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1960
VOLUME III

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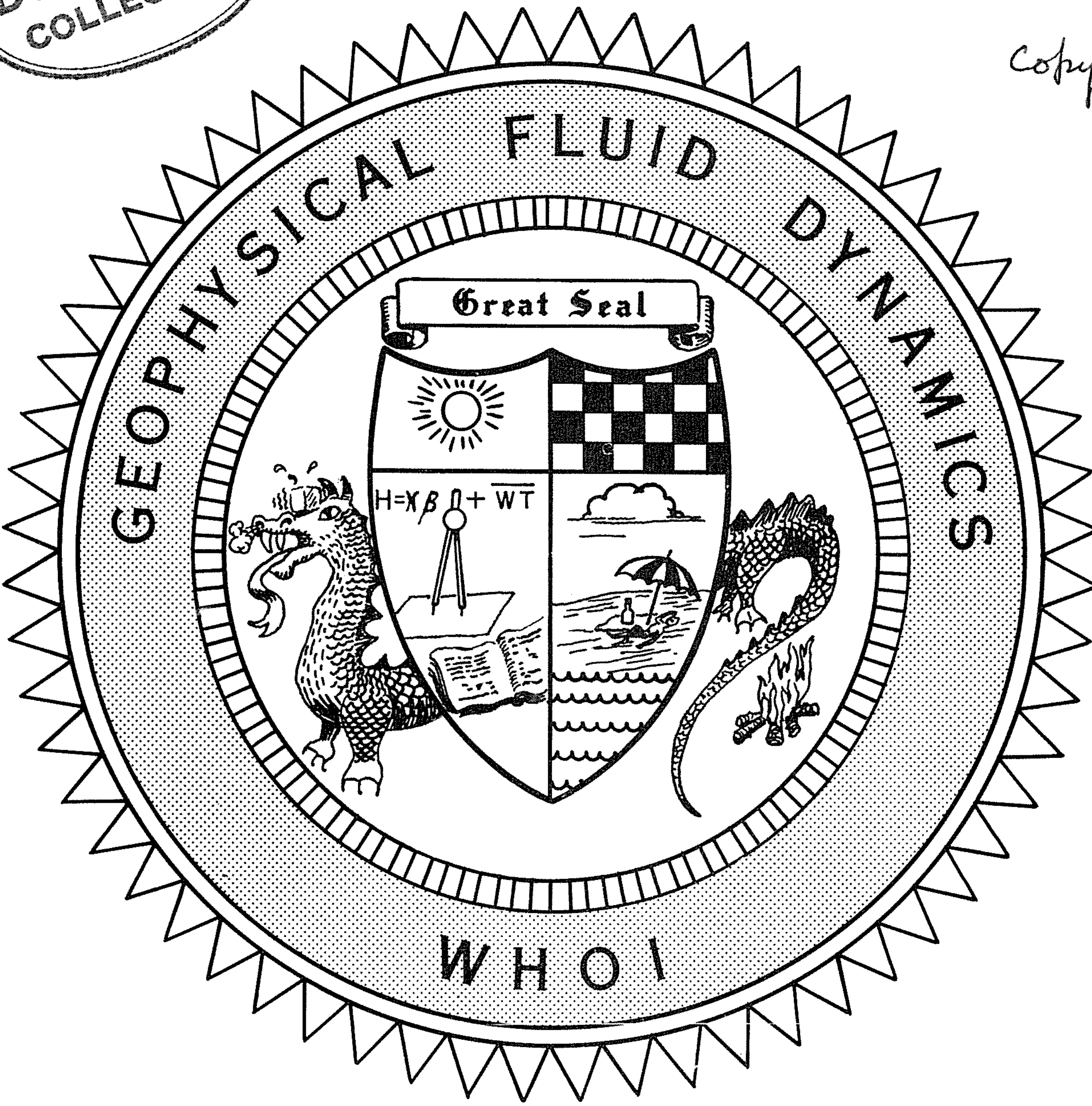

STUDENT LECTURES

Notes on the 1960

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

Edited by

E. A. Spiegel

Institute for Mathematical Sciences

Contents of the Volumes

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Editor's Preface

The past decade has brought an exciting upsurge of interest and research in geophysical fluid dynamics. This development has been particularly manifested by the activities and enthusiasms of a growing number on the staff of the Woods Hole Oceanographic Institution, with the result that many scientists interested in fluid dynamics have become frequent visitors there. In summer especially, the regular staff of the Institution has provided a nucleus for lively gatherings of oceanographers, meteorologists, physicists, mathematicians, and even astrophysicists.

Against this background of meeting and discussion students, sponsored by the Institution's summer fellowship program, have come to take part in the various research programs that develop. As the size of the summer group has increased the danger has arisen that the summer fellowship students might become lost in milieu of high level discussion, and not profit adequately from their efforts. Accordingly, the possibility of providing tangible opportunities for the training of summer fellowship students was explored, and it was decided to institute a summer course in geophysical fluid dynamics.

The first course in geophysical fluid dynamics at Woods Hole was given in the summer of 1959 by staff members of the Institution and some of their summer colleagues. The participants numbered about twenty and included four graduate-student and two postdoctoral fellowship holders provided for by funds from the National Science Foundation. At that time the dragon which adorns the cover of this volume was born. He was created by Prof. Henry Stommel in recognition of the efforts of Prof. Willem Malkus in organizing the course. The success which this first course enjoyed accounts for the reappearance of our dragon in its present position of prominence. For in the summer of 1960, a second course was given whose contents are outlined in the present notes.

These notes were prepared by the students, whose names are given above, with the capable assistance of Mrs. Mary Thayer. They were designed as working notes to be of use during the course. For each series of lectures, two students accepted the responsibility of preparing the notes and it was attempted (with surprising success) to have the notes typed, duplicated and distributed within four days after each lecture. Naturally, such a project could be completed only with rough edges, but the final collection of notes has succeeded very well in presenting the essential content and spirit of the course. They have therefore been assembled in limited number for use by interested persons.

It has seemed worthwhile to divide the notes into three volumes to avoid making them too cumbersome for easy reference. The division of material reflects the structure of the course. In Volume I we have an introduction to the subject as given by the invited lecturer, Professor L.N. Howard. The second volume contains notes on the more specialized lectures given by various staff members of the Institution. Finally, the manuscripts summarizing the student research lectures are reproduced in

Volume III. The topics discussed by the students were either selected by them or suggested by staff members.

Those of us from other institutions who have participated in this course have been treated to an abundant bill of fare, as a look at these notes will attest. For this, we can but express our gratitude to Dr. Willem Malkus and the other staff members of the Oceanographic Institution for their extensive efforts. We are also indebted to the Institution itself for its hospitalities and facilities. Finally, we should like to thank the National Science Foundation for providing funds for student fellowships and the support of an invited lecturer.

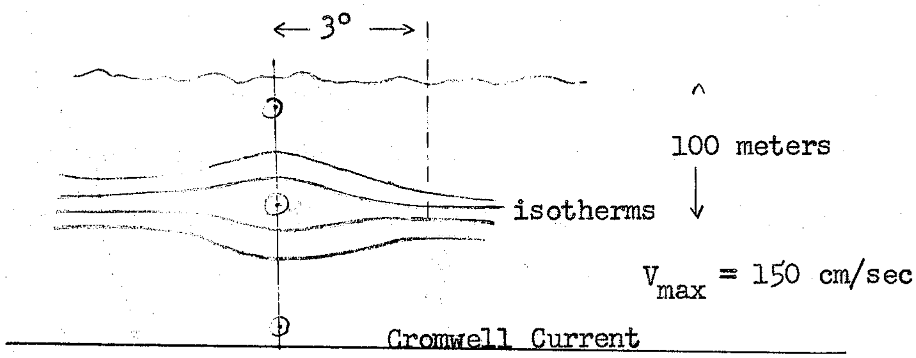
E. A. Spiegel
Inst. for Mathematical Sciences
Sept. 1960.

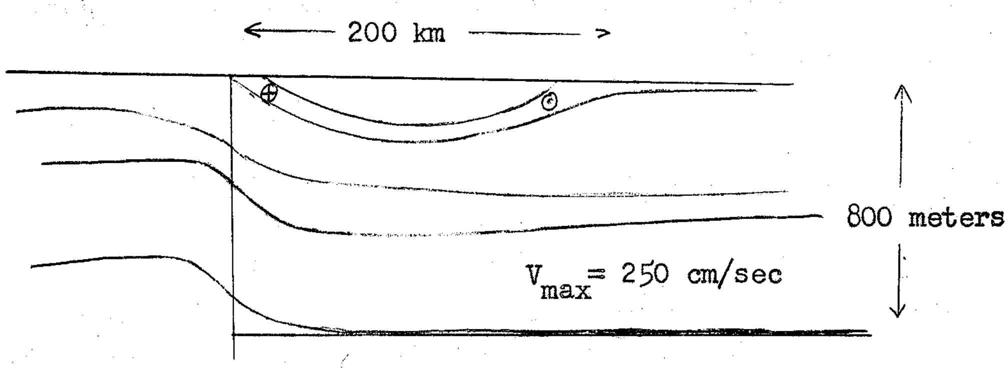
Contents of Volume III

Student Lectures

Ocean Current Models using Potential Vorticity

Cross sections of the Gulf Stream and Cromwell Currents are seen in Fig. 1.





These sections show the isotherms which to a large degree parallel the lines of constant density. In this paper methods are outlined which enable one to calculate the density and velocity cross sections using as input data density sections along two vertical lines.

The conservation of potential vorticity is derived in Stommel's book, The Gulf Stream, on page 108. Assuming eqns. of the form:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_1}\frac{\partial p}{\partial x} + X$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_1}\frac{\partial p}{\partial y} + Y \qquad (1)$$

Then if a parcel of water flows from point to point in a layer of density ρ_1, thickness D, we get:

$$\frac{d}{dt}\,\frac{f+\zeta}{D} = \frac{1}{D}\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad (2)$$

Stommel's model of the Gulf Stream then assumes that v and $\frac{\partial v}{\partial x}$ vanish far to the right of the stream. At such a point the isotherms are parallel and so, neglecting friction, if the fluid moves slowly across the stream,

$$\frac{f + \frac{\partial v}{\partial x}}{h} = \frac{f}{h_o} \qquad (3)$$

we are to have a level of no motion below the moving layer and so, noting Fig. 2:

$$(D - h)\rho_2 = D\rho_1$$

Now, below the free surface,

$$\frac{\partial p}{\partial x} = g\rho_1 \frac{\partial h}{\partial x}$$

If the velocity is only in the y direction and uniform, all non-linear $(\vec{u}\cdot\nabla)\vec{u}$ terms vanish and the geostrophic eqn. $\rho f v = \frac{\partial p}{\partial x}$ is valid.

