})_m + \Delta (\overline{FT'})_m , \qquad (7)$$

where the dashes denote quantities non-dimensionalised as in ref. 4 and where R is the Rayleigh number and

$$\Delta = \frac{\gamma d^5}{\rho_m c_p \nu K^2} , \qquad (8)$$

and where G is the time dependent portion of β — G will not enter the equations in the steady state and so will not be considered here.

The velocity power integral is unaltered and is thus

$$\frac{1}{2}\frac{\partial}{\partial t} (\overline{v'^2})_m = \sigma \left\{ (\overline{w'T'})_m - \left(\frac{\partial v_i'}{\partial x_j} \frac{\partial v_i'}{\partial x_j} \right)_m \right\} \qquad (9)$$

We shall now drop the dashes — no confusion will arise, since dimensionless quantities will be used from now on.

As trial functions we take the solution for rolls given in ref. 4. — thus we take

$$\left.\begin{aligned} \omega &= P\ 2\cos\pi\alpha x\ \sin\pi z \\ T &= Q\ 2\pi^2\delta^2\cos\pi\alpha x\ \sin\pi z \\ U &= -\frac{P}{\alpha}\ 2\sin\pi\alpha x\ \cos\pi z \end{aligned}\right\} \qquad (10)$$

where $\delta^2 = \dfrac{(1+\alpha^2)^2}{\alpha^2}$. (11)

Thus

$$\left.\begin{aligned} (\overline{\omega T})_m &= PQ.\,\pi^2\delta^2 \\ (\overline{\nabla T\cdot\nabla T})_m &= Q^2.\,\pi^6\delta^4(\alpha^2+1) \\ (\overline{\omega T^2})_m &= P^2Q^2.\,\tfrac{3}{2}\,\pi^4\delta^4 \\ \left(\overline{\frac{\partial v_i}{\partial x_j}\ \frac{\partial v_i}{\partial x_j}}\right)_m &= P^2.\,\pi^2\delta^2 \end{aligned}\right\} \qquad (12)$$

We must now consider explicit forms for the function $F(\omega)$. In fact, we confine our attention to two especially simple cases.

Case (1)

$$F = -F_o(\omega + |\omega|) \qquad (13)$$

This gives radiation proportional to ω when ω is positive and zero radiation when ω is negative.

Now for the trial functions (10) (or indeed for any symmetrical trial functions), $(\overline{|\omega| T})_m = 0$ so that

$$(\overline{FT})_m = -F_o(\overline{\omega T})_m \qquad (14)$$

Thus the temperature power integral is identical with that for the radiation free case, except that the coefficient of $(\overline{\omega T})_m$ is changed from R to R - $F_o\Delta$. This suggests that the only effect of the radiation on the critical Rayleigh number will be to increase it if $F_o > 0$ (cooling by radiation) or decrease it if $F_o < 0$ (heating by condensation). This seems reasonable physically - if elements are cooled on rising by radiation they will have an increased tendency to sink back and a greater temperature difference will be needed to establish convection.

Case (2)

$$F = -\frac{F_o}{2}\left(\frac{\omega}{|\omega|} + 1\right) \tag{15}$$

This gives radiation F_o when $\omega > 0$ and zero when $\omega < 0$.

We now find

$$(\overline{F\,T})_m = -4\,F_o\,Q\,\delta^2$$

where, in order that the sign of w be determined unambiguously by χ —— we have taken $P > 0$ in (10).

The steady-state power integrals now assume the form

$$0 = R P Q - R_o Q^2 - \frac{4}{\pi^2}\,Q\,F_o\Delta - \frac{1}{2}\,\pi^2\,\delta^2\,P^2\,Q^2 \quad , \tag{16}$$

$$0 = P Q - P^2 \quad , \tag{17}$$

where $R_o = \pi^4\delta^2(\alpha^2 + 1)$ is the critical Rayleigh number in the radiation free case.

Equation (17) shows that either $P = Q = 0$ or

$$P = Q \tag{18}$$

Thus (16) becomes

$$0 = R\rho^2 - R_o\rho^2 - \frac{4}{\pi^2}\rho F_o\Delta - \frac{1}{2}\pi^2\delta^2\rho^4$$

or since we reject $\rho = 0$

$$\frac{\pi^2\delta^2}{2}\rho^3 - (R - R_o)\rho + \frac{4}{\pi^2}F_o\Delta = 0 \tag{19}$$

where $\rho > 0$. If, for given R, (19) has a positive root ρ^* this will correspond to steady convection. If, for the given R, (19) has no positive roots, the motion will be stable at this R.

It is immediately clear that if $F_o < 0$ (19) always has a positive root since the left-hand side of (19) is positive at $\rho = \infty$ and negative at $\rho = 0$.

Thus for $F_o < 0$ the fluid is <u>unstable for all values of R , however small</u>.

Thus we consider now $F_o \geq 0$

Let
$$\rho = \left(\frac{4 F_o\Delta}{\pi^4\delta^2}\right)^{1/3} s \tag{20}$$

Then (19) becomes

$$s^3 + 2 = gs \tag{21}$$

where

$$g = \frac{2(R - R_o)}{\pi^2\delta^2}\left(\frac{4 F_o\Delta}{\pi^4\delta^2}\right)^{-2/3} \tag{22}$$

Now $s > 0 \iff \rho > 0$ and it is known that (21) has a positive root if and only if $g \geq 3$. Thus

$$R - R_o \geq \frac{3}{2}\pi^2\delta^2\left(\frac{4F_o\Delta}{\pi^4\delta^2}\right)^{2/3}$$

or $\qquad R - R_o \geq \delta^{2/3} B \qquad (23)$

where $\qquad B = \frac{3}{2} \pi^{-2/3} (4 F_o \Delta)^{1/3} \quad . \qquad (24)$

If $B = 0$ we recover the condition $R \geq R_o$ appropriate to the radiation-free problem, but if $B > 0$ we find that the critical Rayleigh number R_c (that is the value of R below which, for given α, convection cells cannot be maintained) is given by

$$R_c = R_o + B \delta^{2/3} \qquad (25)$$

Moreover, the disturbance at this critical Rayleigh number is <u>finite</u> and so $P \not\to 0$ as $R \to R_c + 0$. ! In fact one easily finds that

$$P_c = \left(\frac{4F_o \Delta}{\pi^4 \delta^2} \right)^{1/3} . \qquad (26)$$

Thus if $F_o > 0$ we have an example of a situation in which infinitesimal disturbances are stable, and disturbances must be of a certain size before they can extract enough energy from the mean flow to maintain themselves.

These conclusions are confirmed by examination of the unsteady power integrals, which, for infinitesimal disturbances, take the form

$$\frac{dQ^2}{dt} \propto - \Delta F_o Q \qquad (27)$$

so that for $F_o > 0$ all infinitesimal disturbances are damped whilst for $F_o < 0$ all infinitesimal disturbances grow.

That the radiation should decide the behavior of the infinitesimal disturbances is to be expected, since the radiation law embodied in (15) gives radiation independent of the disturbance amplitude, whilst the viscous dissipation etc. are proportional to the amplitude.

It remains to find the effect of the radiation on the critical cell-size when $B > 0$. Now

$$R_c = \pi^4 \frac{(1+\alpha^2)^3}{\alpha^2} + B \frac{(1+\alpha^2)^{2/3}}{\alpha^{2/3}} \quad . \tag{28}$$

*Setting $\frac{dR_c}{d\alpha^2} = 0$ we find

$$\pi^4 \frac{(1+\alpha^2)^2}{\alpha^4} (2\alpha^2 - 1) = \frac{B(1-\alpha^2)}{3(\alpha^2)^{4/3}(1+\alpha^2)^{1/3}} \tag{29}$$

If $B = 0$ we get $\alpha^2 = 1/2$ as for the free-free case without radiation; for $B > 0$, (29) shows that

$$1/2 < \alpha_c^2 < 1 \ , \tag{30}$$

so that the cell size is reduced by the radiation. Further progress must be made graphically.

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*This argument is due to Dr. S. Fritz.

Reference:

(4) W.V.R. Malkus and G. Veronis, Finite Amplitude Convection. Contribution No.(943) from WHOI; also published in J.Fluid Mech. 4:225:1958.

III. Variation of Cell Size with Instability Parameter for large Rayleigh Number

by S. Fritz and F.B. Lipps.

If one computes the Rayleigh number, using molecular values for ν and κ, for a layer of air one kilometer thick, he finds

$$R \approx 10^{18} \quad \text{when} \quad \Delta T_o = 1^{\circ}C.$$

Since this is very far from the value $R_c = 1700$, it is of interest to investigate the variation of α for large values of R. Moreover, since unstable modes participate in the motion when R is large, it is of interest also to examine the variation of s with α, when $s > 0$. A similar problem was studied by Ledoux, Schwarzschild and Spiegel [3] for the solar atmosphere.

To do this we start with the non-dimensional equation (see eq. 41V, p.39).

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w - R\,\nabla_1^2 w = 0 \tag{1}$$

Let

$$w = e^{st}\, g(x,y)\, h(z) \tag{2}$$

and
$$\nabla_1^2\, g(x,y) = -a^2\, g(x,y) \tag{3}$$

Substituting eqs. (2) and (3) into eq. (1), yields

$$\left(\frac{1}{\sigma}s + a^2 - D^2\right)(s + a^2 - D^2)(-a^2 + D^2)h = -a^2 Rh \tag{4}$$

Now assume,

$$h(z) = \sin \pi z \qquad 0 \leq z \leq 1 \qquad (5)$$

and substitute into eq. (4), so that

$$s^2 + (a^2 + \pi^2)(1+\sigma)s + (a^2+\pi^2)^2\sigma = \frac{Ra^2\sigma}{a^2+\pi^2} \qquad (6)$$

on solving the quadratic eq. (6) for s, we find

$$s = -\tfrac{1}{2}(a^2+\pi^2)(1+\sigma) \pm \tfrac{1}{2}\sqrt{(a^2+\pi^2)^2(1+\sigma)^2 - 4\sigma\left[(a^2+\pi^2)^2 - \frac{a^2}{(a^2+\pi^2)} R\right]} \qquad (7)$$

We may examine eq. (7) for different ranges of a.

If

$$(a^2+\pi^2)^2 = \frac{a^2}{a^2+\pi^2} R, \qquad R = \frac{(a^2+\pi^2)^3}{a^2} \qquad (8)$$

$$s = 0 \qquad \text{or } s < 0$$

Eq.(8) is therefore the condition for neutral stability. If in eq. (8)

$$a^2 \gg \pi^2, \qquad R = a^4$$

If

$$a^2 \ll \pi^2, \qquad R = \frac{\pi^6}{a^2}$$

Thus for $R = 10^{18}$, $s = 0$ occurs at $a^2 = 10^9$ and $a^2 = 10^{-15}$, taking $\pi^6 \approx 10^3$.

Now assume, in eq. (7), that

$$\frac{a^2 R}{a^2+\pi^2} \gg (a^2+\pi^2)^2 \qquad \text{or} \qquad (9)$$

$$R \gg \frac{(a^2+\pi^2)^3}{a^2} \qquad (9a)$$

and take $\sigma \approx 1$. Then

$$s = \sqrt{\frac{a^2}{a^2+\pi^2} R\sigma} > 0 \text{ (amplification)}, \qquad (10a)$$

when

$$\frac{\pi^6}{R} << a^2 << \sqrt{R} \qquad (10b)$$

This result in this range of "a" is exactly the solution to the invicid problem of the stability of an unstable temperature gradient.

We first note that our S is non-dimensional i.e.

$S = S_{Dim} \cdot (\text{time})$

$$\text{or} \qquad S = \frac{S_{Dim}}{\chi/d^2}$$

Thus, after some algebra, we can transform eq. (10a) into,

$$\frac{S_{Dim}}{\sqrt{\beta\gamma}} = \sqrt{\frac{a^2}{\pi^2+a^2}} \qquad (10c)$$

which is independent of κ and ν.

To prove that (10c) is the invicid result we write down eq.(45,49V) with χ and ν set equal to zero.

$$\frac{\partial T}{\partial t} - \beta W = 0$$

$$\frac{\partial}{\partial t} \nabla^2 W = \gamma \nabla_1^2 T$$

We then let W and T be of the form $g(x,y)\sin|\pi z| e^{S_{Dim}t}$, and obtain (10c).

Thus we see that the solution of the general linear stability problem for a wide range of "a", with $R \approx 10^{18}$, is the same as the solution to the invicid stability problem. Only for very large and very small "a" does viscosity and heat conduction play any appreciable role. From eq. (10b) it is evident that as $R \longrightarrow \infty$, the vicid heat conducting solution approaches the invicid solution for all positive a^2.

For $a^2 << \pi^2$, eq. (10a)becomes

$$s = \sqrt{R\sigma}\sqrt{\frac{a^2}{\pi^2}}$$

For $a^2 >> \pi^2$, eq. (10a)indicates that

$$s \rightarrow \sqrt{R\sigma} \approx 10^9$$

Apparently, s will have a maximum for $a^2 >> \pi^2$, since s = 0 at $a^2 = 10^9$ and $s \rightarrow \sqrt{R\sigma}$ for $a^2 >> \pi^2$.

We can also find a^2 for s_{max}. If we differentiate with regard to a^2, and set the result equal to zero, we find

$$R = a^2(\pi^2+a^2)^3 \; \frac{1}{\pi^4} \; \frac{(1+\sigma)^2}{\sigma} \tag{11}$$

If now $a^2 >> \pi^2$, as was found earlier for $s=s_{max}$, and if $\sigma = 1$, eq.(11) becomes

$$a^8 = \frac{\pi^4}{4} R , \qquad a^2 = 2.25\, R^{\frac{1}{4}}$$

$$a = 1.5\, R^{1/8} \tag{12}$$

Since the horizontal scale in eq. (1) had been non-dimensionalized by

$$x = d\, x'$$

it follows because of eq. (3), that the horizontal scale, L, is given by

$$a \frac{L}{d} = 2\pi$$

or using eq.(12) with $R = 10^{18}$

$$\frac{L}{d} = \frac{1}{30} .$$

This shows that for $s = s_{max}$, and $R = 10^{18}$, α is large and the horizontal dimension is 30 times <u>smaller</u> than the vertical dimension; i.e., instead of being large as in the TIROS picture, the cell size is very small in horizontal dimension relative to the vertical dimension.

Figure 3 shows a plot of $\frac{s}{\sqrt{R\sigma}}$ vs a^2 for $\sigma = 1$.

The abscissa is a logarithmic scale; that is it is linear in n where $a^2 = 10^n$ and $R = 10^{18}$ has been used. The figure shows that s has small values for low values of "a", and then rises fairly rapidly for values of a^2 between 1 and 10. At $a^2 \approx R^{\frac{1}{4}}$, $s = s_{max}$, and then s falls to zero at $a^2 = \sqrt{R}$.

Thus if other modes than the one corresponding to R_c were somehow excited, the one which grows fastest, would correspond to a cell size of $\frac{L}{D} = \frac{1}{30}$. We note that "s" is actually large for a wide range of "a", although its actual maximum occurs near $a^2 = R^{\frac{1}{4}}$.

<u>Reference</u>: [3] P.Ledoux, M. Schwarzschild, and E.A.Spiegel; On the Spectrum of Turbulent Convection. Astrophys.J., 133:184-197: Jan.1961.

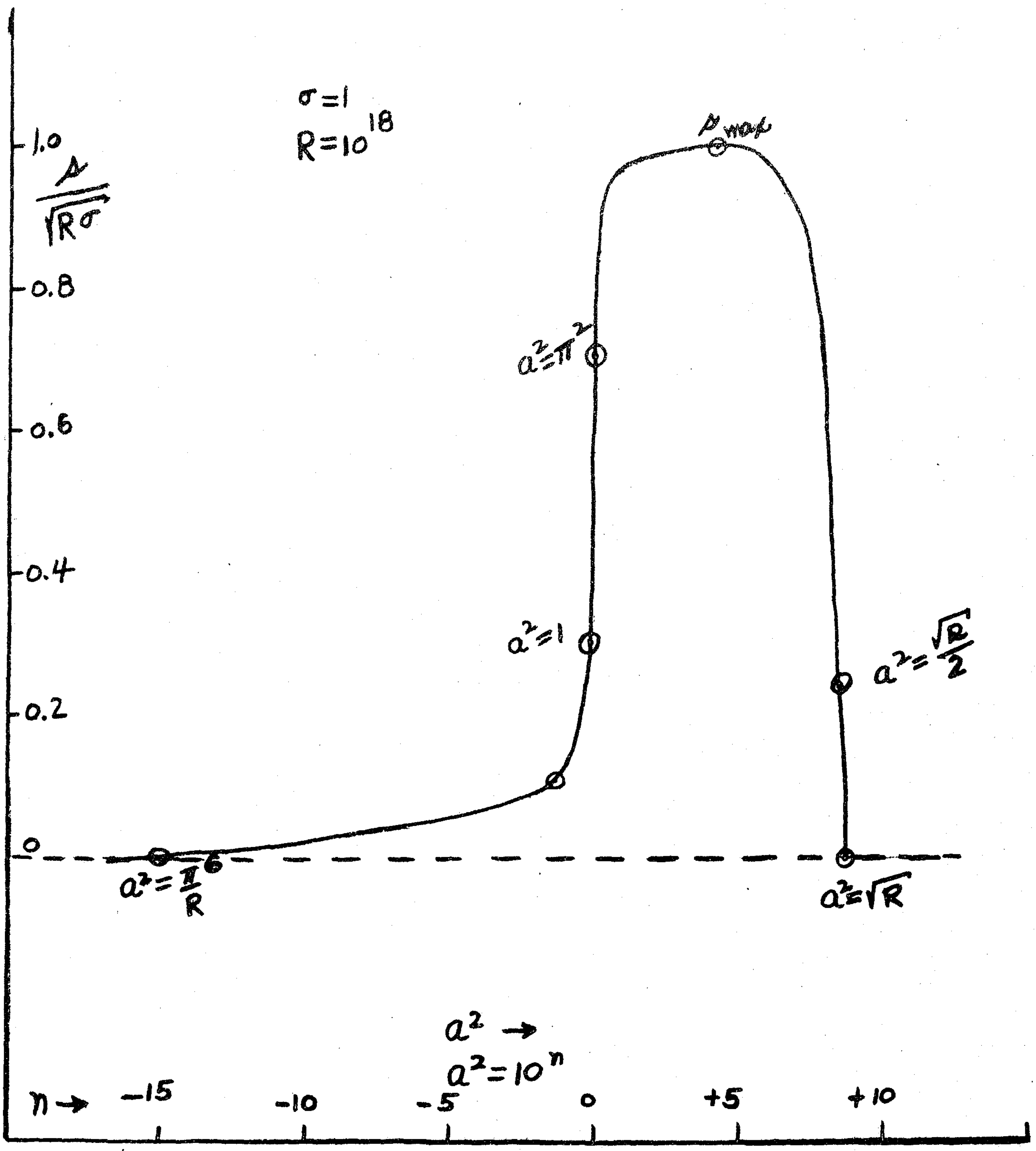

Fig. 3 - Variation of s with a^2

Observations on Convection in Water Cooled from Below.

A. Furumoto and C. Rooth

Introduction.

The density anomaly of fresh water, with a maximum at $4^\circ C$ provides an opportunity to study under laboratory conditions a convective system bounded by a stable fluid layer above. Furthermore theoretical analysis by Veronis, unpublished, indicates that a finite amplitude instability should occur in a water layer with linear temperature stratification including the point of density maximum, subject of course to a Rayleigh number criterion.

The present series of measurements were intended to clarify the heat transport properties of the system in the surroundings of the critical point in order to establish or disprove the finite amplitude instability. Equipment difficulties prohibited the completion of this task within the available time. We can only state at the moment, that the evidence available lends support to the notion that the non-linear effects are important in the whole convective range.

Equipment and Instrumentation.

A diagram of the equipment is given in Figure 1. The system comprises a cooling bath, a trough of distilled water for convection, heating wires immersed in the water, and a set of thermistors to measure the temperature of the convecting water.

The cooling bath at the bottom contained sugar solution. Heat was removed from the bath by circulating a mixture of automobile anti-freeze (glycol) and water through coils of copper tubing. The anti-freeze mixture was in turn cooled by a refrigerator to a temperature of about -20°C. In this way ice was forced to form on the cooling coils. By circulating the sugar solution vigorously by means of two stirrers, the temperature of the bath was maintained essentially at the freezing point of the solution.

The square trough containing the distilled water had dimensions 50 cm x 50 cm. The walls were of fiber glass sheets padded on the outside with styrofoam to secure insulation. The bottom of the trough was an aluminum plate 1/2 inch thick. The water level in the trough was kept at about 8 cm.

At about 5 cm from the bottom of the trough coils of heating wire were suspended horizontally. By means of a variable transformer the current in the coils could be controlled and hence the rate of heat supply to the upper part of the water. A thermometer was laid across the coils for easy checking of the temperature.

The trough was covered with styrofoam boards, to minimize effects of drafts etc. in the room.

Thermistors were suspended from above by means of pulleys. By hooking the pulleys to a small electric motor, the vertical position of the thermistors could be changed at will at a steady velocity. Arrangements were also made so that the horizontal positions of the thermistors could be changed. To measure the temperature of the bottom plate a separate thermistor was inserted into a hole drilled horizontally ca. 5 cm into the plate.

The thermistors were connected to a Variac recorder for convenient reading. The thermistor-recorder combination was calibrated by using thermometers accurate to a tenth of a degree.

The relative vertical position of the thermistors was measured by means of a cathetometer placed one meter away. The distance between any two vertical positions of the thermistors could be measured to a tenth of a millimeter.

Description

A pictorial description of the convective motion of water by cooling from below has been given in a previous report (Furumoto, 1960). In that experiment $AgNO_3$ tracers were used to outline the motion. Although that experiment dealt with a transient case while the present attempts were to consider quasi-steady states, it was felt that the qualitative picture does not very much.

In the present experiments no tracers were used. Visual observation was given up in favor of thermal insulation. Observation was limited to temperature profiles.

Temperature profiles were taken only when there was reasonable assurance that the system of convecting water had reached a steady state.

A hasty calculation indicated that six hours was sufficient time for the system to reach equilibrium, for any change of the controlling factors such as the current through the coils, addition of more water or temperature of the bath. Hence six hours were allowed to elapse before a profile was taken after any change made. And after a profile two or three hours were allowed to elapse again to see whether the system remained steady. The spot check for the steady state was done by checking the temperature or the thermometer lying over the heating coils and the temperature of the water bath.

The heat transport from the heating coils above to the sink below was assumed to be such that the heat flux for the various layers was the same. Because of the padded walls and of the large ratio of width to depth in the system, it was considered safe to neglect the thermal effects of the walls. To further minimize effects of the walls, profiles were taken in the central area of the trough.

Figure 2 illustrates a typical temperature profile when there is convection due to cooling from below. The uppermost layer A was one of convection caused by the heating coils. In this layer temperature and temperature gradient fluctuated from place to place. As this layer acted only as a heat source, temperature profiles of this layer were rarely taken. The transition from layer A to the next layer B was gradual but seemed to be uniform horizontally.

Layer B in Figure 2 was characterized by a linear temperature profile giving a constant gradient. The interpretation is that heat transport in this layer was due entirely to conduction. The heat flux of the system in steady state can then be readily calculated by measuring the temperature gradient of this layer.

The convective motion in layer C was maintained by the downward heat flux in the range of anomalous density variation. The temperature profile varied from place to place, the two lines in figure 2 were delineated by thermistors located at different positions ca. 4 cm apart. Usually the temperature of the layer varied from 3.4°C to 4°C.

Adjacent to the rigid bottom was a boundary layer D where apparently the heat transport was due to conduction only. The gradient had the same value as that of the stable conducting layer B. The temperature of the bottom plate as measured by the thermistor in the plate was taken as the temperature of zero height.

Furumoto noted in the previous report that intense horizontal motions occurred at the top of the convective layer. Later tracer studies showed that these are connected with an inverse cell on top of the main convective cell.

The temperature profile for conduction only in the lower layers was much simpler. There was an upper convecting layer due to the heat source. Below that was a stable conducting layer extending all the way to the bottom. Such a condition was obtained by raising the current in the heating coils.

At first three thermistors were used for simultaneous measurements at various points on a given horizontal plane of the fluid. But as little fluctuations were noticed, and especially as the stable conducting layers yield identical temperature gradients irrespective of horizontal positions, it was considered that the profile from one thermistor was sufficient for the immediate purpose of determining the heat transport data for the system.

Analysis of Data

In examining heat transport due to convection a useful method of analysis is to plot the Nusselt number as a function of Rayleigh number (Silveston 1958). For the present experiment the Rayleigh number needs modification.

The usual definition of Rayleigh number R is

$$R = \frac{g \alpha \Delta T}{\kappa \nu} h^3$$

where g is acceleration due to gravity,

α is the coefficient of thermal expansion of the liquid under examination,

ΔT is the change in temperature from the bottom to the upper boundary,

h is the characteristic height of the convecting liquid,

κ is the thermal conductivity,

ν is the kinematic viscosity.

This number R is applicable where the density variation of the liquid is linear with temperature and α is constant. But for the present case the density of water has a parabolic behavior with a maximum at $4^{\circ}C$.

From the discussion of Beronis (1961), a modified Rayleigh number was taken as

$$R_T = \frac{\mathcal{Q}\, g \alpha (\Delta T)^2}{\kappa \nu} h^3 \left(\frac{h}{d}\right)$$

where d is the distance from the bottom of the $4^{\circ}C$ layer in the undisturbed (conducting) profile. The other symbols have the same meaning as for the usual Rayleigh number. The characteristic height was so chosen that for any profile

$$h = 3\,d.$$

This meant that h will be the height corresponding to a temperature somewhere near $12^{\circ}C$. This choice was made because the $12^{\circ}C$ plane was well near the middle of the stable layer where neither the convecting motion from above due to the heating coils nor that from below due to cooling could affect the fluid, and it was expected that the R-degree depth would be sufficient for asymptotic behaviour.

The numerical values of the other parameters were taken as follows in c.g.s. units.

$$g = 981$$

$$\alpha = 8 \times 10^{-6}$$

$$\kappa = 1.35 \times 10^{-3}$$

$$\nu = 1.6 \times 10^{-2}$$

ΔT was taken as $(4 - T_p)^{\circ}C$,

where T_p is the temperature of the bottom plate. Then h, the characteristic height, was the height of the $(12 - 2\ T_p)^{\circ}$ layer.

The Nusselt number is defined as the ratio of the total heat flux H to the heat flux due to conduction. Symbolically this is

$$Nu = \frac{H}{\kappa \beta_m}$$

where β_m is the mean temperature gradient. The total heat flux can be found by the measurement of the temperature gradient in the stable conducting layer $\left(\frac{dT}{dz}\right)_s$, as it is assumed that heat flux is the same through all the layers. On the other hand the mean temperature gradient β_m is

$$\frac{(12 - 2\ T_p) - T_p}{h} = \frac{12 - 3\ T_p}{h}$$

Conveniently the Nusselt number becomes

$$Nu = \frac{\left(\frac{dT}{dz}\right)_s}{\frac{12 - 3T_p}{h}}$$

For the conducting case the Nusselt number is unity.

Table 1 lists the pertinent data. Figure 3 is a plot of the Nusselt number vs. the modified Rayleigh number.

Consider Figure 3. The points along Nu = 1 were the cases where only conduction occurred. The convecting cases start with R_T of about 150,000 with Nu = 1.5. The distribution of the points of the convecting cases appear to be linear with a slope roughly of 1/3. This value is characteristic of the fully developed turbulent state, with heat transfer limited only by the boundary layers. The fact that this state is attained at once in the convective range strongly supports the importance of the non-linear effects in the system. In spite of allowances for observational errors the extensions of possible slopes for the convection points will intersect the Nu = 1 line such that there would always be a conduction point to the right of the intercept.

A reasonable interpretation of the manner of intersection of the lines is that the transition from a conductive heat transport to a convective motion involves a discrete jump rather than a gradual transition. Such was the result hoped for to establish the theory of finite amplitude instability.

On the other hand, as the highest R_T for the conductive regime is 100,000 while the lowest R_T for the convective regime is ca 150,000, there is still the possible alternative that the transition is gradual, although in a rapid manner, so that finite amplitude instability need not be invoked.

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Platzman and Kuo. 1960. An Investigation of the Nonlinear Solution of the Rayleigh Convection Problem by means of Characteristic Functions. Dept. Met., University of Chicago.

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Table I

Profile	ΔT_s	T_p	$12-2T_p$	h_1	h	$(4-T_p)^2$	h^3	$\frac{h\nabla T_s}{12-3T_p}$	$(4-T_p)^2 h^3$
22	7.80	.15	11.2	1.34	1.46	14.8	3.10	0.99	46
21	8.44	.30	11.4	1.20	1.32	13.7	2.30	1.00	31.5
20	8.43	.25	11.5	1.22	1.34	14.0	2.12	1.01	32.4
19	8.70	.25	11.5	1.16	1.28	14.0	2.05	.99	29.7
18	8.90	.25	11.5	1.15	1.27	14.0	2.05	1.00	28.8
17	9.20	0.4	11.2	1.64	1.76	13.0	5.42	1.50	70.5
16	9.05	0.0	12	2.13	2.25	16.0	11.4	1.71	182
15	9.3	-0.1	12.2	2.10	2.22	16.8	10.9	1.72	183
14	9.2	0.1	11.8	2.12	2.24	15.2	11.2	1.76	170
13	9.0	0.2	11.6	1.94	2.06	14.4	8.70	1.63	125
12	8.11	0.4	11.2	1.70	1.85	13.0	2.30	1.39	82
11	7.9	0.3	11.4	2.08	2.20	13.2	10.6	1.57	145
10	8.23	.25	11.5	2.10	2.22	14.0	10.9	1.62	152
9	7.5	.15	11.7	1.92	2.04	14.8	8.5	1.32	126
8	8.43	0.2	11.6	2.17	2.29	14.4	12.0	1.69	172
7	8.4	0.2	11.6	2.46	2.58	14.4	18.2	1.90	254
6									
5	8.2	0.2	11.6	3.50	3.62	14.4	47.2	2.60	680
4	7.1	0.1	11.8	7.40	3.52	15.2	43.6	2.14	658

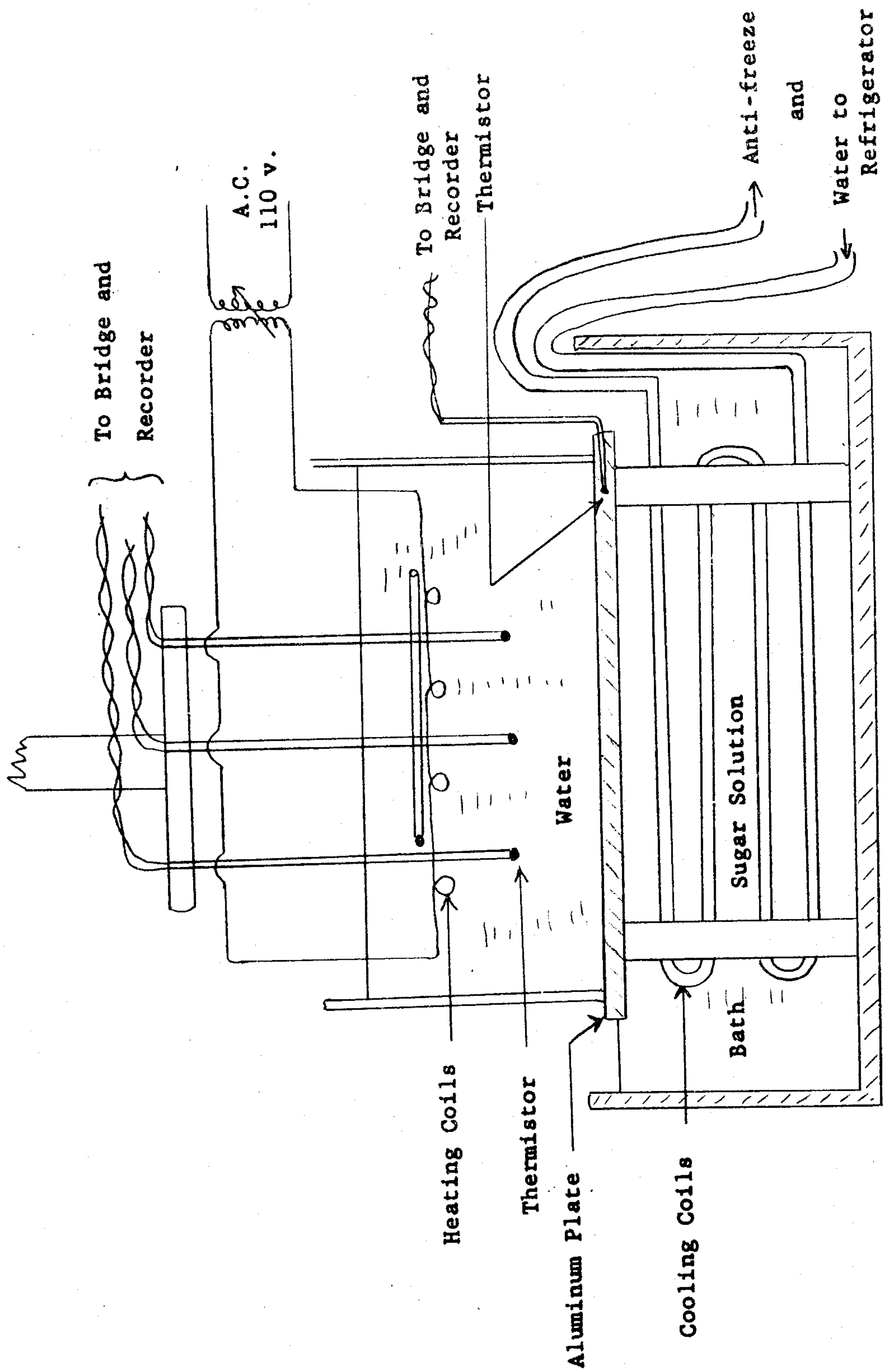
To Bridge and Recorder
A.C. 110 v.
To Bridge and Recorder
Thermistor
Anti-freeze and Water to Refrigerator
Water
Sugar Solution
Bath
Heating Coils
Thermistor
Aluminum Plate
Cooling Coils

Fig. 1.

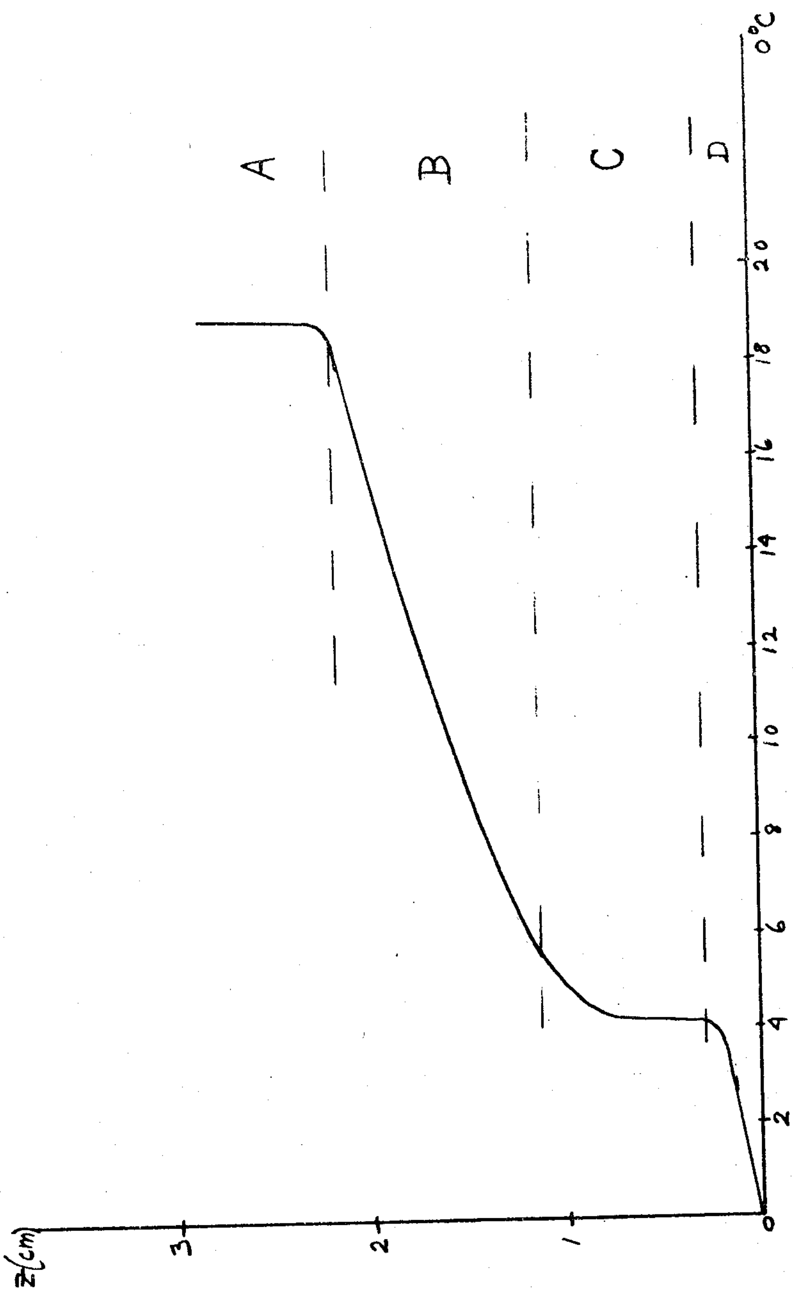

Figure 2.

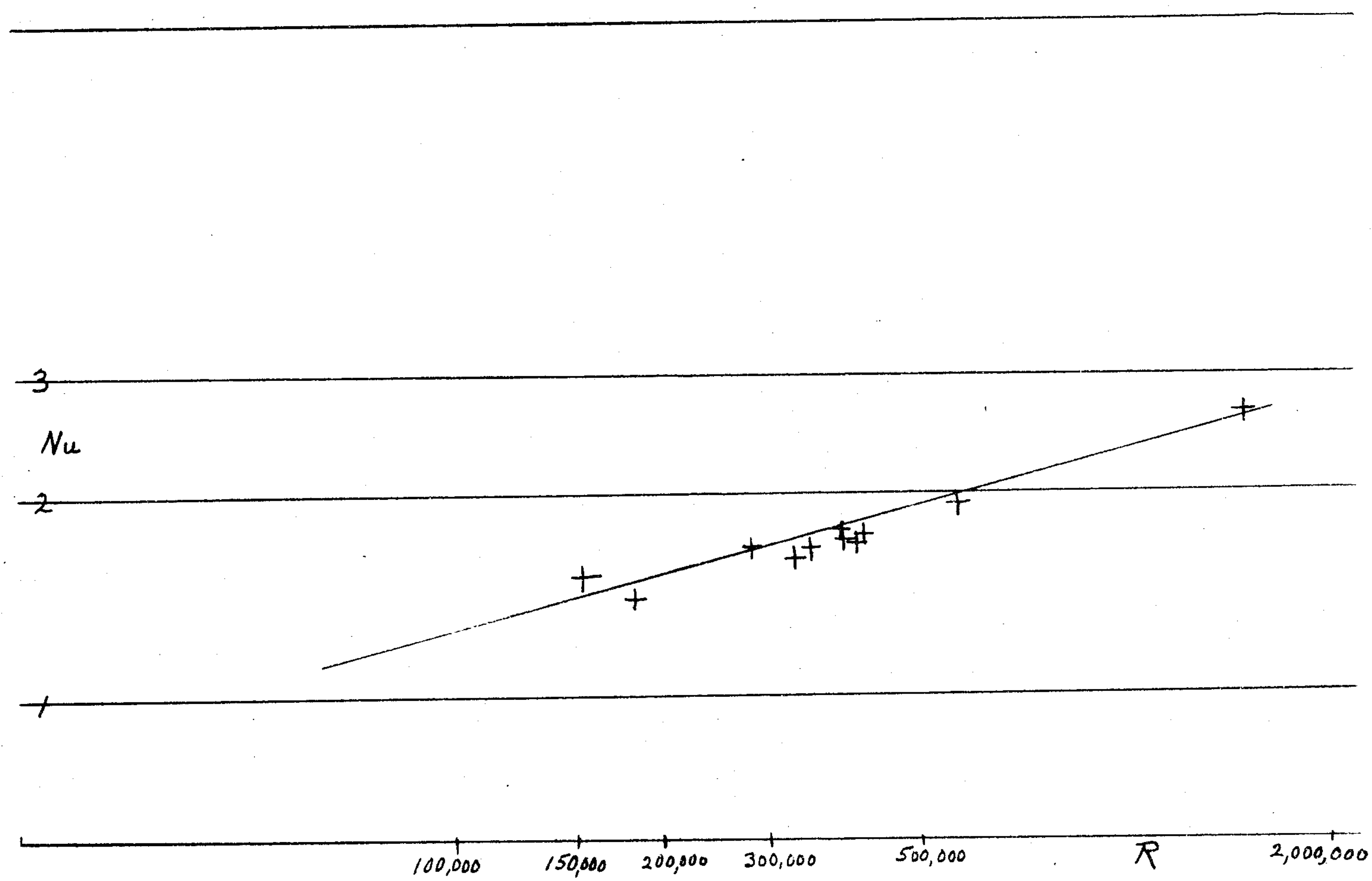

Figure 3.