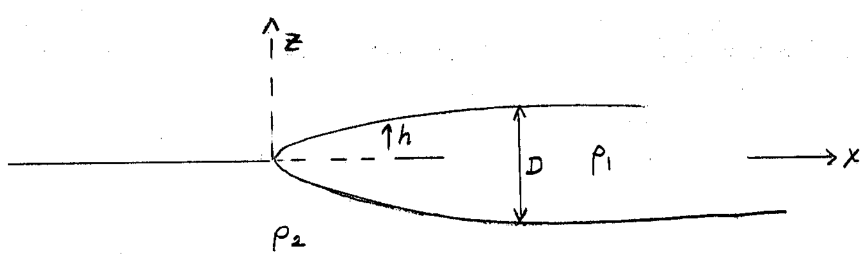


Fig. 2.

Combining the last three equations we get

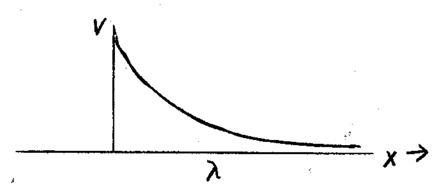
$$fv = g' \frac{\partial D}{\partial x} \qquad g' = g\,\frac{\rho_2 - \rho_1}{\rho_2} \qquad (4)$$

Combining (4) and (3) we get

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{\lambda^2}(D - D_o) \qquad \lambda^2 = \frac{g' D_o}{f^2}$$

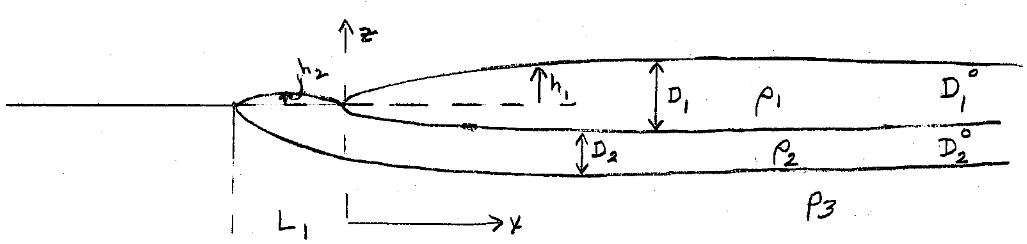
Imposing the boundary condition at infinity:

$$D = D_o(1 - e^{-x/\lambda}) \qquad v = \sqrt{g' D_o}\, e^{-x/\lambda}$$



Introducing realistic values gives a maximum velocity of 4 meters/second.

Two layer models



(3)

If we have two layers, the level of no motion gives the requirement:

$$(D_2 - h_2)\rho_3 = D_1\rho_1 + D_2\rho_2$$

We have also:

$$\frac{\partial p_1}{\partial x} = g\frac{\partial h_1}{\partial x}\rho_1 \qquad \frac{\partial p_2}{\partial x} = g\frac{\partial h_1}{\partial x}\rho_1 + g\frac{\partial h_2}{\partial x}(\rho_2 - \rho_1)$$

Combining these slightly more complicated equations as before together with the approximation $\quad \dfrac{\rho_2 - \rho_1}{\rho_2} \approx \dfrac{\rho_2 - \rho_1}{\rho_3} \quad$ etc we get

$$v_1 = \frac{g}{f}\left[\Delta\rho_1 \frac{\partial D_1}{\partial x} + \Delta\rho_2 \frac{\partial D_2}{\partial x}\right] \qquad \Delta\rho_1 = \frac{\rho_3 - \rho_1'}{\rho_3}$$

$$v_2 = g/f\left[\Delta\rho_2 \frac{\partial D_1}{\partial x} + \Delta\rho_2 \frac{\partial D_2}{\partial x}\right] \qquad (5)$$

and

$$\gamma \frac{\partial^2 D_1}{\partial x^2} + \frac{\partial^2 D_2}{\partial x^2} = \frac{R_2^2}{\beta}(D_1 - \beta D_2^\circ) \qquad R_2^2 = \frac{f^2}{\Delta\rho_2 g D_1^\circ}$$

$$\frac{\partial^2 D}{\partial x^2} + \frac{\partial^2 D_2}{\partial x^2} = R_2^2(D_2 - D_2^\circ) \qquad \frac{D_1^\circ}{D_2^\circ} = \beta \qquad (6)$$

$$\frac{\Delta\rho_1}{\Delta\rho_2} = \gamma$$

In Fig. 3, to the left of x equal 0 we have the same one-layer equations as before. To solve the system it will first be necessary to solve the system of mixed simultaneous equations, and then to impose boundary conditions. First we solve the simultaneous system. This is easily done: e.g. Hildebrand Advanced Calculus for Engineers p. 21.

The soln of the homogeneous equations is of the form:

$$D_1 = a_1 e^{\partial_1 x} + a_2 e^{-\partial_1 x} + a_3 e^{\partial_2 x} + a_4 e^{-\partial_2 x}$$

$$D_2 = b_1 e^{\partial_1 x} + b_2 e^{-\partial_1 x} + b_3 e^{\partial_2 x} + b_4 e^{-\partial_2 x}$$

where the ∂'s satisfy the determinental requirement:

$$\begin{vmatrix} \gamma\partial^2 - R_2^2/\beta & \partial^2 \\ \partial^2 & \partial^2 - R_2^2 \end{vmatrix} = 0$$

or;

$$(\gamma - 1)\partial^2 - R_2^2(\gamma + 1/\beta) + \frac{R_2^4}{\beta} = 0$$

to determine the connections between the 8 unknown coefficients a_1, b_1 we substitute the above solns into eqn. (6) and set the coeff. of each different exponential equal to 0. If we also insist that the soln remain finite at x equal infinity we get: $a_1 = b_1 = a_3 = b_3 = 0$

$$a_2 = \frac{R_2^2 - \partial_1^2}{\partial_1^2} b_2 \qquad a_4 = \frac{R_2^2 - \partial_2^2}{\partial_2^2}$$

Then the soln in the region of 2 layers is, adding the particular soln D_2^0 or D_1^0:

$$D_1 = b_2 \frac{R_2^2 - \partial_1^2}{\partial_1^2} e^{-\partial_1 x} + b_4 \frac{R_2^2 - \partial_2^2}{\partial_2^2} e^{-\partial_2 x} + D_1^0$$

$$D_2 = b_2 e^{-\partial_1 x} + b_4 e^{-\partial_2 x} + D_2^0 \tag{7}$$

The solution to the left of x equal 0 is: if $D_2 = 0$ @ $x = -L_1$

$$D_2 = B\left[e^{-R_2 x} - e^{R_2(2L_1 + x)}\right] + D_2^0\left[1 - e^{R_2(L_1 + x)}\right]$$

Now what boundary conditions shall we impose?

First suppose that the bottom layer never reaches the surface to the left, but instead extends to infinity. $L_1 = -\infty$. Then we impose v equal 0 at x equal $-\infty$. We also impose continuity of v_2 and D_2 at x equal 0 in the bottom layer, and D_1 equal 0 at x equal 0.

If we prescribe as typical values:

$\Delta\rho_1 = \Delta\rho_2 = 10^{-3}$ $\quad D_1^0 = D_2^0 = 400$ meters

$f = 10^{-4}_{sec^{-1}}$, $\quad g = 10$ meters/sec^2

Then as a soln we get:

$$D_2^{left} = \tfrac{1}{2}\left[188.9\, e^{x/20000} + 800\right] \text{ meters} \tag{8}$$

$$D_2^{right} = \tfrac{1}{2}\left[494.5\, e^{-1.618x/20,000} - 305.5\, e^{-.618x/20000} + 800\right] \text{ meters}$$

$$D_1^{right} = \tfrac{1}{2}\left[-305.5\, e^{-1.618x/20000} - 454.5\, e^{-.618x/20000} + 800\right] \text{meters}$$

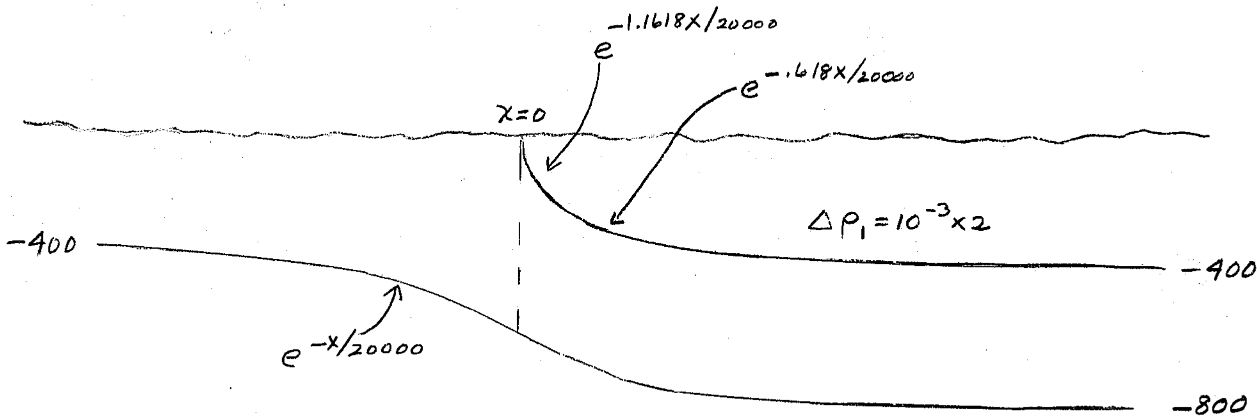
$$v_2^L = \frac{94.5}{200}\, e^{x/20000} \text{ meters/sec}$$

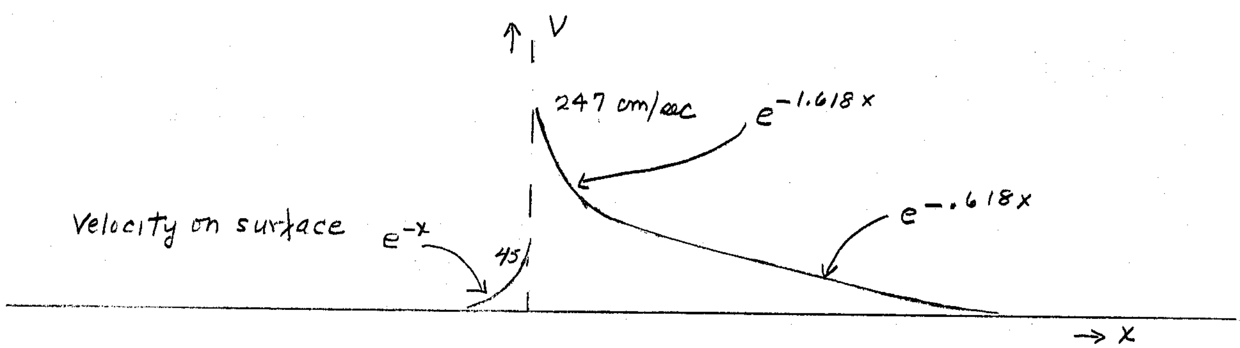
(8 cont')