The arms of the crosses represent estimated uncertainty in the data. N.B. a line of slope 1/3 can be drawn to fit all the convective points within those limits.

The Stability of a Two-Level Model of the Gulf Stream

by

Frank Lipps

I. Introduction

The motivation of this investigation is to examine the stability of the Gulf Stream. This current is a narrow jet of water which runs parallel to the East Coast of the United States from Florida to Cape Hatteras, and then heads out to mid-ocean travelling in a more or less easterly direction. It has a maximum velocity of 200 cm/sec and a total width of about 100 km (Stommel, 1958). In addition, there is a strong density gradient with warmer water in the surface layers and to the right of the stream. The strongest flow is near the surface, but recent measurements by Swallow show that there is a weak countercurrent below the Gulf Stream. The maximum velocity in this layer is about 30 cm/sec.

We will construct a two-layer model of the Gulf Stream using these general features. A similar model of ocean circulations was discussed in the lecture series. Our main interest is in the stability of the Gulf Stream after it leaves the coast, heading in an easterly direction. Thus we will consider a two-layer model with the basic flow in the west-east direction. We hope to learn by this analysis what are the relevant parameters which limit the strength of the current before it becomes unstable. It should also be possible to learn what disturbances are most unstable and how fast they grow.

2. <u>The Two Layer Model.</u>

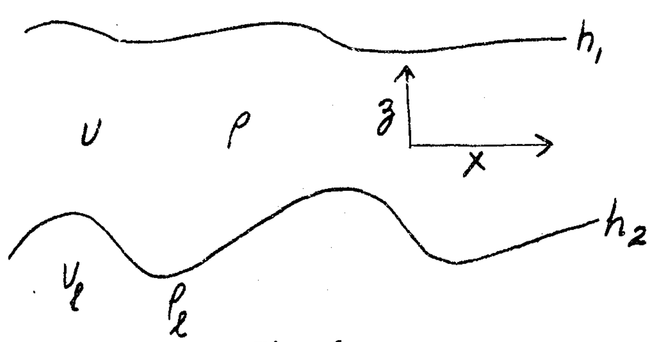

Fig. 1

We consider a two-layer model of the Gulf Stream in which the depth of the upper layer is finite and the depth of the lower layer is infinite. In each layer the horizontal velocities and density are independent of depth. We have the basic velocity U and density ρ in the upper layer, and the corresponding quantities U_ℓ and ρ_ℓ in the lower layer. The coordinates are directed as shown with y pointing into the paper. The motion is taken to be hydrostatic and geostrophic. Thus in the upper layer we get:

$$v = \frac{g}{f}\frac{\partial h_1}{\partial x} \qquad u = -\frac{g}{f}\frac{\partial h_1}{\partial y}$$

and in the lower layer:

$$v = \frac{g}{f}\frac{\partial \psi}{\partial x} \qquad u = -\frac{g}{f}\frac{\partial \psi}{\partial y}$$

where $\psi = h_1 + \frac{\Delta\rho}{\rho} h_2$. In these equations we take $\rho \approx \rho_\ell$ and $\frac{\Delta\rho}{\rho} = \frac{1}{500}$.

In the upper layer we require that the potential vorticity be conserved, i.e.

$$\frac{d}{dt} \frac{\zeta + f}{h} = 0$$

Here ζ is the relative vorticity, f is the Coriolis parameter and h, the depth of the fluid, is $h_1 - h_2$.

In the lower layer, since the depth is infinite, we require conservation of absolute vorticity $\zeta + f$, i.e.

$$\frac{d}{dt}(\zeta + f) = 0$$

For the basic motion we take a current flowing in the west-east direction. In order to examine the stability of this basic flow we superimpose infinitesimal perturbations of the form:

$$h_1 = e^{ik(x-ct)} h_1(y), \quad \psi = e^{ik(x-ct)} \psi(y)$$

In the following $S = \frac{f^2}{g\bar{h}\Delta\rho/\rho}$, where $\bar{h} = \bar{h}(y)$ is the basic thickness of the upper layer, a prime represents a differentiation with respect to y. Also h_p is the perturbation thickness of the upper layer (i.e. $h = \bar{h}(y) + h_p$), and $\beta = \frac{df}{dy}$. From the conservation of potential vorticity in the upper layer we find:

$$(v-c)(h_1'' - K^2 h_1) + (\beta - v'')h_1 + \frac{\Delta\rho}{\rho} S\left(1 - \frac{v'}{f}\right)\left((v-c)h_p + \frac{g}{f} h_1 \bar{h}'\right) = 0 \qquad (1)$$

But h_p is related to h_1 and ψ, and $\bar{h}'$ is related to $V - V_\ell$. We define $S_T = S\left(1 - \frac{v'}{f}\right)$. Thus we obtain:

$$h_1'' - (K^2+S_T)h_1 + \frac{\beta - U'' + S_T(U-U_\ell)}{U-c} h_1 + S_T\psi = 0 \qquad (2a)$$

In the lower layer we obtain the usual equation for conservation of absolute vorticity.

$$\psi'' - K^2\psi + \frac{\beta - U''}{U-c}\psi = 0 \qquad (2b)$$

In this layer we have pure barotropic flow.

For boundary conditions we require that h_1 and $\psi \to 0$ as $y \to \pm\infty$.

The disturbances which satisfy (2a) and (2b) can be divided into two types.

a. Upper layer baroclinic disturbances

For these disturbances $\psi = 0$, which implies $h_2 = -\frac{\ell}{\Delta\rho} h_1$. There is no perturbation motion in the lower layer. The potential vorticity equation for the upper layer becomes:

$$h_1'' - (K^2+S_T)h_1 + \frac{\beta - U'' + S_T(U-U_\ell)}{U-c} h_1 = 0 \qquad (2c)$$

b. Lower layer barotropic disturbances

Here $\psi \neq 0$. There is barotropic motion in the lower layer which forces the perturbations in the upper layer. The equations for both layers remain as previously given.

3. Baroclinic Upper Layer Disturbances

First we derive some general results, and then we consider the stability of a given flow pattern. We will derive a sufficient condition for the stability of the flow. If we set

$h_1 = h_{1r} + i h_{1i}$ and $c = c_r + i c_i$, we can divide equation (2c) into real and imaginary parts:

$$h_{1i}'' - (K^2 + S_T)h_{1i} + \frac{Q}{|U-c|^2}(U-c_r)h_{1i} + \frac{Q}{|U-c|^2}c_i h_{1r} = 0$$

$$h_{1r}'' - (K^2 + S_T)h_{1r} + \frac{Q}{|U-c|^2}(U-c_r)h_{1r} - \frac{Q}{|U-c|^2}c_i h_{1i} = 0$$

where $Q = \beta - U'' + S_T(U - U_\ell)$. Now multiply the first of these equations by h_{1r}, multiply the second by h_{1i}, subtract and integrate. We finally find:

$$c_i \int_{-\infty}^{\infty} \frac{Q}{|U-c|^2} |h_1|^2 dy = 0 \tag{3a}$$

Thus $c_i = 0$ if Q is of one sign throughout the fluid. We see that a countercurrent toward the west in the lower layer tends to stabilize the flow if $S_T > 0$. This result follows since $S_T(U - U_\ell)$ is then positive and tends to make Q positive everywhere in the fluid.

By a similar derivation we can find $\frac{\partial c}{\partial s}$, where $s = -K^2$. For this result we differentiate equation (2c) with respect to s. We multiply the resulting equation by h_1. Then equation (2c) is multiplied by $\frac{\partial h_1}{\partial s}$ and subtracted from the above. After integrating we obtain:

$$\frac{\partial c}{\partial s} = \frac{\int_{-\infty}^{\infty} h_1^2 dy}{\int_{\infty}^{\infty} \frac{U'' - \beta - S_T(U - U_\ell)}{(U-c)^2} h_1^2 dy} \tag{3b}$$

This equation is useful if we know h_1 and c for a particular K. Then the values of c for nearby values of K can be estimated from $\frac{\partial c}{\partial s}$.

We now consider a basic velocity of the form:

$$U = A\,\mathrm{sech}^2\, by \qquad \text{(3c)}$$
$$U_\ell = -\alpha A\,\mathrm{sech}^2\, by$$

We non-dimensionalize the variables and the parameters of the problem

$$x^* = bx \quad y^* = by \quad t^* = Abt \quad R_o = \frac{Ab}{f} \qquad \text{(3d)}$$
$$\beta^* = \frac{\beta}{Ab^2} \quad K^* = \frac{K}{b} \quad C^* = \frac{c}{A} \quad F_i^2 = \frac{f^2}{g\bar{h}\frac{\Delta\rho}{\rho}b^2}$$
$$U^* = \mathrm{sech}^2 y^* \quad h_1^* = S^{1/2}h_1 \quad \psi^* = S^{1/2}\psi$$

Equation (2c) becomes after dropping the stars.

$$h_1'' - \left(K^2 + F_i^2(1 - R_o U')\right)h_1 + \frac{\beta - U'' + F_i(1 - R_o U')(1+\alpha)U}{U - C}h_1 = 0 \qquad \text{(3e)}$$

In the following in order to obtain a solution in simple form we take $F_i^2(1 - RU') \approx F_i^2$. It will be seen that this assumption is not justified for a jet as narrow as the Gulf Stream. For typical values characteristic of this current we have $\frac{\Delta\rho}{\rho} = 1/500$, $A = 200$ cm/sec, $b = 1/30$ km, $f = 10^{-4}\,\mathrm{sec}^{-1}$, $\bar{h} = 500$ m, and $\beta = 1.75 \times 10^{-13}\,\mathrm{cm}^{-1}\mathrm{sec}^{-1}$. From these values we obtain $F_i^2 = 9/10$, $\beta = .008$ and $R_o = 2/3$. Thus we are not justified in setting $R_o U' \sim 0$. The large value of the Rossby number is embarrassing in another respect. We have assumed the motion to be geostrophic in both layers. This state only exists if there is a balance between the pressure and Coriolis terms in the equations of motion, i.e. if the inertial terms can be neglected. Hence we need $R_0 << 1$ for the geostrophic relations to hold. In the following we take $\bar{h}$ equal to a constant when it appears in F_i^2. This assumption is also poor in regard to the Gulf Stream. Actually $\bar{h}$ varies from 200 m to 900 m across the current. (See Stommel, 1958; Fig. 33.) Thus the analysis below is lacking in the two major respects discussed above. Namely, the flow in the vicinity of the Gulf Stream is not geostrophic and $\bar{h}$ is not a constant across the stream. However it is hoped that the following work will give some insight into the factors that determine the stability of the Gulf Stream.

From the above we see that β = .008 and can be neglected. If we neglect β and make the other assumptions above, equation (3e) becomes:

$$h_1'' - (K^2 + F_i^2) h_1 + \frac{-U'' + F_i^2(1+\alpha) U}{U - c} h_1 = 0 \tag{3f}$$

Now we solve for the neutral waves for which $U = c$ at some point in the fluid. It will be seen below that these waves are the waves that separate the stable from the unstable disturbances.

It can be shown that the only values of c for which $U = c$ at some point $y = y_c$ in the fluid are those for which Q = 0 at the same point $y = y_c$. (See Foote and Lin, 1950.) Thus c is determined. We find c = 0 and $c = \frac{2}{3}\left(1 - \frac{F_i^2(1+\alpha)}{4}\right)$. In either case we get a differential equation of the form:

$$h_1'' + [6 \operatorname{sech}^2 y - \mu^2] h_1 = 0$$

If we set $z = \tanh y$, this equation becomes:

$$(1-z^2)\frac{d^2}{dz^2}h_1 - 2z\frac{d}{dz}h_1 + \left[6 - \frac{\mu^2}{1-z^2}\right]h_1 = 0 \tag{3g}$$

The above is a particular case of Legendre's Equation. The only solutions meeting the boundary conditions are the Legendre Polynomials P_2^1 and P_2^2.

We consider P_2^2 first. In this case $\mu^2 = 4$. If $c = \frac{2}{3}\left(1 - \frac{F_i(1+\alpha)}{4}\right)$, we have $\mu^2 = 4 = K^2 + F_i^2$, or:

$$K^2 = 4 - F_i^2$$

If $c = 0$, we have $4 = K^2 + F_i^2 - (F_i^2(1+\alpha) - 4)$, or $K^2 = \alpha F_i^2$. This situation is shown in Fig. 2. The neutral waves are shown by the solid lines on the diagram. From $\frac{\partial c}{\partial \Delta}$ it can be shown that the area inside the solid lines represents the region of the (F_i^2, K^2) plane where unstable disturbances occur. Outside this region are only stable waves. There are no unstable waves for $F_i^2 \geq \frac{4}{1+\alpha}$. From the form of Q, we see that this result is just the condition for stability as given by (3a).

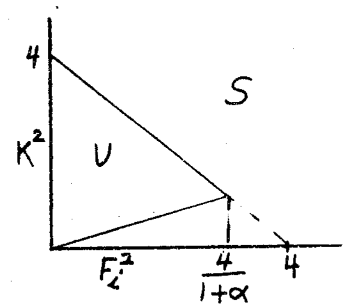

Fig.2

We now consider the neutral solution P_2^1. In this case $\mu^2 = 1$. If $c = \frac{2}{3}\left(1 - \frac{F_i^2(1+\alpha)}{4}\right)$, we have

$$\mu^2 = 1 = K^2 + F_i^2$$

or $K^2 = 1 - F_i^2$

Thus we have the situation shown in Fig. 3. From $\frac{\partial c}{\partial \alpha}$ it can be shown that the area inside the triangle represents the region of the (F_i^2, K^2) plane where unstable waves occur. There are no unstable disturbances for $F_i^2 \geq 1$.

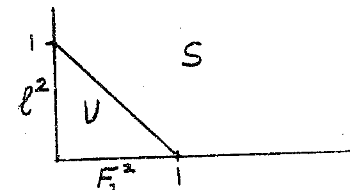

Fig.3

From the values of $\frac{\partial c}{\partial \alpha}$ given below it is expected that the maximum amplification rate, $(Kc_i)_{max}$, is considerably less for the disturbances in Fig. 3 than for those in Fig. 2. However the long waves are unstable for $0 < F_i^2 < 1$. In Fig. 2 they are not.

Since we know the neutral solutions we can calculate $\frac{\partial c}{\partial \alpha}$. For the P_2^2 case $h_1^2 \operatorname{sech}^2 y$, and we can calculate $\frac{\partial c}{\partial \alpha}$ explicitly. If we set $z = \tanh y$, we find for $\frac{\partial c}{\partial \alpha}$ along the upper line in Fig. 2:

$$\frac{\partial c}{\partial \alpha} = \frac{4/3}{-8-12c-6c^2 \int_{-1}^{1} \frac{1}{1-z^2-c}\, dz}$$

We have two singularities to integrate around. These occur at the values of y where $U = c$. According to the criterion given by Foote and Lin we must integrate above the singularity where $U' < 0$ and below the singularity where $U > 0$. Thus we finally obtain:

$$\frac{\partial c}{\partial \alpha} = \frac{\frac{2}{3}\left[-4-6c+\frac{6c^2}{2\sqrt{1-c}} \log \frac{1-\sqrt{1-c}}{1+\sqrt{1-c}} - \frac{6c}{2\sqrt{1-c}} \pi i\right]}{\left[4+6c-\frac{6c}{2\sqrt{1-c}} - \log \frac{1-\sqrt{1-c}}{1+\sqrt{1-c}}\right]^2 + \left[\frac{6c^2}{2\sqrt{1-c}} \pi^2\right]^2}$$

In this problem we take $\alpha = 1/6$ as the maximum velocity in the upper layer is about 200 cm/sec, and in the lower layer it is about 30 cm/sec. A convenient value of c

to put in the above expression for $\frac{\partial c}{\partial \lambda}$ is $c = 1/2$. For this value of c we have $F_i^2 = 6/7$ and $\ell^2 = 2.95$. We find $\frac{\partial c}{\partial \lambda} = -.065 + .025\, i$. Lessen and Fox have solved the above stability problem by numerical integration for $F_i^{-2} = 0$. They have plotted c_i and c_r as functions of K and have calculated $\frac{\partial c}{\partial \lambda}$ for the neutral waves at $K^2 = 4$ (Fig. 2) and $K^2 = 1$ (Fig. 3). For the neutral wave at $K^2 = 4$ they find $\frac{\partial c}{\partial \lambda} = -.0421 + i\,.02771$. For the other neutral wave at $K^2 = 1$ they find $\frac{\partial c}{\partial \lambda} = .0119 + i\,.09021$. In the general case when $F_i^2 \neq 0$ it is noted that $\frac{\partial c_i}{\partial \lambda} \sim c^2$. Thus $\frac{\partial c_c}{\partial \lambda}$ dies off very quickly as $c \to 0$. Hence as $F_i^{-2} \to \frac{4}{1+\alpha}$ and $c \to 0$ along the upper line in Fig. 2 we find that $\frac{\partial c_i}{\partial \lambda} \to 0$ quickly.

4. <u>Barotropic Lower Layer Disturbances</u>.

We previously found $\beta = .008$. Here, in the lower layer, $U'' \sim \frac{1}{6}$ since $\alpha = \frac{1}{6}$. Thus we can still neglect β and the work of Foote and Lin applies. It should be noted that the assumptions made for the upper level baroclinic disturbances that are not justified do not need to be made here. It is true that we assumed the motion to be geostrophic in the lower layer as well as in the upper layer. However, this assumption is not necessary. Since the flow is non-divergent, we can always define a stream function and this stream function will satisfy equation (2b). Hence it would appear that the results obtained for the lower layer barotropic disturbances should have more validity than those obtained for the upper level baroclinic disturbances.

5. Numerical Results

First we consider the upper layer baroclinic disturbances. In Fig. 2 the long neutral wave is given by $K^2 = \alpha F_i^2 = \frac{3}{20}$ or $\lambda = 770 \, km$ Next we calculate the amplification rate of a typical very unstable wave. This result can be expected to be only somewhat better than an order of magnitude argument. We have $\frac{\partial c}{\partial s}$ = $-.065 + .025\,i$ for $c = 1/2$ at $K^2 = 2.95$ along the upper line in Fig. 2. We take $\Delta S = 1$ or $\Delta K^2 = -1$. Therefore $\Delta c_c = .025$. The value of K^2 for the unstable wave is 1.95. Thus $K = 1.4$. We find $K c_i = .035$, or in dimensional terms we get $AbKc_i = .23 \times 10^{-5} sec^{-1}$. Therefore this wave takes about 4.3 days to increase in amplitude by a factor of e. The wavelength is 130 km.

Finally we consider the lower layer barotropic disturbances. From the Lessen and Fox paper we find that the most unstable wave is at $K = 1$. Here $K c_i = .158$. After putting into dimensional terms we get $\alpha A \ell c_i = .176 \times 10^{-5} sec^{-1}$, i.e. it takes about 6 days for this wave to increase in amplitude by a factor of e. The wavelength is 180 km.

6. Conclusions

We find that a countercurrent in the deep lower layer tends to make the flow more stable with respect to the upper layer baroclinic disturbances. The condition that the flow is stable with respect to these disturbances is that $F_{i}^{2} \geq \frac{H}{1+\alpha}$. Thus the larger the α the more stable the flow. For the example studied it is found that both types of disturbances are roughly equally unstable with the most unstable waves having a wavelength between about 100-200 km. We also find that there can be fairly rapid time variations in the lower layer as indicated by the barotropic instability there.

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"Long" Waves on a Rotating Globe

by

L. W. MacMillan

The β-plane concept has been extensively used in dealing with planetary waves, and rightly so, since it is the simplest scheme which allows one to take account of the variation of Coriolis parameter with latitude, the latter being essential to the definition of these waves. However, it is not always clear whether results deduced on the basis of the β-plane concept mirror similar results which could be deduced from a more general scheme, or whether certain results are merely artifacts due to the peculiarity of the β-plane system itself. We will consider this question here with regard to the linearized, or small-amplitude, equations, and will conclude that the β-plane provides a very good guide to the behaviour of the more general system.

For a derivation of the β-plane equations we refer to the summer lecture notes on the wind driven ocean circulation (ref. 1). The essential steps in the approximation are, first, to neglect certain terms in the equations of motion expressed in spherical coordinates which do not involve derivatives, and second, to linearize the Coriolis parameter about a chosen latitude. The first step leads to the restriction that we cannot consider latitudes greater than 30-40^{o}N or S, while the second step allows us to introduce local rectangular coordinates, and, in effect, limits us to a small strip on the chosen latitude. We also mention that neglect of the Coriolis acceleration due to vertical motions restricts us to latitudes at least several degrees away from the equator. If these equations are now linearized, to yield a description of small-amplitude disturbances, we immediately see that the basis for the restriction that we stay away from the poles is removed, so that the linearized equations apply right up to the poles.

In view of this, we will begin with an initial linearization, and thus avoid the need of the arguments given in ref. 1. In the Euler equations,

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \vec{\Omega} \times \vec{u} - \vec{g} = - \frac{\nabla p}{\rho} , \tag{1}$$

$$\nabla \cdot \vec{u} = 0 , \tag{2}$$

we substitute $p = p_0 + p_1, \rho = \rho_0$; then, to zero order,

$$\nabla p_0 = + \vec{g} \rho_0 , \tag{3}$$

and to first order

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{\rho_0} \nabla p_1 + \vec{\Omega} \times \vec{u} = 0 \tag{4}$$

$$\nabla \cdot \vec{u} = 0 . \tag{5}$$

The zero-order equation is simply the hydrostatic equation, and is integrated to yield

$$P_0 = -g \rho_0 r + \text{constant} , \tag{6}$$

assuming r is along $-\vec{g}$.

Transforming the first-order equations to spherical coordinates (ref.1), we have

$$\frac{\partial u}{\partial t} + \Omega w \cos\phi - \Omega v \sin\phi = \frac{1}{\rho_0 r \cos\phi} \frac{\partial p_1}{\partial \lambda} , \tag{7}$$

$$\frac{\partial v}{\partial t} + \Omega u \sin\phi = \frac{1}{\rho_0 r} \frac{\partial p_1}{\partial \phi} , \tag{8}$$

$$\frac{\partial w}{\partial t} - \Omega u \cos\phi = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial r} , \tag{9}$$

$$\frac{\partial}{\partial r}\left(r^2 \cos\phi\, w\right) + r\frac{\partial}{\partial\phi}\left(\cos\phi\, v\right) + r\frac{\partial u}{\partial \lambda} = 0 . \tag{10}$$

The third equation of motion will be ignored, that is, all its terms will be considered very small. This implies ($\partial p_1/\partial r \sim 0$) that we have a "shallow" layer of fluid, of depth D $<<$ R, the earth's radius. The neglect of $\Omega u \cos\phi$ in the third equation is consistent with the neglect of $\Omega w \cos\phi$ in the first equation. This leads to the restriction that we stay at least a few degrees away from the equator (ref.1). Thus, we have the simplified set of equations:

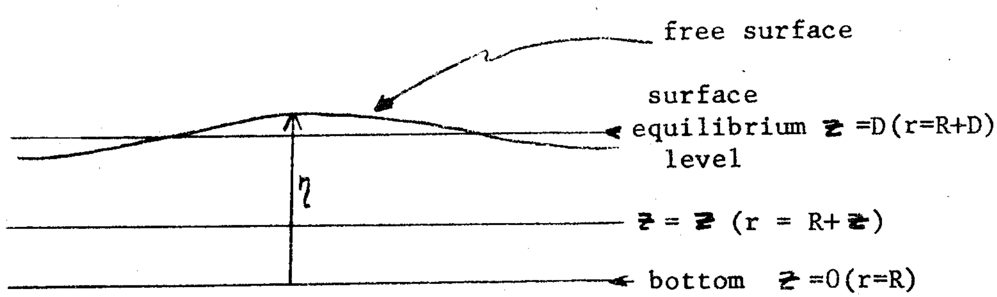
$$\frac{\partial u}{\partial t} - f v = -\frac{1}{\rho_0 R \cos\phi} \frac{\partial p_1}{\partial \lambda} , \tag{11}$$

$$\frac{\partial v}{\partial t} + f u = -\frac{1}{\rho_0 R} \frac{\partial p_1}{\partial \phi} , \tag{12}$$

$$R^2 \cos\phi \frac{\partial w}{\partial r} + R\frac{\partial}{\partial \phi}\left(\cos\phi\, v\right) + R\frac{\partial u}{\partial \lambda} = 0 , \tag{13}$$

with $f = \Omega \sin\phi$.

Since we are dealing with a "thin" layer of fluid, in which the perturbation pressure, p_1, is the same throughout the layer ($\partial p_1/\partial r \sim 0$), we can transform from p_1 to surface elevation, η .



We have, since p_1 doesn't depend on r,

$$p(\lambda, \phi) = p_o + p_1(\lambda, \phi)$$

$$= -g\rho_o r + \text{constant} + p_1$$

$$= -g\rho_o(R + z) + \text{constant} + p_1 \quad (14)$$

Now, from the diagram,

$$p_1 = g\rho_o(\eta - 1)) , \quad (15)$$

so

$$p(\lambda,\phi) = -g\rho_o R - g\rho_o z + \text{constant} + g\rho_o(\eta - D)$$

$$= g\rho_o(\eta - z) , \quad (16)$$

with constant = $g\rho_o(R+D)$ then

$$\frac{\partial p_1}{\partial \lambda} = \frac{\partial p}{\partial \lambda} = g\rho_0 \frac{\partial \eta}{\partial \lambda} \; , \; \frac{\partial p_1}{\partial \phi} = \frac{\partial p}{\partial \phi} = g\rho_0 \frac{\partial \eta}{\partial \phi} \, . \tag{17}$$

Using the foregoing we can also transform the continuity equation, by integrating over the depth of the layer, and using $w = \partial\eta/\partial t$ at the surface, and $w = 0$ at the bottom,

$$\int_{r=R}^{r=\text{surface}} \frac{\partial w}{\partial r} dr + \frac{1}{R\cos\phi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi}(\cos\phi\, v)\right] \int_{r=R}^{r=\text{surface}} dr = 0$$

or
$$\frac{\partial \eta}{\partial t} + \frac{D}{R\cos\phi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi}(\cos\phi\, v)\right] = 0 \tag{18}$$

We thus have the equations

$$\frac{\partial u}{\partial t} - fv = -\frac{g}{R\cos\phi}\frac{\partial \eta}{\partial \lambda} \; , \tag{19}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{g}{R}\frac{\partial \eta}{\partial \phi} \; , \tag{20}$$

$$\frac{\partial \eta}{\partial t} + \frac{D}{R\cos\phi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi}(\cos\phi\, v)\right] = 0 \tag{21}$$

We now transform coordinates to μ, λ from ϕ, λ. μ is defined by $d\phi = \cos\phi\, d\mu$. μ, λ are angular Mercator coordinates (ref.2), and $\mu = \int \sec\phi\, d\phi = -\ln(\sec\phi - \tan\phi)$ $= \ln(\sec\phi + \tan\phi)$, so that $e^{\mu} = \sec\phi + \tan\phi$, $e^{-\mu} = \sec\phi - \tan\phi$. Thus we have

$$\frac{e^{\mu} - e^{-\mu}}{e^{\mu} + e^{-\mu}} = \tan h\mu = \frac{2\tan\phi}{2\sec\phi} = \sin\phi .$$

Similarly $\cos\phi = \operatorname{sech}\mu$, and so on. The transformation can be visualized by considering an arc along the equator, of length $R\,d\lambda$. At latitude ϕ, the arc enclosed by $d\lambda$ has length $R\cos\phi\, d\lambda$. The Mercator chart makes the length at latitude ϕ equal to that at the equator, that is the distance $R\cos\phi\, d\lambda$ is increased by a factor $1/\cos\phi$ to $Rd\lambda$. Further, the Mercator chart now demands the north-south arc length enclosed by ϕ , namely $R\phi$, be increased in the same ratio as the east-west arc length, that is, $R\phi$ is increased to $R\mu = R\phi/\cos\phi$ on the Mercator chart. These relations are local, and should be written $R\,d\phi/\cos\phi = R\,d\mu$, as above.

We then have, writing $\tanh\mu = \tau$, $\operatorname{sech}\mu = \sigma$, and using $\partial/\partial\phi = \frac{1}{\sigma}\,\partial/\partial\mu$,

$$\frac{\partial u}{\partial t} - v\Omega\tau = -\frac{g}{R\sigma}\frac{\partial\eta}{\partial\lambda} , \tag{22}$$

$$\frac{\partial v}{\partial t} + u\Omega\tau = -\frac{g}{R\sigma}\frac{\partial\eta}{\partial\mu} , \tag{23}$$

$$\frac{\partial\eta}{\partial t} + \frac{D}{\sigma R}\left[\frac{\partial u}{\partial\lambda} + \frac{1}{\sigma}\frac{\partial}{\partial\mu}(\sigma v)\right] = 0 \tag{24}$$

with $f = \Omega\tau$. We note the more symmetrical appearance of these equations as compared with the set before transformation. This is a sign that the equations will yield results similar to those obtained from the simpler β-plane equations.

We can derive an energy conservation law from the foregoing equations. We have

$$u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} - fuv + fuv = -\frac{g}{R\sigma}u\frac{\partial\eta}{\partial\lambda} - \frac{g}{R\sigma}v\frac{\partial\eta}{\partial\mu}$$

$$\text{or } \frac{\partial}{\partial t}\frac{(u^2+v^2)}{2} = -\frac{g}{R\sigma}\left[\frac{\partial u\eta}{\partial\lambda} + \frac{\partial v\eta}{\partial\mu} - \eta\frac{\partial u}{\partial\lambda} - \eta\frac{\partial v}{\partial\mu}\right]$$

$$= -\frac{g}{R\sigma}\left[\frac{\partial u\eta}{\partial\lambda} + \frac{\partial v\eta}{\partial\mu} + \frac{v\eta}{\sigma}\frac{\partial\sigma}{\partial\mu} - \eta\frac{\partial u}{\partial\lambda} - \eta\frac{\partial v}{\partial\mu} - \frac{v\eta}{\sigma}\frac{\partial\sigma}{\partial\mu}\right]$$

$$= -\frac{g}{R\sigma}\left[\frac{\partial(u\eta)}{\partial\lambda} + \frac{1}{\sigma}\frac{\partial}{\partial\mu}(\sigma v\eta)\right] + \frac{g\eta}{R\sigma}\left[\frac{\partial u}{\partial\lambda} + \frac{1}{\sigma}\frac{\partial}{\partial\mu}(\sigma v)\right]$$

$$= -\frac{g}{R\sigma}\left[\frac{\partial(u\eta)}{\partial\lambda} + \frac{1}{\sigma}\frac{\partial}{\partial\mu}(\sigma v\eta)\right] + \frac{g\eta}{R\sigma}\left(-\frac{\partial\eta}{\partial t}\right)\frac{\sigma R}{D} \quad ,$$

So

$$\frac{\partial}{\partial t}\left[\frac{(u^2+v^2)D}{2} + \frac{g\eta^2}{2}\right] + \frac{gD}{R\sigma}\left[\frac{\partial(u\eta)}{\partial\lambda} + \frac{1}{\sigma}\frac{\partial}{\partial\mu}(\sigma v\eta)\right] = 0,$$

or $$\frac{\partial E}{\partial t} + div_H \vec{F} = 0 \qquad div_H = \text{horizontal divergence.} \qquad (25)$$

This is of the form of an energy conservation law, with E being the energy density, kinetic plus potential, and $\vec{F}$ being the energy flux. The velocity with which energy is transferred is $\vec{U}_e = \overline{\vec{F}}/\overline{E}$ cm/sec, $\overline{}$ = time average. This is usually equal to the group velocity, defined by $d\omega(k)/d\omega = U_g$, although this equality has to be demonstrated in each case.

If we restrict ourselves to a small range of latitude, about a fixed latitude φ_0, then $R\sigma d\lambda \doteq R\sigma_0 d\lambda \doteq dx$, $R\sigma d\mu = dy$, and $f = f_0 + \beta y$, with $\beta = \frac{\Omega\sigma}{R}$(ref.1). We thus recover the linearized β-plane equations:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial\eta}{\partial x} \quad , \qquad (26)$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial\eta}{\partial y} \quad , \qquad (27)$$

$$\frac{\partial\eta}{\partial t} + D\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0 \,. \qquad (28)$$

Also, $$\frac{\partial E}{\partial t} + gD\left[\frac{\partial}{\partial x}(u\eta) + \frac{\partial}{\partial y}(u\eta)\right] = 0 \,. \qquad (29)$$

These equations have been used to investigate time-dependent motions in the ocean (refs. 3, 4). Of particular interest are solutions in the form of standing waves, and transient solutions. We consider only the former here. It turns out that, using the β-plane equations, standing wave solutions in a rectangular basin cannot be obtained, at least not in an obvious way (ref. 4). This is due to the fact that, while north-south waves can be combined to satisfy boundary conditions ($v = 0$) at zonal boundaries, only westward propagating waves exist, so the boundary condition ($u = 0$) at a meridional boundary cannot be satisfied. In spite of this, Arons and Stommel (ref.3) were able to construct solutions for a basin unbounded in the north-south direction, by taking advantage of the fact that the group velocity of the westward-moving waves can be either westward or eastward. If the energy velocity is calculated for these solutions it is found not to coincide with the group velocity, and in fact to lead to north-south propagation of energy for combinations of westward travelling waves. The situation here is unsatisfactory, and we would like to know to what extent it depends on the use of the β-plane equations. To elucidate this we use the more general equations deduced above.

Assume

$$u = \mathrm{Re}\, \frac{g}{\omega R \sigma} e^{i\omega t} e^{i s m \lambda} X(\mu) = \mathrm{Re}\, \mathcal{U}, \qquad (30)$$

$$v = Re \frac{g}{i\omega R\sigma} e^{i\omega\tau} e^{ism\lambda} Y(\mu) = Re\, V \,, \tag{31}$$

$$\eta = Re \quad e^{i\omega t} e^{ism\lambda} Z(\mu) =, R H \,, \tag{32}$$

with $s = \pm 1$, m 0 and real, and substitute into equations (22), (23), (24). Then we find

$$X + a\tau Y = -sm Z \,, \tag{33}$$

$$Y + a\tau X = -\frac{dZ}{d\mu} \,, \tag{34}$$

$$smX - \frac{dY}{d\mu} = -\sigma^2 L^2 Z \,, \tag{35}$$

with $a = \frac{\Omega}{\omega}$, $L^2 = \frac{\omega^2 R^2}{gD} = \frac{\omega^2 R^2}{c^2}$. Note $a\tau = \frac{f}{\omega}$

We can solve for X, Y from (33), (34),

$$X = \left[sm\omega Z - f \frac{dZ}{d\mu} \right] \frac{\omega}{f^2 - \omega^2} \tag{36}$$

$$Y = \left[-smfZ + \omega \frac{dZ}{d\mu} \right] \frac{\omega}{f^2 - \omega^2} \tag{37}$$

These expressions for X, Y can be substituted into (35) to yield an equation for Z. This equation has not yielded a solution yet. If we could find solutions, then we would be in a position to fit boundary conditions and determine if standing wave patterns can be found, by taking combinations of the $\mathcal{U}$ and $\mathcal{I}m\,\mathcal{U}$, the V and $\mathcal{I}m\,V$, and $Re\,H$ and $\mathcal{I}m\,H$. We note that the energy flux has components (time averaged), for the solutions (30), (31), (32), assuming X, Y, Z are real,

$$c^2\,\overline{u\eta} = R_e\, c^2\,\frac{U^{*}H}{2} = \frac{c^2 g}{2\omega R\sigma}\,X(\mu)\,Z(\mu)$$

$$= \frac{c^2 g}{2\omega R\sigma}\left[smZ^2 - fZ\frac{dZ}{d\mu}\right]\frac{\omega}{f^2-\omega^2}\quad, \tag{38}$$

$$c^2\,\overline{v\eta} = R_e\, c^2\,\frac{V^{*}H}{2} = R_e\,\frac{c^2 g}{-2i\omega R\sigma}\,Y(\mu)\,Z(\mu) \tag{39}$$

$$= 0.$$

Since we cannot obtain general solutions for Z, let us assume $Z = Z_o e^{-n\mu}$, so that $dZ/d\mu = -nZ$. n may be complex. Then, from (36), (37)

$$X = \left[sm\omega + fn\right]\frac{\omega Z}{f^2-\omega^2}\quad, \tag{40}$$

$$Y = \left[-smf - \omega n\right]\frac{\omega Z}{f^2-\omega^2}\quad. \tag{41}$$

In order to satisfy (35) we must have

$$\left[s^2m^2\omega + smfn\right]\frac{\omega Z}{f^2-\omega^2} + \left[smf + \omega n\right]\frac{\omega}{f^2-\omega^2}(-nZ) \tag{42}$$

$$+\left[smf + \omega n\right]\frac{\omega Z(-1)}{(f^2-\omega^2)^2}\cdot 2f\frac{df}{d\mu} + \frac{\omega Z}{f^2-\omega^2}\,sm\frac{df}{d\mu} = -\sigma^2 L^2 Z \tag{43}$$

or, assuming $f \gg \omega$,

$$\frac{s^2m^2\omega^2}{f^2} - \frac{\omega^2 n^2}{f^2} - 2\,\frac{sm\omega}{f^2}\,\frac{df}{d\mu} - 2\,\frac{\omega^2 n}{f^3}\,\frac{df}{d\mu} + \frac{sm\omega}{f^2}\,\frac{df}{d\mu} = -\sigma^2 L^2 Z$$

Finally, with $\lambda^2 = c^2/f^2$, $K = sm/R\sigma$, $\ell = n/R\sigma$, $\beta = \frac{1}{R\sigma}\frac{df}{d\mu}$,

$$\lambda^2 K^2 - \lambda^2\ell^2 - \frac{K\beta\lambda^2}{\omega} - 2\,\frac{\ell\beta\lambda^2}{f} = -1$$

or

$$\omega = \frac{K\beta\lambda^2}{1 - 2\frac{\ell\beta\lambda^2}{f} + \lambda^2(K^2-\ell^2)} \tag{44}$$

For $\ell = 0$, we obtain $\omega = \dfrac{k\beta\lambda^2}{1+\lambda^2 k^2}$, a result which shows that ω and k always have the same sign, and identical to that obtained by Arons and Stommel (ref.3). This indicates that, for solutions of the type assumed, the procedure of Arons and Stommel, for generating solutions which satisfy boundary conditions ($u = 0$) at meridional walls, can also be applied on the sphere. Further, by computing $c^2\overline{u\eta}$, assuming n real so we can use (38),

$$c^2\overline{u\eta} = \frac{c^2 g}{2\omega R\sigma}\left[sm Z_0^2 e^{-2n\mu} + fn Z_0^2 e^{-2n\mu}\right]\frac{\omega}{f^2-\omega^2}$$
$$= \frac{1}{2} g Z_0^2 \lambda^2 \left[\omega k + f\ell\right] e^{-2n\mu}, \qquad (45)$$

We obtain Fofonoff's result (ref.4), so that the same difficulties will occur in the present case as occur in the β-plane. We note that the group velocity, $U_g = d\omega/dk$, is not equal to the energy velocity $U_e = \dfrac{c^2\overline{u\eta}}{E}$, since for $n = 0$, $\omega \ll f$,

$$\bar{E} = \frac{\overline{(u^2+v^2)}}{2}D + \frac{g\overline{\eta^2}}{2} = Re\,\frac{UU^*}{2}\frac{D}{2} + Re\,\frac{VV^*}{2}\frac{D}{2} + \frac{g}{2} Re\,\frac{HH^*}{2}$$
$$= \frac{D}{4}\frac{g^2}{\omega^2 R^2\sigma^2}X^2 + \frac{D}{4}\frac{g^2}{\omega^2 R^2\sigma^2}Y^2 + \frac{g}{4}Z^2$$
$$= \frac{g c^2}{4\omega^2 R^2\sigma^2}\frac{s^2 m^2 \omega^2\omega^2 Z^2}{(f^2-\omega^2)^2} + \frac{g c^2}{4\omega^2 R^2\sigma^2}\frac{s^2 m^2 f^2\omega^2 Z^2}{(f^2-\omega^2)^2} + \frac{g}{4}Z^2$$
$$= \frac{gZ^2}{4}\left[\frac{k^2\omega^2 c^2}{f^2 f^2} + \frac{k^2c^2}{f^2} + 1\right]$$
$$= \frac{gZ^2}{4}\left[\frac{k^2c^2}{f^2} + 1\right],$$

so that $$|\vec{U}_e| = \frac{c^2\overline{u\eta}}{\bar{E}} = \frac{\frac{1}{2} g Z_0^2\lambda^2\omega k}{\frac{1}{2} g Z_0^2\left(\frac{k^2c^2}{f^2}+1\right)\frac{1}{2}}$$
$$= \frac{1}{2}\frac{\lambda^2\omega k}{\lambda^2 k^2+1} = \frac{1}{2}\frac{\omega^2}{\beta}, \qquad \text{from (44)},$$

while $\frac{d\omega}{dK} = U_g = -\beta\lambda^2 \frac{(K^2\lambda^2 - 1)}{(K^2\lambda^2+1)^2}$.