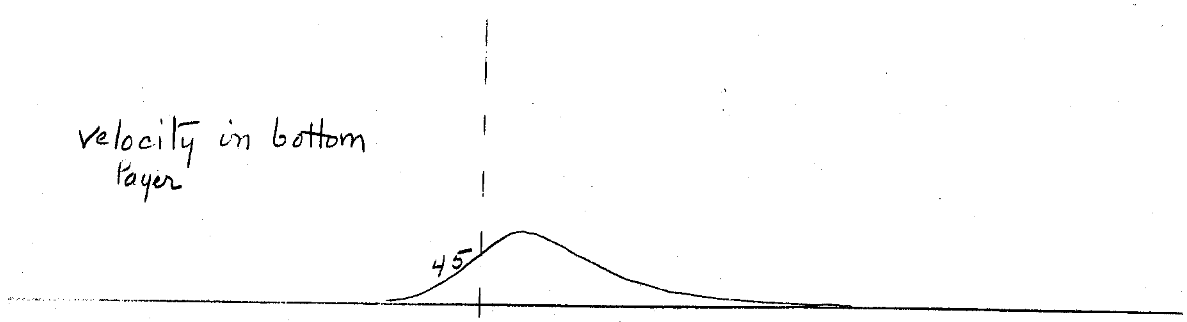
$$v_1^R = \frac{1}{200}\left[94.5\, e^{-1.618/20\overset{x}{0}00} + 400\, e^{-.618x/2000}\right]$$

$$v_2^R = \frac{1}{200}\left[-153\, e^{-1.618x/20000} + 247.2\, e^{-.618x/20000}\right]$$

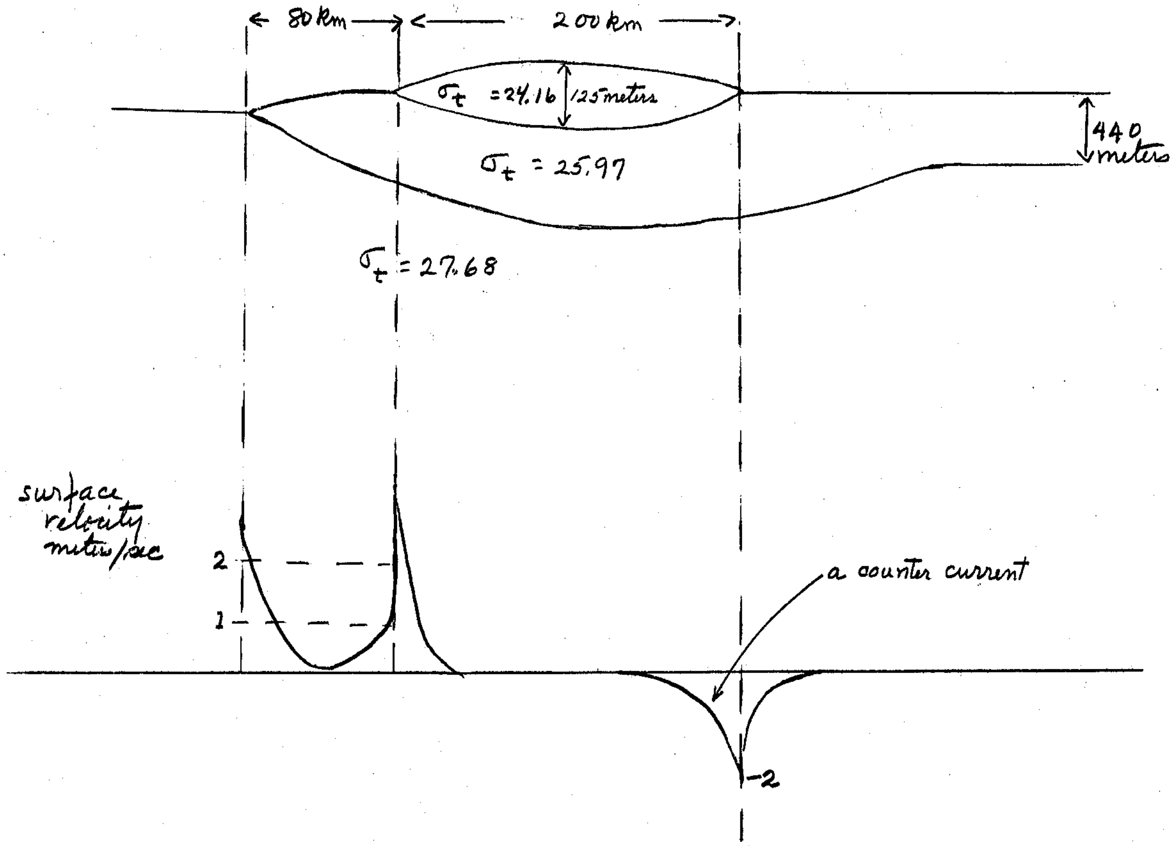
The maximum velocity is 2.47 meters per second, there is a velocity to the left of x equal 0, and the maximum velocity in the lower layer is displaced slightly to the right. Figure 4 shows graphs of this solution. I have shown by slightly tedious inspection of the formal solution of this problem that it is <u>impossible</u> to get a countercurrent in this model unless the density decreases downward.







If we next construct along these lines a model as pictured below, we get the results as graphed. The densities are integrated averages for the regions shown from the data of Worthington.

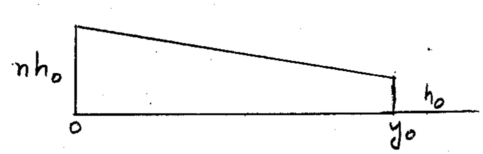


<u>Cromwell Current Models</u>

Perhaps along layers of equal density in the Cromwell Current we also have conservation of potential vorticity. We write:

$$\frac{\beta y - \partial u/\partial y}{h} = \frac{\beta y_0}{h_0} \tag{9}$$

Now looking at Fig. 1, we see that the isotherms are nearly straight lines so we conjecture that the thickness varys as in the graph below:



i.e.

$$h = nh_0 - y/y_0 (n-1) h_0 \tag{10}$$

(I should mention that of course potential vorticity cannot be conserved across the equator since on one side it is positive and on the other negative. But perhaps, we say, there is no mixing across the equator.)

Inserting (10) into (9) we get:

$$\frac{\partial u}{\partial z} = -\beta n \,(y-y_o)$$

Integrating and let u equal 0 at y equal y_o, get:

$$u = \frac{\rho(n)}{2}(y-y_o)^2$$

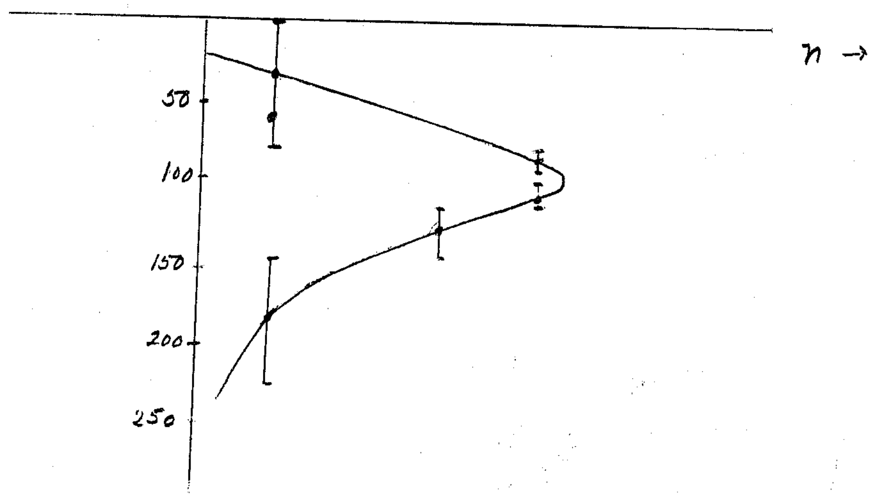
Looking at Knauss' data we see that at y equal three degrees u is approximately 0. Using Knauss' data also to determine n as a function of z we find:

z_o		n
0		
	------	.6?
62		
	------	3.1
90		
	------	3.1
108		
	------	2.1
120		
	------	.6
220		

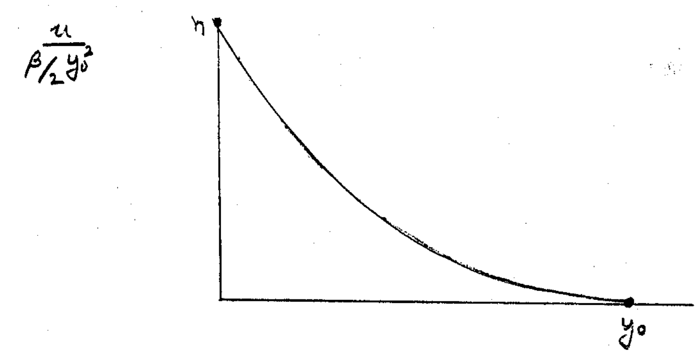
The maximum velocity then is expected to be

$$\frac{1}{2}\, n_{max} \frac{2\Omega}{R} y_o^2 = \frac{1}{2}\, 3.1 \times \frac{2\times 7.29\times 10^{-5}\times(180\times 1.85)^2}{6.36\times 10^3} \times 10^5 \text{cm/sec} = 416 \text{cm/sec}$$

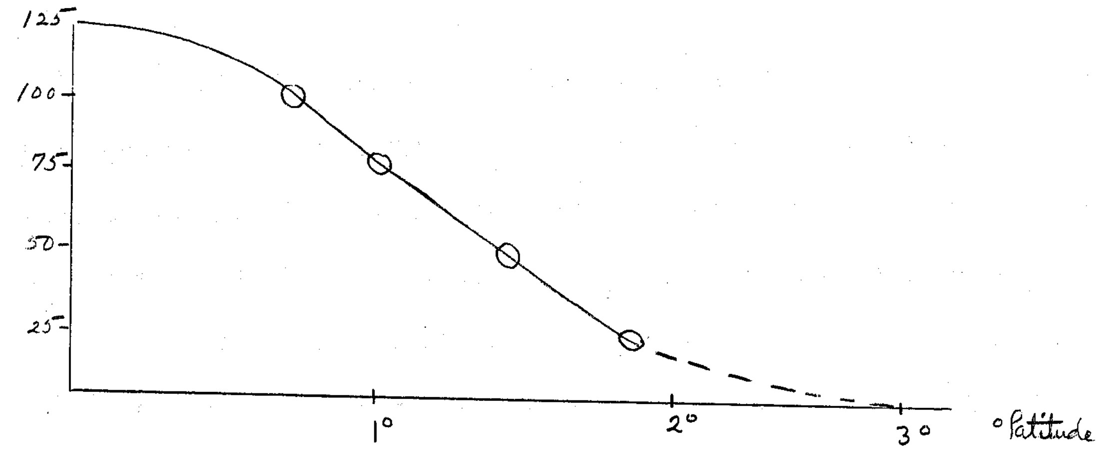
The vertical profile of the velocity should look like:

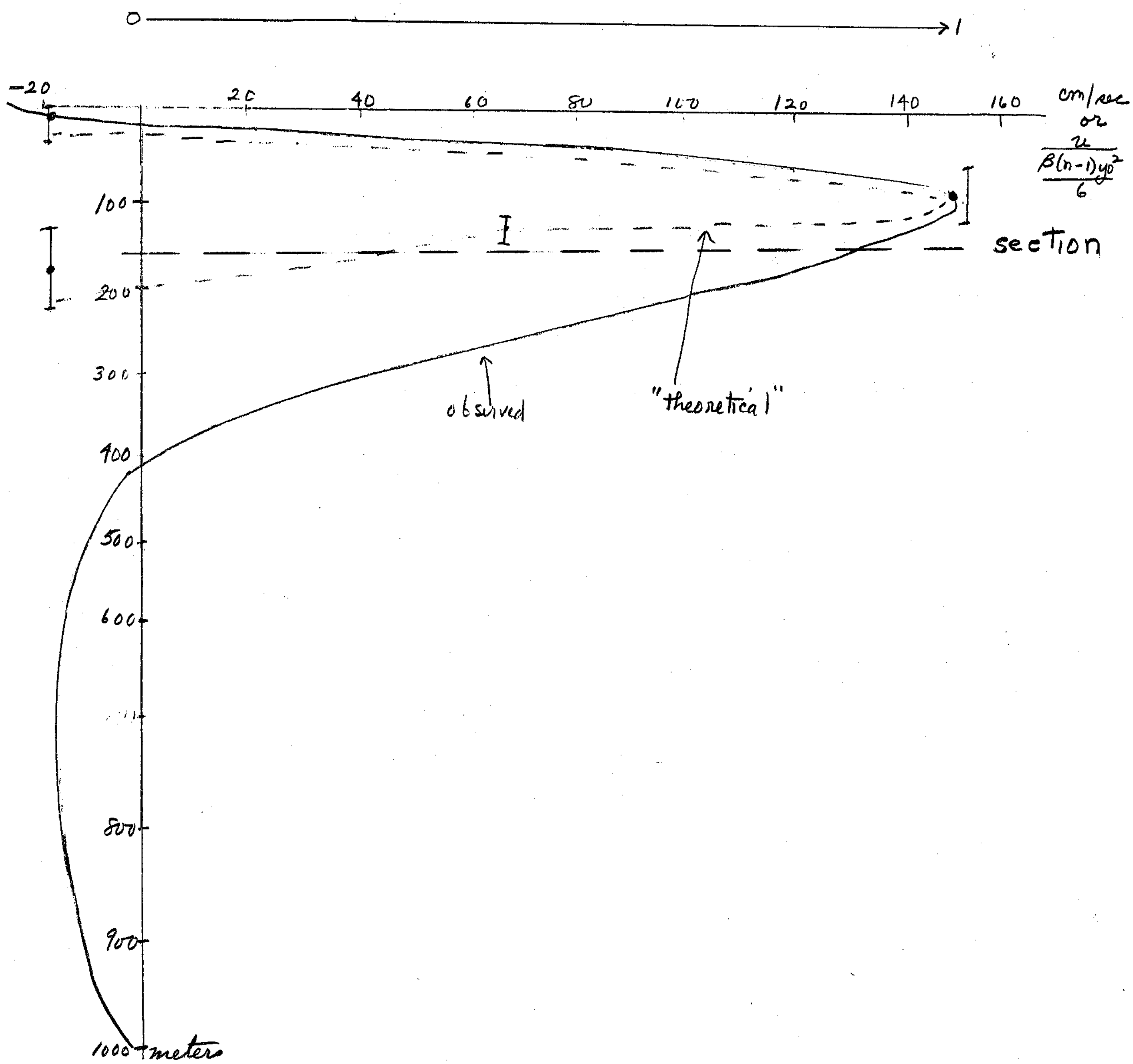


While the horisontal profile along a line of constant density should look like:



The form of the vertical velocity profile is in good agreement with observation except that it gives no countercurrent and of course the velocities are much too large. The horizontal velocity profile instead of having $\frac{\partial u}{\partial y} = 0$ at y equal 0, has a strong slope, showing the discontinuity of potential vorticity. Also it is more concave than the observed cross-stream velocity profile. The observed graphs are sketched on the following page. The horizontal section is taken about 10 meters below the core.


125
100
75
50
25
1°
2°
3°
° latitude


0
1
-20
20
40
60
80
100
120
140
160
cm/sec
or
$\dfrac{u}{\frac{\beta(n-1)y_0^2}{6}}$
100
200
300
400
500
600
800
900
1000 meters
section
observed
"theoretical"

What are we to do about the observed disagreement? For the following step there exist a number of unconvincing heuristic, semi-mathematical reasons. One notes that friction acting on the right side of eqn.(2) could reduce the potential vorticity. Also, one conjectures that the potential vorticity at 0 degrees might be zero"because it is a mixture of two waters of opposite vorticity". Perhaps a "steady-state" diffusion of vorticity exists from right to left. In any event, write:

$$\frac{\beta y - \frac{\partial u}{\partial y}}{h} = \frac{\beta y}{h_0} \tag{11}$$

Introducing as before:

$$h = n h_0 - y/y_0 \,(n-1)\, h_0$$

get:

$$\frac{\partial u}{\partial y} = \beta (n-1)\left(\frac{y^2}{y_0} - y\right)$$

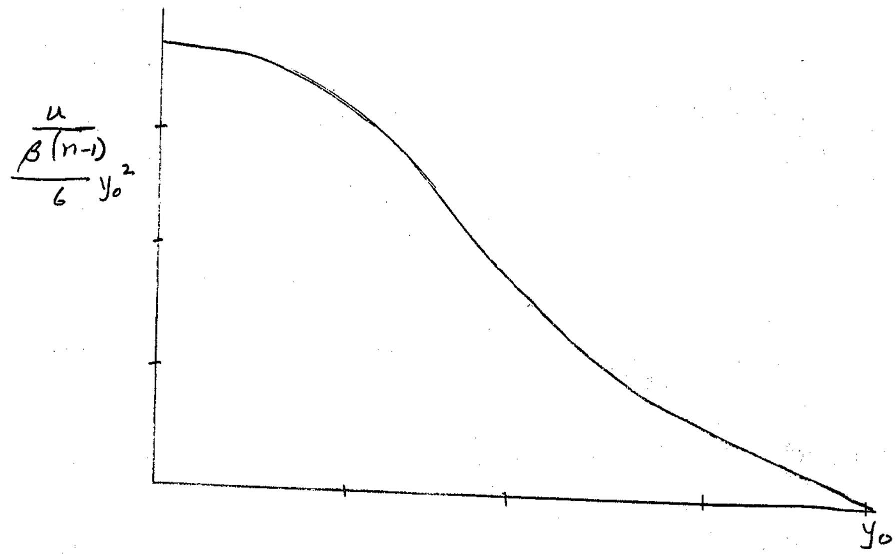
Integrating and set u equal 0 at y equal y_0. Get

$$u = \frac{\beta (n-1)}{6}\left[y_0^2 + 2\, y^3/y_0 - 3 y^2 \right] \tag{12}$$

Now maximum velocity is:

$$\frac{\beta (n-1)}{6} y_0^2 = 93.9 \text{ m/sec}$$

Horizontal profile looks like:



This shows better agreement, $\frac{\partial u}{\partial y}$ is 0 at y equal 0. I can believe that the other discrepancies could be brought into perfect agreement if I had the original data. They could also be thrown completely off, -- . Now we get a counter-current and a more nearly correct velocity cross section. I have tried introducing a parabolic dependence of h and it only changes the maximum velocity by 25%. Thus the form detailed of h is not too important.

We might try to do a one and/or two-layer model of the Cromwell current as we did for the Gulf Stream, I have constructed a one-layer model. A two-layer model would be a simple extension.

Write

$$\frac{\beta y - \frac{\partial u}{\partial y}}{h} = \frac{\beta y}{h_o}$$

and

$$\beta y u = -g' \frac{\partial h}{\partial y}$$

combining, we get:

$$\frac{\partial^2 h}{\partial y^2} - \frac{1}{y}\frac{\partial h}{\partial y} - \frac{\beta^2 y^2}{g' h_o} h = -\frac{\beta^2 y^2}{g'} \qquad (13)$$

introduce the non-dimensional variables

$$h \equiv \frac{h}{h_o} \qquad x = y\alpha^{1/4} \qquad \alpha = \frac{\beta^2}{g'h_o}$$

the eqn becomes:

$$x^2 \frac{\partial^2 h}{\partial x^2} - x\frac{\partial h}{\partial x} - x^4 h = -x^4 \qquad (14)$$

we have boundary conditions at $x = 0$ and $x = y_o \alpha^{\frac{1}{4}} = a$

if we set y_o equal $3°$ we find: $a = 1.56$

The solution to equation (14) is:

$$h = C_1 \sinh x^2/2 + C_2 \cosh x^2/2 + 1$$

There are two possible sets of boundary conditions:

1. $u = 0$ @ $x = a$ + $h = n$ @ $x = 0$

2. $h = 1$ @ $x = a$ + $h = n$ @ $x = 0$

For set 1. we get

$$h = -(n-1)\,h_o \left\{\tanh a^2/2 \sinh x^2/2 - \cosh x^2/2\right\} + h_o$$

$$u = \sqrt{g'h_o}\,(n-1) \left\{\tanh a^2/2 \cosh x^2/2 - \sinh x^2/2\right\} \qquad (15)$$

For set 2. we get:

$$h = -(n-1)h_o \left\{\coth a^2/2 \sinh x^2/2 - \cosh x^2/2\right\} + h_o$$

$$u = \sqrt{g'h_o}\,(n-1) \left\{\coth a^2/2 \cosh x^2/2 - \sinh x^2/2\right\} \qquad (16)$$

Note that for 1. h does not go to h_o at x equal a while for 2. u does not go to zero @ $x = a$. In fact 1. $h(a) = 2.1\, h_o$

2. u(a) = 120 cm/sec. Since this u is impossibly large it seems best to require 1. as the boundary conditions. In this case the maximum velocity is: 120 ± 50 cm/sec. The thickness goes down somewhat like a cubic from 3.1 h_o to about 2.1 h_o at x equals a, which is clearly artificial and rather strange. This single layer seems to be unable to satisfy all experimental conditions at once. A similar situation, although perhaps not so bad would surely arise in a two-layer model. g' was evaluated by using as $\Delta\rho$ the sum of the differences between the central layer and the upper and lower layers of the Cromwell Current.