We note that $\vec{U}_e$ is always to the westward, while U_g can be to west or east. The difficulties with setting up standing wave solutions which were encountered by using the β-plane thus persist when the spherical globe is allowed for, the primary symptcm of these difficulties being the non-equality of energy velocity and group velocity.

We may note that for L = 0, our equations can be reduced to a tabulated form. Thus, we may eliminate X to obtain

$$\frac{dy}{d\mu} + sma\tau Y = (\sigma^2 L^2 - s^2 m^2) Z,$$

$$\frac{dZ}{d\mu} - sma\tau Z = (a^2\tau^2 - 1) Y.$$

Then if we put L = 0, we may eliminate Z to obtain Legendre's Associated Equation for Y (ref.2). The condition that the solutions be finite yields in the very well-known way the condition (ref.2),

$$sma = \ell(\ell+1),$$

ℓ being the order of the Legendre polynomial used. Since a, m, ℓ are all positive, s is positive, that is the waves travel to the west always. One can show in the cases of interest (large ℓ) that the energy velocity and group velocity are always to the west also. Thus, it appears that the difficulties encountered above are associated with the parameter $L^2 = \omega^2 R^2/gH$. The general equations will have to be solved before the problem can be further elucidated.

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Two-Dimensional Motions in a Homogeneous Ocean

by

Derek W. Moore

Introduction

Since the work of Stommel (1948) it has been recognized that any dynamical theory of wind-driven ocean circulation must include the effect of the variation with latitude of the normal component of the earth's rotation if it is to have any chance of explaining the most striking of the observed features - the western boundary current. It is not difficult to modify the equations of motion to include this effect, but even if one makes the customary approximation of representing the turbulence as an eddy viscosity, one is still faced with an intractable problem. Theories of ocean circulation can, in fact, be characterized by their method of overcoming the intractibility of the full Navier-Stokes equations and two types may be distinguished. In the first place one has the frictional theories of Stommel (1948) and Munk (1950) in which the non-linear inertia terms are discarded completely and the β term balanced against the viscous stresses. In the second place one has the inertial theories (Fofonoff (1954), Morgan (1956), Charney (1955), Carrier and Robinson (1961)) in which the frictional terms are dropped from the equations. These theories were compared by Dr. Veronis and the reader is referred to his discussion for details. Mention may also be made of interesting critical accounts by Stommel (1958), Fofonoff (1961) and Carrier and Robinson (1961).

In view of these contrasting theories it seems worthwhile to examine a situation in which both the inertia terms and the frictional terms can be included exactly in the solution. In section two of this report the flow in the neighbourhood of a latitude of vanishing wind stress curl is examined on the basis of the exact equations. It is shown that the problem can be reduced to that of solving an ordinary third order non-linear equation. In the sense that such an equation can be integrated numerically in a straightforward fashion an exact solution has been found but, rather than examine this, attention has been concentrated on determining the relative importance of viscous and inertia forces in this restricted situation. The results are described in terms of a Reynolds number R = $U^{3/2}/\beta^{1/2}\gamma$, where U is the velocity scale of east-west flow and γ an eddy viscosity. If R $<<$ 1 the results are shown to coincide with Munk's theory, whilst if R $>>$ 1 the inertial theories are valid except in viscous sub-layers on the continental walls. The nature of these is examined and it is shown that if the external flow is that of Fofonoff (1954), no viscous sublayer can exist on the eastern boundary.

In § 3 the exact equations are examined in more detail in the case of no wind stress and some exact solutions derived. It is shown that if the flow far from the continental boundaries is west to east solutions representing damped Rossby waves exist. Dr. Fofonoff, who has found similar waves in a slightly different situation, has suggested that these Rossby waves may play a role in ocean circulation and in § 4 of this report a model of ocean circulation involving these waves is constructed. The flow found is east to west in the southern portion with a boundary current on the southern half of the western boundary whilst in the northern portion of the basin there is no boundary current but instead a system of damped Rossby waves which decay on a basic west to east flow.

§ 2. Flow near a latitude of vanishing wind stress curl

The ocean is taken to be of uniform depth and the motion is two-dimensional and parallel to the surface. Thus the wind stress must be regarded as a body force distributed uniformly through the depth of the ocean and bottom friction must be ignored. Let OX, OY be rectangular coordinate axes and that OX is in the west to east direction and OY in the south to north direction. Then the Navier-Stokes equations assume the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + W + r\nabla^2 u, \tag{2.1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + r\nabla^2 v, \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

In these equations (u, v) is the fluid velocity at (x, y), f the Coriolis parameter, p the pressure and γ a coefficient of kinematic viscosity. W is the wind force which, as is customary, has been taken to be in the east-west direction. It is now further assumed that the latitudinal variation of the Coriolis parameter is given by the linear approximation

$$f = f_0 + \beta y, \tag{2.4}$$

where β is a constant. Now (2.3) can be integrated by means of a stream function $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{2.5}$$

and using the relations

$$\frac{\partial}{\partial x}(f\psi) = f\frac{\partial \psi}{\partial x} = -fv, \tag{2.6}$$

$$\frac{\partial}{\partial y}(f\psi) = f\frac{\partial \psi}{\partial y} + \psi\frac{\partial f}{\partial y} = fu + \beta\psi. \tag{2.7}$$

One has finally

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + W + \gamma\nabla^2 u, \tag{2.8}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \psi\beta = -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \gamma\nabla^2 v, \tag{2.9}$$

where

$$p' = p + \rho f\psi. \tag{2.10}$$

The equations in this form show clearly that it is the <u>variation</u>

of the Coriolis parameter which is dynamically important.*

Suppose that y = 0 is the boundary between two regions of disconnected motion of the ocean. Then since neither mass nor momentum is to be transferred across y = 0 one must have

$$v = 0, \quad \frac{\partial u}{\partial y} = 0 \qquad \text{or} \qquad y = 0 \tag{2.11}$$

Thus, under mild restrictions of analyticity, the stream function $\psi(x, y)$ must possess a power series expansion near y = 0 of the form

$$\psi(x,y) = y f(x) + y^2 f_2(x) + \dots \tag{2.12}$$

Sufficiently close to y = 0 the flow will be described by the stream function yf(x) and the higher terms will not be considered in this analysis. If it is supposed that the wind stress term can be expanded in the form

$$w(x,y) = K_0(x) + y K_1(x) + y^2 K_2(x) + \dots \tag{2.13}$$

then substitution into (2.8) leads to the equation

$$f f' = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + K_0(x) + y K_1(x) + y^2 K_2(x) + \dots r f'' \tag{2.14}$$

Thus

$$\frac{p'}{\rho} = C(y) + \chi(x) + y \int K_1(x)\,dx + y^2 \int K_2(x \tag{2.15}$$

where $C(y)$ is an unknown function of integration and $\chi(x)$ a known function of χ. On substituting this expression into (2.9) one has

*If β = 0 one recovers G.I.Taylor's (1917) result that rotating a two-dimensional viscous flow alters only the pressure field.

$$-yff'' + yf'^2 - \beta y f = -\frac{dC}{dy} - \int K_1(x)\,dx - 2y\int K_2(x) + \cdots - \gamma y f''' . \tag{2.16}$$

Thus by inspection of the powers of y in (2.16)

$$-\frac{dC}{dy} = Cy, \text{ say, and } K_1(x) = 0 \tag{2.17}$$

The last condition implies that $\frac{\partial w}{\partial y} = 0$ on $y = 0$, that is to say the wind stress curl must vanish on the ocean boundary as defined by (2.11)). Thus one has, defining*

$$K(x) = -\frac{1}{2}\int K_2(x)\,dx , \tag{2.18}$$

the third order equation

$$\gamma f''' - c - K(x) = ff'' - f'^2 + \beta f . \tag{2.19}$$

So far nothing has been said about boundary conditions. If it is supposed that $x = 0$ and $x = L$ are rigid continental boundaries then the boundary conditions on the function $f(x)$ are

$$\left.\begin{aligned} f(0) = f'(0) &= 0 \\ f(L) = f'(L) &= 0 \end{aligned}\right\} \tag{2.20}$$

It is convenient at this point to introduce dimensionless coordinates and a dimensionless stream function. The choice of the scales involved is quite arbitrary and will not affect the

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*Note that the constant of integration can be absorbed in the unknown constant C in (2.19).

results, but it is partly the object of this analysis to combine the viscous and inertial theories it is simpler to choose scales arising from one of them and in fact the length scale $\ell = \left(\frac{U}{\beta}\right)^{1/2}$ and velocity scale U which arise naturally in the inertial theories will be chosen. U is the westward velocity far from the continental boundaries and $\left(\frac{U}{\beta}\right)^{1/2}$ is the thickness of the inertial boundary layer. Thus one defines

$$-UF = f, \left(\frac{U}{\beta}\right)^{1/2} \underline{X} = x, EV = -C, \underline{\overline{K}}(x) = \frac{K(x)}{\overline{U}}, \qquad (2.21)$$

whence

$$\left(\frac{1}{R}\right) F^{(3)} = F'^2 - FF'' + F + E - \underline{\overline{K}}(x) \qquad (2.22)$$

$$\left.\begin{aligned} F(0) = F'(0) = 0 \\ F\left(\frac{\beta^{1/2}L}{U^{1/2}}\right) = F'\left(\frac{\beta^{1/2}L}{U^{1/2}}\right) = 0 \end{aligned}\right\} \qquad (2.23)$$

where the Reynolds number R is defined by

$$R = U^{3/2}/r\beta^{1/2}. \qquad (2.24)$$

The solution is thus determined essentially by two parameters R and $\left(\frac{\beta}{U}\right)^{1/2} L$ and in general nine possible cases can arise. However $\left(\frac{\beta}{U}\right)^{1/2} L$, which is the ratio of the ocean breadth to the inertial boundary layer thickness, is large and one need only consider the variation of R . There are two cases to consider

Case 1: $R << 1$.

In this case the left-hand side of (2.22) will be very large and viscous and inertia forces will not balance unless the third derivative of F is small, say $O\left(\frac{1}{\delta^3}\right)$ where δ is the length scale of x variation in the dimensionless system. Then $\frac{1}{R\cdot\delta^3} = O(1)$ so that $\delta = O(R^{-1/3})$; furthermore the non-linear terms on the right-hand side of (2.22) are $O(R^{2/3})$ and can be neglected. Thus (2.22) takes the form

$$\frac{1}{R} F^{(3)} = F + \underline{F} - \underline{\bar{K}}(x) \tag{2.25}$$

If one has $\delta << \left(\frac{\beta}{U}\right)^{1/2} L$ the solution represents a slowly varying interior flow with thin boundary layers on the continental boundaries and, indeed, if one takes $\underline{\bar{K}}(x) = W_0 X$, so that the wind force is independent of longitude, one recovers the boundary layer form of Munk's equations.

<u>Case II   R >> 1</u>

Inspection of (2.22) shows that in any region where x derivatives are of O(1) on the left-hand side is negligible, so that in the interior of the ocean one has

$$0 = F'^2 - FF'' + F + \underline{F} - \underline{\bar{K}}(x) \tag{2.26}$$

However, the solutions of this inertial equation will not in general satisfy all the boundary conditions (2.23) and regions where the viscous term becomes important will exist near the boundaries. Since the solutions of (2.26) are themselves of inertial boundary layer character, owing to the assumption $\left(\frac{\beta}{U}\right)^{1/2} L >> 1$, these thin viscous regions will be called the viscous sub-layer. The nature of these viscous sub-layers will now be examined for the case of Fofonoff's (1954) free inertial solutions. Thus $\underline{\bar{K}}(X) = 0$ and one may verify that a solution of (2.26) is $\underline{F} = -1$

$$F = \left(1 - e^{-X}\right) \tag{2.27}$$

near $X = 0$ and

$$F = \left(1 - e^{X - L\left(\frac{\beta}{\nu}\right)^{1/2}}\right) \tag{2.28}$$

near $X = L\left(\frac{\beta}{\nu}\right)^{1/2}$. These solutions fail to satisfy the conditions $F'(0) = 0$ and $F'\left(\frac{\beta^{1/2}}{\nu^{1/2}} L\right) = 0$ which state that the tangential velocity is 0 so that viscous sub-layers will arise. Considering now these layers let $F = \lambda F^*$ and let $\underline{\bar{X}} = \delta x^*$ near the western boundary and let $\delta x^* = L\,\frac{\beta^{1/2}}{\nu^{1/2}} - \underline{\bar{X}}$ near the eastern boundary. Then (2.22) becomes

$$\pm \frac{\lambda}{R\delta^3} F^{*(3)} = \frac{\lambda^2}{\delta^2}\left(F^{*\prime 2} - F^* F^{*\prime\prime}\right) + \lambda F^{*} - 1 \tag{2.29}$$

where the upper sign refers to the western boundary whilst the boundary conditions on the boundaries are in both cases $F^*(0) = F^{*\prime}(0) = 0$. Following the usual boundary layer procedure, one imposes on F* the condition that the y component of velocity should tend as $x^* \to \infty$ to the value given by the inertial solution. Reference to (2.27) and (2.28) shows that in both cases this yields the boundary condition

$$\frac{\lambda}{\delta} F^{*\prime} \to 1 \text{ as } x^* \to \infty \tag{2.30}$$

and since the point of the scaling is to have F* of O(1) one chooses

$\lambda = \delta$. Furthermore if viscous and inertia terms are to balance $\frac{\lambda}{R\delta^3} = O(1)$ so that $\delta = R^{-1/2}$ say. The linear term in (2.29) is now seen to be only $R^{-1/2}$ and can be neglected, so that finally one has

$$\pm F^{*(3)} = F^{*\prime 2} - F^{*}F^{*\prime\prime} - 1 \qquad (2.31)$$

$$F^{*}(0) = F^{\prime *}(0) = 0, F^{\prime *}(X^{*}) \rightarrow 1 \text{ as } X^{*} \rightarrow \infty \qquad (2.32)$$

The equations when the upper sign is taken are identical with those for the boundary layer near the forward stagnation point of a cylinder (Goldstein 1938), but when the lower sign is taken the equations can be reduced to those for the boundary layer near the rear stagnation point and it is known that, since the flow is rapidly decelerating, the boundary layer equations have no solution in this case. Thus no viscous sub-layer can exist on the southern part of the eastern boundary in the present case - presumably if a viscous sub-layer were to be established farther north, where the flow is less rapidly decelerating, it would separate before the southern boundary was reached.

A more mathematical argument to demonstrate that no solutions exist when the lower sign is taken in given in Appendix One.

3. Exact solutions of the free equations

If one takes $K(x) = 0$, so that there is no wind stress (2.19) takes the form

$$\gamma f''' - c = f f'' - f'^2 + \beta f \tag{3.1}$$

one can easily verify that

$$f = -U\left(1 - e^{-\alpha x}\right) \tag{3.2}$$

is a solution of this equation provided that

$$C = \beta U \quad \text{and}$$

$$\gamma \alpha^3 - U\alpha^2 + \beta = 0 \tag{3.3}$$

Thus, since (3.3) has, in general, three distinct roots, three distinct exact solutions have been found. However, since equation (3.1) is non-linear, they cannot be added to construct more general solutions. Furthermore, no choice of U, save the trivial one $U = 0$, will allow $f'(0)$ also to be zero, so that the exact solutions cannot satisfy the required boundary conditions at a continental boundary, i.e., they represent states of motion in the interior of the ocean.

If one puts $\gamma = 0$ one has

$$-U\alpha^2 + \beta = 0 \tag{3.4}$$

so that $\alpha = \pm\left(\frac{\beta}{U}\right)^{1/2}$, in agreement with Fofonoff's inertial theory. If $U > 0$ the roots are real and taking the positive root one has a flow which decays to a uniform east to west flow as $x \to \infty$ If $U < 0$ the roots are pure imaginary and no uniform state is achieved as $X \to \infty$ - this is, of course, just Fofonoff's result that a uniform inertial flow can only be from east to west. It is of interest to generalise these results to the full equation (3.3). It is shown in Appendix 2 that if

$$U \geq \left(\frac{27}{4}\right)^{1/3} \beta^{1/3} \gamma^{-2/3} \tag{3.5}$$

(3.3) has real roots whilst in the contrary case it has a pair of conjugate complex roots. If $\alpha_1 \alpha_2 \alpha_3$ are the roots one has

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= +\frac{U}{\gamma} \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 &= 0 \\ \alpha_1\alpha_2\alpha_3 &= -\beta/\gamma \end{aligned} \tag{3.6}$$

If the roots are all real, then by (3.5) $U > 0$ and it follows from the first and last equations of (3.6) that one root is negative and two are positive. Now suppose that the roots are not all real, so $\alpha_2 = p + iq$, $\alpha_3 = p - iq$

Then

$$\alpha_1(p^2 + q^2) = -\beta/\gamma \tag{3.7}$$

$$2\alpha_1 p + (p^2 + q^2) = 0 \tag{3.8}$$

Equation (3.7) shows that $\alpha_1 < 0$ and then (3.8) shows that $p > 0$. Thus in either case, two of the roots represent solutions which decay as $X \uparrow$ and the other root represents a solution which decays as $X \downarrow$. Thus in contrast to the inertial case one has solutions which decay to a uniform flow as $X \uparrow$ whatever the sign of u, but if $U < \left(\frac{27}{4}\right)^{1/3} \beta^{1/3} \gamma^{-2/3}$ the decay is oscillatory. If $p << q$ the solutions represent slowly damped standing waves on a basically west to east flow. In general it is necessary to solve the cubic numerically to determine p and q, but if $\frac{\beta \gamma^2}{U^3} << 1$ one can easily show that

$$q = \left(\frac{\beta}{U}\right)^{1/2}, \quad p = \frac{1}{2} \frac{\gamma \beta}{U^2} \tag{3.9}$$

A graph of q and p as functions of u, obtained from the numerical solution of the cubic, is given in Figure 1, where the values $\gamma = 10^7$ and $\beta = 10^{-13}$ were adopted.

Dr. Fofonoff has suggested that a model of ocean circulation in a rectangular basin might be constructed which had a basic east to west flow in its southern half and basic west to east flow with superimposed damped Rossby waves in its northern half. The existence of exact solutions of damped Rossby wave type is encouraging and in the next section a model with this idea as its basis is constructed.

4. A model of ocean circulation

In this section the wind-driven circulation in a rectangular basin of uniform depth is considered. The wind $W(y)$ is independent of x and $\frac{dW}{dy} = 0$ when y = 0 and when y = L' (the axes are orientated as in § 2).

On eliminating the pressure terms from (2.8) and (2.9) one finds that one must solve the equation

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)+\beta\frac{\partial\psi}{\partial x}=\frac{\partial W}{\partial y}+r\nabla^4\psi \qquad (4.1)$$

with the boundary conditions

$$\psi,\ \frac{\partial\psi}{\partial x}=0 \quad \text{on } x=0,\ x=L; \qquad (4.2)$$

$$\psi,\ \frac{\partial^2\psi}{\partial y^2}=0 \quad \text{on } y=0,\ y=L'. \qquad (4.3)$$

The problem posed by (4.1), (4.2) and (4.3) is, of course, quite intractable and, as described in the introduction, two types of approximation have, in previous work, been made, namely, to neglect the viscous term on the right-hand side of (4.1) or to neglect the non-linear terms on the left-hand side. Now the damped Rossby waves which are the basis of Dr. Fofonoff's suggestion for model for ocean circulation arise only through the interaction of viscous <u>and</u> inertia forces, so that an approximation which neglects either cannot possibly reproduce them. Thus any scheme of approximation which is arrived at exploring the above suggestion must retain the inertia terms in some form or other. An obvious choice is the Oseen approximation in which the non-linear terms $\underline{u}\cdot\nabla\underline{u}$ are replaced by $\underline{V}\cdot\nabla\underline{u}$ where $\underline{V}$ is a constant vector field. In the present case $\underline{V}$ is taken to be a uniform east to west flow V and the Oseen equation is

$$-V\frac{\partial}{\partial x}(\nabla^2\psi)+\beta\frac{\partial\psi}{\partial x}=\frac{dW}{dy}+r\nabla^4\psi. \qquad (4.4)$$

It is further assumed that the y derivatives of ψ are everywhere smaller than the x derivatives (so that the only boundary layers occur on the western and eastern continental boundaries) and then (4.4) becomes

$$-U\frac{\partial^3\psi}{\partial x^3}+\beta\frac{\partial\psi}{\partial x}=\frac{dW}{dy}+\gamma\frac{\partial^4\psi}{\partial x^4} \qquad (4.5)$$

If the wind stress is $W=-W_0\cos\frac{\pi y}{L'}$, and one writes $\psi=\psi^*\sin\frac{\pi y}{L'}$ one notes that (4.3) are automatically satisfied and that the y dependence factors out leaving

$$-U\frac{\partial^3\psi^*}{\partial x^3}+\beta\frac{\partial\psi^*}{\partial x}=\frac{W_0\pi}{L'}+\gamma\frac{\partial^4\psi^*}{\partial x^4} \qquad (4.6)$$

with the boundary conditions

$$\psi^*,\ \frac{\partial\psi^*}{\partial x}=0 \quad \text{on} \quad x=0,\ x=L \qquad (4.7)$$

The solution of (4.6) is

$$\psi^*=\frac{W_0\pi x}{\beta L'}+\sum_{i=1}^{3}A_i e^{-\lambda_i x}+A_0 \qquad (4.8)$$

where A_i are constants and λ_i are the roots of the cubic

$$\gamma\lambda^3-U\lambda^2+\beta=0 \qquad (4.9)$$

It will be noted that (4.9) is identical with the cubic (3.3) obtained for the exact solutions of § 3, so that the discussion of the nature of the roots given there may be carried over. There are thus two cases to consider.

Case 1, $U\geq\left(\frac{27}{4}\right)^{1/3}\beta^{1/3}\gamma^{-2/3}$

In this case there are two positive roots λ_1, λ_2 say and one negative root, λ_3. The boundary conditions (4.7) lead to the equations

$$\left.\begin{aligned} &A_0 + A_1 + A_2 + A_3 = 0 \\ &\frac{W_0\pi}{\beta L'} - \lambda_1 A_1 - \lambda_2 A_2 - \lambda_3 A_3 = 0 \\ &A_0 + \frac{W_0 \pi L}{\beta L'} + A_1 e^{-\lambda_1 L} + A_2 e^{-\lambda_2 L} + A_3 e^{-\lambda_3 L} = 0 \\ &\frac{W_0\pi}{\beta L'} - \lambda_1 A_1 e^{-\lambda_1 L} - \lambda_2 A_2 e^{-\lambda_2 L} - \lambda_3 A_3 e^{-\lambda_3 L} = 0 \end{aligned}\right\} \quad (4.10)$$

The solution of this system is greatly simplified if $\lambda_1 L$, $\lambda_2 L$, $-\lambda_3 L >> 1$, (so that the boundary layer thicknesses are small compared to the breadth of the basin) and in this case one finds that

$$\psi^* = \frac{W_0\pi}{\beta L'}\left\{x + \frac{(\lambda_2\lambda_3 L + \lambda_2 - \lambda_3)}{\lambda_3(\lambda_2 - \lambda_1)} e^{-\lambda_1 x} + \frac{(\lambda_1\lambda_3 L + \lambda_1 - \lambda_3)}{\lambda_3(\lambda_1 - \lambda_2)} e^{-\lambda_2 x} + \frac{1}{\lambda_3} e^{\lambda_3 (L-x)} - L - \frac{1}{\lambda_3}\right\} \quad (4.11)$$

As in the Munk solution the eastern boundary layer is "invisible", since the term giving rise to it is $O\left(\frac{1}{L\lambda_3}\right)$ of the other contributions to ψ^* near $x = L$.

<u>Case 2</u>: $U < \left(\frac{27}{4}\right)^{1/3} \beta^{1/3} r^{-2/3}$

In this case the roots are $\lambda_1 = p + iq$, $\lambda_2 = p - iq$ and $\lambda_3 < 0$, where $p > 0$. Thus the general solution (4.8) may be written

$$\psi^* = \frac{W_0 \pi x}{\beta L'} + A_0' + A_1' e^{-px} \cos qx + A_2' e^{-px} \sin qx + A_3' e^{-\lambda_3 x} \quad (4.12)$$

and assuming that pL, $-\lambda_3 L >> 1$ the solution is

$$\psi^* = \frac{W_0 \pi}{\beta L'}\left\{x + \left(L + \frac{1}{\lambda_3}\right)e^{-px}\cos qx + \left(-\frac{1}{q} + \frac{pL}{q} + \frac{p}{q\lambda_3}\right)e^{-px}\sin qx - L - \frac{1}{\lambda_3} + \frac{1}{\lambda_3}e^{\lambda_3(L-x)}\right\} \tag{4.13}$$

The flow near the eastern boundary is similar to the previous case, but the flow in the western half consists of damped oscillations.

So far nothing has been said about the choice of U in the Oseen equation. Now the actual x-component of velocity in the mid-ocean is easily found to be $-\frac{W_0 \pi}{\beta}\cos\left(\frac{\pi y}{L'}\right)(L-x)$ and this is from east to west if $y < \frac{1}{2}L'$ and from west to east if $y > \frac{1}{2}L'$, that is to say east to west in the southern half and west to east in the northern half. Thus it is tempting to take for U not a constant but a function of y say

$$U = +U_0 \cos\frac{\pi y}{L'} \tag{4.14}$$

which has the same y variation as the actual x-component of flow. If one refers back to the derivation one sees that (4.11) and (4.13) will still satisfy (4.6) even if the λ_i are functions of y (since there are no y derivatives in this equation) but the solution no longer satisfies the condition $\frac{\partial^2\psi}{\partial y^2} = 0$ on y = 0, y = L'. This is not a serious violation of the boundary conditions. However the extent to which (4.4) can be regarded as a model of the Navier-Stokes equations is in doubt if u varies, for possibly effects of importance are suppressed when the terms involving derivatives of $U(y)$, which would arise in a systematic approximation, are omitted.

Inclusion of these terms would greatly complicate the problem and it seems worthwhile to study the present heuristic model before attempting a more realistic model. To this end the basic cubic has been solved numerically for a single value of Υ.

An eddy viscosity $\Upsilon = 10^7$ was chosen as it seemed to yield reasonable damping for the waves. The values of p and q obtained are shown in Figure one. It is noticeable that p, the damping, decreases rapidly as U becomes more negative. Thus the Rossby waves, which are damped out after one or two maxima near the middle of the basin, penetrate for much greater distances in the northern quarter of the basin. Thus the streamlines will be oscillating in the northern portion. Farther south the main flow will be confined to a boundary layer near the wall, though an outer region of reversed flow, rather like that in Munk's (1950) solution, will develop as the middle of the basin is approached. Velocity profiles for the north-south velocity are shown for two latitudes in Figures two and Figure three, for the case of $U_0 =$ 10 cm sec^{-1}. When y/L' is equal to 0.33 viscous sub-layer is very marked, the velocity reaching its maximum quite close to the boundary. When $y/L' = 0.55$ the character of the profile has changed completely. It is now oscillatory and deviations from geostrophic flow extend farther into the ocean. For larger values of y many more oscillations will appear, and the deviations from geostrophic flow will extend right across the ocean basin. This indicates that a larger value of Υ might perhaps have been more appropriate. Unfortunately a full examination of the flow patterns when U_0 and r are varied is rather laborious and the work is not complete at the time of writing.

It is a pleasure to record my gratitude to Dr. N. Fofonoff, Dr. A. Robinson, Prof. H. Stommel, Dr. G. Veronis and Dr. P. Welander for their valuable comments on the work I have described.

Appendix I

The equation to be considered is

$$\pm F^{*'''} = F^{*'^2} - F^* F^{*''} - 1, \tag{1}$$

with the boundary conditions

$$F^*(0) = F^{*'}(0) = 0, \tag{2}$$

$$F^{*'} \to 1 \text{ as } X^* \to \infty \tag{3}$$

where upper sign gives equation for western boundary and the lower sign gives equation for eastern boundary. It will be shown that in the latter case (1) has no solution which satisfies (3).

Let $F^* \sim X^* + g$ as $X^* \to \infty$ (4)

so that $g^* \to 0$ as $X^* \to \infty$.

Then g must satisfy the equation

$$\pm g^{(3)} = 2g^{(1)} - X^* g^{(2)}, \tag{5}$$

where second order terms in g have been omitted. If $h = g^{(1)}$ one has finally

$$h^{(2)} + x h^{(1)} - 2h = 0 \quad \text{on western boundary} \tag{6}$$

$$h^{(2)} - x^* h^{(1)} + 2h = 0 \quad \text{on eastern boundary} \tag{7}$$

First, consider (6). If

$$h(x^*) = K(x^*) \exp\left\{-\tfrac{1}{4} x^{*2}\right\}, \tag{8}$$

then K satisfies

$$\frac{\partial^2 K}{dx^{*2}} + \left\{-3 + \frac{1}{2} - \frac{1}{4}x^{*2}\right\} K(x) = 0 \qquad (9)$$

whose general solution is

$$K = AD_{-3}(x^*) + BD_2(jx^*), \qquad (10)$$

where A, B are constants and D_n is the parabolic cylinder function (Whittaker and Watson 1958). Now as $y \to \infty$ for $|\arg y| < 3/4\,\pi$

$$D_n(y) \sim e^{-y^2/4}y^n, \qquad (11)$$

and, on using this result one finds that

$$h \sim e^{\frac{-x^{*2}}{2}} x^{*-3} \text{ or } x^{*2} \text{ as } x^* \to \infty, \qquad (12)$$

the first term being the unique acceptable solution at ∞.

Now consider (7). If

$$h^*(x) = K(x^*) \exp\left\{+\frac{1}{4}x^{*2}\right\}, \qquad (13)$$

then K satisfies

$$\frac{d^2 K}{dx^{*2}} + \left\{2 + \frac{1}{2} - \frac{1}{4}x^{*2}\right\} K = 0, \qquad (14)$$

whose general solution is

$$K = AD_2(x^*) + BD_{-3}(x^*). \qquad (15)$$

Hence, one has

$$h^* \text{ or } x^{*2} \text{ or } e^{\frac{x^{*2}}{2}} x^{*-3},$$

neither of which is acceptable as $x^* \to \infty$. Thus (1) has no solution which satisfies (3) when the lower sign is taken.

Appendix II

The cubic to be examined is

$$\gamma \alpha^3 - \nu \alpha^2 + \beta = 0 \tag{1}$$

or

$$\alpha^3 - \frac{\nu}{\gamma}\alpha^2 + \frac{\beta}{\gamma} = 0 \tag{2}$$

Let

$$K^3 = -\left(\frac{\nu}{3\gamma}\right)^3 + \frac{\beta}{2\gamma} \tag{3}$$

and

$$\alpha = Ky + \frac{\nu}{3\gamma} \tag{4}$$

Then (2) is reduced to the standard form

$$y^3 + 2 = 3py\,, \tag{5}$$

where

$$p = \frac{\nu^2}{9\gamma^2}\left\{\frac{\beta}{2\gamma} - \left(\frac{\nu}{3\gamma}\right)^3\right\}^{-2/3} \tag{6}$$

The roots of (5) as functions of p are tabulated by Jahnke and Emde (1945).

Let

$$x = \frac{\nu}{3\gamma}\,, \quad a = \frac{\beta}{2\gamma} \tag{7}$$

Then

$$p = \frac{x^2}{(a - x^3)^{2/3}} \tag{8}$$

The graph of p as a function of x is sketched below.

If $p < 1$ the cubic has two conjugate complex roots and one negative root and, from the sketch, this is true if

$$x < \left(\tfrac{1}{2}\right)^{1/3} a^{1/3}\,, \tag{9}$$

which is equivalent to (3.5) of the text.

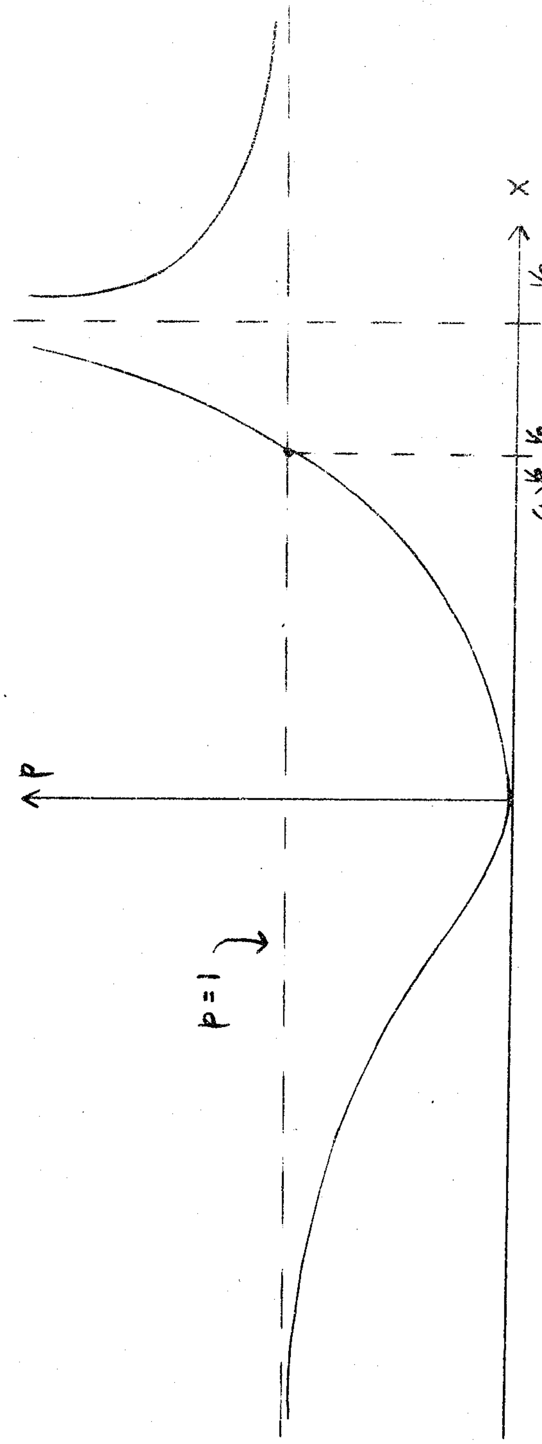
P
x
$a^{1/3}$
$(\frac{1}{2})^{1/6} a^{1/3}$
P = 1

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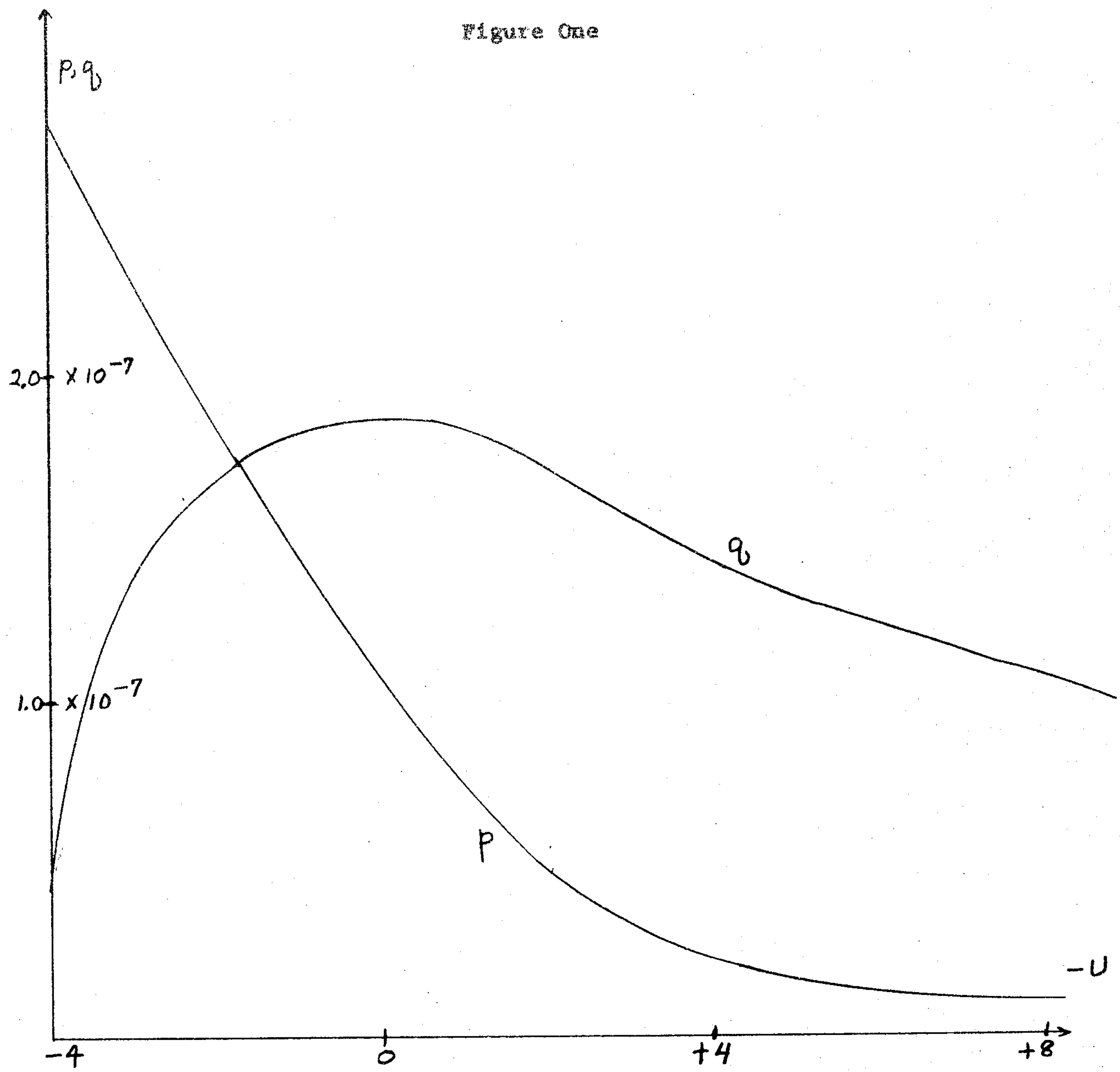
Figure One



p and q as functions of $-U$ when $\beta = 10^{-13}$ and $\gamma = 10^{7}$.