Continuous Models

The following work is a minor extension of some work done by Rossby in 1937. (J.Mar.Res. Vol. 1, p.239). Imagine that we have a stratified ocean with greatest density ρ_b and least density ρ_h.
We define a vertical coordinate r such that:

$$r = \frac{\rho_b - \rho}{\rho_b - \rho_h}$$

we also define κ such that:

$$\rho_b - \rho_h = 2\kappa\rho_b$$

and observe:

$$\rho = \rho_b(1 - 2\kappa r)$$

Remembering the conservation of potential vorticity: $\frac{f + \frac{\partial u}{\partial x}}{h} = \frac{f}{h_o}$

Rewriting this: $\frac{\partial u}{\partial x} = f\left(\frac{h - h_o}{h_o}\right)$

For the continuous case, we replace h by: $h = \frac{-\partial z}{\partial\rho} d\rho$

get: $$\frac{\partial u}{\partial x} = \left(\frac{\partial z/\partial\rho - \left(\frac{\partial z}{\partial\rho}\right)_o}{\left(\frac{\partial z}{\partial\rho}\right)_o}\right)$$

define: $$z_o = g_o(\rho_b(1 - 2\kappa r)) = g_o(r)$$

so:
$$\frac{\partial v}{\partial x} = \left[\frac{\frac{\partial z}{\partial r} - \frac{\partial g_0}{\partial r}}{\partial g_0 / \partial r} \right]$$

define:
$$\Delta(r) = z(r) - g_0(r)$$

so
$$\frac{\partial v}{\partial x} = f \frac{\frac{\partial \Delta}{\partial r}}{\partial g_0 / \partial r} \tag{18}$$

Now the pressure at Z is:
$$p = \int_z^{surface} \rho \, dz$$

Integrating by parts:
$$p = g\rho_s h - g\rho_z z - g \int_\rho^{\rho_s} z \, d\rho$$

Using the fact that:
$$\left(\frac{\partial p}{\partial x}\right)_\rho = \left(\frac{\partial p}{\partial x}\right)_z - g\rho \frac{\partial z(\rho)}{\partial x}$$

We get:
$$\left(\frac{\partial p}{\partial x}\right)_z = g\rho_s \frac{\partial h}{\partial x} + 2k\rho_b g \left\{ \int_r^{r_s} \frac{\partial \Delta}{\partial x} \, dr + z_s \frac{\partial r_s}{\partial x} \right\} \tag{18a}$$

For the Gulf Stream
$$\rho f v = \left(\frac{\partial p}{\partial x}\right)_z$$

so
$$\left(\frac{\partial v}{\partial x}\right)_\rho = \frac{1}{\rho f} \left(\frac{\partial}{\partial x}\right)_\rho \left(\frac{\partial p}{\partial x}\right)_z$$

or
$$\frac{\partial v}{\partial x} = \frac{g}{f} \frac{\partial^2 h}{\partial x^2} + \frac{2k\rho_b g}{f} \left\{ \int_r^{r_s} \frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial \Delta}{\partial x} \frac{\partial r_s}{\partial x} + z_s \frac{\partial^2 r_s}{\partial x^2} + \frac{\partial z_s}{\partial x} \frac{\partial r_s}{\partial x} \right\} \tag{18b}$$

$$\frac{\partial^2 v}{\partial x \partial r} = \frac{\partial \zeta}{\partial r} = \frac{2\kappa\rho_b}{f}\left\{-\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial x \partial r}\frac{\partial r_s}{\partial x}\right\} \tag{19}$$

but from (18)

$$\frac{\partial^2 v}{\partial x \partial r} = \frac{f\frac{\partial^2 \Delta}{\partial r^2}}{\partial g_o/\partial r} - \frac{f\frac{\partial \Delta}{\partial r}\frac{\partial^2 g_o}{\partial r^2}}{\left(\frac{\partial g_o}{\partial r}\right)^2} \tag{20}$$

so

$$\boxed{\frac{\partial^2 \Delta}{\partial r^2} + \frac{2\kappa\rho_o g_o'}{f^2}\left\{\frac{\partial^2 \Delta}{\partial x^2} - 1\frac{\partial^2 \Delta}{\partial x \partial r}\frac{\partial r_s}{\partial x}\right\} - \frac{g_o''}{g_o'}\frac{\partial \Delta}{\partial r} = 0} \tag{21}$$

Now to get the boundary condition on the <u>free</u> <u>surface</u>

Remember $\Delta = z(r) - g_o(r)$

$@\ r = r_s\ ;\ z = h$

Using (18a)

$$fv = g\frac{\partial h}{\partial x} + 2\kappa g h\frac{\partial r_s}{\partial x}$$ since the integral vanishes

but $\dfrac{\partial v}{\partial x} = \dfrac{f\frac{\partial \Delta}{\partial r}}{g_o'}$

so $$\frac{f\frac{\partial \Delta}{\partial r}}{g_o'} = \frac{g}{f}\frac{\partial^2 h}{\partial x^2} + \frac{2\kappa g}{f}\left(\frac{\partial h}{\partial x}\frac{\partial r_s}{\partial x} + h\frac{\partial^2 r_s}{\partial x^2}\right)$$

On surface

$$\Delta_s = h - g_o[r_s(x)] \quad \text{or} \quad h = \Delta_s + g_o[r_o(x)]$$

This gives: from previous eqn

$$\frac{f\frac{\partial \Delta}{\partial r}}{g_o'} = \left\{\frac{g}{f}\left[\frac{\partial^2 \Delta_s}{\partial x^2} + \frac{\partial^2 r_s}{\partial x^2}\frac{\partial g_o}{\partial r} + \left(\frac{\partial r_s}{dx}\right)^2\frac{\partial^2 g_o}{\partial r^2}\right] + \frac{2\kappa g}{f}\left[\frac{\partial r_s}{\partial x}\left(\frac{\partial \Delta_s}{\partial x} + \frac{\partial r_s}{\partial x}\frac{\partial g_o}{\partial r}\right) + (\Delta_s + g_o)\frac{\partial^2 r_s}{\partial x^2}\right]\right\} \tag{22}$$

Introduce $\Delta' \to \frac{\Delta}{h_o}$ $\quad g_1 = \frac{g_o}{h_o}$

$x' \to \frac{x}{\sqrt{\frac{2\kappa g h_o}{f^2}}}$ $\quad h_o$ = depth from $z_o(\rho_n)$ to $z_o(\rho_b)$

get

$$\boxed{\begin{aligned} &\frac{\partial^2\Delta}{\partial r^2} + g_1'\left\{\frac{\partial^2\Delta}{\partial x^2} - \frac{\partial^2\Delta}{\partial r\partial x}\frac{\partial r_s}{\partial x}\right\} - \frac{g_1''}{g_1'}\frac{\partial\Delta}{\partial r} = 0 \qquad (21a)\\ &\frac{\partial\Delta_s}{\partial r} = \frac{1}{2\kappa}\left\{\frac{\partial^2\Delta_s}{\partial x^2} + \frac{\partial^2 r_s}{\partial x^2}\frac{\partial g_1}{\partial r} + \left(\frac{\partial r_s}{\partial x}\right)^2\frac{\partial^2 g_1}{\partial r^2}\right\} \qquad (22a)\\ &+\left\{\frac{\partial r_s}{\partial x}\left(\frac{\partial\Delta_s}{\partial x} + \frac{\partial r_s}{\partial x}\frac{\partial g_1}{\partial r}\right) + (\Delta_s + g_1)\frac{\partial^2 r_s}{\partial x^2}\right\}\end{aligned}}$$

we seek a simple expression for the velocity. Assume a level of no motion at r equal r_b.

Then for r equal r_b

$$\int_r^{r_s} \rho\, dz + \rho_b g z_b = \text{cost},$$

Consider the variation of this quantity along a line of constant density.

Integrating by parts and differentiating:

$$\left(\frac{\partial}{\partial x}\right)_\rho = g\rho_s\frac{\partial h}{\partial x} + 2\kappa\rho_o g\left\{\int_r^{r_s}\frac{\partial\Delta}{\partial x}dr + z_s\frac{\partial r_s}{\partial x}\right\}$$

if $r = r_b$, setting $\frac{\partial}{\partial x} = 0$

$$g\rho_s\frac{\partial h}{\partial x} + 2\kappa\rho_b g z_s\frac{\partial r_s}{\partial x} = -2\kappa\rho_b g\int_{r_b}^{r_s}\frac{\partial\Delta}{\partial x}dr$$

Using (18a) we get

$$\left(\frac{\partial p}{\partial x}\right)_z = -2\kappa\rho_b g\int_{r_b}^{r}\frac{\partial\Delta}{\partial x}dr \qquad (23)$$

$$\therefore \quad v = -\frac{2kg}{f}\int_{r_b}^{r}\frac{\partial\Delta}{\partial x}\,dr \tag{24}$$

or, introducing dimensionless numbers $\Delta + x$

$$\boxed{v = -\sqrt{2kgh_0}\int_{r_b}^{r}\frac{\partial\Delta}{\partial x}\,dr}$$

Introducing assumption (23) into (18a) enables us to simplify our final equation and surface boundary condition.

We have

$$\rho f v = -2k\rho g\int_{r_b}^{r}\frac{\partial\Delta}{\partial x}\,dr$$

so $\dfrac{\partial^2 v}{\partial x\,\partial r} = -\dfrac{2kg}{f}\dfrac{\partial^2\Delta}{\partial x^2} \qquad \left(\dfrac{\partial\rho_b}{\partial x}=0\right)$

In non-dimensional form (21) becomes:

$$\boxed{\frac{\partial^2\Delta}{\partial r^2} + g_1'\frac{\partial^2\Delta}{\partial x^2} - \frac{g_1''}{g_1'}\frac{\partial\Delta}{\partial r} = 0} \tag{21'}$$

and (22) becomes or we see below

$$\boxed{g_1'\frac{\partial^2\Delta}{\partial x\,\partial r}\frac{\partial r_s}{\partial x} + \frac{\partial^2\Delta}{\partial r^2} - \frac{\partial\Delta}{\partial r}\frac{g_1''}{g_1'} = 0} \tag{22'}$$

Proof of (22)'

$$\left(\frac{\partial v}{\partial x}\right)_s = -\frac{2kg}{f}\left\{\int_{r_b}^{r_s}\frac{\partial^2\Delta}{\partial x^2} + \frac{\partial\Delta}{\partial x}\frac{\partial r_s}{\partial x}\right\} = \frac{f\frac{\partial\Delta}{\partial r}}{g_0'}$$

take $\dfrac{\partial}{\partial r}$ of both sides and get (22)' since integral vanishes.

To get the bottom boundary condition in the case of no motion at the bottom, we write:

$$V = \int_{r_b}^{r} \frac{\partial \Delta}{\partial x}\, dr \quad \left(-\sqrt{2Kgh_0}\right)$$

$$\text{so} \quad \frac{\partial v}{\partial x} = \int_{r_b}^{r} \frac{\partial^2 \Delta}{\partial x^2} \left(-\sqrt{2Kgh_0}\right) \qquad \text{at } r_b,\ \frac{\partial v}{\partial x} = 0$$

but from (18)

$$\frac{\partial v}{\partial x} = \frac{F \frac{\partial \Delta}{\partial r}}{g_0'}$$

$$\therefore \frac{\partial \Delta}{\partial r} = 0 \quad @ \; r = r_b$$

For the boundary condition in the first case, it is seen that the bottom is no different from the rest of the water mass and so the boundary condition is just the differential equation. This seems strange and the question is still not resolved in my mind. In any event, in practice the level of no motion will surely be used most of the time. One conjectures that replacing our free boundary condition by $\Delta(r) = -g_0(r)$ would not change the velocity values very much, even though such a boundary condition sets h equal to 0 at the free surface, because the velocity is calculated by integrating $\frac{\partial \Delta}{\partial x}$ from top to bottom and one feels that this would not be changed very much by the above change in boundary condition. However, this is something to be checked.

For the simplest possible application of these results we consider a Gulf Stream with a linear density gradient, a level of no motion, isotherms coming to the surface in a linear manner over a finite region. The situation is as depicted below.

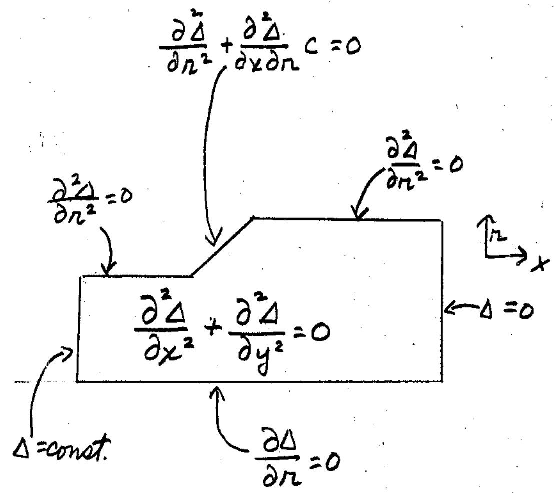
$$\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial y^2} = 0 \quad \text{in eqn.}$$

$$\frac{\partial^2 \Delta}{\partial x \partial r} \frac{\partial r_s}{\partial x} + \frac{\partial^2 \Delta}{\partial r^2} = 0 \quad \text{surface B.C.}$$

$$\frac{\partial \Delta}{\partial r} = 0 \qquad \text{bottom B.C.}$$

$$\Delta = 0 \qquad \text{Right hand B.C.}$$

$$\Delta = \text{const} \quad \text{Left hand BC}$$



$$V = -\sqrt{2 \kappa q h_0} \int_{r_b}^{r} \frac{\partial \Delta}{\partial x} \, dr$$

Perhaps the top B.C. could be replaced by

$$\Delta = -g_1(r) = \text{ in this case to } -r.$$

A Comparison of Steady Fluid Motion maintained by a Non-Uniform Wind Stress Distribution and Steady Motion maintained by a Non-Uniform Temperature Distribution

by

William R. Holland

1. Introduction

The problem in which we are interested is based on a simple model proposed by Lineykin in 1955 and further explored by Stommel and Veronis in 1957. We think of a fluid layer heated from above and cooled from below so that a large vertical thermal gradient is present and the fluid is in stable equilibrium.

This model is first to be explored when the surface is subjected to an infinitesimal wind stress distribution and then when it is subjected to an infinitesimal temperature perturbation. We will be interested in the form and vertical extent of the motions.

In particular we will consider only the case in which the motions are small and in which the disturbances in the temperature field are small compared to the basic uniform temperature field. Under these circumstances we shall use the following perturbation equations:

$$-f\rho_0 v = -\frac{\partial p}{\partial x} + \rho_0 \nu \frac{\partial^2 u}{\partial z^2} \qquad (1)$$

$$f\rho_0 u = -\frac{\partial p}{\partial y} + \rho_0 \nu \frac{\partial^2 v}{\partial z^2} \qquad (2)$$

linearized equations of motion

$$g\rho = -\frac{\partial p}{\partial x} \qquad (3)$$

hydrostatic equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$ continuity equation using Boussinesq approx. (ρ constant except when it interacts with g) (4)

$$w\frac{\partial \bar{\theta}}{\partial z}=\kappa\frac{\partial^2\theta}{\partial z^2}$$ vertical advection balanced by conductive perturbed heat transport (5)

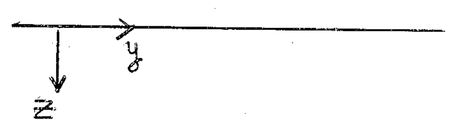
$$\rho=\rho_0(1-\alpha\theta)$$ simple equation of state (6)

The horizontal extent of the motions is considered to be very large compared to the vertical so only the z-component of the ∇ operator is preserved.

2. No Rotation

In the case of no rotation we can simplify the above equations. Let us consider a two-dimensional model (solutions independent of x) with f=0. Then the above equations become

$$\frac{\partial p}{\partial y}=\nu\rho_0\frac{\partial^2 v}{\partial z^2}$$



$$g\rho=-\frac{\partial p}{\partial z}$$

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$wb=\kappa\frac{\partial^2\theta}{\partial z^2}$$

$$\rho=\rho_0(1-\alpha\theta)$$

where b = mean vertical temperature gradient.

A. Surface wind stress distribution $T_y=\rho_0\nu\left(\frac{\partial v}{\partial z}\right)_{z=0}=T'\sin ky$

Eliminating all dependent variables except v, we find

$$\nu\rho_0\frac{\partial^3 v}{\partial z^3}=\rho_0 g\alpha\frac{\partial\theta}{\partial y}$$

$$\nu\rho_0\frac{\partial^5 v}{\partial z^5}=\rho_0 g\alpha\frac{b}{\kappa}\frac{\partial w}{\partial y}$$

$$\nu\rho_0\frac{\partial^6 v}{\partial z^6}=-\frac{\rho_0 g\alpha b}{\kappa}\frac{\partial^2 v}{\partial y^2}$$

Let the solution be of the form $v = V(z) \sin ky$. Then

$$\frac{d^6V}{dz^6} = \frac{g\alpha bk^2}{k\nu} V \tag{7}$$

If $F^6 = \frac{g\alpha bk^2}{k\nu}$, the solution has the form

$v = \sin ky \sum C_i e^{p_i Fz}$ where p_i are the 6th roots of unity.

Allowing only those solutions for which v vanishes as z gets very large, we find

$$v = \sin ky \left\{ C_1 e^{-Fz} + e^{-\frac{1}{2}Fz}\left(C_2 \cos \frac{\sqrt{3}}{2} Fz + C_3 \sin \frac{\sqrt{3}}{2} Fz\right)\right\} \tag{8}$$

The boundary conditions at z = 0 are:

$$\frac{\partial v}{\partial z} = \frac{T}{\rho_0 \nu} \sin ky$$

$$w = 0 \Rightarrow \frac{\partial^5 v}{\partial z^5} = 0$$

$$\theta = 0 \Rightarrow \frac{\partial^3 v}{\partial z^3} = 0$$

$\frac{\partial^5 v}{\partial z^5} = 0$ leads to $C_1 + \frac{1}{2}C_2 + \frac{\sqrt{3}}{2} C_3 = 0$

$\frac{\partial^3 v}{\partial z^3} = 0$ leads to $C_1 = C_2$

Combining these we find $C_1 = -\frac{\sqrt{3}}{3} C_3$.

Then

$$\frac{\partial v}{\partial z} = \frac{T}{\rho_0 \nu} \sin ky = -F C_1 \sin ky \left\{1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \frac{3}{\sqrt{3}}\right)\right\}$$

$$C_1 = -\frac{T}{3F\rho_0 \nu}$$

Hence

$$\boxed{v = -\frac{T}{3F\rho_0\gamma}\sin ky\left\{e^{-Fz} + e^{-\frac{1}{2}Fz}\left(\cos\frac{\sqrt{3}}{2}Fz - \sqrt{3}\sin\frac{\sqrt{3}}{2}Fz\right)\right\}}$$

From the equation of continuity we can find w:

$$\boxed{w = \frac{Tk}{3F^2\rho_0\gamma}\cos ky\left\{-e^{-Fz} + e^{-\frac{1}{2}Fz}\left(\cos\frac{\sqrt{3}}{2}Fz + \frac{3\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}Fz\right)\right\}}$$

We can also find θ from equation (5):

$$\boxed{\theta = -\frac{TF^2}{3\rho_0^2 g\alpha k}\cos ky\left\{e^{-Fz} + e^{-\frac{1}{2}Fz}\left(-\cos\frac{\sqrt{3}}{2}Fz + \sqrt{3}\sin\frac{\sqrt{3}}{2}Fz\right)\right\}}$$

B. Surface Temperature Distribution $\theta = \theta_o \cos ky$

Now if we eliminate all dependent variables except θ we find

$$\frac{d^6\Theta}{dz^6} = F^6\Theta \qquad \text{where } \theta = \Theta(z)\cos ky$$

This is the same differential equation as before and so the solution is of the form

$$\theta = \cos ky\left\{C_1 e^{-Fz} + e^{-\frac{1}{2}Fz}\left(C_2\cos\frac{\sqrt{3}}{2}Fz + C_3\sin\frac{\sqrt{3}}{2}Fz\right)\right\}$$

The boundary conditions at $z = 0$ are:

$$\theta = \theta_o \cos ky$$

$$w = 0 \implies \frac{\partial^2\theta}{\partial z^2} = 0$$

$$\frac{\partial v}{\partial z} = 0 \implies \frac{\partial^4\theta}{\partial z^4} = 0$$

$$\frac{\partial^2\theta}{\partial z^2} \text{ leads to } C_1 - \tfrac{1}{2} C_2 - \frac{\sqrt{3}}{2} C_3 = 0$$

$$\frac{\partial^4\theta}{\partial z^4} \text{ leads to } C_1 - \tfrac{1}{2} C_2 + \frac{\sqrt{3}}{2} C_3 = 0$$

$$2 C_1 - C_2 = 0$$

$$C_1 = \tfrac{1}{2} C_2$$

$$C_3 = 0$$

$$\theta = \theta_o \cos ky = \cos ky \, (C_1 + C_2) = 3C_1 \cos ky$$

$$C_1 = \frac{\theta_o}{3}$$

Hence

$$\boxed{\theta = \frac{\theta_o \cos ky}{3} \left\{ e^{-Fz} + 2e^{-\frac{1}{2}Fz} \cos\frac{\sqrt{3}}{2} Fz \right\}}$$

Using equation (5):

$$\boxed{w = \frac{KF^2\theta_o}{3b} \cos ky \left\{ e^{-Fz} + e^{-\frac{1}{2}Fz}\left(-\cos\frac{\sqrt{3}}{2} Fz + \sqrt{3} \sin\frac{\sqrt{3}}{2} Fz\right) \right\}}$$

Using the continuity equation:

$$\boxed{v = -\frac{KF^3\theta_o}{3bk} \sin ky \left\{ -e^{-Fz} + 2e^{-\frac{1}{2}Fz} \cos\frac{\sqrt{3}}{2} Fz \right\}}$$

C. <u>Discussion of Results</u>

Let us examine the form and vertical extent of the motions and temperature perturbation (See figures 1 and 2).

We see in the wind stress case a <u>deep</u> perturbation in the temperature field with the maximum at $Fz = 2.5$. In the other case, however, we have a quite different behavior, the perturbation falling off quite rapidly to zero (at about $F = 3.5$).