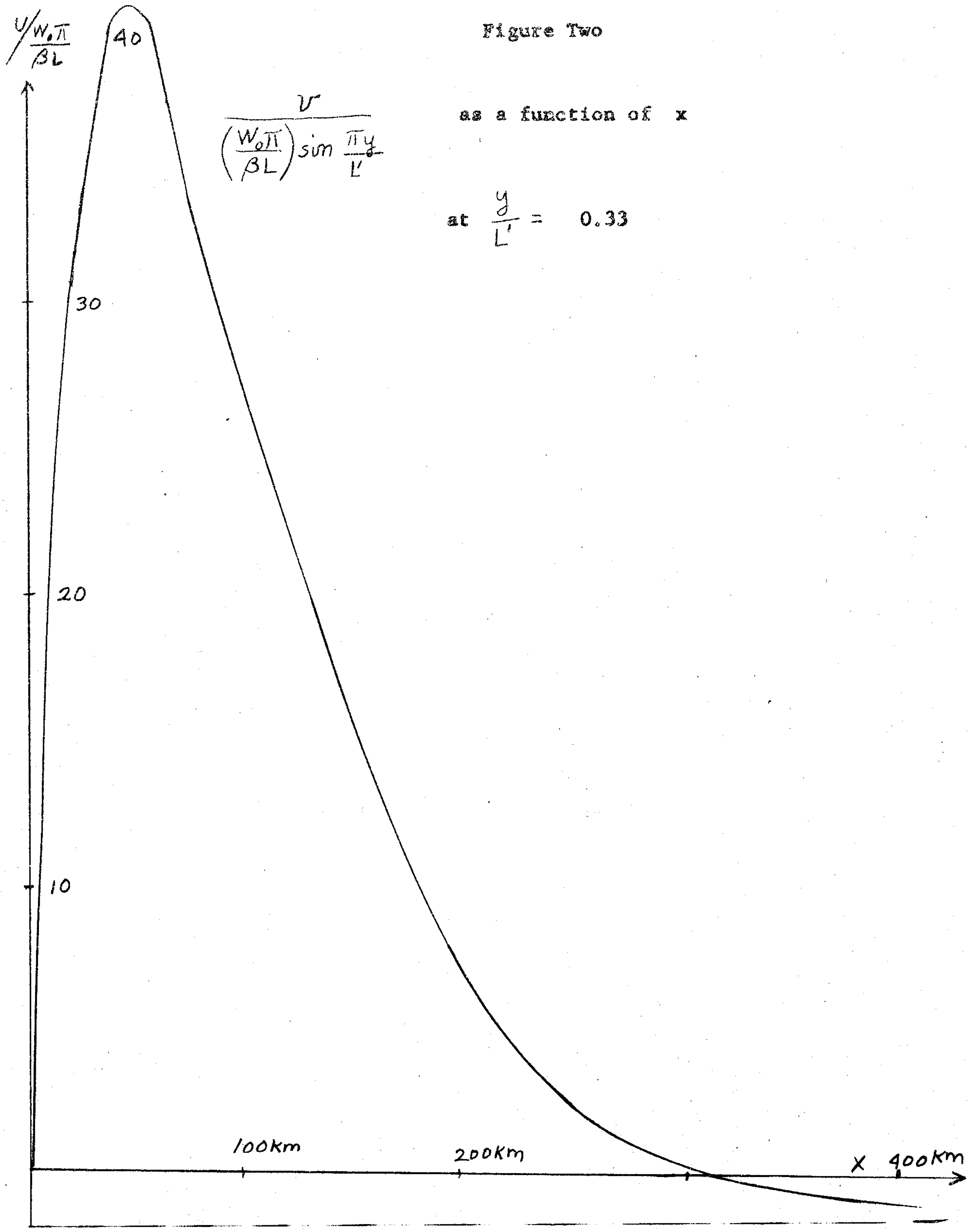
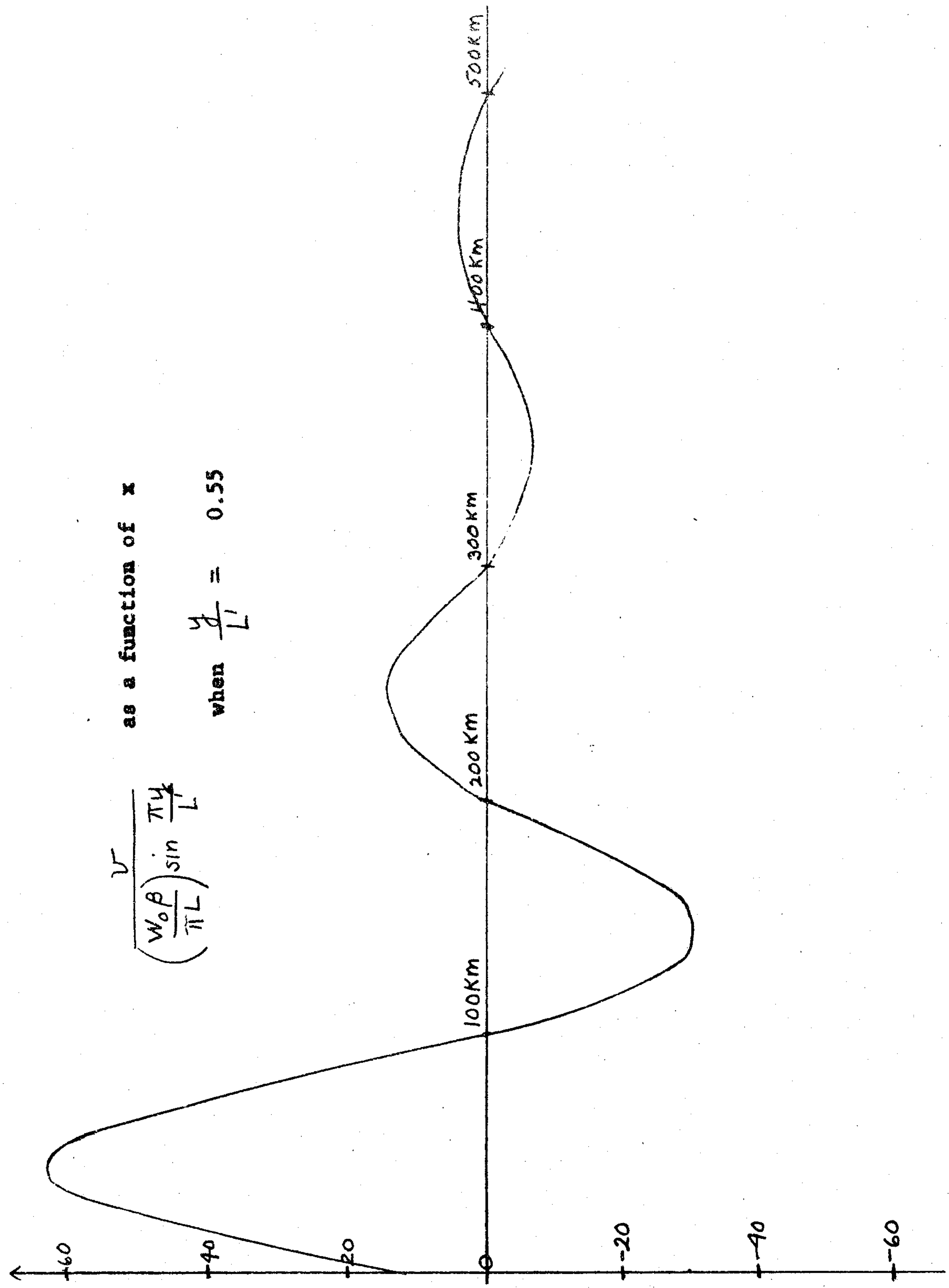


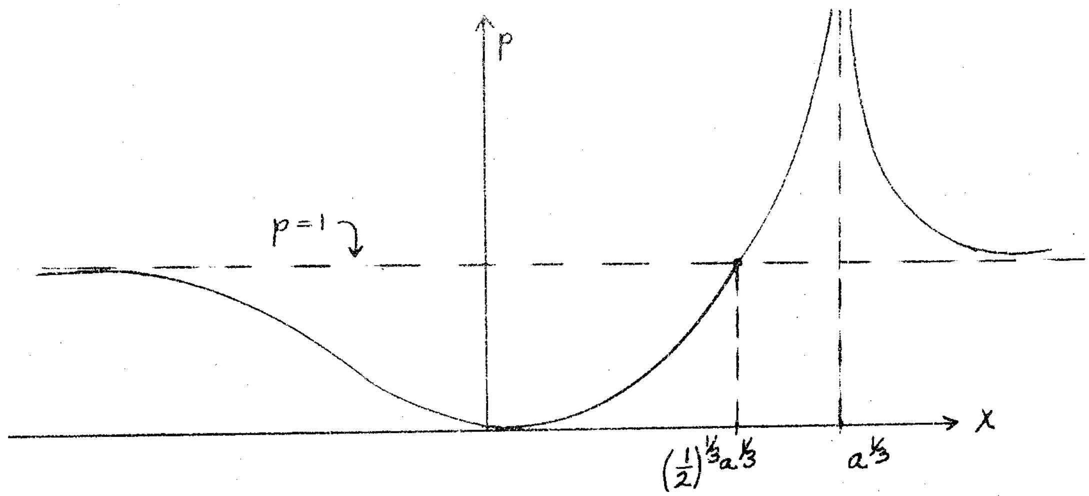
Figure Two

$$\frac{v}{\left(\frac{W_0\pi}{\beta L}\right)\sin\frac{\pi y}{L'}}$$ as a function of x

at $\frac{y}{L'} = 0.33$

Figure Three





If $p < 1$ the cubic has two conjugate complex roots and one negative root and, from the sketch, this is true if

$$x < \left(\tfrac{1}{2}\right)^{1/3} a^{1/3}, \tag{9}$$

which is equivalent to (3.5) of the text.

Circulation in a Rectangular Basin

derived from a Source and Sink Model

by

Martin Mork

1. It has been pointed out that a model with distributions of sources and sinks can be made equivalent to a wind stress distribution as far as the motion in the deep frictionless layer is concerned. The non-uniformity of wind stress causes a convergence or divergence within the Ekman-layer, and that is again associated with a vertical transport.

The Ekman layer is very thin compared to the vertical scale of the geostrophic currents. It reaches down to a depth D where the frictional forces are negligible. On the assumption that within this layer there is a balance between the Coriolis force and the wind stress, Charney found a relationship between the wind stress and the vertical Ekman transport. The equations are

$$\int_{-D}^{0} f(\vec{K} \times \vec{V}) \rho \, dz = \vec{\tau} \qquad (1)$$

$$\int_{-D}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = -\Delta w \equiv -w_E \qquad (2)$$

which gives

$$W_E = \vec{K} \cdot \nabla \times \frac{\vec{\tau}}{\rho f} \qquad (3)$$

W_E is equivalent to $\frac{D\eta}{dt}$ in the source and sink model. If we consider a wind-stress distribution with only an x-component as a function of y alone we obtain:

$$W_E = -\frac{\partial}{\partial y}\left(\frac{\tau}{\rho f}\right) \tag{4}$$

Since the geostrophic currents are small $\frac{D\eta}{dt}$ can be approximated by $\frac{\partial\eta}{\partial t} = \dot{\eta}$. A constant $\dot{\eta}$ then gives a wind stress distribution:

$$\tau = \rho f\,[C - \dot{\eta}\,y] \tag{5}$$

We will now consider the circulation in a closed basin according to the restrictions:

1. Steady motion
2. Constant density, $\rho = 1$
3. Uniformly raising or lowering of the surface $\frac{\partial\eta}{\partial t} = \dot{\eta} = \text{const.}$
4. Constant mean depth, H = const.

The equations are:

$$u u_x + v u_y - f v = -p_x + f r. \tag{6}$$

$$u v_x + v v_y + f u = -p_y + f r. \tag{7}$$

$$p = g(\eta - z) \tag{8}$$

$$u_x + v_y + w_z = 0 \tag{9}$$

Assuming geostrophy in the interior and integrating from bottom to surface we obtain:

$$-fV = -gH\frac{\partial\eta}{\partial x} \tag{10}$$

$$f\,\mathcal{U} = -gH\frac{\partial \eta}{\partial y} \tag{11}$$

$$\mathcal{U}_x + V_y = -\dot{\eta} \tag{12}$$

where $\mathcal{U} = \int_{-H}^{\eta} u\,dz$ and $V = \int_{-H}^{\eta} v\,dz$ (13)

By cross-differentiating and using the continuity equation we find:

$$V = \frac{f}{\beta}\dot{\eta} \tag{14}$$

$$\mathcal{U}_x = -2\dot{\eta} \tag{15}$$

which integrates to give

$$\mathcal{U} = -2\dot{\eta}x + G(y) \tag{16}$$

If u and v are functions of x and y alone we get with high degree of accuracy:

$$u = -\frac{2\dot{\eta}x}{H} + G_1(y) \tag{17}$$

$$v = \frac{f\dot{\eta}}{\beta H} \tag{18}$$

Solving for η we find:

$$\eta = \frac{\dot{\eta}f^2}{\beta g H}x - \int^{y}\frac{f}{g}\,G(y)\,dy \tag{19}$$

Now we will apply these solutions to a rectangular basin. We can satisfy boundary conditions at one boundary.

I. $u = 0$ at western boundary.

$$u = -\frac{2\dot{\eta}}{H}x \qquad v = \frac{f\dot{\eta}}{\beta H} \tag{20}$$

$$\eta = \frac{\dot{\eta} f^2}{\beta g H} x \tag{21}$$

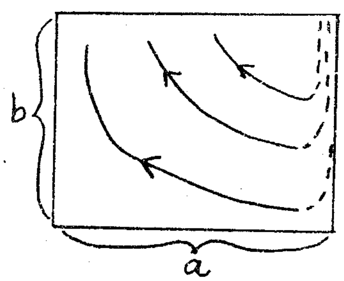
II. $u = 0$ at eastern boundary.

$$u = -\frac{2\dot{\eta}}{H}(x-a) \qquad v = \frac{f\dot{\eta}}{\beta H} \tag{22}$$

$$\eta = \frac{\dot{\eta} f^2}{\beta g H}(x-\ell) \tag{23}$$

Subject to I and II there are four types of flows to consider.

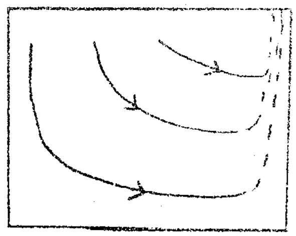
I_1



Source: $\dot{\eta} > 0$

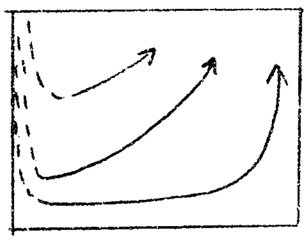
In order to satisfy boundary conditions at eastern boundary we must introduce a boundary current. The assumed boundary flow is dotted. This flow is possible if the absolute vorticity is conserved along the stream-lines. Near the boundary the relative vorticity ζ is approximately given by $\zeta = v_x$. v_x is negative and increasing along the stream-line while f is decreasing so $\zeta + f$ can remain constant. Later on we will give this point a mathematical treatment.

I_2 Sink $\dot{\eta} < 0$



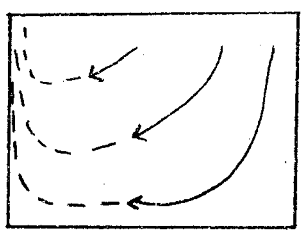
In this case the absolute vorticity is not conserved along the dotted lines because both f and ζ are increasing in the flow direction.

II_1 Source $\dot{\eta} > 0$



By the same argument as earlier we can see that it is impossible to have a strong boundary current.

II_2 Sink $\dot{\eta} < 0$



This flow is possible and it will be treated mathematically.

Now we have ruled out the cases I_2 and II_1 and we will consider the two cases I_1 and II_2 in more detail.

The eastern and western boundary currents.

In this treatment we will neglect the viscous terms. It is possible to join the inertial solution with the inertial-viscous solution in the very thin layer near the wall.

Using equations (6) and (7) and cross-differentiating we get:

$$u\frac{\partial}{\partial x}(\zeta+f)+v\frac{\partial}{\partial y}(\zeta+f)=0 \qquad (24)$$

where $\zeta = v_x - u_y$ and where we have neglected frictional terms.

In order to have a stream function to work with, we now redefine v.

$$v = v^* - \frac{f\dot{\eta}}{\beta H} \qquad (25)$$

Then we have $u_x + v_y^* = 0$ and $u = -\frac{\partial\psi}{\partial y}$ and $v^* = \frac{\partial\psi}{\partial x}$
Since $|v^*| >> \left|\frac{f\dot{\eta}}{\beta H}\right|$ in the boundary layer, we obtain from equation (24):

$$\nabla^2\psi + f = R(\psi) \qquad (26)$$

which is the equation of conservation of potential vorticity.

In the interior the relative vorticity is zero and we have:

$$f = R(\psi_i) \qquad (27)$$

$$v_i^* = \frac{2f\dot{\eta}}{\beta H} \qquad (28)$$

From these we obtain, since $v_i^* = \frac{\partial\psi_i}{\partial x}$ (29)

$$\psi_i = \frac{2f\dot{\eta}}{\beta H}(x+K) \qquad (30)$$

and
$$R(\psi_i) = \frac{\beta H}{2\dot{\eta}(x+K)}\psi_i' \qquad (31)$$

For the case with boundary layer at eastern wall K = 0 and since $x \approx a$ near the boundary we get the equation to solve:

$$\nabla^2\psi - \frac{\beta H}{2\dot{\eta}a}\psi = -f \qquad (32)$$

We can now dimensionalize by putting $\psi = \bar{\psi}\hat{\psi}$ $\quad x = x'a$ $\quad y' = y'a$ $\quad f = f_1\hat{f}$. Dividing through by f_1 and choosing $\bar{\psi} = \frac{2\dot{\eta}a}{\beta H}f_1$ we get

$$\varepsilon\nabla^2\hat{\psi} - \hat{\psi} = -\hat{f} \qquad (33)$$

where $\varepsilon = \frac{2\dot{\eta}}{\beta H a}$ and using data from actual ocean basin we find $\varepsilon \sim 10^{-4}$.

We know that in the boundary layer there must be a balance between the two first terms in the equation (33) and it is natural to assume that the changes in ψ are greater normal to the boundary than along the flow. If we can neglect $\hat{\psi}_{yy}$ relative to $\hat{\psi}_{xx}$ we then have that $\hat{\psi}_{xx}$ must be of order ε^{-1}. Putting $x'-1 = \varepsilon^{1/2}\xi$ we get

$$\hat{\psi}_{\xi\xi} - \hat{\psi} = -\hat{f} \qquad (34)$$

The boundary layer solution is

$$\hat{\psi} = A(y')e^{\xi} + \hat{f}$$

and subject to the condition that $\xi = 0$ must be a streamline which we take to be $\hat{\Psi} = 0$ we obtain:

$$\hat{\Psi} = \hat{f}(1 - e^{\xi})$$

With dimensions:

$$\Psi_{EB} = \frac{2\dot{\eta} a f}{\beta H}\left(1 - e^{\varepsilon^{-1/2}\frac{x-a}{a}}\right)$$

Here we can replace a by x since $x \simeq a$ in the boundary layer.

$$\Psi = \frac{2\dot{\eta} x f}{\beta H}\left[1 - e^{\varepsilon^{-1/2}\frac{x-a}{a}}\right] \tag{35}$$

This stream function is valid both in the boundary layer and in the interior. The velocities are given by

$$u = -\frac{\partial\psi}{\partial y} = -\frac{2\dot{\eta} x}{H}\left[1 - e^{+\varepsilon^{-1/2}\frac{x-a}{a}}\right]$$

$$v = \frac{\partial\psi}{\partial x} - \frac{f\dot{\eta}}{\beta H} = \frac{2\dot{\eta} f}{\beta H}\left[\frac{1}{2} - e^{+\varepsilon^{-1/2}\frac{x-a}{a}}\right] - \frac{2\dot{\eta} x f}{\beta H a}\varepsilon^{-1/2} e^{\varepsilon^{-1/2}\frac{x-a}{a}}$$

The velocity along the wall is about one hundred times bigger than the velocity in the interior.

In the same way we can solve the equation for the western boundary layer. The equation is:

$$\varepsilon \nabla'^2 \hat{\Psi} - \hat{\Psi} = -\hat{f}$$

where $\varepsilon = \frac{2s}{\beta H a}$ and $s = -\dot{\eta}$.

The solution for ψ is:

$$\Psi_{WB} = \frac{2sa}{\beta H} f\left[1 - e^{-\varepsilon^{-1/2}\frac{x}{a}}\right]$$

Here we can replace a by $(a-x)$ in order to have a solution for the whole basin.

$$\psi = \frac{2S(a-x)}{\beta H} f\left[1-e^{-\varepsilon^{-1/2}\frac{x}{a}}\right]$$

The velocities are

$$u = -\frac{2S(a-x)}{H}\left[1-e^{-\varepsilon^{-1/2}\frac{x}{a}}\right]$$

$$v = -\frac{2Sf}{\beta H}\left[\frac{1}{2}-e^{-\varepsilon^{-1/2}\frac{x}{a}}\right]+\frac{2S(a-x)}{\beta H}f e^{-\varepsilon^{-1/2}\frac{x}{a}}$$

The current along the northern boundary.

Since $y=b$ is a boundary the flow must be along the northern boundary. The equations for the inertial flow are

$$fu = -g\eta_y \tag{36}$$

$$\frac{u^2}{2}+g\eta = F(\psi) \tag{37}$$

with $u = -\dfrac{\partial\psi}{\partial y}$

Combined, these give

$$\frac{1}{2}\left(\frac{\partial\psi}{\partial y}\right)^2+f\psi-\beta\int\psi\,dy = F(\psi) \tag{38}$$

In the interior the square of velocity is negligible compared to $g\eta_i$, so we have approximately:

$$g\eta_i = F(\psi_i)$$

In order to have a stream function we can redefine u.

$$u = u^* - \frac{\dot{\eta}x}{H}$$

$\psi_i = \dfrac{f\dot{\eta}}{\beta H}x$ for the flow with eastern boundary layer and

$\psi_i = \frac{f\dot{\eta}}{\beta H}(x-a)$ for the flow with western boundary layer.

For the last case

$$u = -\frac{\partial \psi_i}{\partial y} - \frac{\dot{\eta}}{H}(x-a)$$

In both cases we have $g\eta_i = f\psi_i$

$$F(\psi_i) = f\psi_i$$

Equation (38) then yields

$$\frac{1}{2}\left(\frac{\partial \psi}{\partial y}\right)^2 - \beta\int \psi\, dy = 0 \qquad (39)$$

Taking the derivative with respect to y and multiplying by $\frac{\partial \psi}{\partial y}$ gives

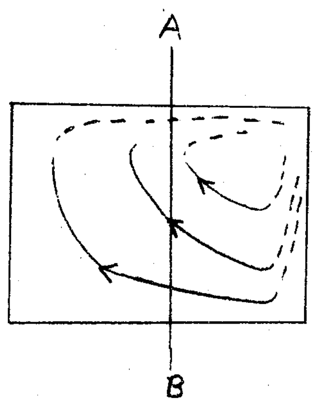
$$\left(\frac{\partial \psi}{\partial y}\right)^2 \frac{\partial^2 \psi}{\partial y^2} - \beta\psi\frac{\partial \psi}{\partial y} = 0$$

This equation can be integrated to give

$$\frac{1}{3}\left(\frac{\partial \psi}{\partial y}\right)^3 - \frac{1}{2}\beta\psi^2 = Q(x)$$

It is here more convenient to use $\theta = b - y$ as a new variable. If we take $\psi = 0$ for $\theta = 0$ and $\left[\frac{\partial \psi}{\partial \theta}\right]_{\theta=0} = \mu_1(x)$ we have

$$\frac{\partial \psi}{\partial \theta} = \mu_1\left[1 - \frac{3\beta}{2\mu_1 3}\psi^2\right]^{1/3} \qquad (40)$$



The mass transport into the left-hand side of AB minus the amount which goes into raising the surface through interior flow is T_i. This mass has to leave as boundary transport T_b.

$$T_i = \dot{\eta}\, b\, x + \frac{f_0\dot{\eta}}{\beta}x \equiv \frac{f(b)\dot{\eta}\, x}{\beta}$$

From equation (40) $T_\theta = \sqrt{\frac{2}{3}\,\frac{u_1^3}{\beta}}\; H$

Since $T_i = T_b$ $\qquad u_1 = \alpha_1 x^{\frac{2}{3}}$

where $\alpha_1 = \left[\frac{\frac{3}{2} f^2(b)\,\dot{\eta}^2}{\beta H^2}\right]^{1/3} \sim 5\cdot 10^{-4}$

With this information we can see from equation (40) that the solution does not give a narrow boundary current but rather a broad flow which affects the whole interior flow, and the assumptions we made are then violated.

We can approximate the solution by putting

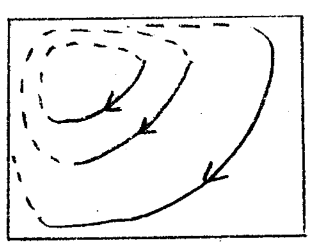
$$\frac{\partial \psi}{\partial \theta} \approx u_1\left[1 - \frac{\beta}{2u_1^3}\,\psi^2\right]$$

$$\psi_b = \sqrt{\frac{2u_1^3}{\beta}}\;\tanh \sqrt{\frac{\beta}{2u_1}}\;\theta$$

The velocity along the wall is

$$u_b = \frac{u_1}{\cosh^2\sqrt{\frac{\beta}{2u_1}}\;\theta}$$

The flow is getting broader as it approaches the eastern boundary.



For the flow along the northern wall fed by the western boundary current we have the same equation to solve.

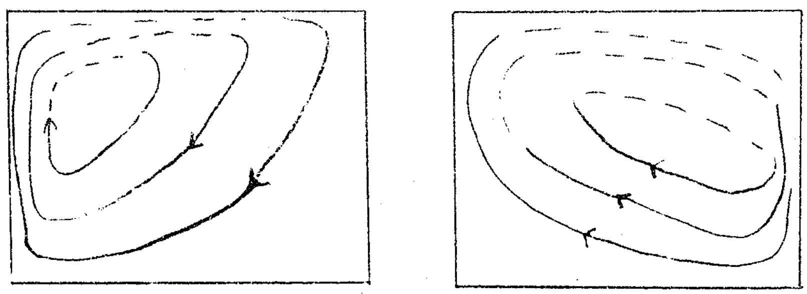
$$\frac{\partial \psi}{\partial \theta} = u_2(x)\left[1 - \frac{3\beta}{2u(x)^3}\,\psi^2\right]^{1/3}$$

By equating transports we get

$$u_2 = \alpha_1 (a-x)^{\frac{2}{3}}$$

with the same $\alpha_1 \sim 5 \cdot 10^{-4}$

The flows obtained can be sketched



The northern current has to get thinner as the flow is diverging in order to conservate the absolute vorticity.

The viscous-inertial boundary layer.

We want to study the effect of the viscous forces upon the eastern and western boundary current. And we will join the solution in terms of the stream function with the interior form of Ψ near the boundaries.

$$\Psi_i = \frac{2f\dot{\eta}}{\beta H} \mathcal{L}(x)$$

Equations (6), (7) and (8) are in terms of ψ

$$\Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} - f\Psi_x = -p_x - K_H \nabla_1^2 \Psi_y$$

$$\Psi_x \Psi_{xy} - \Psi_y \Psi_{xx} - f\Psi_y = -p_y + K_H \nabla_1^2 \Psi_x$$

$$p = g(\eta - z)$$

We non-dimensionalize by putting:

$$\psi = \bar{\Psi}\hat{\psi} \quad x = ax' \quad y = ay' \quad f = \beta a \hat{f} \quad \eta = \bar{\eta}\hat{\eta}$$

We choose $\bar{\Psi} = \frac{g\bar{\eta}}{\beta a}$ and in terms of parameters $\varepsilon = \frac{g\bar{\eta}}{(\beta a^2)^2}$ and $\delta = \frac{K_M}{\beta a^3}$ the equations are:

$$\varepsilon(\psi_y \psi_{xy} - \psi_x \psi_{yy}) - f\psi_x = -\eta_x - \delta \nabla_1^2 \psi_y$$

$$\varepsilon(\psi_x \psi_{xy} - \psi_y \psi_{xx}) - f\psi_y = -\eta_y + \delta \nabla_1^2 \psi_x$$

where hats and primes are omitted.

The estimated values of ε and δ from typical data are:

$$\varepsilon \sim 10^{-5}$$

For $10 < K_H < 10^3 \ m^2 \ sek^{-1}$; $5 \cdot 10^{-8} < \delta < 5 \cdot 10^{-6}$

In order to have derivatives of the same order in both directions we must stretch x coordinate

$$x = \delta^n \xi$$

Cross-differentiating equations gives when obvious small terms are dropped

$$\varepsilon \delta^{-2n} \left[\psi_\xi \psi_{\xi y} - \psi_y \psi_{\xi\xi}\right]_\xi + \psi_\xi = \delta^{1-3n} \psi_{\xi\xi\xi}$$

ψ_ξ includes the β effect. If the term on the right-hand side is of the same order of magnitude, then $n = \frac{1}{3}$. It follows that $\Lambda = \varepsilon \delta^{-2/3}$ lies between 1 and 10^{-1}. Integrating once we get:

$$\Lambda \left[\psi_\xi \psi_{\xi y} - \psi_y \psi_{\xi\xi}\right] + \psi = \psi_{\xi\xi\xi} + C(y)$$

Assuming separability $\psi = X(\xi) Y$ and taking $C(y) =$

C, Y we obtain

$$Y_y = \text{const} = \alpha$$

$$r\alpha(\chi'^2 - \chi\chi'') + \chi = \chi''' + C_1$$

Here we can try the solution $\chi = A + B e^{\lambda\xi}$

Choosing $A = C_1$ we see that the roots of λ are given by the cubic equation

$$\lambda^3 + R\lambda^2 - 1 = 0 \qquad \text{where } R = \alpha r C_1$$

There are 3 cases to consider:

1. $R > 3(4)^{-1/3}$ $\quad \lambda_1$ real $\quad \lambda_{2,3} = c \pm i d$

   where $c < 0$

2. $R = 3(4)^{-1/3}$ $\quad \lambda_1 = \lambda_2 = -\frac{2}{3}R \quad \lambda_3 = \frac{1}{3}R$

3. $R = 3(4)^{-1/3}$ $\quad \lambda_1 < 0 \quad \lambda_2 < 0 \quad \lambda_3 > 0$

   and the roots are real.

Since the differential equation is non-linear we cannot combine the solutions to satisfy all the boundary conditions. But we have an exact solution which satisfies the condition $\mu = 0$ at the eastern or western boundary. If we write ψ with dimensions again and match the boundary solution with the interior solution we have:

$$\psi = \frac{2f\dot{\eta}}{\beta H} \mathcal{L}(x_1)\left[1 - e^{\lambda\xi}\right]$$

$\xi = \dfrac{\delta^{1/3}x}{a}$ and $\mathcal{L} = -a$ as western boundary and

$\xi = \frac{\delta^{1/3}(x-a)}{a}$ and $\mathcal{L} = a$ as eastern boundary.

Negative roots of λ give a boundary solution as western boundary and positive roots of λ satisfy the eastern boundary solution. It follows that we only have a damped wave solution for the flow along western boundary.

$$R = \delta^{-2/3} \frac{2\dot{\eta}}{a\beta H} \sim 10^{2/3} \text{ for } \delta \sim 10^{-7}$$

and we see that we can have either of the 3 cases according to the assumed value of δ.

Notation.

τ		wind stress
f	=	$\beta y + f_0$ Coriolis parameter
η	.	variation of surface from the mean depth
$\dot{\eta}$	=	$\frac{\partial \eta}{\partial t}$
H		mean depth

Sizes of the parameters.

$a \sim$ 5000 km = $5 \cdot 10^6$m

$b \sim$ 3000 km = $3 \cdot 10^6$m

$\dot{\eta} \sim$ 1 cm/day = 10^{-7} m/sec

$H \sim$ 400 m

$\beta \sim 4 \cdot 10^{-12}$ m^{-1} sec^{-1}

$f \sim 10^{-5}$

$g \sim$ 10 m/sec^2

$\bar{\eta} \sim 10^{-2}$m

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Interaction of Mean and Fluctuating Fields in

Turbulent Thermal Convection

by

S. Nagarajan

1. Introduction

The non-linear interactions associated with the turbulent motion of fluids may be divided into two classes: interaction of fluctuating field quantities among themselves and interaction between fluctuating and mean field quantities. In isotropic turbulence, the mean fields vanish, and only the fluctuating-fluctuating interaction remains. In turbulent transport problems, however, the interaction between the mean fields and the fluctuating fields plays a crucial role.

The aim of the present investigation is to obtain some insight into the mathematical structure of the interaction between mean and fluctuating fields for the problem of turbulent thermal convection in the Boussinesq approximation at very high Prandtl number. The reason for taking the high Prandtl number case is that the coupled equations for the temperature and velocity fields may then be reduced to a single scalar equation for the temperature field alone.

We make the familiar (but artificial) assumption of free boundary conditions on the top and bottom surfaces of a layer of fluid of infinite horizontal extent which is heated from below.

Then we express the dynamical equations in terms of a Fourier decomposition of the deviation of the temperature field from the constant-gradient field which would exist in the absence of convection. We make a division of the total dynamical interaction of the Fourier modes into the mean-field-fluctuating-field interaction and fluctuating field self-interaction mentioned above. The dynamical justification given for this division is that both types of interaction individually conserve an integral quantity which is formally analogous to energy. We then explore the structure of the mean-field-fluctuating-field interaction by seeking exact (non-perturbative) solutions for the steady-state behavior of truncated systems consisting of only a few of the Fourier modes. Very little in the way of final results have been obtained so far. The purpose of this preliminary report is principally to present the motivation of the study and the method of formulating the problem.

2. Equations of the Problem

Following Boussinesq, we write the equations for the local temperature $T(\underset{\sim}{x}, t)$ and velocity $\underset{\sim}{u}(\underset{\sim}{x}, t)$ of the fluid as

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = -\frac{1}{\rho_0}\nabla p + \gamma(T - T_0)\hat{k} + \nu \nabla^2 \underset{\sim}{u}, \qquad (1)$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla) T = K \nabla^2 T, \qquad (2)$$

and $$\nabla \cdot \underset{\sim}{u} = 0, \qquad (3)$$

where ρ_0 is the mean density, ν is the kinematic viscosity,

p is the pressure, K the thermometric conductivity and $\gamma = \alpha g$, with α the coefficient of linear expansion of the fluid and g the acceleration due to gravity. The symbol $\hat{k}$ represents a unit vector in the vertical z direction. The medium is assumed to be homogeneous and to extend to infinity in the horizontal direction. It has the dimension D in the vertical. This geometry of the problem, together with the basic asymmetry in the vertical direction due to the imposed temperature difference $T_{bottom} - T_{top} = \beta_0 D$, suggests that we introduce variables which make clear the distinction between the quantities associated with the vertical and horizontal directions. Therefore we write

$$\left.\begin{aligned} \underset{\sim}{u} = w\hat{k} + \tilde{\sigma} \; &: \; \tilde{\sigma} = \hat{\imath}\,u + \hat{\jmath}\,v \\ &: \; \nabla' = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} \\ T - T_0 &= -\beta_0 z - \psi \\ \chi &= p - \frac{\gamma\beta_0 z^2}{2} \end{aligned}\right\} \tag{4}$$

where u and v are the velocities of the fluid in the x and y directions and ψ is the deviation of the temperature field from the constant-gradient profile which would exist in the absence of convection.

In terms of the new variables, the equations of motion are

$$\frac{\partial \tilde{\sigma}}{\partial t} = -\frac{1}{\rho_0}\nabla'\chi + \nu\nabla^2\tilde{\sigma} - \omega\frac{\partial}{\partial z}\tilde{\sigma} - (\tilde{\sigma}\cdot\nabla')\tilde{\sigma}, \tag{5}$$

$$\frac{\partial \omega}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \chi}{\partial z} + \nu\nabla^2\omega - \gamma\psi\hat{k} - \omega\frac{\partial \omega}{\partial z} - (\tilde{\sigma}\cdot\nabla')\omega, \tag{6}$$

$$\frac{\partial \psi}{\partial t} = K\nabla^2\psi - \omega\frac{\partial \psi}{\partial z} - (\tilde{\sigma}\cdot\nabla')\psi - \beta_0\omega, \tag{7}$$

and

$$(\nabla'\cdot\tilde{\sigma}) + \frac{\partial \omega}{\partial z} = 0 \tag{8}$$

The term in equation (7) containing the constant gradient β_0 represents a driving term for the ψ-field. Our separation of the role of the mean-gradient part from the remainder of the temperature field leads to some useful conservation properties, as we shall see shortly.

We non-dimensionalize according to the transformations

$$\left.\begin{aligned} (x,y,z) &\longrightarrow D(x,y,z) \\ t &\longrightarrow \frac{\nu}{\gamma\beta_0 D^2}(t) \\ \psi &\longrightarrow \beta_0 D(\psi) \end{aligned}\right\} \tag{9}$$

with

$$\begin{aligned} &\text{Grashoff number} \quad G = \frac{D^4\gamma\beta_0}{\nu^2} \\ &\text{Rayleigh number} \quad \lambda = \frac{\gamma\beta_0 D^4}{K\nu} \\ \text{and} \quad &\text{Prandtl number} \quad \sigma = \lambda/R = \nu/K \end{aligned} \tag{10}$$

We will restrict ourselves to the large Prandtl number case and neglect all terms involving G compared to terms involving λ. In this case the equations of motion become

$$-\frac{1}{\rho_0}\frac{\partial \chi}{\partial z} + \nabla^2\omega - \psi = 0 \tag{11}$$

$$-\frac{1}{\rho_0}\nabla'\chi + \nabla^2\tilde{\sigma} = 0 \tag{12}$$

$$\frac{\partial \psi}{\partial t} + (\sigma \cdot \nabla')\psi + \omega \frac{\partial \psi}{\partial z} = -\omega + \frac{1}{\lambda}\nabla^2 \psi \tag{13}$$

$$(\nabla' \cdot \tilde{\sigma}) + \frac{\partial \psi}{\partial z} = 0 \tag{14}$$

We shall specialise on the case of free-free boundaries and shall expand ω, $\tilde{\sigma}$, χ and ψ in suitable Fourier series in a box of non-dimensional size $\ell \left(= L/_D\right)$ in the horizontal direction and unity in the vertical direction. (We will ultimately be interested in the limit $(\ell \to \infty.)$ We write

$$\left.\begin{aligned}
\tilde{\sigma}(x,y,z,t) &= \frac{1}{\ell}\sum_{m=1}^{\infty}\sum_{\underset{\sim}{K}=\pm 1}^{\infty} \tilde{\sigma}_m(\underset{\sim}{k},t)\cos m\pi z\, e^{\pi i \underset{\sim}{k}\cdot \underset{\sim}{r}},\\
\omega(x,y,z,t) &= \frac{1}{\ell}\sum_{m=1}^{\infty}\sum_{\underset{\sim}{K}=\pm 1}^{\infty} \omega_m(\underset{\sim}{k},t)\sin m\pi z\, e^{\pi i \underset{\sim}{k}\cdot \underset{\sim}{r}},\\
\psi(x,y,z,t) &= \frac{1}{\ell}\sum_{m=1}^{\infty}\sum_{\underset{\sim}{K}\pm 1}^{\infty} \psi_m(\underset{\sim}{k},t)\sin m\pi z\, e^{\pi i \underset{\sim}{k}\cdot \underset{\sim}{r}},\\
\chi(x,y,z,t) &= \frac{1}{\ell}\sum_{m=1}^{\infty}\sum_{\underset{\sim}{K}\pm 1}^{\infty} \chi_m(\underset{\sim}{k},t)\cos m\pi z\, e^{\pi i \underset{\sim}{k}\cdot \underset{\sim}{r}},
\end{aligned}\right\} \tag{15}$$

where $\underset{\sim}{r} = \hat{i}x + \hat{\gamma}y.$

With these substitutions, we get from equations (11-14)

$$\left.\begin{aligned}
-\alpha^2 \tilde{\sigma}_m(\underset{\sim}{k},t) &= i\pi \underset{\sim}{k}\, \chi_m(k,t)\\
-\alpha^2 \omega_m(\underset{\sim}{k},t) &= \psi_m(\underset{\sim}{k},t) - m\pi\, \chi_m(\underset{\sim}{k},t)\\
i\pi \underset{\sim}{k}\cdot \tilde{\sigma}_m(\underset{\sim}{k},t) &+ m\pi\, \omega_m(\underset{\sim}{k},t) = 0
\end{aligned}\right\} \tag{16}$$

where $\alpha^2 = (m^2 + k^2)\pi^2$

From these we can eliminate $\chi_m(\underset{\sim}{k},t)$ to find

$$\left.\begin{aligned}
\omega_m(\underset{\sim}{k},t) &= -\pi^2 k^2 \frac{1}{\alpha^4}\psi_m(\underset{\sim}{k},t)\\
\omega_m(\underset{\sim}{k},t) &= -i\pi \underset{\sim}{k}\left(\frac{1}{\alpha^4}\right) m\pi\, \psi_m(\underset{\sim}{k},t)
\end{aligned}\right\} \tag{17}$$

Substituting equations (15) and (17) into equation (13) we get for the Fourier coefficient $\Psi_m(\underset{\sim}{k},t)$ the equation

$$\left(\frac{d}{dt}+\frac{\alpha^2}{\lambda}-\frac{\pi^2 k^2}{\alpha^4}\right)\psi_m(\underset{\sim}{k},t)$$
$$=\frac{1}{\ell}\sum_{\underset{\sim}{k}'}\pi^3 \underset{\sim}{k}'\left[\sum_{m'=1}^{m-1}\frac{m\underset{\sim}{k}'-m'\underset{\sim}{k}}{\alpha^4}\psi_{m'}(k',t)\,\psi_{m-m'}(\underset{\sim}{k}-\underset{\sim}{k}',t)\right.$$
$$-\sum_{m'=m+1}^{\infty}\frac{m\underset{\sim}{k}'-m'\underset{\sim}{k}}{\alpha'^4}\psi_{m'}(\underset{\sim}{k}',t)\,\psi_{m'-m}(\underset{\sim}{k}-\underset{\sim}{k}',t)$$
$$\left.-\sum_{m'=1}^{\infty}\frac{m\underset{\sim}{k}'+m'\underset{\sim}{k}}{\alpha'^4}\psi_{m'}(\underset{\sim}{k}',t)\,\psi_{m+m'}(\underset{\sim}{k}-\underset{\sim}{k}',t)\right] \qquad (18)$$

Equation (18) is the exact equation in the Boussinesq approximation for thermally-driven turbulence at high Prandtl numbers.

3. Interaction through the Mean Field

The Ψ's can be divided into two categories 1) the $\Psi_m(0,t)$, which corresponds to zero horizontal wave numbers, and 2) the $\Psi_m(\underset{\sim}{k},t)\,[\underset{\sim}{k}\neq 0]$, corresponding to non-zero horizontal wave numbers. The $\Psi_m(0)$'s are horizontally-averaged mean temperature deviations from the constant gradient. On the basis of this division we can rearrange the interaction terms between the Ψ's into two categories:

1) A mode $\Psi_m(\underset{\sim}{k})$ interacting with another mode $\Psi_{m'}(k)$ through a mean field mode $\Psi_{m''}(0)$. We call this the mean-fluctuating interaction or the interaction through the mean field (MF). In particular, the diagonal interaction of $\Psi_m(k)$ with itself (corresponding to m' = m and m" = 2m) is included here.

2) A mode $\Psi_m(k)$ interacting with another mode $\Psi_m(k')$ through a third mode $\Psi_{m''}(k'')$ ($\underset{\sim}{k}$, k', and $k''\neq 0$). We call this the fluctuating-

fluctuating interaction.

This division of the interactions of the Ψ's into the MF and FF categories is mutually exclusive and hence we can write down dynamically consistent equations of motions for the Ψ's with either MF or the FF interactions discarded. For the case where only FF interactions are retained, the equations are the same as equation (18) with only a further restriction on the summation over $\underset{\sim}{k}$ which excludes $\underset{\sim}{k} = 0$. The case with only MF interactions retained is given by

$$
\begin{aligned}
\left(\frac{d}{dt} + \frac{\alpha^2}{\lambda} - \frac{\pi^2 k^2}{\alpha^4}\right) \psi_m(\underset{\sim}{k}, t)
&= \pi^3 k^2 \Bigg[\sum_{m'=1}^{m-1} \frac{m-m'}{\alpha^4}\, \psi_{m'}(\underset{\sim}{k}, t)\, \psi_{m-m'}(0, t) \\
&\quad + \sum_{m'=m+1}^{\infty} \frac{m'-m}{\alpha'^4}\, \psi_{m'}(\underset{\sim}{k}, t)\, \psi_{m'-m}(0, t) \\
&\quad - \sum_{m'=1}^{\infty} \frac{m+m'}{\alpha'^4}\, \psi_{m'}(\underset{\sim}{k}, t)\, \psi_{m+m'}(0, t)\Bigg]
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
\left(\frac{d}{dt} + \frac{m^2\pi^2}{\lambda}\right) \psi_m(0, t) &= \frac{\pi^3}{\ell} \sum_K m\, k^2 \Bigg[\sum_{m'=1}^{m-1} \frac{\psi_{m'}(\underset{\sim}{k}, t)\, \psi^*_{m-m'}(\underset{\sim}{k}, t)}{\alpha'^4} \\
&\quad - \sum_{m'=m+1}^{\infty} \frac{\psi_{m'}(\underset{\sim}{k}, t)\, \psi_{m'-m}(\underset{\sim}{k}, t)}{\alpha'^4} - \sum_{m'=1}^{\infty} \frac{\psi_m(\underset{\sim}{k}, t)\, \psi_{m+m'}(\underset{\sim}{k}, t)}{\alpha'^4}\Bigg]
\end{aligned}
$$

The similarity of equations (19) and (20) and, though less apparent, of equation (18) to the equations of coupled oscillators is suggestive. The fluctuating field equations with either MF, FF or the complete interactions exhibit an effective viscous damping $= \frac{\alpha^2}{\lambda} - \frac{\pi^2 k^2}{\alpha^4} = \nu_m(k)$ which may be positive or negative. If all these damping terms were zero, there would be neither dissipation nor input. The mean field equations also have damping terms, but they are always positive (dissipative). In the absence of all these dissipative terms, these equations would show the conservation properties.

$$\frac{d}{dt}\left[\sum_{m=1}^{\infty}\sum_{k=q\pm 1}^{\infty}\left(\psi_m(k)\right)^2\right]=0 \tag{21}$$

(MF and FF both retained)

$$\frac{d}{dt}\left[\sum_{m=1}^{\infty}\left\{\psi_m^2(0)+\sum_{k=\pm 1}^{\infty}\left(\psi_m(k)\right)^2\right\}\right]=0 \tag{22}$$

(only MF retained)

and

$$\frac{d}{dt}\left[\sum_{m=1}^{\infty}\sum_{k=\pm 1}^{\infty}\left(\psi_m(k)\right)^2\right]=0 \tag{23}$$

(only FF retained).

Thus in the absence of dissipation, a quantity which is analogous to the energy in a coupled oscillator problem, is conserved. We will refer to this as 'energy' hereafter. The dynamical interactions MF and FF have the property that they conserve both separately and together the 'energy' of the system. This suggests that the roles played by these two interactions may meaningfully be considered in isolation.

Another interesting feature which is common both to the MF and FF interactions is the triad nature of the detailed coupling between the ψ's. The elementary coupling between any three ψ's (either MF or FF) is again conservative, in the sense defined as above.

The conservation property for the MF interactions suggests a way of picturing the dynamical coupling between the ψ's. The imposed temperature difference drives those fluctuating field components $\psi_m(\underset{\sim}{k})$ which are unstable upon the associated mean gradient β_0. Then, through the MF interaction with $\psi_m(k)$, the mean field component $\psi_{2m}(0)$ is excited. [In equations (22) and (23), it is easy to see that even in the absence of coupling with all $\psi_{m'}(0)$'s and $\psi_{m'}(k')$ the mode $\psi_m(k)$ is 'diagonally' coupled to $\psi_{2m}(0)$]. The mean field modes $\psi_{2m}(0)$, once generated, then couple the various $\psi_m(k)$ to each other. Thus the 'energy' which is fed into the low-wave-number $\psi_m(k)$'s, due to their instability on the constant-gradient part of the mean profile, can effectively be fed into higher ψ's. A steady state may be reached in which the amplitudes of the higher $\psi_m(k')$'s and $\psi_m(0)$'s, which in isolation would all be stable on the constant-gradient part of the mean profile, can be supported by the interaction with the lower wave number fluctuating component $\psi_m(k)$. This suggests that the diagonal coupling of the $\psi_m(k)$'s to the $\psi_{2m}(0)$'s perhaps plays a more basic role than the 'off-diagonal' couplings of model $\psi_m(k)$ to modes $\psi_{m'}(k)(m \pm m')$.

In a statistically steady state, the dynamical picture just presented has the following interpretation in terms of the balance of the conserved quantity we have called "energy".

"Energy" is fed into certain low wave number fluctuating modes $\psi_m(k)$. (There is no direct input into the mean field modes through the mean gradient β_0.) The 'energy' then flows by interaction into mean field modes and higher fluctuating modes and eventually is dissipated as represented by terms involving ν and K.

The present treatment is based upon a hope that the total dynamics may be dominated in some significant sense by the MF interaction.(Some support for this hope is provided by the fact that Malkus (1954, 1961) has obtained approximate quantitative agreement with experiment by analysis of the thermal fluctuation problem using variational criteria in which the explicit form of the FF coupling does not enter. Further support is given by the finite-amplitude result of Malkus and Veronis (1958)). In what follows we therefore omit the FF interaction entirely, with the eventual intention of reintroducing it as a perturbation upon the MF interaction. The dynamical consistency of this procedure (but not of course the quantitative validity of the results) is supported by the separate conservation properties for the MF and FF interactions.

4 Diagonal Coupling

We visualise the MF system as composed of an infinite number of sub-systems. Each of these sub-systems includes a $\psi_m(k)$ and the associated second harmonic mean field component

$\psi_{2m}(0)$. This we have called the diagonal part of the MF interaction. These sub-systems are again coupled to one another through a non-diagonal MF term.

We will first consider the $\psi_m(k)-\psi_{2m}(0)$ system in isolation and discuss its properties in the statistically-steady state. The dynamical equations for this hypothetical case are

$$\left[\frac{d}{dt} + \nu_m(k) + \frac{2mk^2}{\pi(m^2+k^2)^2}\phi_{2m}\right]\psi_m(k) = 0 \tag{24}$$

and $$\left(\frac{d}{dt} + \frac{m^2\pi^2}{\lambda}\right)\phi_{2m} = \frac{1}{\ell\pi}\sum_K 2mk^2 \frac{|\psi_m(k)|^2}{(m^2+K^2)^2} \tag{25}$$

where $$\nu_m(k) = \frac{\pi^2(m^2+K^2)}{\lambda} - \frac{K^2}{(m^2+K^2)^2\pi^2} \tag{26}$$

and $\psi_{2m}(0) = \phi_{2m}$.

In the steady state ϕ_{2m} is a constant and if $\psi_m(k) \sim e^{-i\omega t}$

$$\left(-i\omega + \nu_m(k) + \frac{2mk^2}{\pi(m^2+k^2)^2}\phi_{2m}\right) = 0 \quad \text{if } \psi_m(k) \neq 0 \tag{27}$$

If we impose the condition that

$$\left.\begin{array}{l} I_m\,\omega < 0 \quad \text{for all } k \text{ except } k_0 \\ \qquad I_m\,\omega(k_0) = 0 \end{array}\right\} \tag{28}$$

equations (27) and (28) imply that

$$\left.\begin{array}{l} R_e\ \omega(k) = 0 \\ I_m\,\omega(k_0) = \left[\dfrac{d}{dk} I_m\,\omega(k)\right]_{K=k_0} = 0 \\ \text{and} \quad \left[\dfrac{d\lambda}{dk^2} I_m\,\omega(k)\right]_{k=k_0} > 0 \end{array}\right\} \tag{29}$$

It is easy to manipulate these equations and get solutions for ϕ_{2m} and the steady state wave number k_o .

$$\phi_{2m} = \frac{1}{2\pi m}\left[1 - \frac{\pi^4}{\lambda}\left(\frac{27}{4}m^2\right)\right] \tag{30}$$

and

$$k_o^2 = \frac{1}{2}m^2 \tag{31}$$

These give for the mean square amplitude of $\psi_m(k)$

$$\left(\psi_m(k)\right)^2 = \frac{9}{4}\frac{m^3}{\lambda}\pi^2\left[1 - \frac{\pi^4}{\lambda}\left(\frac{27}{4}m^2\right)\right] \tag{32}$$

This shows the existence of a critical λ , above which the steady state solutions are possible.

5. <u>Coupled Sub-systems</u>

That there exists a steady state for the diagonal system is not unexpected. The isolated normal modes of the system have been previously treated in the literature. Malkus and Veronis, 1958, consider the dependence of the heat transport on the Rayleigh number, through a minimization procedure assuming steady state time independent normal modes which are uncoupled. They find a $\lambda^{1/4}$ law for the 'eddy conductivity'. The experimental law on the other hand is $\lambda^{1/3}$ for large λ . If our assumption that the FF interaction can be neglected is justified, then probably the off-diagonal coupling in the MF interaction is responsible for this difference.

The complete infinite set of sub-systems, with all off-diagonal couplings included, is extremely difficult to treat. We adopt here the procedure of examining a class of simpler problems obtained by retaining only a few of the Fourier modes, in the dynamical equations and treating exactly the interactions (both diagonal and off-diagonal) among them. It is hoped that this will give insights into the full problem.

The dynamical justification for this procedure is again the fact that the conservation properties are true not only for the complete **MF** or **FF** systems, but they are true for any triad of dynamically-coupled modes.

As a first step in our investigation, we study the pairs $(\psi_1(K)-\phi_2)$, $(\psi_2(K)-\phi_4)$, and $(\psi_3(K)-\phi_6)$. The equations of motion in this case are

$$\left[\frac{\partial}{\partial t}+\frac{1}{\pi^2}\eta_1(K)\right]\psi_1(K)=\frac{1}{\pi^2}C_{13}\psi_3(K) \tag{33}$$

$$\left[\frac{\partial}{\partial t}+\frac{1}{\pi^2}\eta_2(K)\right]\psi_2(K)=0 \tag{34}$$

$$\left[\frac{\partial}{\partial t}+\frac{1}{\pi^2}\eta_3(K)\right]\psi_3(K)=\frac{1}{\pi^2}C_{31}\psi_1(K) \tag{35}$$

$$\left(\frac{\partial}{\partial t}+\alpha_2\right)\phi_2=\frac{1}{\ell\pi}\sum_{\underset{\sim}{K}}2K^2\left[\frac{\psi_1(K)\psi_1^*(K)}{(1+K^2)^2}-\frac{\psi_3(K)\psi_1^*(K)}{(9+K^2)^2}\right] \tag{36}$$

$$\left(\frac{\partial}{\partial t}+\alpha_4\right)=\frac{1}{\ell K}\sum_{\underset{\sim}{K}}4K^2\left[\frac{\psi_2(K)\psi_2^*(K)}{(4+K^2)^2}+\frac{\psi_1(K)\psi_3(K)}{(1+K^2)^2}+\frac{\psi_3(K)\psi_1^*(K)}{(9+K^2)^2}\right] \tag{37}$$

and

$$\left(\frac{\partial}{\partial t}+\alpha_6\right)\phi_6=\frac{1}{\ell\pi}\sum_{K}6K^2\left(\frac{\psi_3(K)\psi_3^*(K)}{(9+K^2)^2}\right) \tag{38}$$

where $\eta_m(k)=\pi^2\nu_m(K)+\frac{K^2m}{(m^2+K^2)^2}2\pi\phi_{2m}$

$$\alpha_m=\frac{m^2\pi^2}{\lambda}$$

$$\left.\begin{aligned} C_{13} &= \frac{2\pi K^2}{(9+K^2)^2}\left(\phi_2 - 2\phi_4\right) \\ \text{and} \qquad C_{31} &= \frac{2\pi K^2}{(1+K^2)^2}\left(\phi_2 - 2\phi_4\right) \end{aligned}\right\} \tag{39}$$

We make some further substitutions to simplify the algebra

$$\begin{aligned} a &= 1 - 2\pi\phi_2 \\ b &= 1 - 6\pi\phi_6 \\ d &= (1 - 4\pi\phi_4) \\ c &= 2\pi(\phi_2 - 2\phi_4) \end{aligned} \tag{40}$$

and $$\lambda_0 = (\lambda/\pi^4)$$

In the steady state, the mean fields and consequently a, b, c and d are constants and the $\psi_m(K)$ have a time dependence $\sim \operatorname{Exp}\left(-i\frac{\omega}{\pi^2}t\right)$. The non-triviality condition for the amplitudes $\psi_1(K)$, $\psi_2(K)$ and $\psi_3(K)$ is

$$(-i\omega+\eta_2)\left\{(-i\omega+\eta_1)(-i\omega+\eta_3) - C_{13}C_{31}\right\} = 0 \tag{41}$$

That is either

$$(-i\omega+\eta_2) = 0 \quad \ldots\ldots\ldots \tag{42}$$

or $$(-i\omega+\eta_1(K))(-i\omega+\eta_3(K)) - C_{13}C_{31} = 0 \tag{43}$$

or both.

The first condition (42) is an artificial decoupled solution for the $(\psi_2(K) - \phi_4)$ system, with $(\psi_1(K) - \phi_2)$ and $(\psi_3(K) - \phi_6)$ having a trivial solution. This reduces to the trivial first order set we considered earlier. We will consider

(43) instead. It can be rewritten as

$$\omega^2 + i\omega(\eta_1 + \eta_3) - \left(C_{13}C_{31} - \eta_1(k)\,\eta_3(k)\right) = 0 \tag{44}$$

If we impose the conditions on the roots ω_1 and ω_2

$$\left.\begin{array}{ll} \mathrm{Im}\,\omega_1(k) \leq 0 & \text{Equality at } k = k_0 \\ \mathrm{Im}\,\omega_2(k) < 0 & \text{for all } k \end{array}\right\} \tag{45}$$

we get

$$\mathrm{Re}(\omega_1) = 0 \qquad \mathrm{Re}\,\omega_2 = 0 \quad \text{for all } k.$$

From equations (44) and (45) we get

$$\eta_1\eta_3 - C_{13}C_{31} \geq 0 \tag{46}$$

$$\left[\frac{d}{dk}(\eta_1\eta_3 - C_{13}C_{31})\right]_{k=k_0} = 0 \tag{47}$$

$$\left[\frac{d^2}{dk^2}(\eta_1\eta_3 - C_{13}C_{31})\right]_{k=k_0} > 0 \tag{48}$$

and

$$(\eta_1 + \eta_3) > 0 \tag{49}$$

If we assume that there exists only one k_0 satisfying these conditions, the mean field equations take a simpler form. We make this assumption - just to explore the possibility. The mean field equations become

$$\phi_2 = \frac{\lambda_0\pi}{2}\, X_0 \left[\frac{1}{(1+X_0)^2} - \left(\frac{1}{(9+X_0)^2} + \frac{1}{(1+X_0)^2}\right)\epsilon_0\right]\psi_1^2 \tag{50}$$

$$\phi_4 = \frac{\lambda_0\pi}{4}\, X_0 \left[\frac{1}{(1+X_0)^2} + \frac{1}{(9+X_0)^2}\right]\psi_1^2\epsilon_0 \tag{51}$$

$$\phi_6 = \frac{\lambda_0\pi}{6}\, X_0 \,\frac{1}{(9+X_0)^2}\,\psi_1^2\epsilon_0^2 \tag{52}$$

where $$\psi_3(x_0) = \psi_1(x_0)\,\epsilon(x_0) \tag{53}$$

and $$x_0 = k_0^2 \tag{54}$$

Incorporating the substitutions (40) and carrying through the elimination of ψ_1^2, we get from equations (46-52)

$$\frac{2-2a-c}{1-b} = \frac{z^2}{y^2}\,\frac{1}{\epsilon_0^2} \tag{55}$$

$$\frac{1-a-c}{1-b} = \frac{z^2+y^2}{y^2}\,\frac{1}{\epsilon_0} \tag{56}$$

$$\epsilon_0 = \frac{z^2}{y^2}\,\frac{y^3-x_0\lambda_0 a}{x_0\lambda_0 c} \tag{57}$$

$$= \frac{z^2}{y^2}\,\frac{x_0\lambda_0 c}{(z^3-x_0\lambda_0 b)} \tag{58}$$

$$\frac{2(5+x_0)z^3y^3}{\lambda_0^2} - \frac{1}{\lambda_0}\left[a(9-7x_0)z^3 + b(9+17x_0)y^3\right] - 2x_0(ab-c^2)(x_0^2-9) = 0 \tag{59}$$

where $y = 1+x_0 \;:\; z = 9+x_0$.

c is the term involving the non-diagonal coupling and equations (55-58) show that the solutions would bear a singular dependence on c as $c \longrightarrow 0$. In the case $c \equiv 0$, we get the decoupled solutions

$$a = \frac{y^3}{x_0\lambda_0} \qquad x_0 = \tfrac{1}{2} \qquad \lambda_0 > \frac{27}{4}$$

$$b = \frac{z^3}{x_0\lambda_0} \qquad x_0 = \tfrac{9}{2} \qquad \lambda_0 > \frac{27}{4}\,9$$

Already with only two sub-systems effectively coupled, the system shows interesting complexities. Investigation of this and more complex systems is presently continuing.

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Integral Relations in a Variable Density Medium

by

P. P. Niiler

a) Introduction

The attention of this investigation has been directed toward the study of some integral properties of a fully developed statistically steady, turbulent convection. The particular model under consideration is a layer of perfect gas, in a constant gravity field, between two mass-impervious boundaries, and heated from below and cooled from above. Moderate success has been attained in describing this model by statistical (integral) methods, in a system where simplifications have been made in the pertinent equations. Some particular integral relations of the simplified system have been useful in relating: a) the heat transport and kinetic energy balances; b) determining the "most stable" solution of the temperature field and heat transport from among many which would satisfy the statistically steady equations and boundary conditions; c) equating extremum heat transport integral properties of the system to entropy production rates. This study is an attempt to generalize these methods of analysis to a more general model. The question we want to answer is this:

Is the method of obtaining information (which was derived from integral relations) in the Boussinesq system (3) unique to that system or can it be also applied to the variable density model?

b) The Boussinesq Approximation

The most widely used approximation is that which was proposed by Boussinesq (4). The essence of this approximation is two-fold. The depth of the system is to be much smaller than the depth required for the change in a thermodynamic property of the fluid to be of the order of the mean value of that property in the system. The characteristic accelerations which occur in the system are much smaller than the accelerations which are due to gravity.

This formulation, however, is realistic only in terrestrial geophysics; i.e., natural convection in the ocean and in a very few models of the atmosphere. For most problems related to the atmosphere of celestial bodies the vertical gradients of the thermodynamic properties are very large, and, consequently, the Boussinesq approximation does not apply.

c) Outline of the investigation

This investigation will embark upon the following course: We will look at natural convection described by the Navier-Stokes equations for a perfect gas and formulate the system, in a statistical sense, in terms of quantities which can be equated to positive definite integrals. Our purpose will be to ascertain and/or catalogue:

i) the form of the total heat transport integral and the effect of compressibility on it.

ii) the form of some of the power integrals in as much of an analogous form to the Boussinesq system as possible.

iii) the feasibility of deriving a "relative stability" criterion similar to that derived in the Boussinesq system.

d) Formulation of the general problem

The Navier-Stokes equations for a perfect gas in a gravitational field are, (in the usual notation)

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\vec{v}\vec{v}\rho) + \nabla p = -g\rho\vec{k} + \mu\left[\nabla^2 \vec{v} + \tfrac{1}{3}\nabla(\nabla \cdot \vec{v})\right] \qquad (1).$$

$$C_v \frac{\partial \rho T}{\partial t} + C_v \nabla \cdot \rho T \vec{v} + p\,\nabla \cdot \vec{v} = K\nabla^2 T - \tfrac{2}{3}\mu(\nabla \cdot \vec{v})^2 + \frac{\mu}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]^2 \qquad (2)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \qquad (3)$$

$$p = \rho R T \qquad (4)$$

$$C_p - C_v = R \qquad (5)$$

A thermodynamic variable in this system will be separated into two components: one which is dependent upon the vertical direction only (the static variation), and one which depends on the motions of the system and, therefore, is a general function of space and time.

If f is a thermodynamic variable, then it is defined as

$$f = f_0(z) + f'(\vec{x}, t)$$

From the above equations it follows that

$$\frac{\partial p_0}{\partial z} = -g\rho_0 \qquad (6)$$

$$\nabla^2 T_0 = 0 \qquad (7).$$

If these static solutions are subtracted from the total equations and equations (2) and (3) are used, the equations will be in the form which pertains to the system which is in motion,

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \vec{v}\vec{v}\rho + \nabla p' = -g\rho'\vec{k} + \mu\left(\nabla^2 \vec{v} + \frac{1}{3}\nabla(\nabla \cdot \vec{v})\right) \tag{8}$$

$$C_v \frac{\partial \rho T}{\partial t} - C_p T_0 \frac{\partial \rho'}{\partial t} + C_p \nabla \rho T' \vec{v} - \rho' w g - \vec{v} \cdot \nabla p' + \left\{\frac{\partial T_0}{\partial z} + \frac{g}{C_p}\right\} C_p \rho w =$$

$$= k \nabla^2 T' + \frac{\mu}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]^2 - \frac{2}{3}\mu\left[\nabla \cdot \vec{v}\right]^2, \tag{9}$$

e) <u>The Heat Transport Integral</u>

If the system is heated from below and cooled from above, at supercritical Rayleigh No., the heat is transported by conduction and "convection" of energy from the heated to the cooled surface. The total heat transport can be expressed mathematically by means of an integration of equation (9).

If we average (9) in the horizontal (bar denotes the horizontal average) and integrate from zero to z, the heat transport H is,

$$H = K\beta + C_p \overline{\rho T w} - \int_0^z \overline{\rho' w g + \vec{v} \cdot \nabla p'}\, dz' + \int_0^z \mu\left[\frac{2}{3}(\nabla \cdot \vec{v})^2 - \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]^2\right] dz \tag{10}$$

where $\beta = -\frac{\partial \overline{T}}{\partial z}$ and $T' = \overline{T}(z) + T(\vec{x}, t)$

Furthermore, the vertical average of (10) gives the mean heat transport as,

$$H_m = K\beta_m + (C_p \rho T w)_m - \left\{(d-z)\left[(\rho' w g + \vec{v} \cdot \nabla p') - \mu\left(\frac{2}{3}(\nabla \cdot \vec{v}) - \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2\right)\right]\right\}_m \tag{11}$$

where m denotes the space average and d is the depth of the system.

There is an alternate method of constructing the pressure gradient dependence terms of (11) which will indicate more clearly the role played by it in the convective process.

Multiplication of (8) by $\vec{v}_0$ yields the kinetic energy equation,

$$\frac{1}{2}\frac{\partial}{\partial t}(\vec{v}\cdot\vec{v}\rho)+\nabla\cdot\left(\frac{\vec{v}\cdot\vec{v}}{2}\rho\vec{v}\right)=-\vec{v}\cdot\nabla p'-W\rho' g+\mu\left[\vec{v}\cdot\nabla^2\vec{v}+\frac{1}{3}\vec{v}\cdot\nabla(\nabla\cdot\vec{v})\right] \tag{12}$$

We note that the terms of this equation describe three distinct processes. The left-hand side denotes the rate at which kinetic energy is changing in a unit volume. The right-hand side describes the rate at which work is done by pressure and gravity forces and the rate at which energy of motion is being dissipated into heat. Taking the space average of (12) yields

$$-\left[g\rho' w\right]_m-(\vec{v}\cdot\nabla p')_m=\mu\left(\nabla\vec{v}\cdot\nabla\vec{v}+\frac{1}{3}(\nabla\cdot\vec{v})^2\right)_m, \tag{13}$$

and multiplication of (12) by z and integrating over space yields,

$$-\left[zg\rho' w\right]_m-(z\vec{v}\cdot\nabla p')_m=\mu\left(\nabla z\vec{v}\cdot\nabla\vec{v}+\frac{1}{3}(\nabla\cdot z\vec{v})\nabla\cdot\vec{v}\right)_m-\left(\frac{\vec{v}\cdot\vec{v}}{2}\rho w\right)_m \tag{14}$$

The space average of (9) yields,

$$\left[v\cdot\nabla p'+\rho' wg\right]_m=\frac{2}{3}\mu\left(\left[\nabla\cdot\vec{v}\right]^2\right)_m-\frac{1}{2}\left\{\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]^2\right\}_m \tag{15}$$

Substituting (13), (14), (15) into (11) yields the heat transport in the desired form:

$$H_m=K\beta_m+\left\{\left(C_pT\rho+\frac{\vec{v}\cdot\vec{v}}{2}\rho\right)w\right\}_m+\left\{z\,\Phi\right\}_m, \tag{16}$$

where Φ is the viscous dissipation contribution to the total energy equation and is positive in all cases. Its explicit form is,

$$\Phi = \mu \overline{\left[+ \vec{v} \cdot \nabla^2 \vec{v} + \frac{\vec{v}}{3} \cdot \nabla(\nabla \cdot \vec{v}) - \frac{2}{3}(\nabla \cdot v)^2 + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 \right]} \cdot$$

We have, indeed, formulated the general system in a positive, definite, integral form.* Equations (12), (13), (15), (16), describe the energy transfer cycle for a fully developed motion, but since the general system includes many modes of heat transport, e.g. by sound waves, dynamic pressure, transport of mass, it is not possible to relate the relative magnitude of these modes, nor compare their efficiencies of transport.

The information which we can glean from this formulation, as compared with the Boussinesq system is too general and vague. E. Spiegel has been successful in obtaining approximate solutions of the linear stability problem of a compressible fluid (polytropic atmosphere with small time dependent fluctuations of thermodynamic properties). The problem of choosing the preferred temperature gradient (cell shape) from among many which satisfy the equations as well as the boundary conditions in the infinitesimal motion (as well as finite amplitude motion), however, still remains to be solved. More definite information than has been derived so far is needed from the integral equations for the compressible model to enable one to parallel the analysis of the Boussinesq model:

*The second term of (16) is one which "usually" is positive, but the author cannot show this to be so for the general case.

A "relative stability" criterion for the general case similar to that of the Boussinesq system is too difficult (non-linear) for this author to solve, neither is it possible to relate entropy production rates to other positive definite integrals which have relatively simple interpretation. The latter derivation is of importance because minimum entropy production has been accepted by some as a preferred state of motion and has been shown, in the Boussinesq system, to be equivalent to the "most stable" solution for the temperature and velocity fields (3). For the above reasons, it is necessary to consider compressible models where certain approximations have been made.

f) <u>Variable density with small deviations from a local mean (1)</u>

If f is a thermodynamic variable, then it has a local mean f_0 and a fluctuating part f',

$$f = f_0(z) + f'(\vec{x}, t).$$

This approximation is,

$$\frac{|f'|}{f_0} \equiv \epsilon << 1$$

and $\left[\frac{J_f}{\rho_0} \; \partial \rho' / \partial t\right] \leq \epsilon$, where J_f is the characteristic time scale of the system.

Note that "sound waves", as high frequency variations of density, have again been omitted.

The pertinent equations (to $O(\epsilon)$) of motion are,

$$\rho_0 \frac{\partial \vec{v}}{\partial t} + \rho_0 \vec{v} \cdot \nabla \vec{v} + \nabla p' = -\rho' g \vec{k} + \mu\left(\nabla^2 \vec{v} + \frac{\nabla}{3}(\nabla \cdot \vec{v})\right) \tag{17}$$

$$\rho_0 C_v \frac{\partial T'}{\partial t} + C_v \rho_0 u \cdot \nabla T + p_0 \nabla \cdot v = k \nabla^2 T' \tag{18}$$

$$\nabla \cdot v = - \left[\frac{\partial}{\partial t} + u \cdot \nabla\right] \left[\ln \rho_0 + \frac{\rho'}{\rho_0}\right] \tag{19}$$

$$= - w/\rho_0 \, \partial \rho_0 / \partial z$$

$$\frac{p'}{P_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0} \tag{20}$$

$$P_0 = R \rho_0 T_0$$

Indeed, these equations are analogous to the Boussinesq system except there is no limitation on the variation of the local mean.

If we use (19) and (20) and take the horizontal average of (18) and integrate from 0 to z , the heat transport H is

$$H = K\beta_T + C_v \overline{\rho_0 W T} - \int_0^z \overline{v \cdot \nabla' p'} \, dz' \tag{21}$$

where $\beta_T = - \frac{\partial \overline{T}}{\partial z} - \frac{\partial T_0}{\partial z}$

The analogous power integrals of interest are,

$$\left[\vec{v} \cdot \nabla p' + g w \rho'\right]_m = - \left[\mu \nabla \vec{v} \cdot \nabla \vec{v} + \mu \tfrac{1}{3} (\nabla \cdot \vec{v})^2\right]_m \tag{22}$$

Substituting (19) and (20) into (22),

$$\left[g \frac{T'}{T_0} \rho_0 w\right]_m - \left[p' \frac{\partial T_0}{\partial z} \frac{w}{T_0}\right]_m = \mu \left[\nabla \vec{v} \cdot \nabla \vec{v} + \tfrac{1}{3} (\nabla \cdot \vec{v})^2\right]_m \tag{23}$$

The mean square fluctuation temperature integral now also has a manageable form of,

$$-\left[T p_0 \nabla \cdot \vec{v}\right] + \left[C_v \rho_0 \beta_T w T\right]_m = K\left[\nabla T \cdot \nabla T\right]_m \tag{24}$$

Similar difficulties arise in this system, as incurred in the general formulation of the problem, in the attempt to give relative importance to the effects of the additional terms introduced into equations (21) - (24). Therefore, a further stipulation will be made:

$$\frac{p'}{P_0} << \frac{\rho'}{\rho_0} \text{ OR } \frac{T'}{T_0}$$

at each point in the fluid, i.e. the function of the pressure field is only to aid in the release of potential energy and plays no part directly in heat transport. The equations of motion now are,

$$\rho_0 \frac{\partial \vec{v}}{\partial t} + \rho_0 \vec{v} \cdot \nabla \vec{v} + \nabla p' = + g \frac{T'}{T_0} \rho_0 \vec{k} + \mu\left(\nabla^2 \vec{v} + \frac{\nabla}{3}(\nabla \cdot \vec{v})\right) \tag{25}$$

$$\rho_0 C_p \frac{\partial T'}{\partial t} + C_p \rho_0 \vec{v} \cdot \nabla T' + C_p w \rho_0\left[\frac{\partial T_0}{\partial z} + g/C_p\right] = K \nabla^2 T' \tag{26}$$

$$\nabla \cdot \rho_0 \vec{v} = 0 \tag{27}$$

At this point a scale height will have to be introduced in order to derive a "relative stability" criterion similar to the one that was derived for the Boussinesq system.

If h_0 is the depth of the "total atmosphere" of density ρ_0 , above any particular point denoted by ρ_0 , then

$$h_0 = \frac{p_0}{g \rho_0}$$

(For earth's atmosphere, $h \sim 10$ km for $0 < d < 5$ km).

We will stipulate that the variation of the local mean log temperature is limited by:

$$\frac{1}{T_0}\frac{dT_0}{dz'} \sim \frac{d}{D h_0} \text{ OR, integrating } \Delta T_0 \sim \frac{dg}{R}$$

where $z' = z/d$.

The power integrals of equations (25) - (27) now become:

$$\left[g\frac{T}{T_0}\rho_0 w\right]_m = \mu\left[\nabla\vec{v}\cdot\nabla\vec{v} + \frac{1}{3}(\nabla\cdot\vec{v})^2\right]_m \tag{28}$$

$$\left[\rho_0 w T C_p\left(\beta_T - \frac{g}{C_p}\right)\right]_m = K\left[\nabla T\cdot\nabla T\right]_m \tag{29}$$

$$H = K\beta_T + C_p\,\overline{\rho_0 w T} \tag{30}$$

Note that the second term of (23) has been eliminated in (28) (it is of $O(\epsilon)$ as compared with the first),

$$\frac{\left[\frac{p'}{\rho_0}\frac{\partial T_0}{\partial z}\frac{w\rho_0}{T_0}\right]}{\left[\frac{T'}{T_0}g\rho_0 w\right]} \sim \epsilon$$

It is now possible to show that the "relative stability" criterion for this system is identical with the one derived for the Boussinesq system.

The power integrals (28) - (30) are very similar to the Boussinesq system except for some weighing factors, (ρ_0, T_0) which appear in certain terms - the time dependent variables, $\vec{v}$, T , appear in identical form (a few extra terms are added, but these introduce no complications) as in the Boussinesq system. If $\vec{v}'$, $\hat{T}$, are the perturbations on $\vec{v}$ and T', respectively, the power integrals for the mean square of the perturbation fields are:

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\rho_o \vec{v}'\cdot\vec{v}'\right]_m + \left[\rho_o \vec{v}'\cdot\vec{v}'\cdot\nabla\vec{v}\right]_m - \left[g\frac{\hat{T}}{T_o}\rho_o w\right]_m + \mu\left[\nabla\vec{v}'\cdot\nabla\vec{v}' + \frac{1}{3}(\nabla\cdot\vec{v}')^2\right]_m = 0 \quad (31)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\rho_o \hat{T}^2\right]_m + \left[\rho_o C_p \vec{v}'\cdot\hat{T}\nabla T\right]_m - \left[C_p \rho_o w'\hat{T}\left(\beta_T - \frac{g}{C_p}\right)\right]_m + K(\nabla\hat{T}\cdot\nabla\hat{T})_m = 0 \quad (32)$$

Equations (28) - (32) together with the stipulation that the initial time rate of change of the mean square of the perturbation velocity and temperature fields, which are identical to real solutions, be greater than zero, yield the identical "relative stability" criterion as was derived for the Boussinesq system.

We note that the Boussinesq system stipulated that, where T_m was the mean temperature of the system,

$$\frac{1}{T_m}\frac{dT_o}{dz'} \sim \frac{d}{D h_m}$$

or integrating,

$$\Delta T_o \sim \frac{dg}{R}.$$

This is an identical statement with the above in regard to the temperature variation. However, pressure p_o, and density ρ_o, can still have larger variations than have been imposed in the Boussinesq system. The static equation of state and (6) and (7) are the only limiting equations for ρ_o and p_o variations. The entropy production integral has a relatively simple interpretation as the ratio of the square of the fluctuation to the local mean temperature. Where (σ) is the internal entropy production,

$$(\sigma)_m \equiv K\left(\frac{\nabla T' \cdot \nabla T'}{T^2}\right)_m$$

$$= \frac{(K\beta^2 + K\,\overline{\nabla T \cdot \nabla T})_m}{T_0^2} + O(\epsilon)\ldots \qquad (33)$$

From (18)

$$K\,\overline{\nabla T \cdot \nabla T} = \overline{C_v \rho_0 w\, T \beta_T} - \frac{\partial}{\partial z}\overline{\frac{T^2}{2} C_v \rho_0 w} - \overline{T \rho_0 \nabla \cdot \vec{v}}$$

and substituting into (33), and using (21),

$$(\sigma)_m = H\left[\frac{\beta_T}{T_0^2}\right]_m - \left[\left(\frac{T}{T_0}\right)^2 \left\{\frac{C_v \rho_0 w}{T_0} \frac{\partial T_0}{\partial z}\right\} + \frac{T}{T_0^2} \rho_0\, \nabla \cdot \vec{v}\right]_m \qquad (34)$$

$$= H\left[\frac{\beta_T}{T_0^2}\right]_m - \left[\left(\frac{T}{T_0}\right)^2 \left\{\frac{C_v \rho_0 w}{T_0} \frac{\partial T_0}{\partial z}\right\} + \frac{T}{T_0} \rho_0\, R\, \overline{\nabla \cdot \vec{v}}\right]_m$$

If $\Delta T_0 \sim \frac{dg}{R}$ then

the entropy production has the familiar interpretation of the Boussinesq system,

$$(\sigma)_m \propto H \beta_m$$

g) Incompressible system with variable density and small fluctuations from the local mean.(2)

There is another model which could be investigated. It is one which stipulates an incompressible medium, but allows for "sound wave" propagations in the form of "rapid" variations of density. The equations pertaining to this system are:

$$\rho_0 \frac{\partial \vec{v}}{\partial t} + \rho_0 \vec{v} \cdot \nabla \vec{v} + \nabla \cdot p' = -g \rho' \vec{k} + \mu \nabla^2 \vec{v} \qquad (35)$$

$$C_v \rho_0 \frac{\partial T'}{\partial t} + C_v \rho_0\, \vec{v} \cdot \nabla T = K \nabla^2 T' \qquad (36)$$

$$\frac{\partial \rho'}{\partial t} + w \frac{\partial \rho_0}{\partial z} = 0 \qquad (37)$$

$$\nabla \cdot \vec{v} = 0 \qquad (38)$$

$$C_v = C_p \qquad (39)$$

This system is of interest because the heat transport integral can be written in a form which clearly points out the contribution of large density gradients in density stratified models of convective systems.

$$H_m = K\beta_{T_m} + \left[C_p \overline{T w \left[\rho_0 + z \frac{\partial \rho_0}{\partial z}\right]}\right]_m \qquad (40)$$

Since $\frac{\partial \rho_0}{\partial z} < 0$ in systems under consideration in "natural convection", the effect of large density gradients would be to reduce heat transport, (as compared with the Boussinesq system). The "relative stability" problem here is highly non-linear, and yields nothing interesting.

h) <u>Concluding Remarks</u>

The general system is, indeed, too general to yield pertinent information with the above methods. It has been possible, however, to apply integral methods as were employed in the Boussinesq system to a certain "compressible" (variable density) model of the convective system to derive information analogous to the Boussinesq system. The degree of "compressibility" which can be introduced with successful employment of the above integral methods of analysis, seems to be bounded by the stipulation that fluctuation pressure should be allowed only to aid in the release of potential energy; the pressure field, p', is restricted from drawing energy from the mean field. The temperature difference across the system must be of the same order as in a Boussinesq system.