Looking at the z dependence of the horizontal velocity we find the opposite behavior from the above. At the surface both cases have maximum values for v but in case A the velocity amplitude drops off

θ - temperature perturbation
(wind stress)

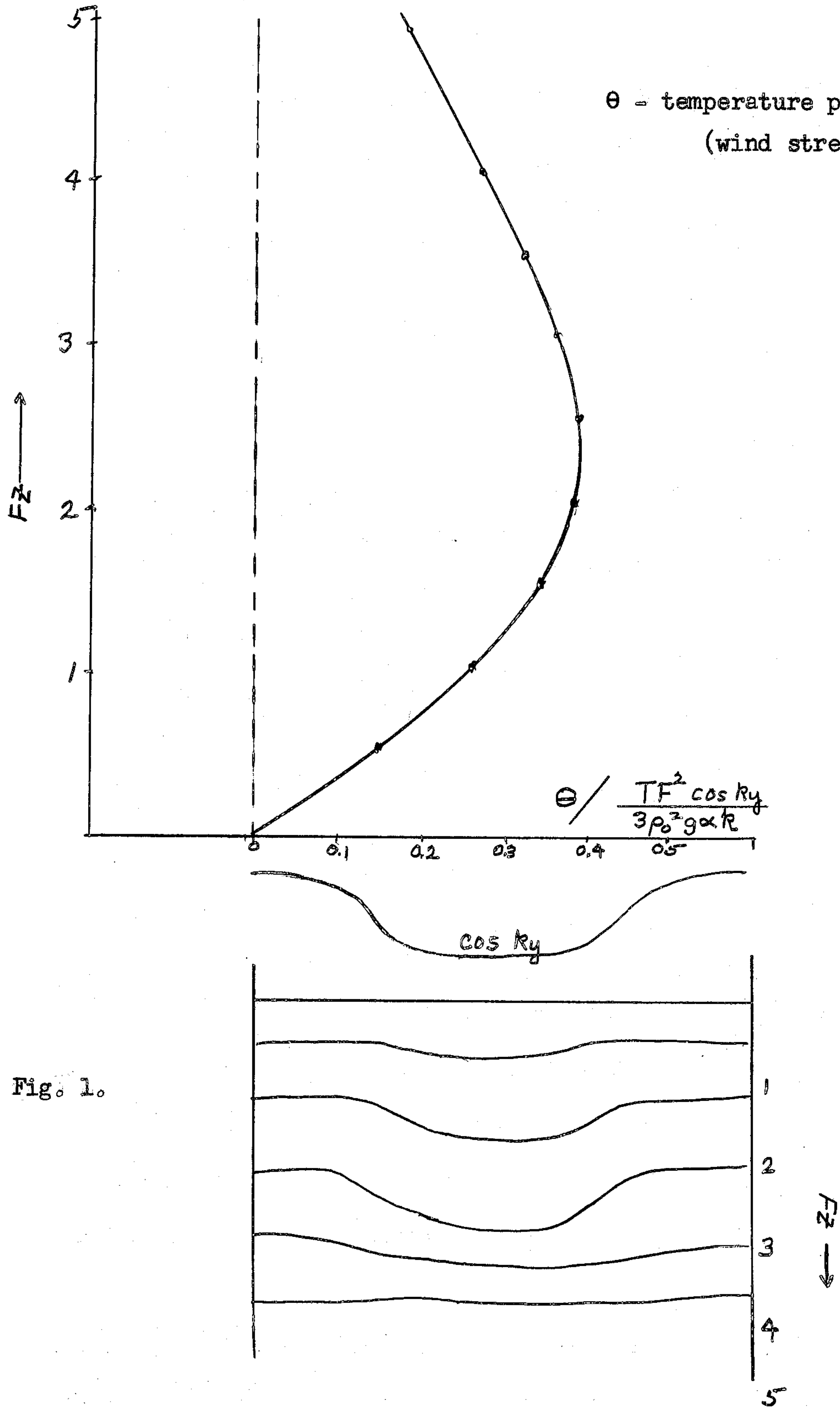


Fig. 1.

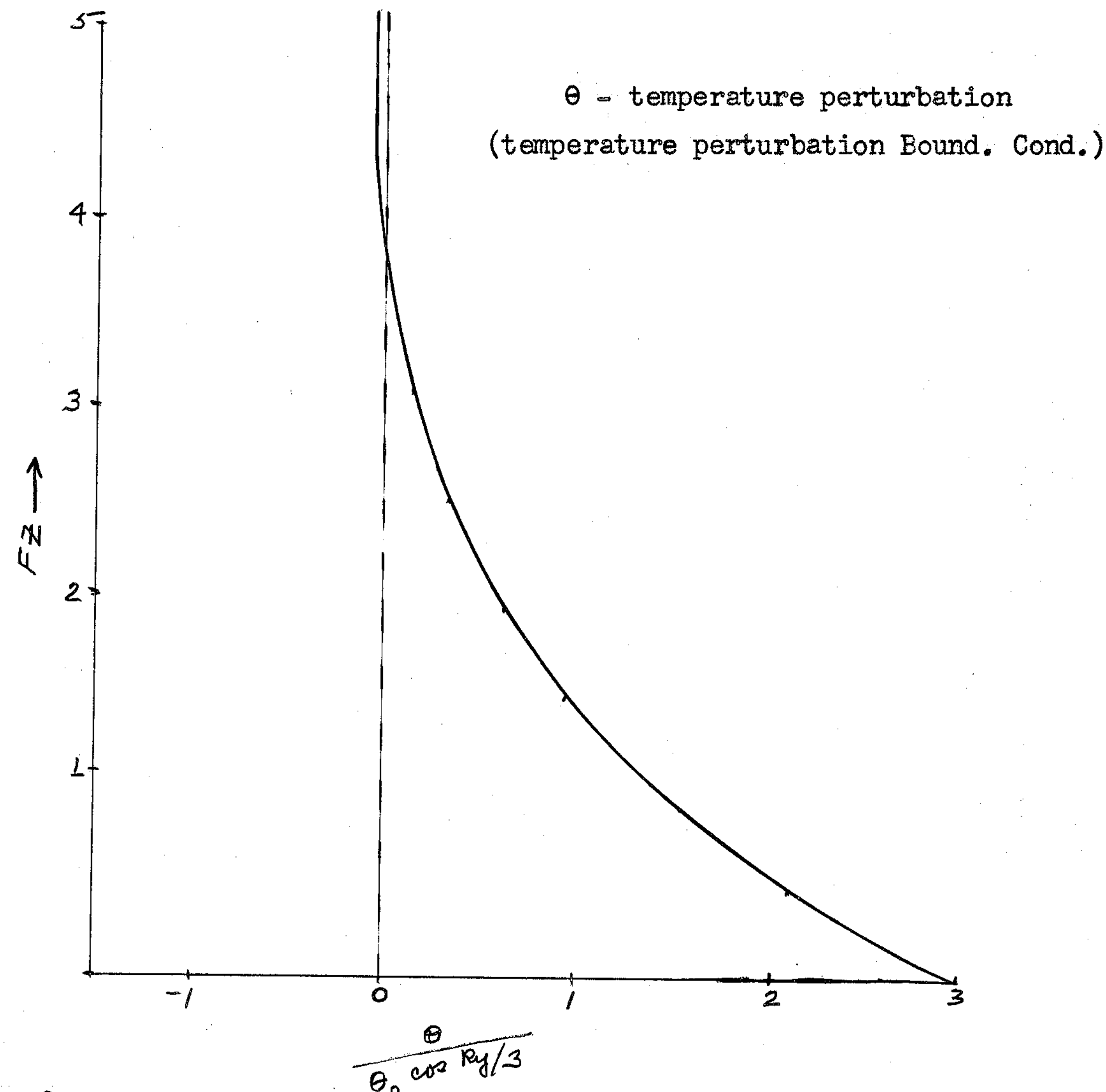
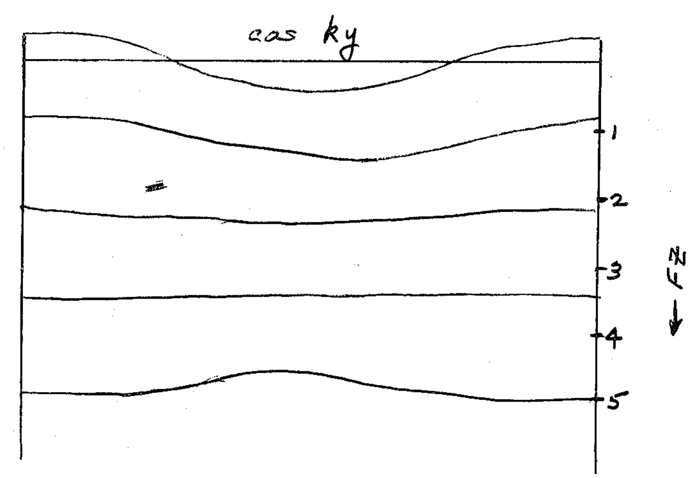
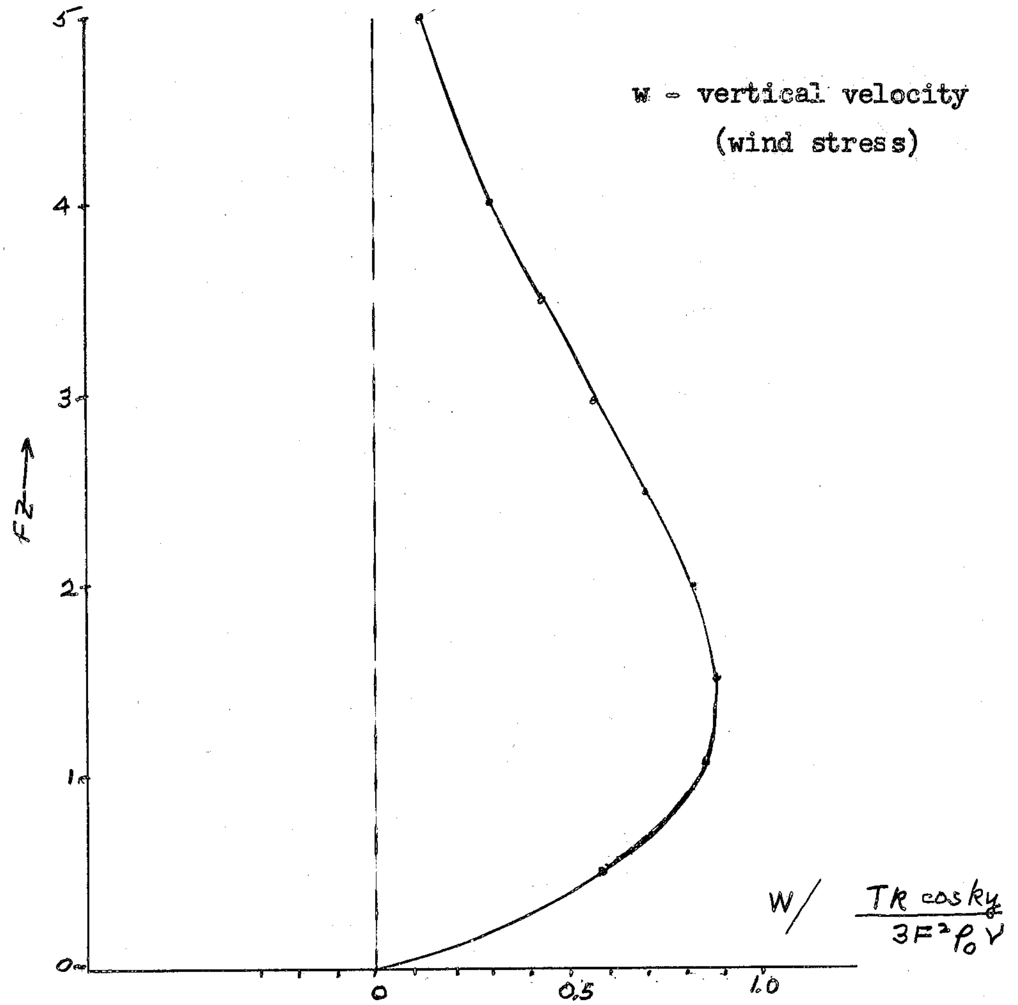