The notable effect of variable density, i.e. negative density gradient of the local mean, ρ_0, is the reduction of heat transport (as compared with the Boussinesq system). This is apparent from equations (30) and (38). The effect of allowing pressure field, p', to draw energy from the mean field will also reduce heat transport, i.e. equation (21).

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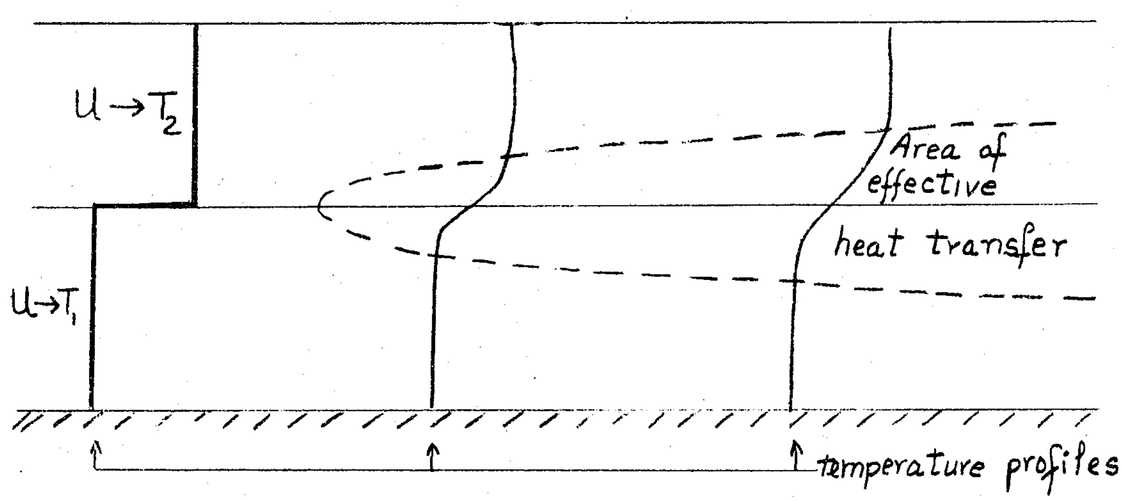
The Effect of Temperature Variations
on Uniform Flows of Water

by

D. Howell Peregrine
August, 1961

A Two-Layer Problem

A stream of water is initially travelling with uniform velocity in two layers, with the lower layer at temperature T, and the upper one at temperature T_2 such that the two layers are stable under gravity. The lower boundary of the flow is a horizontal plane. Its upper surface is free. The flow is considered to be two-dimensional and inviscid. After the flow passes a certain point heat is considered to be able to diffuse between the layers. We wish to know what effect this has on the velocities, and whether or not the non-linear variation of density with temperature makes any significant difference.

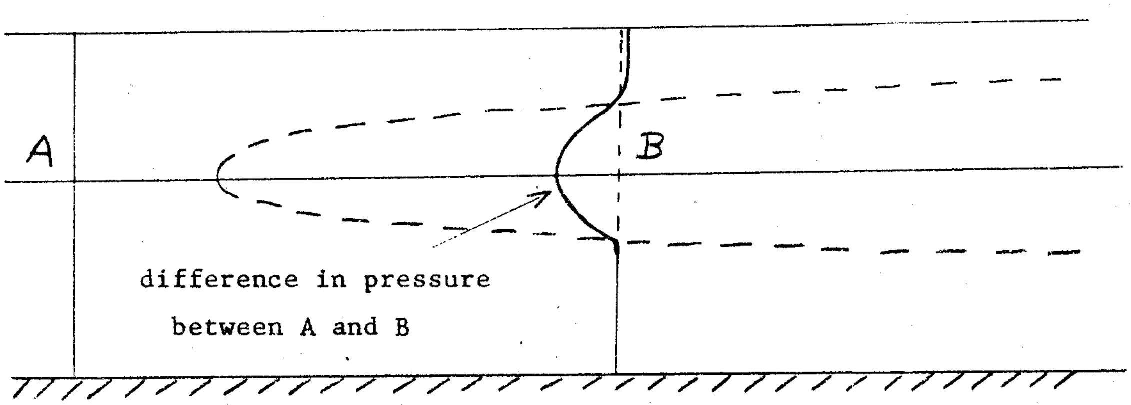


<u>Qualitative Argument</u>

When two masses of water at different temperatures are mixed there is a contraction in total volume because the specific volume of water does not depend linearly on temperature. The specific volume α can be expressed as

$$\alpha = \alpha_o(1 + aT + bT^2) = \frac{1}{\rho} . \qquad (1)$$

Thus as the heat diffuses we would expect the height of the free surface to decrease and the uppermost portion of the flow to accelerate. However, an adverse horizontal pressure gradient is created by the change in density distribution downstream.



This pressure gradient slows the central portions of the flow so that the velocity profile would look similar to the above curve for the pressure difference. This decrease in velocity implies a reduction in total flow, so the surface must be lowered to create a positive horizontal pressure gradient to keep a constant flow rate.

Approximate Mathematical Formulation of the Problem

Preliminary Definitions.

Define a mass coordinate m such that

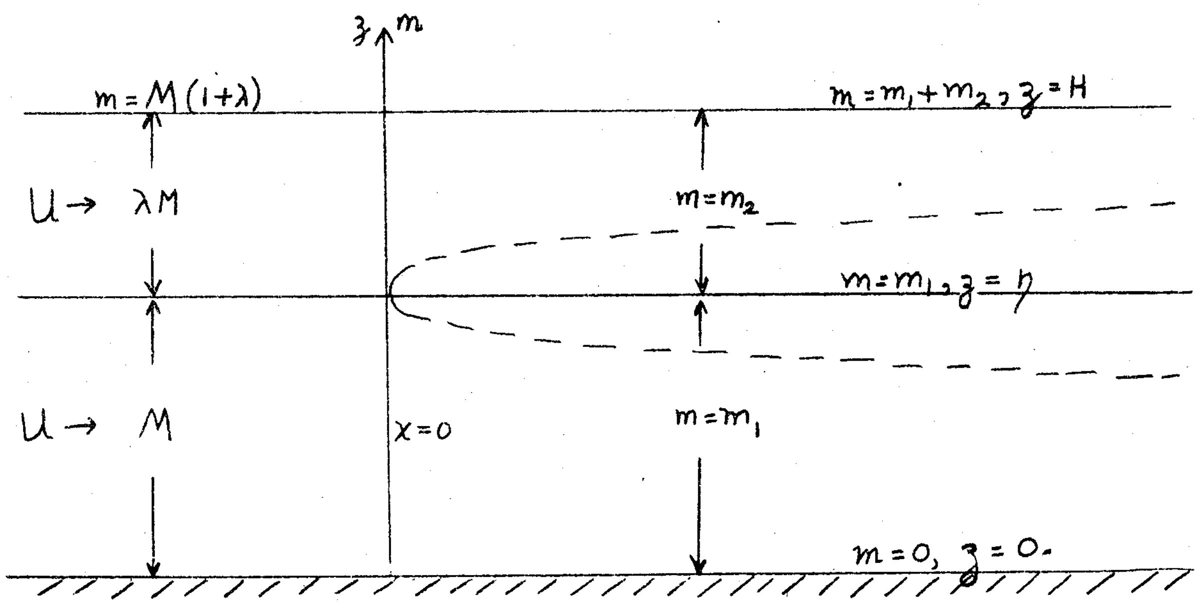
$$\rho\, dz = dm, \quad \text{or} \quad dz = \alpha\, dm \tag{2}$$

with $m = 0$ when $z = 0$.

m_1 and m_2 are the "masses" of the lower and upper layers respectively. Initially

$$m_1 = M, \qquad m_2 = \lambda M$$

and the velocity is U in both layers.



Temperature Distribution

Our first approximation is to specify the temperature distribution. In order to have a guide we consider the heat equation.

$$(\underline{u}\cdot\nabla)T = \kappa\nabla^2 T. \tag{3}$$

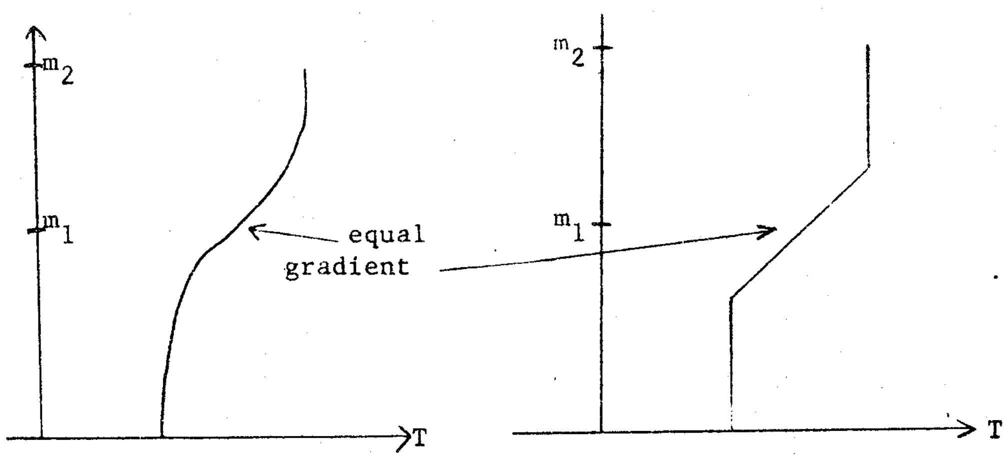
Since the initial velocity is U in both layers we simplify (3) to

$$U\frac{\partial T}{\partial x} = \kappa\frac{\partial^2 T}{\partial z^2} = \frac{\kappa}{\alpha^2}\frac{\partial^2 T}{\partial m^2}. \tag{4}$$

If we assume that the water extends to infinity in all directions the solution of (4) for $x > 0$ is

$$T = \frac{T_1+T_2}{2} + \frac{T_2-T_1}{2}\,\mathrm{erf}\left[\frac{(m-m_1)\alpha}{2}\sqrt{\frac{U}{\kappa x}}\right]. \tag{5}$$

We shall approximate this temperature distribution by straight line segments, thus



(5) approximation.

From (5)

$$\left(\frac{\partial T}{\partial m}\right)_{m=m_1} = \frac{(T_2-T_1)}{2}\,\alpha\sqrt{\frac{U}{\pi\kappa x}}, \tag{6}$$

and is the same in the approximation.

This specified temperature distribution is

$$T = T_1 \qquad \text{for} \quad 0 \leq m \leq m_1(1-\delta), \tag{7}$$

$$T = \frac{T_1 + T_2}{2} + \frac{(T_2 - T_1)}{2}\,\frac{(m - m_1)}{m_1 \delta}$$

$$\text{for} \quad m_1(1-\delta) \leq m \leq m_1(1+\delta), \tag{8}$$

and $$T = T_2 \qquad \text{for} \quad m_1(1+\delta) \leq m \leq m_2, \tag{9}$$

where $$\delta = \frac{1}{\alpha m_1}\sqrt{\frac{\pi K X}{u}} \simeq \frac{1}{\bar{\alpha} M}\sqrt{\frac{\pi K X}{u}}. \tag{10}$$

<u>Density Distribution</u>

We now simplify the expression of density variations. We shall use specific volume, and its variation as expressed in (1).

Define $$\alpha_1 = \alpha_0(1 + aT_1 + bT_1^2), \tag{11}$$

$$\alpha_2 = \alpha_0(1 + aT_2 + bT_2^2), \tag{12}$$

$$\bar{\alpha} = \tfrac{1}{2}(\alpha_1 + \alpha_2) \tag{13}$$

and $$A = \frac{1}{\bar{\alpha}}(\alpha_2 - \alpha_1), \tag{14}$$

so that $$\alpha_1 = \bar{\alpha}(1 - \tfrac{1}{2}A) \tag{15}$$

and $$\alpha_2 = \bar{\alpha}(1 + \tfrac{1}{2}A). \tag{16}$$

We now express α for the region $m_1(1-\delta) \leq m \leq m_1(1+\delta)$ as simply as possible.

Consider $\alpha - \bar{\alpha}\left[1 + \frac{A(m - m_1)}{2 m_1 \delta}\right]$. When the dependence on temperature (1) is substituted and (8) is used to eliminate T we find

$$\alpha - \bar{\alpha}\left[1 + \frac{A(m-m_1)}{2m_1\delta}\right] = \frac{\alpha_0 b(T_2-T_1)^2}{4}\left[\frac{(m-m_1)^2}{m_1^2\delta^2} - 1\right]. \tag{17}$$

Thus we have

$$\alpha = \bar{\alpha}\left(1 - \tfrac{1}{2}A\right) \text{ for } \quad 0 \leq m \leq m_1(1-\delta), \tag{18}$$

$$\alpha = \bar{\alpha}\left\{1 + \frac{A(m-m_1)}{2m_1\delta} + \frac{B}{4}\left[\frac{(m-m_1)^2}{m_1^2\delta^2} - 1\right]\right\}$$

$$\text{for } \quad m_1(1-\delta) \leq m \leq m_1(1+\delta), \tag{19}$$

and $\alpha = \bar{\alpha}\left(1 + \tfrac{1}{2}A\right)$ for $m_1(1+\delta) \leq m \leq m_2$, (20)

where $$A = \frac{\alpha_0}{\bar{\alpha}}(T_2 - T_1)\left[a + b(T_1 + T_2)\right] \tag{21}$$

and $$B = \frac{\alpha_0}{\bar{\alpha}} b(T_2 - T_1)^2. \tag{22}$$

It can be shown that

$$A \geq B \tag{23}$$

for all stable stratifications.

Numerical values

In order to show the order of magnitude of the various quantities introduced, their values will be calculated from the following numbers and any further values will be given from these numbers without comments.

$a = 6 \times 10^{-4}\ {}^\circ C^{-1}$, $b = 5 \times 10^{-6}\ {}^\circ C^{-2}$, $T_1 = 5^\circ C$

$T_2 = 15^\circ C$, $K = .1\ cm^2 sec^{-1}$, $g = 980\ cm\ sec^{-2}$;

$\bar{\alpha} = \alpha_0 = 1\ cm^3 gm^{-1}$, $U = 10\ cm\ sec^{-1}$,

$M = 10^5$ gm cm^{-2} (this is equivalent to a depth of 1 km).

This makes $A = 7\times10^{-3}$, $B = 5\times10^{-4}$

and $\delta = 3\times10^{-6}\sqrt{x}$ x in cm

or $\delta = 10^{-3}\sqrt{y}$ where $y = 10^5 x$ and therefore is measured in kilometers.

Equations of Motion

In the x-direction the equation for the conservation of momentum is

$$\rho u \frac{\partial u}{\partial x} + \rho w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = 0 \tag{24}$$

In the z-direction we shall use the hydrostatic pressure equation

$$\frac{\partial p}{\partial z} = -g\rho, \tag{25}$$

so that

$$p = \int_z^H g\rho\, dz = g\int_m^{m_1+m_2} dm$$

$$= g(m_1+m_2-m). \tag{26}$$

The equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{27}$$

We now non-dimensionalize as follows:

$$m' = \frac{m}{M},\ u' = \frac{u}{U},\ w' = \frac{w}{U},\ x' = \frac{x}{M\bar{\alpha}},\ z' = \frac{z}{M\bar{\alpha}},$$

$$p' = \frac{p}{Mg},\ \alpha' = \frac{1}{\rho'} = \frac{\alpha}{\bar{\alpha}} = \frac{1}{\rho\bar{\alpha}}. \tag{28}$$

Therefore, dropping dashes, the equations of motion become

$$F^2\left(\rho u \frac{\partial u}{\partial x} + \rho w \frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial x} = 0, \tag{29}$$

$$p = m_1 + m_2 - m, \tag{30}$$

and
$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho w) = 0, \tag{31}$$

where
$$F^2 = \frac{U^2}{g\bar{\alpha} M} = 10^{-6}. \tag{32}$$

If we use (31), (29) can be rearranged

$$\frac{\partial}{\partial x}\left(F^2 \rho u^2 + p\right) + F^2 \frac{\partial}{\partial z}(\rho u w) = 0. \tag{33}$$

We integrate (33) with respect to z from $z = 0$ to $z = H$, obtaining

$$\int_0^H \frac{\partial}{\partial x}\left(F^2 \rho u^2 + p\right) dz + F^2 \left[\rho u w\right]_0^H = 0 \tag{34}$$

Rearranging the first integral gives us

$$\frac{\partial}{\partial x}\int_0^H \left(F^2 \rho u^2 + p\right) dz - \left(F^2 \rho u^2 + p\right)_{z=H} \frac{\partial H}{\partial z} + F^2\left[\rho u w\right]_0^H = 0 \tag{35}$$

The boundary conditions are, $w = 0$ when $z = 0$, and

$$u \frac{\partial H}{\partial x} = w, \quad p = 0 \qquad \text{when} \quad z = H.$$

By using these boundary conditions in (35) we get

$$\frac{\partial}{\partial x}\int_0^H \left(F^2 \rho u^2 + p\right) dz = 0, \tag{36}$$

or
$$\int_0^{m_1+m_2} \left(F^2 u^2 + \alpha p\right) dm = S = \text{constant}. \tag{37}$$

(37) states that the momentum flow is constant.

By integrating (33) from $z = 0$ to $z = \eta$ we find

$$\frac{\partial}{\partial x}\int_0^{m_1}\left(F^2u^2+\alpha p\right)dm - p_2 = \eta\frac{\partial\eta}{\partial x} = 0 \tag{38}$$

If we substitute $p = m_1 + m_2 - m$ and $\eta = \int_0^{m_1}\alpha\, dm$ into (38), it simplifies to

$$\frac{\partial}{\partial x}\int_0^{m_1}\left(F^2u^2+\alpha(m_1-m)\right)dm + \eta\frac{\partial m_2}{\partial x} = 0 \tag{39}$$

Both the top and the bottom remain at their initial temperature as long as δ is less than the least of 1 and $\frac{m_2}{m_1}$, so we may apply Bernoulli's equation. At the bottom

$$\tfrac{1}{2}F^2u^2 + \alpha_1 p = R_1, \tag{40}$$

and at the top

$$\tfrac{1}{2}F^2u^2 + H = R_2. \tag{41}$$

The total flow in each layer is constant, therefore

$$\int_0^{m_1} u\, dm = Q_1 \tag{42}$$

and

$$\int_{m_1}^{m_1+m_2} u\, dm = Q_2. \tag{43}$$

We shall now put

$$u = u_1(x) + v_1(x,m) \tag{44}$$

in the lower layer, where u_1 is the velocity of the water which is still at its initial temperature. Similarly we put

$$u = u_2(x) + v_2(x,m) \tag{45}$$

in the upper layer. Thus

$$Q_1 = m_1 u_1 + m_1 \delta \bar{v}_1 \quad , \tag{46}$$

$$Q_2 = m_2 u_2 + m_1 \delta \bar{v}_2 \, , \tag{47}$$

$$\int_0^{m_1+m_2} u^2 dm = m_1 u_1^2 + m_2 u_2^2 + 2m_1 \delta u_1 \bar{v}_1 + 2m_1 \delta u_2 \bar{v}_2 + m_1 \delta \bar{v}_1^2 + m_1 \delta \bar{v}_2^2 \tag{48}$$

and $$\int_0^{m_1} u^2 dm = m_1 u_1^2 + 2m_1 \delta u_1 \bar{v}_1 + m_1 \delta \bar{v}_1^2 . \tag{49}$$

Other integrals are

$$\int_0^{m_1+m_2} \alpha p dm = \tfrac{1}{2}(m_1+m_2)^2 + \tfrac{1}{2}A\left(\tfrac{1}{2}m_2^2 - m_1 m_2 - m_1^2 + \tfrac{1}{3}m_1^2\delta^2\right) - \tfrac{1}{3}Bm_1 m_2 \delta . \tag{50}$$

$$\int_0^{m_1} \alpha (m_1 - m) dm = \tfrac{1}{2}m_1^2 - \tfrac{1}{4}Am_1^2\left(1 - \tfrac{1}{3}\delta^2\right) - \tfrac{1}{16}Bm_1^2\delta^2 \tag{51}$$

$$H = m_1 + m_2 + \tfrac{1}{2}A(m_2 - m_1) - \tfrac{1}{3}Bm_1\delta . \tag{52}$$

$$\eta = m_1 - \tfrac{1}{2}Am_1\left(1 - \tfrac{1}{2}\delta\right) - \tfrac{1}{6}Bm_1\delta . \tag{53}$$

If we use these integrals in (37) and (39), use (46) and (47) to eliminate $\bar{v}$ and put S, Q_1, Q_2, R_1 and R_2 equal to their initial values, we have the following four equations from which to find m_1, m_2, u_1 and u_2 with $\overline{v^2}$ also unknown.

$$F^2\left[2u_1 + 2\lambda u_2 - m_1 u_1^2 - m_2 u_2^2 + m_1\delta\left(\overline{v_1^2} + \overline{v_2^2}\right)\right] + \tfrac{1}{2}(m_1+m_2)^2 + \tfrac{1}{2}A\left(\tfrac{1}{2}m_2^2 - m_1 m_2 - m_1^2 + \tfrac{1}{3}m_1^2\delta^2\right) - \tfrac{1}{3}Bm_1 m_2\delta = \tfrac{1}{2}(1+\lambda)^2 + F^2(1+\lambda) + \tfrac{1}{2}A\left(\tfrac{1}{2}\lambda^2 - \lambda - 1\right). \tag{54}$$

$$\frac{\partial}{\partial x}\left\{F^2\left(2u_1 - m_1 u_1^2 + m_1\delta\overline{v_1^2}\right) + \tfrac{1}{2}m_1^2 - \tfrac{1}{4}Am_1^2\left(1 - \tfrac{1}{3}\delta^2\right) - \tfrac{1}{16}Bm_1^2\delta^2\right\} + \left[m_1 - \tfrac{1}{2}Am_1\left(1 - \tfrac{1}{2}\delta\right) - \tfrac{1}{6}Bm_1\delta\right]\frac{\partial m_2}{\partial x} = 0 \tag{55}$$

$$F^2 u_1^2 + (2-A)(m_1+m_2) = F^2 + (2-A)(1+\lambda). \tag{56}$$

$$F^2 u_2^2 + 2(m_1+m_2) + A(m_2-m_1) - \tfrac{2}{3} B m_1 \delta = F^2 + 2(1+\lambda) + A(\lambda-1) \tag{57}$$

Further Approximation

The approximations made so far are

(i) prescribing the temperature field,

(ii) using the hydrostatic pressure equation.

We now make further approximations. Let

$$m_1 = 1 + \mu_1, \qquad m_2 = \lambda + \mu_2. \tag{58}$$

We assume that μ_1, μ_2, F^2, A and B are all small quantities and neglect their squares and products, and also neglect $\overline{v^2}$. Equations (54), (55), (56) and (57) are now, (after integrating (55)),

$$F^2(1-u_1)^2 + F^2\lambda(1-u_2)^2 = (\mu_1+\mu_2)(1+\lambda) + \tfrac{1}{6}A\delta^2 - \tfrac{1}{3}B\lambda\delta, \tag{59}$$

$$F^2(1-u_1)^2 = \mu_1+\mu_2 + \tfrac{1}{12}A\delta^2 - \tfrac{1}{16}B\delta^2, \tag{60}$$

$$F^2(1-u_1^2) = 2(\mu_1+\mu_2), \tag{61}$$

and

$$F^2(1-u_2^2) = 2(\mu_1+\mu_2) - \tfrac{2}{3}B\delta. \tag{62}$$

These four equations have only three unknowns in them, u_1, u_2, and $(\mu_1+\mu_2)$. However, these equations are inconsistent, so there is a fault in our assumptions. If we consider F^2 to be of order 1 we do get a solution.

$$\mu_1 = \frac{B\lambda\delta}{3F^2(1+\lambda-F^2)} - \frac{B\delta^2}{16F^2} - \frac{A\delta^2(1-\lambda+F^2)}{12F^2(1+\lambda-F^2)}. \tag{63}$$

$$\mu_2 = \frac{B\lambda\delta(1-F^2)}{3F^2(1+\lambda-F^2)} + \frac{B\delta^2}{16F^2} + \frac{A\delta^2(1-\lambda+F^2)}{12F^2(1+\lambda-F^2)}. \tag{64}$$

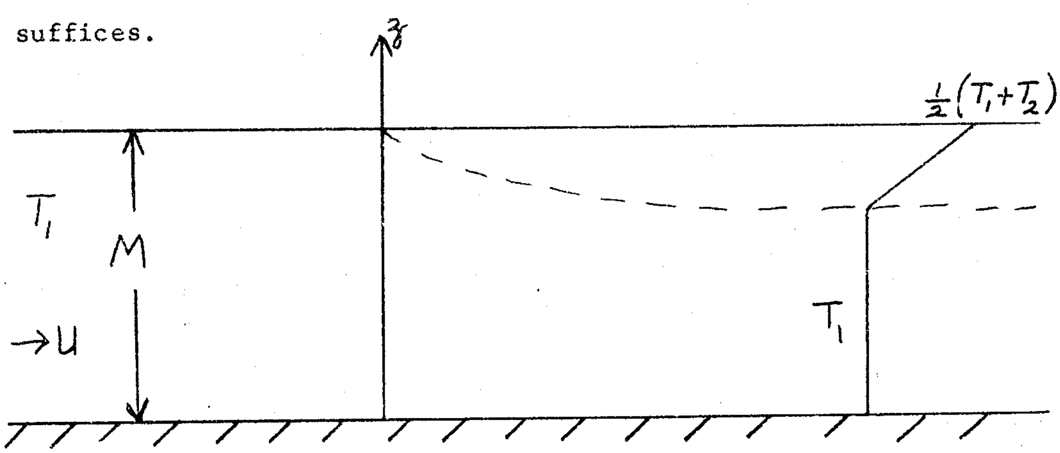
F^2 of order 1 is not of much interest in ocean currents. On the other hand if F^2 and δ are both very small

$$\mu_1 = \frac{2B\lambda\delta}{3A(1-2\lambda)}, \qquad \mu_2 = -\mu_1. \tag{65}$$

Neither of these solutions correspond to examples we are interested in. To try and find out why we have inconsistent equations in the intermediate case we turn to a simpler system.

One Layer Problem.

In this simpler problem, a flow of water is heated from above. The prescribed temperature and density distributions are the same as those for the lower layer in the previous problem. We shall use the same notation as before except for suffices.



The equations equivalent to (46), (40) and (37) are

$$Q = 1 = mu + m\delta\bar{v}. \tag{66}$$

$$R = \tfrac{1}{2}F^2 + 1 - \tfrac{1}{2}A = \tfrac{1}{2}F^2u^2 + (1-\tfrac{1}{2}A)_m \tag{67}$$

$$S = \frac{1}{2} + F^2 - \frac{1}{4}A = F^2(mu^2 - 2m\delta\bar{v} + m\delta\overline{v^2}) + m^2(\tfrac{1}{2} - \tfrac{1}{4}A + \tfrac{1}{4}Y), \qquad (68)$$

where, for ease of writing

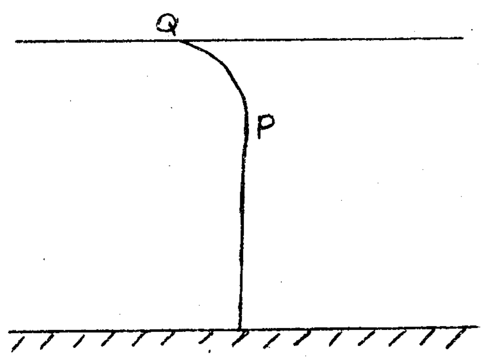
$$Y = \delta^2\left(\tfrac{1}{3}A - \tfrac{1}{4}B\right) = 2 \times 10^{-9} y. \qquad (69)$$

Since Y is the only term B enters into, it is clear that the non-linear expansion of water has no special effect in this problem.

$\overline{v^2}$ turns out not to be negligible, so we shall put

$$\overline{v^2} = r\,\bar{v}^2. \qquad (70)$$

r may be a function of x. A likely velocity profile has the form



and if the part PQ were parabolic, $\overline{v^2}$ would equal $\bar{v}^2$. It is possible, therefore, that $r \simeq 1$.

By using (66) and (70) we may eliminate $\bar{v}$ and $\overline{v^2}$ from (68). (67) may then be used to eliminate u and this gives us a sextic equation f(m) = 0.

$$
\begin{aligned}
& m^6\Big[\tfrac{25}{4}\delta^2 - 10\delta r + A r^2 + A\left(-\tfrac{25}{4}\delta^2 + 10\delta r - 4r^2\right) + \gamma\left(\tfrac{5}{4}\delta^2 - \delta r\right) \\
& \quad + A^2\left(\tfrac{25}{16}\delta^2 - \tfrac{5}{2}\delta r + r^2\right) + A\gamma\left(-\tfrac{5}{8}\delta^2 + \tfrac{1}{2}\delta r\right) + \tfrac{1}{16}\gamma^2\delta^2\Big] \\
& + m^5\Big[-10\delta^2 + 18\delta r - 8r^2 + F^2\left(-5\delta^2 + 9\delta r - 4r^2\right) + A\left(10\delta^2 - 18\delta r + 8r^2\right) + \gamma\left(-\delta^2 + \delta r\right) \\
& + F^2A\left(\tfrac{5}{2}\delta^2 - \tfrac{9}{2}\delta r + 2r^2\right) + F^2\gamma\left(-\tfrac{1}{2}\delta^2 + \tfrac{1}{2}\delta r\right) + A^2\left(-\tfrac{5}{2}\delta^2 + \tfrac{9}{2}\delta r - 2r^2\right) + \tfrac{1}{2}A\gamma\delta(\delta - r)\Big] \\
& + m^4\Big[\tfrac{3}{2}\delta^2 - 6\delta r + 4r^2 + F^2\left(-\delta^2 - 4\delta r + 4r^2\right) + A\left(-\tfrac{3}{2}\delta^2 + 6\delta r - 4r^2\right) - \tfrac{1}{4}\gamma\delta^2 \\
& + F^4\left(\delta^2 - 2\delta r + r^2\right) + F^2A\left(\tfrac{1}{2}\delta^2 + 2\delta r - 2r^2\right) - \tfrac{1}{2}F^2\gamma\delta^2 + A^2\left(\tfrac{3}{8}\delta^2 - \tfrac{3}{2}\delta r + r^2\right) + \tfrac{1}{8}A\gamma\delta^2\Big] \\
& + m^3\Big[2\delta^2 - \delta r + F^2\left(13\delta^2 - 16\delta r + 4r^2\right) + A\left(-2\delta^2 + 2\delta r\right) \\
& + F^4\left(2\delta - \delta r\right) + F^2A\left(-\tfrac{13}{2}\delta^2 + 8\delta r - 2r^2\right) + \tfrac{1}{2}F^2\gamma\delta r + A^2\left(\tfrac{1}{2}\delta^2 - \tfrac{1}{2}\delta r\right)\Big] \\
& m^2\Big[\tfrac{1}{4}\delta^2 + F^2\left(-7\delta^2 + 12\delta r - 4r^2\right) - \tfrac{1}{4}A\delta^2 \\
& + F^4\left(-3\delta^2 + 6\delta r - 2r^2\right) + F^2A\left(\tfrac{7}{2}\delta^2 - 6\delta r + 2r^2\right) + \tfrac{1}{16}A^2\delta^2\Big] \\
& + m\Big[-F^2\delta r - 2F^4\delta r + \tfrac{1}{2}F^2A\delta r\Big] + F^4r^2 = 0
\end{aligned}
\tag{71}
$$

We now consider F^2, A, γ to be small quantities. (71) has the following properties.

To the zero order in F^2, A, γ

$$f(1) = 0 \,, \tag{72}$$

$$f'(1) = 0 \,, \tag{73}$$

$$f''(1) = 2(3\delta - 2r)^2 \,. \tag{74}$$

To the first order in F^2, A, Y

$$f(1) = 0, \tag{75}$$

$$f'(1) = -4F^2\delta(\delta - r) + \tfrac{3}{2}Y\delta^2. \tag{76}$$

To the second order in F^2, A, Y

$$f(1) = -F^2 Y\delta(\delta - r) + \tfrac{1}{16}Y^2\delta^2. \tag{77}$$

This means that if we want a root of (71) near $m = 1$ and of the order F^2, A or Y we must have

$$f(1+\mu) = 0 = f(1) + \mu f'(1) + \tfrac{1}{2}\mu^2 f''(1). \tag{78}$$

In other words we have the quadratic in μ

$$\mu^2(3\delta - 2r)^2 - 2\mu\left[2F^2\delta(\delta - r) - \tfrac{3}{4}Y\delta^2\right] - F^2Y\delta(\delta - r) + \tfrac{1}{16}Y^2\delta^2 = 0 \tag{79}$$

Put

$$Z = F^2\delta(\delta - r) \tag{80}$$

for ease of writing. We have two roots of (79)

$$\mu = \frac{1}{(3\delta - 2r)^2}\left\{2Z - \tfrac{3}{4}Y\delta^2 \pm \left[4Z^2 + 2ZY(3\delta^2 - 6\delta r + 2r^2) + \tfrac{1}{4}Y^2\delta^3(3\delta - r)\right]^{1/2}\right\}. \tag{81}$$

In order to consider these two solutions more easily we will take their values as $\delta \rightarrow 0$. These values are

$$\mu = -\frac{Y}{4} + O(Y\delta) \tag{82}$$

and

$$\mu = \frac{Z}{r^2} + O(Y). \tag{83}$$

If we substitute for Y and Z

$$\mu = -\frac{\delta^2}{4}\left(\frac{A}{3} - \frac{B}{4}\right) \tag{84}$$

or

$$\mu = -\frac{F^2\delta}{r}. \tag{85}$$

For (84) we have

$$u = 1 + \frac{\delta^2}{4F^2}\left(\frac{A}{3} - \frac{B}{4}\right) \tag{86}$$

and

$$\bar{v} = -\frac{\delta}{4F^2}\left(\frac{A}{3} - \frac{B}{4}\right) \simeq -\frac{1}{2}\sqrt{y}\,. \tag{87}$$

(87) shows that the surface is brought to rest in a distance of the same order as the depth of water. From (85) we have

$$u = 1 + \frac{\delta}{r} \tag{88}$$

and

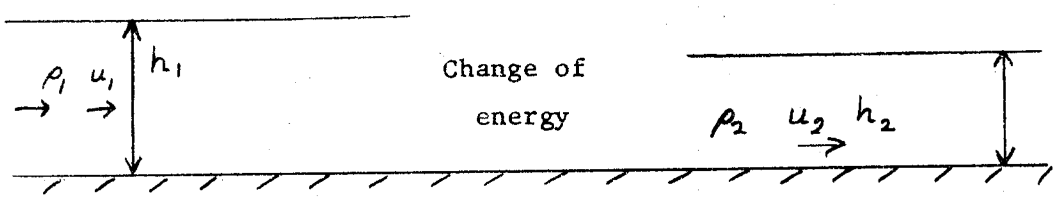
$$\bar{v} = -\frac{1}{r} \tag{89}$$

so that if $r \simeq 1$ the surface is brought to rest instantaneously.

To see the reason for two solutions we look at an even simpler problem.

Simplest One Layer Problem.

We consider a uniform flow which, after suffering a change of energy, becomes another uniform flow.



The rates of flow of mass and of momentum are constant so

$$\rho_1 u_1 h_1 = \rho_2 u_2 h_2 \tag{90}$$

and $\rho_1 u_1^2 h_1 + \frac{1}{2}\rho_1 g h_1^2 = \rho_2 u_2^2$ (91)

If we non-dimensionalize by putting

$$\rho = \frac{\rho_1}{\rho_2} \text{ etc. and let } \frac{u_1^2}{g h_1} = F_1^2 \tag{92}$$

we have

$$\rho u h = 1 \tag{93}$$

and $2F_1^2 + 1 = 2F_1^2 \rho u^2 h + \rho h^2$. (94)

If we eliminate u between (93) and (94)

$$\rho^2 h^3 - (2F_1^2+1)\rho h + 2F_1^2 = 0. \tag{95}$$

The kinetic energy of the flow is $\frac{1}{2}\rho u^2$, which, from (93) equals $\frac{1}{2\rho h^2}$. We put

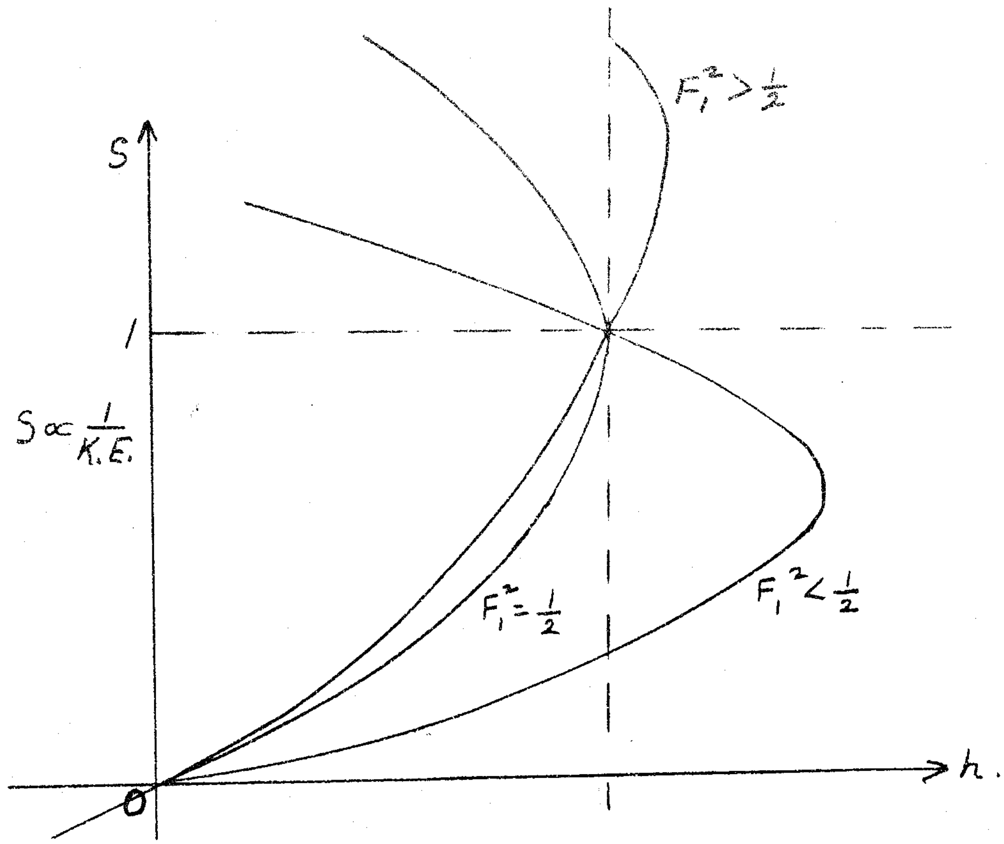
$$S = \rho h^2. \tag{96}$$

S is an inverse measure of kinetic energy. If we multiply (95) by h we have

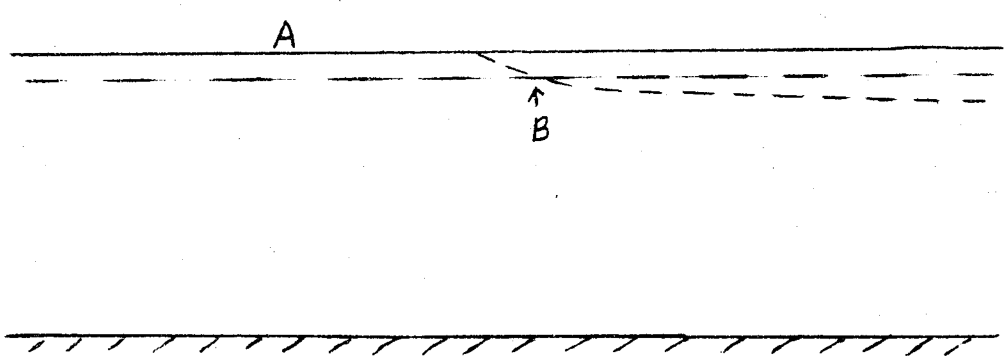
$$S^2 - (2F_1^2+1)S + 2F_1^2 h = 0 \tag{97}$$

By using (97) we can plot S against h for various values of F.

From these curves it is easy to see that for any depth there are in general two possible kinetic energies. This is similar to flow in an incompressible stream which can be supercritical or subcritical for any rate of flow and momentum. In this case there are two possible flows with the same depth, rate



of flow and momentum. We now look at the thin layer of water at the top in the previous problem.



Consider the top layer in the diagram, and its mean properties at A and B. Its thickness is $\alpha M \delta$ so that at A where its velocity is u

$$F_1^2 = \frac{u^2}{g\alpha M\delta} \simeq \frac{F^2}{\delta}. \tag{98}$$

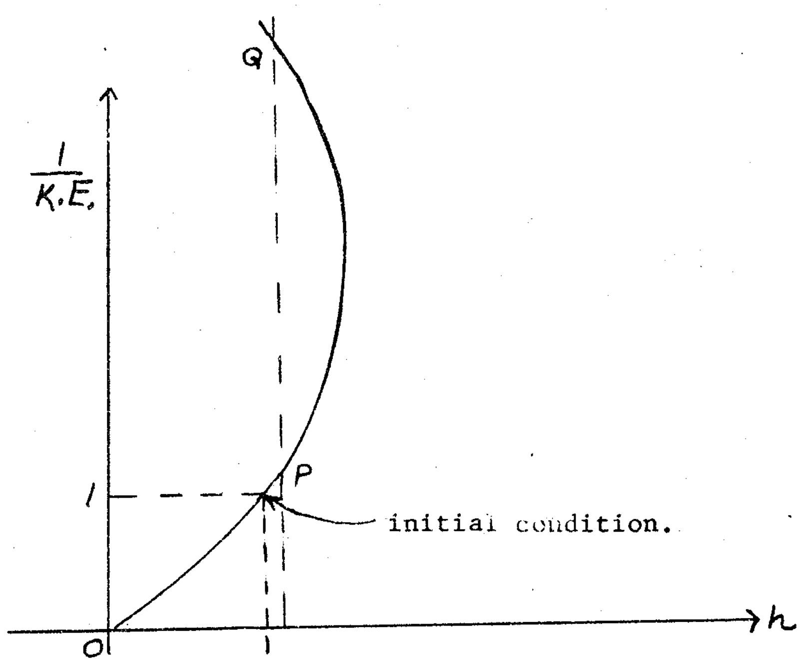
When $\delta \to 0$, so that we may use the approximation giving (84) to (89), F_1^2 becomes greater than ½. At B the total velocity is, in the first case from (86) and (87)

$$u\left[1 - \frac{\delta}{4F^2}\left(\frac{A}{3} - \frac{B}{4}\right)\right], \tag{99}$$

and in the second case from (88) and (89),

$$u\left[1 - \frac{1}{r} + \frac{\delta}{r}\right]. \tag{100}$$

The thickness of the layer increases slightly from A to B so that a plot of S against h gives



From this curve it can easily be seen that P corresponds to (99) and Q to (100). (99) involves a small reduction in kinetic energy, (100) needs a large change.

Conclusions

The one layer model shows reasons why a simple approximation does not give a solution to the two layer problem. There is not a unique solution. One solution requires a much greater energy change than the other but this does not necessarily rule it out; it does make it seem less likely. Both solutions give large changes in the surface velocity. This suggests that the most severe approximation used is that of prescribing the density distribution.

Acknowledgements

I wish to thank Dr. Nicholas P. Foffonoff for suggesting this problem and for helpful discussions. I am most grateful for the generosity of the Fellowship Committee in making my visit to the Woods Hole Oceanographic Institution possible.

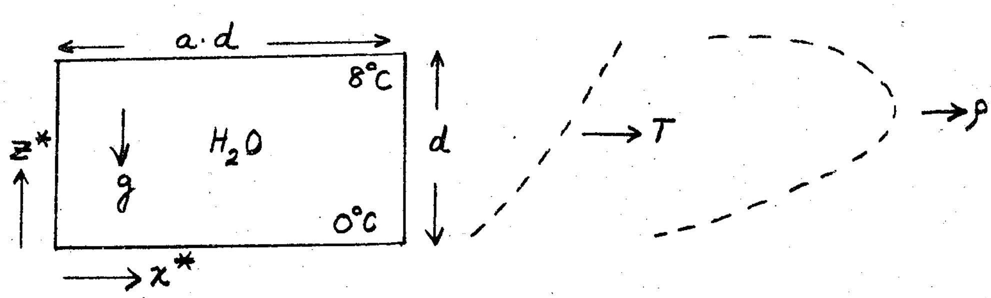
PENETRATIVE CONVECTION

by

S. Rosencrans

This report deals with several aspects of the problem of "symmetric" penetrative convection - the sense of "symmetric" will be explained below. Three separate topics are dealt with: the WKB solution to the stability problem, an unsuccessful approach to the finite amplitude problem using generalized power integrals, and a Fourier series solution to the finite amplitude problem with rigid boundaries.

I. <u>The Problem</u> The system to be considered here is a two dimensional rectangular body of water under the influence of gravity, with the temperatures on the lower and upper boundaries fixed at $0^{o}C$ and $8^{o}C$ respectively.



Thus, before motion has begun, there is an imposed linear temperature distribution in $T = \beta z^*$, as shown with $\beta d = 8^{o}C$. Now the equation of state of water in this temperature range is very nearly

$$\rho = \rho_0\left(1 - \alpha_1 (T - T_0)^2\right)$$

where ρ_0 and α_1 are constants, $T_0 = 4^\circ C$, and T is the temperature in degrees Centigrade. Thus the imposed initial density distribution is parabolic, as shown.

In fact the departure from this law is very small. Fitting a cubic to the tabulated function $\rho(T)$ at $T = 0^\circ$, 3°, 5°, 8° gives

$\rho = 1 - 10^{-7}\left\{-.84(T-T_0)^3 + 80(T-T_0)^2 + 2.34(T-T_0) - .5\right\}$ in cgs units,

and clearly the quadratic term is dominant.

If we define θ^* by

$$T =: \beta z^* + \theta^* ,$$

and define p to be the deviation of the pressure from the hydrostatic pressure, then, according to the Boussinesq approximation, the equation of motion is

$$\frac{\partial \underline{u}}{\partial t^*} + \underline{u}^* \cdot \nabla \underline{u}^* = 2\alpha_1 g\, \theta^* (\beta z^* - T_0)\underline{k} - \nabla p + \nu \Delta \underline{u}^* + \alpha_1 \theta^{*2} g$$

where $\underline{u}^* := (u^*, w^*)$ is the velocity field, $\underline{k} = (0,0,1)$, ν is the kinematic viscosity of water, and g is the acceleration due to gravity.

The heat equation is

$$\frac{\partial \theta^*}{\partial t^*} + \underline{u}^* \cdot \nabla \theta^* = \chi \Delta \theta^* - w^* \beta$$

where κ is the coefficient of thermal diffusivity of water.

We now non-dimensionalize all quantities according to the scheme

$\theta^* = \theta\beta d \qquad t^* = t \cdot \frac{d^2}{\kappa} \qquad \underline{u}^* = \underline{u} \cdot \frac{\kappa}{d} \qquad x^* = x d \qquad z^* = z d$

and define $\sigma := \frac{\kappa}{\nu}$ and $\lambda := \frac{\alpha_1 g d^5 \beta^2}{\nu\kappa}$

Then, after cross-differentiating the momentum equation to eliminate the pressure, we arrive at the system of equations, for $\frac{\partial}{\partial t} = 0$:

$$\left.\begin{aligned} \Delta^2\psi - \lambda\frac{\partial}{\partial x}\left(\zeta\theta + \theta^2\right) &= \sigma \frac{\partial(\Delta\psi, \psi)}{\partial(x,z)} \\ \Delta\theta + \frac{\partial\psi}{\partial x} &= \frac{\partial(\theta,\psi)}{\partial(x,z)} \end{aligned}\right\} \quad \begin{aligned} 0 \le x \le a \\ 0 \le z \le 1 \end{aligned}$$

where a stream function ψ has been introduced

$\left(u = \frac{\partial\psi}{\partial z} \text{ and } w = -\frac{\partial\psi}{\partial x}\right)$ and $\zeta := 2z - 1.$

The boundary conditions to be considered here are: free boundaries at $x = 0, a$ and either rigid boundaries at $z = 0,1$ or free boundaries at $z = 0,1$. The boundaries at $x = 0, a$ are free because we have in mind really an infinite fluid in x, whose motion will be described by alternately duplicating and anti-duplicating the solutions we derive for $0 \le x \le a$. This is the familiar cellular regime of "rolls".

Thus at $x = 0$ and $x = a$, $\psi = \psi_{xx} = \psi_{xxxx} = \theta_x = \theta_{xxx} = 0$. For free boundaries in z, $\psi = \psi_{zz} = \psi_{zzzz} = \theta = \theta_{zz} = \theta_{zzzz} = 0$ at $z = 0$ and $z = 1$. For rigid boundaries in z, $\psi = \psi_z = \psi_{zzzz} + 2\psi_{xxzz} = \theta = \theta_{zz} = \theta_{zzz} + \theta_{xxz} = 0$ at $z = 0$ and $z = 1$.

Because of the imposed parabolic density distribution the fluid is divided into two sub-regions: the layer $0 \leq z \leq \frac{1}{2}$, is gravitationally unstable, whereas the layer $\frac{1}{2} \leq z \leq 1$, is gravitationally stable. The convection to occur is referred to as "penetrative" because the fluid in the unstable layer will penetrate into that in the stable layer. This is "symmetric" penetration in that the two layers are of equal thickness. The problem of non-symmetric convection is much more difficult, and only the stability problem associated with this has been studied (by others - e.g., Chandrasekhar).

II. <u>The WKB Solution to the Stability Problem</u> The stability problem associated with the symmetric water problem has received a great deal of attention, partly because it happens to be the same as the stability problem for flow between oppositely rotating cylinders, for some values of that problem's parameters.

For the case of free boundaries in z, the principle of exchange of stabilities has been proven to hold, hence, we will set $\frac{\partial}{\partial t} = 0$ in the equations. The method of solution discussed in this section does not depend strongly on the type of boundary conditions, and we will allow for the possibility of rigid boundaries in Z, thus assuming that the exchange of stabilities holds in that case also.

The stability equation is

$$\Delta^3 \theta + \lambda z \theta_{xx} = 0$$

We seek a solution of the form $\theta = \cos \pi\alpha x\, \tilde{\theta}(z)$ where $\alpha := \bar{a}'$. Alternatively, one may think of θ as expanded in a cosine series in x, with coefficients functions of z. Then the coefficient $\tilde{\theta}_n(Z)$ of $\cos n\pi\alpha x$ satisfies

$$(D^2 - n^2\pi^2\alpha^2)^3\tilde{\theta}_n - n^2\pi^2\alpha^2\lambda z\,\tilde{\theta}_n = 0, \quad \text{where } D^2 := \frac{d^2}{dz^2}.$$

For each n we will be concerned with the eigenfunction corresponding to the minimum eigenvalue. But the "minimum minimorum" is achieved only with n = 1, hence we consider only

$$(D^2 - \pi^2\alpha^2)^3\tilde{\theta} - \pi^2\alpha^2\lambda z\,\tilde{\theta} = 0.$$

Perhaps the simplest method of solution, no matter what the boundary conditions, is that using Fourier analysis. This will be presented later, as part of the finite amplitude investigation. However certain difficulties occur in this investigation because the solution to the stability problem is not in closed form. These difficulties can be overcome by using an ingenious device of Dr. G. Veronis, but still, a solution to the stability problem in closed form would be a great advantage. For this reason, and because of its mathematical interest, we turn to the WKB method.

The WKB method provides an asymptotic series for the solution to this equation in inverse powers of $\lambda^{1/6}$, valid as $\lambda \rightarrow \infty$. We are, of course, interested in a fixed value of λ, in fact the minimum value that λ attains. The hope is that this minimum value is large enough so that the first term of the

WKB solution is rather a good approximation, when $\lambda = \lambda_c$, to the exact solution. Then the WKB method will have provided closed-form approximate eigenfunctions and an estimate for λ_c.

To carry out the method it is first necessary to cast the equation in a sort of "canonical" WKB form. To do so, define k and ξ by

$$\pi^2\alpha^2\lambda =: \xi^6 \quad \text{and} \quad \pi^2\alpha^2 a = k^2\xi^2$$

Then

$$(D^2 - k^2\xi^2)^3\tilde{\theta} - \xi^6 \cdot z\,\tilde{\theta} = 0.$$

According to the WKB method solutions of this equation have asymptotic expansions, if k is considered a fixed constant, of the form

$$\tilde{\theta} \sim e^{\xi\beta(z)}\left\{a_0(z) + \frac{a_1(z)}{\xi} + \frac{a_2(z)}{\xi^2} + \cdots\right\}$$

valid as $\xi \to \infty$. Further, the differential equations satisfied by $\beta(z)$ and $a_1(z)$ can be determined by inserting for $\tilde{\theta}$ the series to which it is supposed asymptotic, cancelling $e^{\xi\beta(z)}$, and equating to zero coefficients of ξ^S for S = 6, 5, 4, ..., 1, 0, -1, -2, ... We are concerned with only a_0 and β.

A rather lengthy computation shows that

$$\begin{cases} a_0(z) = z^{-1/3}(k^2+z^{1/3})^{-1/4} \\ \beta'^2 = k^2 + z^{1/3} \end{cases} \tag{1}$$

hence

$$\beta(z) = \tfrac{3}{7}(k^2+z^{1/3})^{7/2} - \tfrac{6}{5}k^2(k^2+z^{1/3})^{5/2} + k^4(k^2+z^{1/3})^{3/2}$$

Six independent "solutions" to the equation are obtained by choosing simultaneously a branch of $z^{1/3}$ and a branch of the square root in $(k^2 + z^{1/3})^{\frac{1}{2}}$.

Note that these representations of $\tilde{\theta}$ have branch points at $z = 0$ and $z = -k^6$, and a_0 in fact blows up at these points in accord with the usual results of the WKB method. Note both occur in the range of the problem, $-1 \leq z \leq 1$. In the terminology of quantum mechanics, there are "turning points" at $z = 0$ and $z = -k^6$. These turning points have an unusual and disastrous character. In the first place, since k is very small, they are close together, and this is a most dreaded circumstance because it makes the matching process, to be described below, very delicate.

Secondly, the cube-root branch point is in a sense <u>interior</u> to the square-root branch point, and this apparently never happens in the usual problems to which the method is applied. It is <u>not</u> unusual to have two turning points in a problem; however, they usually occur due to β's taking the form $(f(z))^{\frac{1}{2}}$ where f vanishes at two points in the range of z, but is analytic in z itself. In our case f vanishes at one point but f itself has a branch point at $z = 0$ (i.e., at $z = \frac{1}{2}$).

In order to make β and a_0 analytic in the complex variable z we will from now on work in the z-plane cut along the upper edge of the negative real axis. Thus if C is a complex number we will require $-\pi \leq \arg C < \pi$. This determines completely the branches of $(\ )^{1/3}$ and $(\ )^{1/2}$. The general analytic function "$\mathfrak{z}^{1/3}$" will be denoted $\hat{\mathfrak{z}}^{1/3}$.

Having done this we now define for $\mathfrak{z}$ positive $\beta_i(z)$ for $i = 1,2,3$ to be β with $\hat{\mathfrak{z}}^{1/3} = \omega_i\, \mathfrak{z}^{1/3}$ where the cube root on the right-hand side is that determined in the cut plane, and $\omega_1 = e^{i \cdot \frac{2\pi}{3}}$, $\omega_2 = e^{-i\frac{2\pi}{3}}$, $\omega_3 = 1$. Similarly for $\mathfrak{z}$ negative, define $\beta_i(z)$ by "$\mathfrak{z}^{1/3}$" $= \delta_i |\mathfrak{z}|^{1/3}$ where $\delta_1 = e^{-i\pi/3}$, $\delta_2 = e^{i\pi/3}$, $\delta_3 = e^{i\pi}$.

Then we have WKB solutions in three distinct regions: $-1 \leq \mathfrak{z} < -k^6$, $-k^6 < \mathfrak{z} < 0$, $0 < \mathfrak{z} \leq 1$. Denoting these regions R_3, R_2, R_1, we have:

$$\left.\begin{aligned}
\tilde{\theta} &\sim \sum_{i=1}^{3} E_i^{\pm} \cdot \mathfrak{z}^{-1/3}\left(k^2 + \omega_i \mathfrak{z}^{1/3}\right)^{-1/4} \exp\left(\pm \xi \beta_i(z)\right) \text{ in } R_1 \\
\tilde{\theta} &\sim \sum_{i=1}^{3} G_i^{\pm} |\mathfrak{z}|^{-1/3}\left(k^2 + \delta_i |\mathfrak{z}|^{1/3}\right)^{-1/4} \exp\left(\pm \xi \beta_i(z)\right) \text{ in } R_2 \\
\tilde{\theta} &\sim \sum_{i=1}^{3} F_i^{\pm} |\mathfrak{z}|^{-1/3}\left(k^2 + \delta_i |\mathfrak{z}|^{1/3}\right)^{-1/4} \exp\left(\pm \xi \beta_i(z)\right) \text{ in } R_3
\end{aligned}\right\} \qquad (2)$$

Sums are also over plus and minus signs.

It may be noted that the solution in R_2 is expected to be nowhere valid in R_2, wedged as closely as it is between two turning points. Similarly $\tilde{\theta}(R_1)$ and $\theta(R_2)$ are expected to represent $\tilde{\theta}$ only near $\mathfrak{z} = +1$ ($z = 1$) and $\mathfrak{z} = -1$ ($z = 0$) respectively. That the solutions are valid near these boundaries is fortunate since boundary conditions will be applied there. The usual WKB bag of tricks for most second order eigenvalue problems contains instructions for forming out of Airy functions and $\tilde{\theta}(R_1)$ and $\tilde{\theta}(R_3)$ a uniformly valid WKB solution, and since we shall shortly reduce this problem to a trio of second order problems, something of this kind might be expected. However, the second order problems are exceptional cases which will not have Airy-like solutions in the neighborhood of their turning points - a fact which makes the construction of uniformly valid solutions much more difficult; thus no attack has been made on uniformly valid solutions in this report.

The problem that we must solve next is the matching problem. That is, to determine each $G_i^{\pm}$ as a linear combination of all the $E_i^{\pm}$, and to find each $F_i^{\pm}$ as a linear combination of all the $E_i^{\pm}$. In this way the WKB solution we have found will have only six "arbitrary" constants and our boundary conditions will be applied to determine the constants (with, say, $E_1^{\pm} = 1$) and the eigenvalue spectrum $\{\lambda\}$. Now it is a fact, told to me by Dr. E. A. Spiegel, that these linear combinations mentioned above take a form (for example) like:

$$\begin{Bmatrix} F_1^+ \\ F_2^+ \\ F_3^+ \\ F_1^- \\ F_2^- \\ F_3^- \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ * & 0 & * & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & * \end{bmatrix} \begin{Bmatrix} E_1^+ \\ E_2^+ \\ E_3^+ \\ E_1^- \\ E_2^- \\ E_3^- \end{Bmatrix}$$

where * represents some possibly non-zero constant. This matrix is not intended to represent the actual matrix for this problem - we mean only that the form of the matrix must be the same in the following sense: each row has only two stars and four zeros, and, there are exactly two rows with the same star pattern, and, each column has only two stars.

One can express this more succintly by saying (as Dr. Spiegel did) that the six "waves" in each of the regions R_1, R_2, R_3, match up not only "six to six", but also "two to two". Thus there are two waves in R_1, which evolve to two waves in R_2, which in turn evolve to two waves in R_3 - and there are three such wave pairs in R_1, etc. This suggests the search for a trio of second order differential equations having WKB solutions given by (1), with a different choice of $\hat{z}^{1/3}$ for each equation. The equations are trivially found by finding the WKB solution of $\frac{d^2 v}{dz^2} + f(z)\frac{dv}{dz} + \xi^2 g(z) v = 0$, setting it equal to that given by (1) and solving for f and g.

The result is,

$$\frac{d^2 v}{dz^2} + \frac{2}{3z}\frac{d\theta}{dz} - \frac{1}{4}\xi^2\left(k^2 + \hat{z}^{1/3}\right)\theta = 0$$

There are three equations, one for each choice of $\hat{z}^{1/3}$.

If we make the substitution $\zeta = \hat{z}^{1/3}$ (hence $\zeta = \omega_i y$ for i = 1,2, or 3 and y is real), then the equation for $v(\zeta)$ is

$$\frac{d^2 v}{d\zeta^2} - \frac{9}{4}\xi^2 \cdot \zeta^4\left(k^2 + \zeta\right) = 0 \qquad (3)$$

Unfortunately this equation is apparently not solvable in terms of tabulated functions. However its solution is certainly analytic in the parameter k^2 and the coefficients of k^{2n} easily determined. Now the functions on the right-hand side of (2) are also analytic in k^2 for $z \neq 0$. Hence a procedure that might be tried is to use the coefficient $v_{2n}(\zeta)$ of k^{2n} in the expansion of the solution $v(\zeta;k)$ of (3) to match the corresponding component in the expansions of equations (2). This complete procedure has not been carried out. It has been carried out only for the coefficients of $k^0(=1)$, obtained by setting $k = 0$ in (2) and (3). Note that letting $k \to 0$ collapses R_2 into one point ($z = 0$) and thus coalesces the branch points at 0 and at $-k^6$. We thus obtain a matching matrix between $E_i^{\pm}$ and $F_i^{\pm}$ that would be the true matching matrix if k were zero. A crucial step is then taken: we assume that this matching matrix is very close to the true matrix and in fact use it in applying boundary conditions, etc.

The matrix, which will be derived in an appendix, is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

Thus the waves not only match "two by two", but also "one by one".

This is very surprising, and the simplicity of the entries also makes one doubt the correctness of this procedure.

Note that this method gives no information about the $G_i^{\pm}$. We have only the WKB solution near the boundaries in z. The next step is to apply the boundary conditions. The determinant arising from this, which provides λ_c, has not yet been evaluated. It is displayed in an appendix.

II. Finite Amplitude Investigation

The equations are

$$\left.\begin{aligned} \Delta^2\psi - \lambda z\,\theta_x - \sigma\frac{\partial(\Delta\psi,\psi)}{\partial(x,z)} &= 2\lambda\theta\theta_x \\ \Delta\theta + \psi_x &= \frac{\partial(\theta,\psi)}{\partial(x,z)} \end{aligned}\right\} \qquad (4)$$

These equations are to be solved by Fourier analysis of θ and ψ over the fundamental region $0 \leq x \leq a$, $0 \leq z \leq 1$. It is convenient to extend θ_x and ψ oddly into the range $-a \leq x \leq 0$, $-1 \leq z \leq 0$. Thus

$$\begin{aligned} \psi &\sim \sum{}^* \hat{\psi}_{kj} \sin k\pi\alpha x \sin j\pi z \\ \theta &\sim \sum{}^* \hat{\theta}_{kj} \cos k\pi\alpha x \sin j\pi z \end{aligned} \qquad (5)$$

where by $\sum^*$ is meant the sum over all non-negative integers. We will reserve $\sum$ for sum over <u>all</u> integers. The alternative representations

$$\begin{aligned}\psi &\sim \sum \psi_{\kappa j}\, e^{i\pi(k\alpha x + j\pi z)} \\ \theta &\sim \sum i\,\theta_{\kappa j}\, e^{i\pi(k\alpha x + j\pi z)}\end{aligned} \tag{6}$$

will also be used. The numbers $\psi_{\kappa j}$, $\theta_{\kappa j}$ are defined in terms of $\hat{\psi}_{\kappa j}$ and $\hat{\theta}_{\kappa j}$ by the following equations for κ, j both non-negative.

$$\begin{aligned}\psi_{\kappa j} &= \psi_{-\kappa,-j} = -\psi_{-\kappa j} = -\psi_{\kappa,-j} = -\tfrac{1}{4}\,\hat{\psi}_{\kappa j} \\ \theta_{\kappa j} &= \theta_{-\kappa,j} = -\theta_{\kappa,-j} = -\theta_{-\kappa,-j} = -\tfrac{1}{4}\,\hat{\theta}_{\kappa j}\end{aligned} \tag{7}$$

Because we are dealing with rigid boundaries in z, the series (5) and (6) for ψ do not provide the Fourier cosine series in z for ψ_{zzz}, when differentiated term by term. Instead, it is necessary to introduce the constant $R_{\kappa j}$, defined below, into the series for ψ_{zzz}. It may be evaluated by using the boundary condition $\psi_z = 0$ at $z = 0$, and $z = 1$. The need for $R_{\kappa j}$ is discussed, for example, in Jeffreys and Jeffreys.

$$\left.\begin{aligned}
&\psi_{zzz} \sim \sum \left\{-i\pi^3 j^3 \psi_{\kappa j} + i R_{\kappa j}\right\} e^{i\pi(k\alpha x + j\pi z)} \\
&\text{for } k \text{ and } j \text{ positive,} \\
&R_{\kappa j} = -\tfrac{1}{4}(A_\kappa, B_\kappa)_j = R_{\kappa,-j} = -R_{-\kappa,-j} = -R_{-\kappa,j} \\
&(A_\kappa, B_\kappa)_j := \begin{cases} \tfrac{1}{2} A_\kappa & \text{if } j = 0 \\ A_\kappa & \text{if } j \text{ is even, } \neq 0 \\ B_\kappa & \text{if } j \text{ is odd} \end{cases}
\end{aligned}\right\} \tag{8}$$

Because the equations (4) require only four derivatives of ψ and two of θ, this adjustment is the only one that need be made.

Next the series (6) (using (8) where necessary) is inserted into the left-hand side of (4a) and the entire equation (4b). Into the right-hand side of (4a) is inserted (5). Then the left-side of (4a) is rewritten in its real form, which is a "sin sin" series since each term of the left side of (4a) is odd in both x and z if z is extended evenly. However the right side is a "sin-cos" series because it is odd in x and even in z. To relate the coefficients on either side of (4a), the finite sin transform is taken, over (0,a) and (0,1). Because all terms of (4b) are even in x and odd in z, the coefficients of $e^{i\pi(k\alpha x + jz)}$ can be equated on both sides.

The results of this are:

$$-\frac{\pi}{\alpha}\theta_{kj}(\alpha^2 k^2 + j^2) + k\,\psi_{kj} = \pi A_{kj} \tag{9}$$

$$\frac{(k^2\alpha^2+j^2)^2}{\alpha}\psi_{kj} + \frac{\lambda}{\pi^3}k\sum_{q+\ell=j} a_\ell \theta_{kq} + \sigma B_{kj} - \pi j R_{kj} - \pi\alpha\sigma E_{kj} = \frac{16\lambda}{\pi^3} C_{kj} \tag{10}$$

$$a_\ell := -\frac{2}{\pi^2}\left\{\frac{1-(-1)^\ell}{\ell^2}\right\} \quad a_0 = 0 \tag{11}$$

$$A_{kj} := \sum_{\substack{r+m=k \\ s+n=j}} \psi_{rs}\theta_{mn}(rn - ms) \tag{12}$$

$$B_{kj} := \sum_{\substack{r+m=k \\ s+n=j}} \psi_{rs}\psi_{mn}(rn-ms)(m^2\alpha^2+n^2) \tag{13}$$

$$C_{kj} := \sum\nolimits^{*}_{r} \theta_{mn}\theta_{rs}(\delta_{r+m,k} + \delta_{r-m,k}) D_{jsn} \tag{14}$$

$$D_{jsn} := \frac{jsn \cdot \{(-1)^{j+s+n} - 1\}}{(j^2-(s+n)^2)(j^2-(s-n)^2)}$$ wherever it makes sense, zero otherwise. (15)

$$E_{kj} := \sum_{\substack{r+m=k \\ s+n=j}} r\, \psi_{rs}\, R_{mn} \tag{16}$$

Next the quantities θ_{kj}, ψ_{kj}, λ, and R_{kj} are expanded in a series of the Malkus-Veronis type:

$$\left.\begin{aligned} \theta_{kj} &= \sum_{i=1}^{\infty} \theta_{kj}^{(i)} \epsilon^i \\ \psi_{kj} &= \sum_{i=1}^{\infty} \psi_{kj}^{(i)} \epsilon^i \\ \lambda &= \sum_{i=0}^{\infty} \lambda_i \epsilon^i \\ R_{kj} &= \sum_{i=1}^{\infty} R_{kj}^{(i)} \epsilon^i \\ A_{kj} &= \sum_{i=2}^{\infty} A_{kj}^{(i)} \epsilon^i \text{ etc.} \end{aligned}\right\} \tag{17}$$

This expansion is inserted into the equations and coefficients of powers of ϵ are equated, thus generating a solvable iterative sequence of algebraic equations. At each stage the λ_i are evaluated by simply requiring that the equations have a solution. The criterion needed is the exact algebraic analogue of a criterion used by Malkus and Veronis in their study of the Boussinesq problem. Also at each iteration, and for each "horizontal" integer k, $R_{kj}^{(i)}$ must be evaluated, and this is done by use of $\psi_z = 0$ on the boundaries, which is equivalent to

$$\sum_{j\,\text{even}}^{*} j\,\psi_{kj} = \sum_{j\,\text{odd}}^{*} j\,\psi_{kj} = 0 \text{ for } k \neq 0 \tag{18}$$

The stability problem (the " ϵ -approximation") will now be solved. For reasons mentioned earlier, only $k = 1$ is considered. From (9) and (10),

$$\frac{(\alpha^2+j^2)^3}{\alpha^2}\theta_{1j}^{(1)} + \frac{\lambda_0}{\pi^4}\sum_{q+\ell=j} a_\ell\,\theta_{1q}^{(1)} - j\,R_{1j}^{(1)} = 0. \tag{19}$$

Using (9) with $k = 1$, and (18), we get

$$\sum_{j\,\text{even}}^{*} j(\alpha^2+j^2)\theta_{1j}^{(1)} = \sum_{j\,\text{odd}}^{*} j(\alpha^2+j^2)\theta_{1j}^{(1)} = 0 \tag{20}$$

But the equation (19) can be written in the form:

$$\underline{\underline{\text{j even}}}: j(\alpha^2+j^2)\theta_{1j}^{(1)} + \frac{\lambda_0 j\,\alpha^2}{\pi^4}\frac{1}{(\alpha^2+j^2)^2}\sum_{q+\ell=j} a_\ell\,\theta_{1q}^{(1)} = \frac{j^2\alpha^2 A_1^{(1)}}{(\alpha^2+j^2)^2} \tag{21}$$

$$\underline{\underline{\text{j odd}}}: j(\alpha^2+j^2)\theta_{1j}^{(1)} + \frac{\lambda_0 j\,\alpha^2}{\pi^4}\frac{1}{(\alpha^2+j^2)^2}\sum_{q+\ell=j} a_\ell\,\theta_{1q}^{(1)} = \frac{j\,\alpha^2 B_1^{(1)}}{(\alpha^2+j^2)^2} \tag{22}$$

Now if (21) and (22) are summed over j even and odd respectively, and (20) used, we have an expression for $A_1^{(1)}$ and $B_1^{(1)}$ in terms of $\theta_{1j}^{(1)}$. These are then inserted into (19) producing a set of linear equations for the $\theta_{1j}^{(1)}$ alone. There are, of course, an infinite number of equations. A reasonable procedure is to take $\theta_{1j}^{(1)} = 0$ for $j > S$ for some integer S. This results in a finite set of equations whose solution is approximately the $\theta_{1j}^{(1)}$ for $j = 1, \ldots, S$. The exact equations are given in an appendix.

For s = 2, one finds

$$\begin{Bmatrix} \theta_{11}^{(1)} \\ \theta_{12}^{(1)} \end{Bmatrix} \begin{bmatrix} \frac{(\alpha^2+1)^3}{\alpha^2} & \frac{\lambda_0}{\pi^4}\left\{(a_1-a_3) - \frac{1}{g_o(\alpha)}\left[\frac{a_1-a_3}{(\alpha^2+1)^2} + \frac{3(a_1-a_5)}{(\alpha^2+9)^2} + \frac{5(a_3-a_7)}{(\alpha^2+25)^2} + \cdots\right]\right\} \\ \frac{\lambda_0}{\pi^4}\left\{(a_1-a_3) - \frac{2}{g_e(\alpha)}\left[\frac{2a_1-a_3}{(\alpha^2+4)^2} + \frac{4(a_3-a_5)}{(\alpha^2+16)^2} + \frac{6(a_5-a_7)}{(\alpha^2+36)^2} + \cdots\right]\right\} & \frac{(\alpha^2+4)^3}{\alpha^2} \end{bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{23}$$

where

$$g_o(\alpha) := \sum_{j\ \mathrm{odd}}^{*} \frac{j^2}{(\alpha^2+j^2)^2}$$
$$g_e(\alpha) = \sum_{j\ \mathrm{even}}^{*} \frac{j^2}{(\alpha^2+j^2)^2} \tag{24}$$

Setting equal to zero the determinant of the matrix in (23) gives λ_0, to the 2x2 approximation. The result is, using some computations of Dr. Howard,

$$\lambda_0 \simeq 18{,}640$$
$$\theta_{11}^{(1)} = 1$$
$$\theta_{12}^{(1)} \simeq .31$$
$$\alpha = 4/\pi$$

<u>The ϵ^2-approximation.</u> The equations are

$$\frac{j^4}{\alpha}\psi_{oj}^{(2)} + \sigma\theta_{oj}^{(2)} - \pi j R_{oj}^{(2)} - \pi\alpha\sigma E_{oj}^{(2)} = \frac{16\lambda_0}{\pi^3} C_{oj}^{(2)}$$
$$\frac{-j^2}{\alpha}\theta_{oj}^{(2)} = A_{oj}^{(2)} \tag{25}$$

for $k \neq 0$,

$$\frac{(\alpha^2 k^2 + j^2)^3}{k^2 \alpha^2} \theta_{kj}^{(2)} + \frac{\lambda_0}{\pi^4} \sum_{q+l=j} a_l \theta_{kq}^{(2)} = \frac{16\lambda_0}{k\pi^4} C_{kj}^{(2)} + \frac{\alpha\sigma}{k} E_{kj}^{(2)} + \frac{j}{k} R_{kj}^{(2)} -$$

$$- \frac{\sigma}{k\pi} B_{kj}^{(2)} - \frac{\lambda_1}{\pi^4} \sum_{q+l=j} a_l \theta_{kq}^{(1)} - \frac{(\alpha^2 k^2 + j^2)^2}{\alpha k^2} A_{kj}^{(2)} \qquad (26)$$

Now by actual computation, $B_{oj}^{(2)} = R_{oj}^{(2)} = E_{oj}^{(2)} = 0$

hence

$$\psi_{oj}^{(2)} = \frac{16\lambda_0 \alpha}{j^4} C_{oj}^{(2)} \qquad (27)$$

And thus both $\theta_{oj}^{(2)}$ and $\psi_{oj}^{(2)}$ can be easily computed. Note that $\psi_{oj}^{(2)}$ is not zero for all j, and this is surprising because it does multiply a zero term in the solution, namely sin $0\pi\alpha x$. However it must be computed because it will play a role in future computations.

The next computation is for $k = 1$. The evaluation of λ_i will always occur as part of the $k = 1$ computation because (when $R_{1j}^{(i)}$ has been evaluated in terms of $\theta_{1j}^{(i)}$) the linear operator acting on $\theta_{1j}^{(i)}$ (considered as a vector with components j = 1,2,3, ...) is then singular, and a process to be described will simultaneously ensure existence of a unique solution $\theta_{1j}^{(i)}$ even in this case, and evaluate λ_i.

For $k = 1$, (26) reduces to the following when zero terms are not written down:

$$\frac{(\alpha^2 + j^2)}{\alpha^2} \theta_{1j}^{(2)} + \frac{\lambda_0}{\pi^4} \sum_{q+l=j} a_l \theta_{1q}^{(2)} = j R_{1j}^{(2)} - \frac{\lambda_1}{\pi^4} \sum_{q+l=j} a_l \theta_{1q}^{(1)} \qquad (28)$$

As in the stability problem $R_{1j}^{(2)}$ is now evaluated in terms of the $\{\theta_{1j}^{(2)}\}$ by using (18), which in fact takes the form of (20) with $\theta_{1j}^{(1)}$ replaced by $\theta_{1j}^{(2)}$. This is so because $A_{1j}^{(2)} = 0$. By analogy with the stability solution we see that $R_{1j}^{(2)}$ consists of a linear combination of the $\{\theta_{1j}^{(2)}\}$ plus a constant times λ_1.

Thus (28) can be written

$$\mathcal{L}\left[\theta_{1j}^{(2)}\right] = \lambda_1 \cdot (\text{constant})_j \qquad (29)$$

where $\mathcal{L}$ is the linear operator of the stability problem (which is approximately the matrix in (23)). Now, we know that $\mathcal{L}$ is singular because λ_o has been chosen so that $\det(\mathcal{L}) = 0$. Thus (29) has a solution iff its right-hand side, considered as a vector with j = 1,2,3, ... , is orthogonal to the solution $\vec{v}$ of $\mathcal{L}^T \vec{v} = 0$ where $\mathcal{L}^T$ is the operator whose matrix is the transpose of that of $\mathcal{L}$. Now by actual computation, $\vec{v} =$ (1, .25, ...) and it is clear that this orthogonality cannot hold unless $\lambda_1 = 0$.

Thus $\mathcal{L}\left[\theta_{1j}^{(2)}\right] = 0$. This, of course, implies that $\theta_{1j}^{(2)} = C\,\theta_{1j}^{(1)}$ where C is any constant. The uniqueness condition is now applied: that the solution $\theta_{1j}^{(p)}$ of each p be orthogonal to $\theta_{1j}^{(1)}$. Thus $C = 0$, and

$$\theta_{1j}^{(2)} = 0$$

(This implies that $R_{1j}^{(2)} = 0$ since $R_{1j}^{(2)}$ is linear in $\theta_{1j}^{(2)}$ and λ_1).

The next computation is that of the $\theta^{(2)}_{2j}$. For this purpose we have first to evaluate the $R^{(2)}_{2j}$ using (18). This and later computations have not yet been fully carried out.

The process of evaluating $\theta^{(p)}_{kj}$ should be clear by now. For $k = 1$, λ_p must be evaluated, and for each $k_0 > 0$, $R^{(p)}_{k_0 j}$ must be evaluated as a linear function of the $\theta^{(p)}_{k_0 j}$. If $k \neq 1$, the resulting linear operator on $\theta^{(p)}_{kj}$ is non-singular, hence has a unique inverse.

We previously mentioned that the great difficulty in this method is that only an approximate solution to the stability problem is available - i.e., the first few (two, in our case) terms of the sine series for $\theta^{(1)}$. The difficulty occurs, at each iteration, in the $k = 1$ computation. The "right-hand side" vector which is to be made orthogonal to $\theta^{(1)}_{1j}$ $(j = 1,2,3,4,\ldots)$ is actually made orthogonal only to $(\theta^{(1)}_{11}, \theta^{(1)}_{12}, \theta^{(1)}_{13}, \theta^{(1)}_{14}, \cdots, \theta^{(1)}_{1S}, 0, 0, 0, \cdots)$. Thus unless we restrict ourselves to finding $(\theta^{(p)}_{11}, \theta^{(p)}_{12}, \cdots, \theta^{(p)}_{1S})$ ($S = 2$ in our approximation), we are in trouble, because $\theta^{(p)}_{1j}$ for $j > S$ may not be correctly computed. It turns out that until, say, the ϵ^5-approximation, the straightforward method we have described can be carried out without such difficulties ruining the computation. However, in the fully three-dimensional case a refinement of the method, due to G. Veronis (unpublished), must be used.

III <u>Power Integrals, and others</u>. Some uses of power integrals are described in the paper of Malkus and Veronis. Another use of such integrals is as constraints in variational approaches to various convection problems - one replaces the full equations by some selected set of integrals. Two new integrals are given here - one which is valid also for the Boussinesq problem, and another which depends strongly on the penetrative convection that we have here.

First, for reference, we give our version of the momentum and thermal power integrals used in the paper of Malkus and Veronis (a double bar denotes integration over $0 \leq x \leq a$, $0 \leq z \leq 1$). These, and the other integrals here, are valid at least for free-free boundary conditions in z :

$$\lambda \overline{\overline{\psi_x (z\theta + \theta^3)}} + \overline{\overline{|\nabla \psi_x|^2}} + \overline{\overline{|\nabla \psi_z|^2}} = 0 \tag{30}$$

$$\overline{\overline{|\nabla \theta|^2}} - \overline{\overline{\psi_x \theta}} = 0 \tag{31}$$

The integral depending strongly on <u>penetrative</u> convection, as is easily verified, is

$$\lambda \overline{\overline{\theta^2}} + 2\sigma \overline{\overline{\theta(\psi_{xx}\psi_{zz} - \psi_{xz}^2)}} = 0 \tag{32}$$

which is obtained by taking the dot product of the momentum equation and $\nabla\theta$.

Finally if we multiply the heat equation by

$\theta^n f'(\psi(x,z))$ where $f(0) = f'(0) = 0$ then,

$$-\overline{\overline{\nabla\theta\cdot\nabla(f'(\psi)\theta^n)}} + \overline{\overline{\theta^n \frac{\partial}{\partial x} f(\psi)}} = 0 \tag{33}$$

Taking $n = 0$ and $f(x) = x^2$, then

$$\overline{\overline{\nabla\theta\cdot\nabla\psi}} = 0 \tag{34}$$

which states that in the average, conductive heat flow is parallel to the velocity.

A possibly interesting mathematical remark (suggested by remarks of Dr. Howard) about θ can be had by taking $n = 0$ in (33). Then (33) says that θ is orthogonal to $f'_{\circ}\,\psi$ in the Dirichlet norm, where by $f'_{\circ}\,\psi$ we mean "f' composed with ψ".

Now $f'_{\circ}\,\psi$ is a large subset of all those functions zero on the cell boundary. Iff θ were harmonic it would be orthogonal (à la Dirichlet) to <u>all</u> functions zero on the boundary. Thus θ partly shares one defining property of harmonic functions.

APPENDIX 1 WKB MATCHING WITH $k = 0$.

For $k = 0$ the associated second order equations are

$$\frac{d^2 v}{d\zeta^2} - \frac{9}{4} \xi^2 \zeta^5 v = 0 \quad \text{where } \zeta = \omega_i y \quad i = 1, 2, \text{ or } 3$$

and y is real, $y = z^{1/3}$ in the cut plane for $z > 0$.

Consider this equation for $y > 0$, (hence $z > 0$).

Solutions are

$$v = A_1 X^{1/2} H^{(1)}_{1/7}\left(+\tfrac{3}{7} i \xi X^{7/2}\right) + A_2 X^{1/2} H^{(1)}_{1/7}\left(-\tfrac{3}{7} i \xi X^{7/2}\right)$$

(for $\omega_i = \omega_1$ or ω_2 or ω_3). As an example we will perform the matching afforded by $\omega_i = \omega_1$.

Then it is easily seen that for $y > 0$, this solution is, in the cut plane,

$$e^{\pi i/3} y^{1/2} \left\{ A_1 H^{(1)}_{1/7}\left(\tfrac{3}{7} \xi e^{5\pi i/6} y^{7/2}\right) + A_2 H^{(1)}_{1/7}\left(3/7\, \xi e^{-\pi i/6} y^{7/2}\right) \right\}.$$

If this solution is continued into $y < 0$ in the cut plane, it becomes

$$y < 0: \; e^{-i\pi/6} |y|^{1/2} \left\{ A_1 H^{(1)}_{1/7}\left(3/7\, \xi e^{-2/3 \pi i} |y|^{7/2}\right) + A_2 H^{(1)}_{1/7}\left(3/7\, \xi e^{i\pi/3} |y|^{7/2}\right) \right\}$$

Using the formula for the asymptotic expansion of $H^{(1)}_{1/7}(?)$ for large (?), we obtain asymptotic expansions of these functions of the form:

$$y > 0: \; C|y|^{-5/4} \left\{ A_1 e^{-\frac{1}{12}\pi i} e^{3/7 \xi |y|^{7/2}} v_4 + A_2 e^{\frac{5\pi i}{12}} e^{3/7 \xi |y|^{7/2}} v_1 \right\}$$

$$y < 0: \; C|y|^{-5/4} \left\{ A_1 e^{i\pi/6} e^{3/7 \xi |y|^{7/2}} K_1 + A_2 e^{-i\pi/3} e^{3/7 \xi |y|^{7/2}} K_4 \right\}$$

Where C is a constant, and ν_i , κ_i are the 6 sixth roots of 1 and -1 respectively:

$$\nu_i = e^{i\pi/3},\ e^{i2\pi/3},\ e^{i\pi},\ e^{-2/3\pi i},\ e^{-1/3\pi i},\ 1.$$

$$\kappa_i = e^{-\pi i/6},\ e^{\pi i/6},\ e^{i\pi/2},\ e^{i5\pi/6},\ e^{-i5\pi/6},\ e^{-i\pi/2}.$$

Now we rewrite equations (2) for $k = 0$. For $z > 0$ it is

$$z^{-5/12}\Big\{E_1^+\omega_1^{-1/4}\exp(\tfrac{3}{7}z^{7/6}\xi\nu_1) + E_2^+\omega_2^{-1/4}\exp(\tfrac{3}{7}z^{7/6}\xi\nu_5) + E_3^+\omega_3^{-1/4}\exp(\tfrac{3}{7}\xi z^{7/6}\nu_6)$$
$$+ E_1^-\omega_1^{-1/4}\exp(\tfrac{3}{7}\xi z^{7/6}\nu_4) + E_2^-\omega_2^{-1/4}\exp(\tfrac{3}{7}\xi z^{7/6}\nu_2) + E_3^-\omega_3^{-1/4}\exp(\tfrac{3}{7}\xi z^{7/6}\nu_3)\Big\}$$

and for $z < 0$, it is

$$|z|^{-5/12}\Big\{F_1^+\delta_1^{-1/4}\exp(\tfrac{3}{7}\xi|z|^{7/6}\cdot\kappa_4) + F_2^+\delta_2^{-1/4}\exp(\tfrac{3}{7}\xi|z|^{7/6}\kappa_5) + F_3^+\delta_3^{-1/4}(\tfrac{3}{7}\xi|z|^{7/6}\kappa_6)$$
$$+ F_1^-\delta_1^{-1/4}\exp(\tfrac{3}{7}\xi|z|^{7/6}\cdot\kappa_1) + F_2^-\delta_2^{-1/4}\exp(\tfrac{3}{7}\xi|z|^{7/6}\kappa_2) + F_3^-\delta_3^{-1/4}(\tfrac{3}{7}\xi|z|^{7/6}\kappa_3)\Big\}$$

Now suppose that all the E's are zero except E_1^-. Then for matching in $\mathcal{R}_1$, we require

$$A_1 C e^{\frac{-\pi i}{12}} = E_1^-\omega_1^{-1/4} \qquad A_2 = 0$$

Hence the Hankel function solution continued into $\mathcal{R}_2$ gives $F_1^+ = 0$ and $C_1 A_1 e^{i\pi/6} = F_1^-\delta_1^{-1/4}$

Thus

$$\omega_1^{-1/4}E_1^- e^{i\frac{\pi}{12}} = F_1^-\delta_1^{-1/4}e^{-i\frac{\pi}{6}}$$

$$\text{or} \quad E_1^- = F_1^-.$$

Now suppose all E's zero except E_1^+. Then

$$A_1 = 0 \qquad C A_2 e^{5\pi i/4} = E_1^+ \omega_1^{-1/4}$$

Thus $F_1^- = 0$ and $C A_2 e^{-i\pi/3} = F_1^+ \delta_1^{-1/4}$

Therefore $E_1^+ \omega_1^{-1/4} e^{-5\pi i/4} = e^{i\pi/3} F_1^+ \delta_1^{-1/4}$

or, $E_1^+ = F_1^+$.

The determinant whose zeros provide $\{\lambda\}$ is

$$\begin{vmatrix}
\theta_1^+(+) & \theta_2^+(+) & \theta_3^+(+) & \theta_1^-(+) & \theta_2^-(+) & \theta_3^-(+) \\
\ddot{\theta}_1^+(+) & \ddot{\theta}_2^+(+) & \ddot{\theta}_3^+(+) & \ddot{\theta}_1^-(+) & \ddot{\theta}_2^-(+) & \ddot{\theta}_3^-(+) \\
\dddot{\theta}_1^+(+) & \dddot{\theta}_2^+(+) & \dddot{\theta}_3^+(+) & \dddot{\theta}_1^-(+) & \dddot{\theta}_2^-(+) & \dddot{\theta}_3^-(+) \\
\theta_1^+(-) & -i\theta_2^+(-) & -i\theta_3^-(-) & \theta_1^-(-) & i\theta_2^-(-) & i\theta_3^+(-) \\
\ddot{\theta}_1^+(-) & -i\ddot{\theta}_2^+(-) & -i\ddot{\theta}_3^-(-) & \ddot{\theta}_1^-(-) & i\ddot{\theta}_2^-(-) & i\ddot{\theta}_3^+(-) \\
\dddot{\theta}_1^+(-) & -i\dddot{\theta}_2^+(-) & -i\dddot{\theta}_3^-(+) & \dddot{\theta}_1^-(-) & i\dddot{\theta}_2^-(-) & i\dddot{\theta}_3^+(-)
\end{vmatrix}$$

In this determinant, the term $\ddot{\theta}_3^-(+)$ refers to $\tilde{\theta}$ with exponential factor $\exp(-\beta_3(z)\cdot\xi)$, evaluated at $z = +1$. Similarly $\ddot{\theta}_3^-(-)$ is evaluated at $z = -1$.

APPENDIX II

The exact stability equations are:

$$\frac{(\alpha^2+j^2)}{\alpha^2}\theta_{1j}^{(1)}+\frac{\lambda_0}{\pi^4}\sum_{q+l=j}a_l\theta_{1q}^{(1)}=\frac{j\lambda_0}{\pi^4}\frac{1}{g_e(\alpha)}\sum_{p\,\mathrm{even}}^{*}\frac{p}{(\alpha^2+p^2)^2}\sum_{q+l=p}a_l\theta_{1q}^{(1)} \qquad \underline{\text{for } j \text{ even}}$$

and for j odd, the same equation with sum over odd p, and $g_o(\alpha)$ replacing $g_e(\alpha)$.

The numbers $A_1^{(1)}$ and $B_1^{(1)}$ are given by:

$$A_1^{(1)}=(g_e(\alpha))^{-1}\frac{\lambda_0}{\pi^4}\sum_{j\,\mathrm{even}}\frac{j}{(\alpha^2+j^2)^2}\sum_{q+l=j}a_l\theta_{1q}^{(1)}$$

$$B_1^{(1)}=(g_o(\alpha))^{-1}\frac{\lambda_0}{\pi^4}\sum_{j\,\mathrm{odd}}\frac{j}{(\alpha^2+j^2)^2}\sum_{q+l=j}a_l\theta_{1q}^{(1)}.$$

Combined Shear Flow and Convection

by

Fred C. Shure

Introduction

The onset of thermal convection in a quiet fluid contained between two perfectly conducting rigid horizontal surfaces and heated from below has been thoroughly investigated by Pellew and Southwell (1). They found that the fluid is unstable when the Rayleigh number exceeds the critical value of 1708. The plan form of the growing motions allowed by their theory may be any space-filling lattice (hexagons, rolls, etc.)[1]. Moreover in view of the complete horizontal symmetry, the orientation of this plan form is unrestricted.

It is of interest to consider a natural generalization of Pellew and Southwell's stability problem: the fluid is not at rest, but flowing (with some given velocity profile) horizontally between the plates. It may be conjectured that the shear flow produces a stabilizing effect. A disturbance which varies spatially in the direction of the basic flow is "advected away from itself" since the basic flow varies vertically through the fluid. Of course, this stabilizing effect can not serve to increase the critical Rayleigh number of the

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[1]Bisshopp (2) has shown more recently that if the cells are to be parallelograms, they must in fact be rectangles.

system. Disturbances whose only horizontal spatial variation is perpendicular to the stream[1] are unaffected by the presence of the flow, at least insofar as stability criteria are concerned. Hence down-stream rolls will grow exponentially in time (according to the linearized theory) whenever $R > 1708$. However, if our conjecture is correct, the critical Rayleigh number for disturbances of any other plan form will be somewhat greater than 1708. Under these conditions down-stream rolls appear as a preferred form of cellular convective motion. This may serve to explain their frequent occurrence in cloud and fog formations, as may be seen in some of the striking photographs in the work of Avsec(3).

The basic equations:

In our model, the undisturbed fluid is moving in the x-direction with a linear (Couette) velocity profile.

$$\vec{U} = U(z)\hat{i} = \frac{Uz}{d}\hat{i} \tag{1}$$

z is the vertical direction and d is the vertical spacing between the plates confining the fluid. The bottom surface is maintained at a higher temperature than the top, and we define

$$\beta = \frac{|\Delta T|}{d}, \text{ the (positive) temperature gradient.} \tag{2}$$

We also define $\gamma = g\alpha$, where g is the gravitational constant,

[1]These are "down-stream rolls": rolls whose axes are parallel to the stream.

and α the coefficient of thermal expansion.

The energy-momentum-mass equations (in the Boussinesq approximation) are

$$\text{a.}\quad \frac{\partial \vec{u}}{\partial t} + \vec{U}\cdot\vec{\nabla}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{U} = -\frac{\nabla p}{\rho_m} + \gamma T\hat{k} + \nu\nabla^2\vec{u}$$

$$\text{b.}\quad \frac{\partial T}{\partial t} + \vec{U}\cdot\vec{\nabla}T = \beta w + \kappa\nabla^2 T \qquad (3)$$

$$\text{c.}\quad \vec{\nabla}\cdot\vec{u} = 0$$

p , ρ_m , ν , κ are the pressure, mean density, kinematic viscosity and diffusivity respectively. T and $\vec{u} = (u,v,w)$ are the variations of the temperature and velocity from the mean temperature and velocity fields. Terms quadratic in $\vec{u}$ and T are omitted.

We apply the operator

$$\mathcal{O}_z = (\nabla\times\nabla\times)_z = \frac{\partial^2}{\partial x\partial z}\hat{i} + \frac{\partial^2}{\partial y\partial z}\hat{j} - \nabla_1^2\hat{k} \qquad (4)$$

to the momentum equation.

Note that

$$\mathcal{O}_z\cdot\vec{\nabla} = 0 \qquad \mathcal{O}_z\cdot\vec{u} = -\nabla^2 w \qquad \mathcal{O}_z\cdot T\hat{k} = -\nabla_1^2 T \qquad (5)$$

and, in view of the specified form of $\vec{U}$

$$\mathcal{O}_z\cdot\left[\vec{U}\cdot\vec{\nabla}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{U}\right] = -\frac{U_z}{d}\frac{\partial}{\partial x}\nabla^2 w \qquad (6)$$

Thus the coupled equations for w and T become

$$\text{a.}\quad \left(\frac{\partial}{\partial t} - \nu\nabla^2 + \frac{U_z}{d}\frac{\partial}{\partial x}\right)\nabla^2 w = \gamma\nabla_1^2 T$$

$$\qquad (7)$$

$$\text{b.}\quad \left(\frac{\partial}{\partial t} - \kappa\nabla^2 + \frac{U_z}{d}\frac{\partial}{\partial x}\right)T = \beta w$$

Notice that for the cross-stream rolls previously mentioned, where w and T are independent of x, the advective term drops out of the equations and we are left with the stability equations of Pellew and Southwell. Thus with respect to such disturbances, the system will be unstable whenever the Rayleigh number defined by

$$R = \frac{\gamma\beta d^4}{\kappa\nu} \tag{8}$$

exceeds 1708.

We shall investigate disturbances whose horizontal spatial variation is of the form $\sim e^{i\frac{ax}{d}}$ (i.e. down-stream rolls). Combining eqs. (7a) and 7b) and non-dimensionalizing according to

$$\begin{aligned} l_0 &= d \\ t_0 &= \frac{d^2}{\nu} \end{aligned} \tag{9}$$

we obtain the 6th order equation for w

$$\left[\left(\frac{\partial}{\partial t}-\frac{1}{\sigma}\nabla^2+ia\mathcal{R}z\right)\left(\frac{\partial}{\partial t}-\nabla^2+ia\mathcal{R}z\right)\nabla^2+\frac{Ra^2}{\sigma}\right]w=0 \tag{10}$$

where $\mathcal{R} = \frac{Ud}{\nu}$, the Reynolds number

$\sigma = \frac{\nu}{\kappa}$, the Prandtl number. (11)

<u>The case of low Reynolds Number</u>:

When the Reynolds number is low, we are led naturally to an expansion of our system of equations about the system considered by Pellew and Southwell.

We consider the basic equations (obtained from the non-dimensional form of eqs. (7) by assuming horizontal spatial dependence $\sim e^{i\frac{ax}{d}}$)

$$\left(\frac{\partial}{\partial t}+i\epsilon z-\nabla^2\right)\nabla^2 w=-a^2 T$$
$$\left[\sigma\left(\frac{\partial}{\partial t}+i\epsilon z\right)-\nabla^2\right]T=Rw \tag{12}$$

along with the boundary conditions for rigid surfaces.

$$w\left(\pm\tfrac{1}{2}\right)=Dw\left(\pm\tfrac{1}{2}\right)=T\left(\pm\tfrac{1}{2}\right)=0. \tag{13}$$

$\epsilon=\mathcal{R}a$ will serve as an expansion parameter.

We seek solutions of eqs. (12) for fixed a whose time dependence is of the form e^{st} where s is purely imaginary. The eigenvalue R thus obtained will then determine a point on the marginal stability curve of R vs. a. If the marginal stability curve possesses an absolute minimum, the minimum will be located at the critical Rayleigh number and the accompanying value of a will be the wave number of the most unstable mode.

Our 6th order equation for w

$$\left[\sigma(s+i\epsilon z)-\nabla^2\right]\left[s+i\epsilon z-\nabla^2\right]\nabla^2 w=-Ra^2 w \tag{14}$$

with boundary conditions

$$w\left(\pm\tfrac{1}{2}\right)=Dw\left(\pm\tfrac{1}{2}\right)=\left[s+i\epsilon z-\nabla^2\right]\nabla^2 w\left(\pm\tfrac{1}{2}\right)=0 \tag{15}$$

incorporates as a special case ($\epsilon = 0$) the 6th order equation

$$\nabla^6 w_0=-R_0 a^2 w_0 \tag{16}$$

with boundary conditions

$$w(\pm\tfrac{1}{2}) = Dw(\pm\tfrac{1}{2}) = \nabla^4 w(\pm\tfrac{1}{2}) = 0 \tag{17}$$

treated by Pellew and Southwell. That is to say, although the principle of exchange of stabilities does not apply in this case and S may turn out to be complex, we infer from Pellew and Southwell's results that $S_i \to 0$ as $\epsilon \to 0$ and for solutions on the marginal stability curve we write

$$S = S_1\epsilon + S_2\epsilon^2 + S_3\epsilon^3 + \ldots\ldots S_n \text{ imaginary} \tag{18}$$

We also write

$$\begin{aligned} w &= w_0 + w_1\epsilon + w_2\epsilon^2 + \ldots \\ R &= R_0 + R_1\epsilon + R_2\epsilon^2 + \ldots \end{aligned} \tag{19}$$

with w_0 and R_0 given by Pellew and Southwell.

We now obtain the surprising result that

$$S_n = 0 \qquad \text{all } n.$$

In other words, although we have been unable to prove that the principle of exchange of stability applies when $\epsilon \neq 0$, it is a formal consequence of this perturbation treatment.

We write symbolically:

$$\left[\sigma(S+i\epsilon z)-\nabla^2\right]\left[S+i\epsilon z-\nabla^2\right]\nabla^2 + Ra^2 = \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1\epsilon + \mathcal{L}_2\epsilon^2 + \ldots \tag{20}$$

$$\mathcal{L}_0 = \nabla^6 + R_0 a^2 \tag{20.0}$$

$$\begin{aligned} \mathcal{L}_1 &= -(\sigma+1)S_1\nabla^4 - \sigma i z\nabla^4 - i\nabla^2 z\nabla^2 + R_1 a^2 \\ \mathcal{L}_2 &= -(\sigma+1)S_2\nabla^4 + \sigma(S_1+iz)^2\nabla^2 + R_2 a^2 \end{aligned} \tag{20.1}$$

$$\mathcal{L}_n = -(\sigma+1)S_n \nabla^4 + 2i\sigma S_{n-1} z + R_n a^2 \quad n \geq 3 \tag{20.n}$$

Then to satisfy eq. (14) to all orders in ϵ we have

$$\mathcal{L}_0 w_0 = 0 \tag{21.0}$$

$$\mathcal{L}_0 w_1 = -\mathcal{L}_1 w_0$$

$$\mathcal{L}_0 w_2 = -\mathcal{L}_1 w_1 - \mathcal{L}_2 w_0$$

$$\vdots$$

$$\mathcal{L}_0 w_n = -\mathcal{L}_1 w_{n-1} - \mathcal{L}_2 w_{n-2} \cdots - \mathcal{L}_n w_0 = -\sum_{i=1}^{n} \mathcal{L}_i w_{n-i} \tag{21.n}$$

The boundary conditions become

$$w_n(\pm\tfrac{1}{2}) = D w_n(\pm\tfrac{1}{2}) = 0$$

$$\nabla^4 w_n(\pm\tfrac{1}{2}) = iz\nabla^2 w_{n-1}(\pm\tfrac{1}{2}) + \sum_{i=1}^{n_1} S_i \nabla^2 w_{n-i}(\pm\tfrac{1}{2}) \tag{22.n}$$

By direct calculation we obtain the useful relation:

$$\int T_0 \mathcal{L}_0 w_n dz = \int w_n \underset{\substack{\parallel \\ 0}}{\mathcal{L}_0 T_0} dz - DT_0 \nabla^4 w_n \Big|_{-1/2}^{1/2} \tag{23}$$

$$= -\left\{ DT_0 \left[iz\nabla^2 w_{n-1} + \sum_{i=1}^{n} S_i \nabla^2 w_{n-i} \right] \right\} \Big|_{-1/2}^{1/2}$$

We first show that $R_1 = S_1 = 0$ and w_1 is an odd imaginary function.

Multiplying eq. (21.1) by T_0 and integrate over z

$$iz DT_0 \nabla^2 w_0 \Big|_{-1/2}^{1/2} + S_1 DT_0 \nabla^2 w_0 \Big|_{-1/2}^{1/2} = S_1(\sigma+1)\int T_0 \nabla^4 w_0 \, dz + \int i\sigma T_0 \left[z\nabla^4 + \nabla^2 z \nabla^2 \right] w_0 dz - R_1 a^2 \int T_0 w_0 dz \tag{24}$$

or $$A + BS_1 = CS_1 + D + ER_1$$

Since w_0 and T_0 are even functions A and D vanish. Since w_0 and T_0 are real, B, C and E are real. Hence $R_1 = S_1 = 0$, because S_1 is imaginary and R_1 real.

Eq. (21.2) and (22.2) become

$$\mathcal{L}_0 w_1 = -i\left[\sigma\, z \nabla^4 + \nabla_z^2 \nabla^2\right] w_0 = \text{odd imaginary function.} \tag{25}$$

also $$\nabla^4 w_1\left(\tfrac{1}{2}\right) = \frac{i}{2}\nabla^2 w_0\left(\tfrac{1}{2}\right) = -\nabla^4 w_1\left(-\tfrac{1}{2}\right) \tag{26}$$

Thus $$w_1 = w_1^p + C w_0 \tag{27}$$

where the particular solution is an odd imaginary function. w_0 satisfies the homogeneous equation and boundary conditions. To maintain normalization of w to 1st order in ϵ we must choose $C = 0$. Thus w_1 is odd and imaginary.

Proceeding further, in the same manner we obtain $S_2 = 0$ and w_2 = even, real function. In fact, it is quite simple to prove (by induction) that

$$\begin{aligned} S_n &= 0 \quad \text{all } n. \\ R_n &= 0 \quad \text{all odd } n. \end{aligned} \tag{28}$$

w_n is even and real for all even n.

w_n is odd and imaginary for all odd n.

Determination of R_2.

We have shown that whenever the perturbation expansion is valid, marginally stable solutions ($S_r = 0$) are in fact time independent ($S_i = S_r = 0$). We may then set $S = 0$ to obtain

$$\mathcal{L}w = \left[(\sigma i\epsilon z - \nabla^2)(i\epsilon z - \nabla^2)\nabla^2 + Ra^2\right]w = 0 \tag{29}$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1\epsilon + \mathcal{L}_2\epsilon^2 + \ldots \tag{30}$$

$$\mathcal{L}_0 = \nabla^6 + R_0 a^2$$

$$\mathcal{L}_1 = -i(\sigma z\nabla^2 + \nabla^2 z)\nabla^2$$

$$\mathcal{L}_2 = -\sigma z^2\nabla^2 + R_2 a^2$$

$$\mathcal{L}_3 = \mathcal{L}_5 = \mathcal{L}_7 \ldots = 0$$

$$\mathcal{L}_4 = R_4 a^2$$

$$\mathcal{L}_6 = R_6 a^2$$

$$\vdots$$

As before we have

$$\mathcal{L}_0 w_n = -\sum_{i=1}^{n} \mathcal{L}_i w_{n-i} \tag{21.n}$$

with the simplified boundary conditions.

$$\begin{aligned} w_n(\pm\tfrac{1}{2}) &= Dw_n(\pm\tfrac{1}{2}) \\ \nabla^4 w_n(\pm\tfrac{1}{2}) &= \pm\tfrac{i}{2}\nabla^2 w_{n-1}(\pm\tfrac{1}{2}) \qquad n \geq 1 \\ &= 0 \qquad n = 0 \end{aligned} \tag{31.n}$$

There are two straightforward methods for obtaining the first correction to Pellew and Southwell's results. In neither case have the (rather lengthy) computations been completed, so only the methods will be presented.

One method is by direct solution of the perturbation equations.

Multiplying equation (21.2) by T_0 and integrating over z gives

$$\int T_0 \mathcal{L}_0 w_2 = -\int T_0 \mathcal{L}_1 w_1 - \int T_0 \mathcal{L}_2 w_0 \tag{32}$$

or

$$-iz D T_0 \nabla^2 w_1 \Big|_{-1/2}^{1/2} = i\int T_0 (\sigma z \nabla^2 + \nabla^2 z) \nabla^2 w_1 \, dz + \sigma \int T_0 z^2 \nabla^2 w_0 \, dz - R_2 a^2 \int T_0 w_0 \, dz \tag{33}$$

or,

$$R_2 a^2 = \frac{i\int T_0 (\sigma z \nabla^2 + \nabla^2 z) \nabla^2 w_1 \, dz + \sigma \int T_0 z^2 \nabla^2 w_0 \, dz + iz D T_0 \nabla^2 w_1 \Big|_{-1/2}^{1/2}}{\int T_0 w_0 \, dz} \tag{34}$$

where w_1 is the solution of

$$(\nabla^6 + R_0 a^2) w_1 = i(\sigma z \nabla^2 + \nabla^2 z) \nabla^2 w_0 \tag{35}$$

with boundary conditions given by (31.1)
Because of the inhomogeneous boundary condition, it is impractical to perform the perturbation calculation in the usual manner (by expanding w_1 in terms of the eigenfunctions of the operator ∇^6). It is better to solve the differential equation (35) directly.

As shown in Pellew and Southwell, w_0 is given by

$$w_0 = \sum_{i=1}^{3} \omega_i^2 \frac{\cosh 2\mu_i z}{\cosh \mu_i} \tag{36}$$

where

$$\begin{aligned} w_i &= 1_i^{1/3} \\ (2\mu_i)^2 &= a^2(1 - \lambda \omega_i) \\ \lambda^3 a^4 &= R_0 \end{aligned} \tag{37}$$

The expression for w_0 automatically satisfies

$$w_0(\pm\tfrac{1}{2}) = \nabla^4 w_0(\pm\tfrac{1}{2}) = 0 \tag{38}$$

and applying the boundary condition

$$D w_0(\pm\tfrac{1}{2}) = 0 \tag{39}$$

gives the secular equation determing λ and hence R_0

$$\sum_i \mu_i \omega_i^2 \tanh \mu_i = 0 \tag{40}$$

One would determine w_1 by finding a particular solution w_1^p to eq.(35) and then setting

$$w_1 = w_1^p + \sum_i A_i \sinh 2\mu_i z \tag{41}$$

where the A_i are chosen so that w_1 satisfies the boundary conditions (31.1).

It is found that

$$w_1^p = i\sum_{i=1}^{3} \omega_i^2 \left\{ \frac{(\sigma+1)}{24\mu_i} z^2 \frac{\sinh 2\mu_i z}{\cosh \mu_i} + \left(\frac{(\sigma+1)}{48\mu_i^2} - \frac{(5\sigma+1)}{12} \right) z \frac{\cosh 2\mu_i z}{\cosh \mu_i} \right\} \tag{42}$$

so that the determination of R_2 by this method becomes a rather arduous task.

One could also establish a variational principle to determine the critical Rayleigh number. The method is that used by DiPrima (1961)..

We start with eq. (12) with $\frac{\partial}{\partial t} = 0$

$$\begin{aligned} (i\epsilon z - \nabla^2)\nabla^2 w &= -a^2 T \\ (\sigma i \epsilon z - \nabla^2) T &= R w \end{aligned} \tag{43}$$

If we call $R^{1/2} = \Lambda$ and redefine w as $\frac{ia}{\Lambda} w$ we obtain the more symmetrical form

$$(i\epsilon z - \nabla^2)\nabla^2 w = ia\Lambda T$$
$$(\sigma i\epsilon z - \nabla^2)T = ia\Lambda w \tag{44}$$

The adjoint system is

$$\nabla^2(i\epsilon z - \nabla^2)w^\dagger = ia\Lambda T^\dagger$$
$$(\sigma i\epsilon z - \nabla^2)T^\dagger = ia\Lambda w^\dagger \tag{45}$$

with boundary conditions

$$w^\dagger(\pm\tfrac{1}{2}) = Dw^\dagger(\pm\tfrac{1}{2}) = T^\dagger(\pm\tfrac{1}{2}) \tag{46}$$

then, if we consider the expression

$$I = \int_{-1/2}^{1/2}\left\{w^\dagger(i\epsilon z - \nabla^2)\nabla^2 w + T^\dagger(\sigma i\epsilon z - \nabla^2)T - ia\Lambda\left[w^\dagger T + T^\dagger w\right]\right\}dz \tag{47}$$

we find that I is stationary with respect to arbitrary independent variations of w, $w^\dagger$, T, $T^\dagger$ which satisfy the boundary conditions, so long as w, $w^\dagger$, T, $T^\dagger$ satisfy eq.(44) and (45). Thus we expand

$$w = \sum a_n C_n(z)$$
$$w^\dagger = \sum b_n C_n(z)$$
$$T = \sum c_n E_n(z) \tag{48}$$
$$T^\dagger = \sum d_n E_n(z)$$

The functions $E_n(z)$, $C_n(z)$ are chosen to satisfy the boundary conditions. For example, it is reasonable to write

$$C_n(z) = \begin{cases} \sin n\pi z & n \text{ even} \\ \cos n\pi z & n \text{ odd.} \end{cases} \qquad (49)$$

If the sums chosen for the trial functions consist of N terms each, then the requirement that

$$\frac{\partial I}{\partial a_n} = \frac{\partial I}{\partial b_n} = \frac{\partial I}{\partial c_n} = \frac{\partial I}{\partial d_n} = 0 \qquad (50)$$

gives $4N$ homogeneous equations for the $4N$ unknowns a_n, b_n, c_n, d_n. The condition that the determinant of the coefficient must vanish yields the secular equation for Λ.

<u>The case of high Reynolds number</u>:

We have in mind a situation (such as probably exists in the atmosphere) where viscous and conductive effects are small. In this case it is more appropriate to non-dimensionalize according to

$$\ell_0 = d \qquad t_0 = \frac{d}{dU} \qquad (51)$$

the 6th order equation is

$$\left\{\left[\left(\frac{\partial}{\partial t} + iz\right) - \frac{1}{\mathcal{R}\sigma}\nabla^2\right]\left[\left(\frac{\partial}{\partial t} + iz\right) - \frac{1}{\mathcal{R}}\nabla^2\right]\nabla^2 + \frac{R}{\sigma\mathcal{R}^2}\right\} w = 0 \qquad (52)$$

Dropping the terms in $\frac{1}{\mathcal{R}}$, $\frac{1}{\mathcal{R}\sigma}$ we obtain the "inviscid limit"

$$\left(\frac{\partial}{\partial t} + iz\right)^2 \nabla^2 w = \frac{R}{\sigma\mathcal{R}^2} w \qquad (53)$$

where $\frac{R}{\sigma\mathcal{R}^2} = \frac{\gamma\beta d^2}{U^2} = J$ the Richardson number. (54)

This system is characterized by the Richardson number alone. If it turns out that there is a critical Richardson number below which all disturbances decay with time, then we have the result

$$R_{crit} = \sigma J_{crit} \mathcal{R}^2 \tag{55}$$

i.e. for large Reynolds numbers, the critical Rayleigh number is proportional to the square of the Reynolds number.

Let us look for solutions of eq. (53) whose time dependence is of the form e^{-ict}. Then

$$(z-c)^2 \nabla^2 w - J w = 0 \tag{56}$$

The boundary conditions are

$$w\left(-\tfrac{1}{2}\right) = w\left(\tfrac{1}{2}\right) = 0 \tag{57}$$

Without solving equation (56) explicitly we can show that the allowed values of c lie in the strip $-\frac{1}{2} < \mathrm{Re}\, c < \frac{1}{2}$. We multiply eq. (56) by $\frac{w^*}{(z-c)^2}$ (the $*$ indicates complex conjugation) and integrate over the range of z.

$$\int_{-1/2}^{1/2} w^* \nabla^2 w \, dz - J \int_{-1/2}^{1/2} \frac{|w|^2 (z-c^*)^2 dz}{|z-c|^4} = 0 \tag{58}$$

The real and imaginary parts of this equation are

A) $$c_i \int_{-1/2}^{1/2} \frac{|w|^2 (z-c_r) dz}{|c-z|^4} = 0 \tag{59}$$

B) $$a^2 \int_{-1/2}^{1/2} |w|^2 dz + \int_{-1/2}^{1/2} \left|\frac{\partial w}{\partial z}\right|^2 dz + J \int_{-1/2}^{1/2} \frac{|w|^2 ((z-c_r)^2 - c_i^2) dz}{|c-z|^4} = 0$$

From (59A) we see that whenever $c_i \neq 0$ c_r must lie in the range $-\frac{1}{2} < C_r < \frac{1}{2}$. But if C is assumed real, and the integrals containing the factor $\frac{1}{|c-z|^4}$ exist, then we see from (59B) that no values of C are allowed. Thus the only allowed real values of C lie also in the range $-\frac{1}{2} < C < \frac{1}{2}$ and hence all possible values of C lie in the strip $|Re\, c| < \frac{1}{2}$.

Equation (56) is a form of Bessel's equation and admits solutions

$$w(z) = \sqrt{z-c}\ \mathcal{C}_\nu\left(u\,(z-c)\right) \qquad (60)$$

where $\mathcal{C}_\nu$ is a Bessel function of imaginary argument and $(\nu-\frac{1}{2})(\nu+\frac{1}{2}) = J$.

When J takes on any of the values n^2+n (n an integer) then $\nu = n+\frac{1}{2}$ and solutions are quite simple.

In particular, for $J = 2$ we have

$$w^\pm(z) = \begin{matrix}\cosh\\ \sinh\end{matrix}\left(a(z-c)\right) - \frac{1}{a(z-c)}\begin{matrix}\sinh\\ \cosh\end{matrix}\left(a(z-c)\right) \qquad (61)$$

These are proportional to $\sqrt{z-c}\ I_{\pm 3/2}\left(a\,(z-c)\right)$.

It is more convenient to work with w_1, w_2 defined by

$$\frac{w_1(z) \pm w_2(z)}{2} = w^\pm(z) \qquad (62)$$

i.e.

$$w_1 = e^{a(z-c)}\left(1 - \frac{1}{a(z-c)}\right)$$

$$w_2 = e^{a(z-c)}\left(1 + \frac{1}{a(z-c)}\right) \qquad (63)$$

If we assume a solution of the form

$$w = A\,w_1 + B\,w_2 \qquad (64)$$

and apply the boundary condition, then the secular determinant

$$w_1\left(\tfrac{1}{2}\right) w_2\left(-\tfrac{1}{2}\right) - w_1\left(-\tfrac{1}{2}\right) w_2\left(\tfrac{1}{2}\right) \tag{65}$$

must vanish for such a solution to exist. This leads to the secular equation

$$e^{a}\left(1-\frac{2}{a(1-2c)}\right)\left(1-\frac{2}{a(1+2c)}\right)-e^{-a}\left(1+\frac{2}{a(1+2c)}\right)\left(1+\frac{2}{a(1-2c)}\right)=0 \tag{66}$$

or

$$c^2-\left[\frac{1}{4}-\left(\frac{\coth a}{a}-\frac{1}{a^2}\right)\right]=0 \tag{67}$$

When $\frac{\coth a}{a}-\frac{1}{a^2} < \frac{1}{4}$ there are two real roots. This occurs when $a > 2.399$.

For $a < 2.399$ the roots are complex (in this case imaginary) and the system is unstable for such disturbances.

Thus there is a transition between stable and unstable wave numbers at $a = 2.399$. At this transition point, the solution is time independent: $c = 0$.

From the preceding considerations we get an idea of the behavior of the eigenvalues c, even though we can not solve the secular equation for arbitrary values of J. In fact it is reasonable to expect that when the wave number a passes from a stable to an unstable region, the eigenvalue $c = 0$. This follows from the fact that when $w(z)$ is a solution of (56) with eigenvalue c then $w^*(z)$ and $w(-z)$ are solutions with eigenvalues c^* and $-c$ respectively.

At the transition point $a \sim 2.399$ $c = 0$ if we write

$$w = A w_1 + B w_2 \tag{68}$$

and impose the boundary condition $w(\tfrac{1}{2}) = 0$, we find

$$A = -B \quad \text{which implies} \tag{69}$$

$$w = w^- = \sqrt{z}\; I_{-\frac{3}{2}}(az). \tag{70}$$

Thus, at the transition point, the solution is just one of the two Bessel function solutions, not a linear combination of the two. The boundary conditions are automatically satisfied, since $I_{-\frac{3}{2}}(az)$ possesses a real zero. In fact

$$I_{-3/2}\left(\frac{a}{2}\right) = 0 \Rightarrow \left(e^{a/2} - e^{-a/2}\right) = \frac{2}{a}\left(e^{a/2} + e^{-a/2}\right) \tag{71}$$

which may be rearranged to give

$$\frac{\coth a}{a} - \frac{1}{a^2} = \frac{1}{4} \tag{72}$$

which is just the transition point.

In other words, if there is a transition point at $C = 0$, one of the Bessel function solutions of (56) must have a zero. This may be shown to be true in general. For $C = 0$, the secular equation is

$$I_\nu\left(\frac{a}{2}\right) I_{-\nu}\left(-\frac{a}{2}\right) - I_\nu\left(-\frac{a}{2}\right) I_{-\nu}\left(\frac{a}{2}\right) = 0 \tag{73}$$

but $$I_\nu(-z) = e^{\nu\pi i} I_\nu(z) \tag{74}$$

thus $$\sin \nu\pi i\; I_\nu\left(\frac{a}{2}\right) I_{-\nu}\left(\frac{a}{2}\right) = 0 \tag{75}$$

and except when ν is an integer (when the I's don't form a system of solutions anyway) one of the I's must possess a zero.

To recapitulate: For $J = 2$, the system is unstable but possesses a region of stable wave numbers. Hence for $J_{crit} < J < 2$ we expect this also to be the case. When there is a transition from unstable to stable wave numbers, it is associated with a zero of one of the two Bessel functions. But for I_ν to possess a real zero ν must be ≤ -1[5]. $\nu = -1$ corresponds to $J = 3/4$, which we suggest is the critical value for this problem.

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