θ - temperature perturbation
(temperature perturbation Bound. Cond.)
FZ
5
4
3
2
1
-1
0
1
2
3
Θ
Θ₀ cos ky/3

Fig. 2.


cos ky
1
2
3
4
5
FZ


w - vertical velocity
(wind stress)
Fz
0
1
2
3
4
5
W/ $\frac{T k \cos ky}{3F^2 \rho_0 \nu}$
0
0.5
1.0

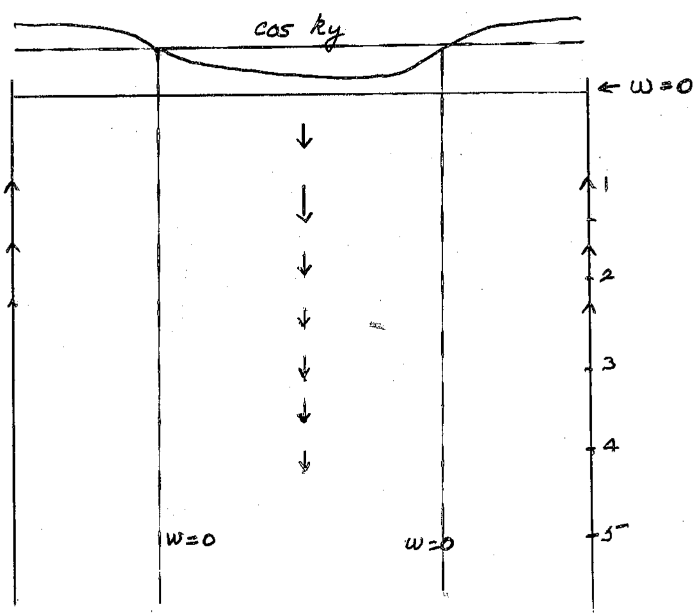

cos ky
← ω=0
1
2
3
4
5
w=0
ω=0

Fig. 3.

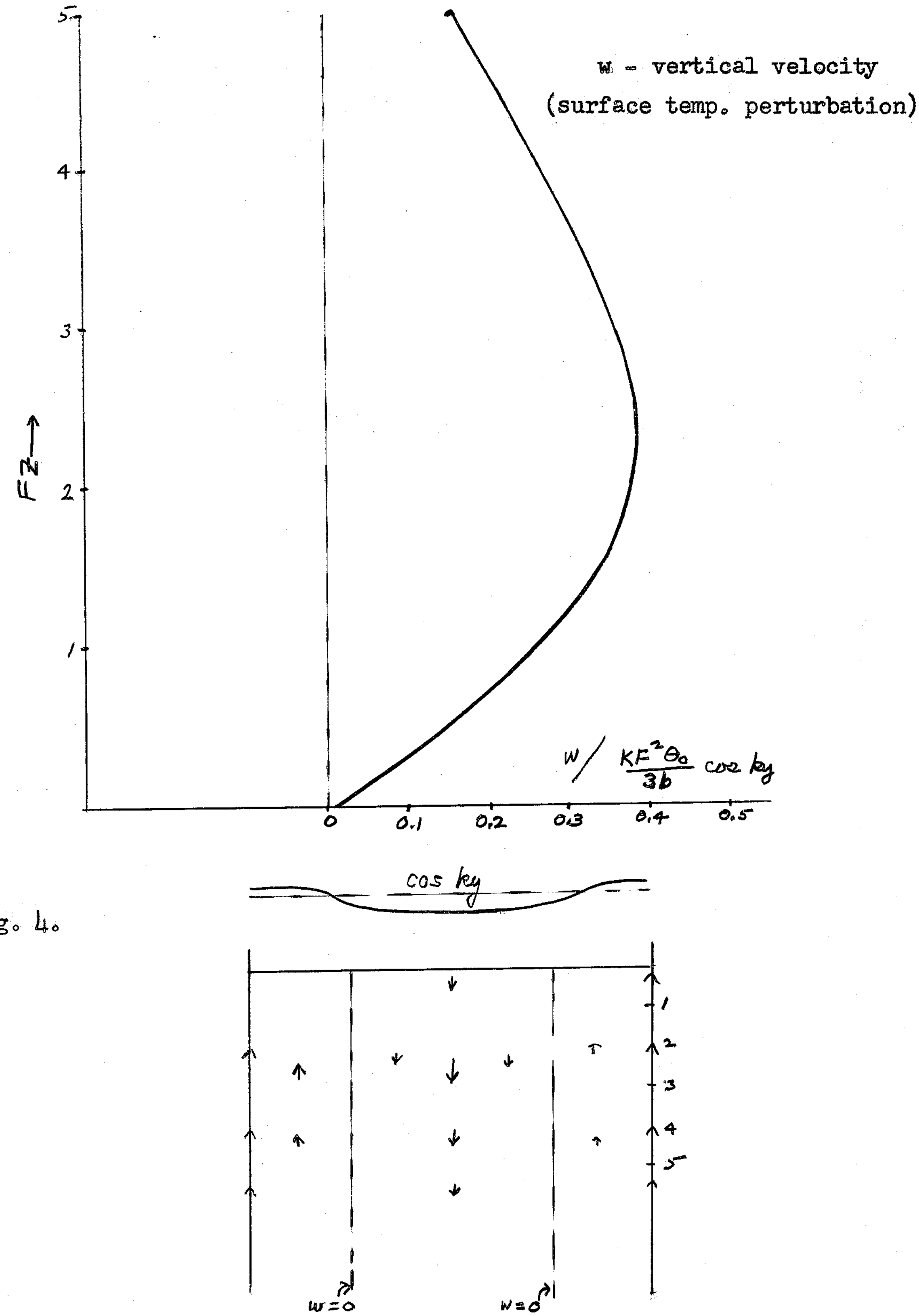

w - vertical velocity
(surface temp. perturbation)
Fz
w / KF²θ₀/3b cos ky
0 0.1 0.2 0.3 0.4 0.5
1 2 3 4 5
cos ky
w=0
w=0

Fig. 4.

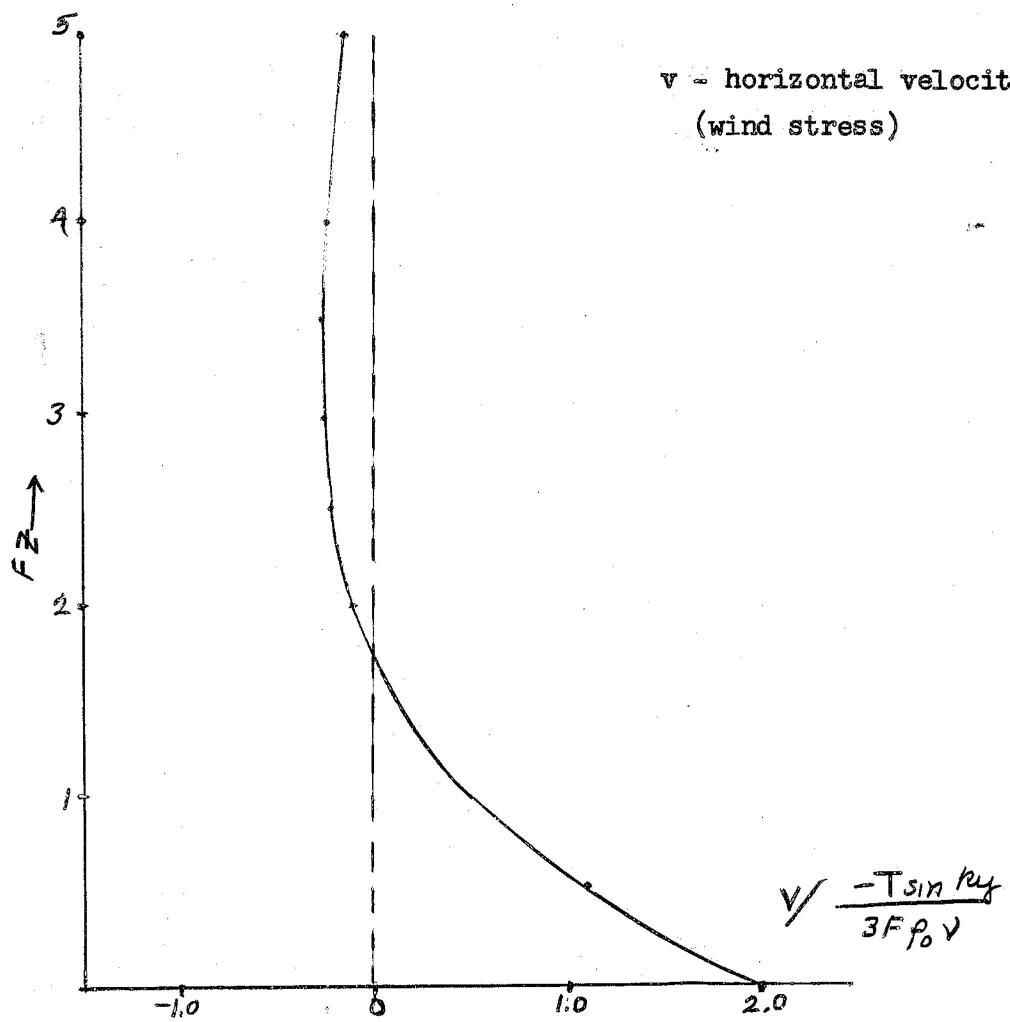
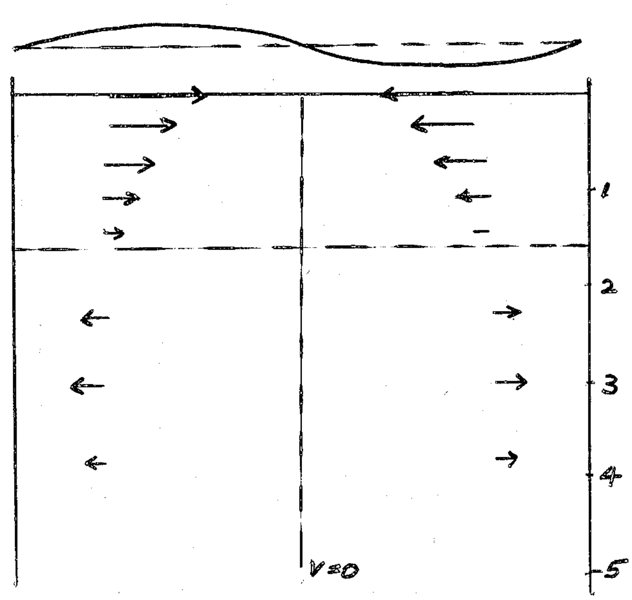

v - horizontal velocity
(wind stress)
5
4
3
2
1
FZ
-1.0
0
1.0
2.0
V/ -T sin ky / 3F ρ0 ν

Fig. 5.


1
2
3
4
5
v=0

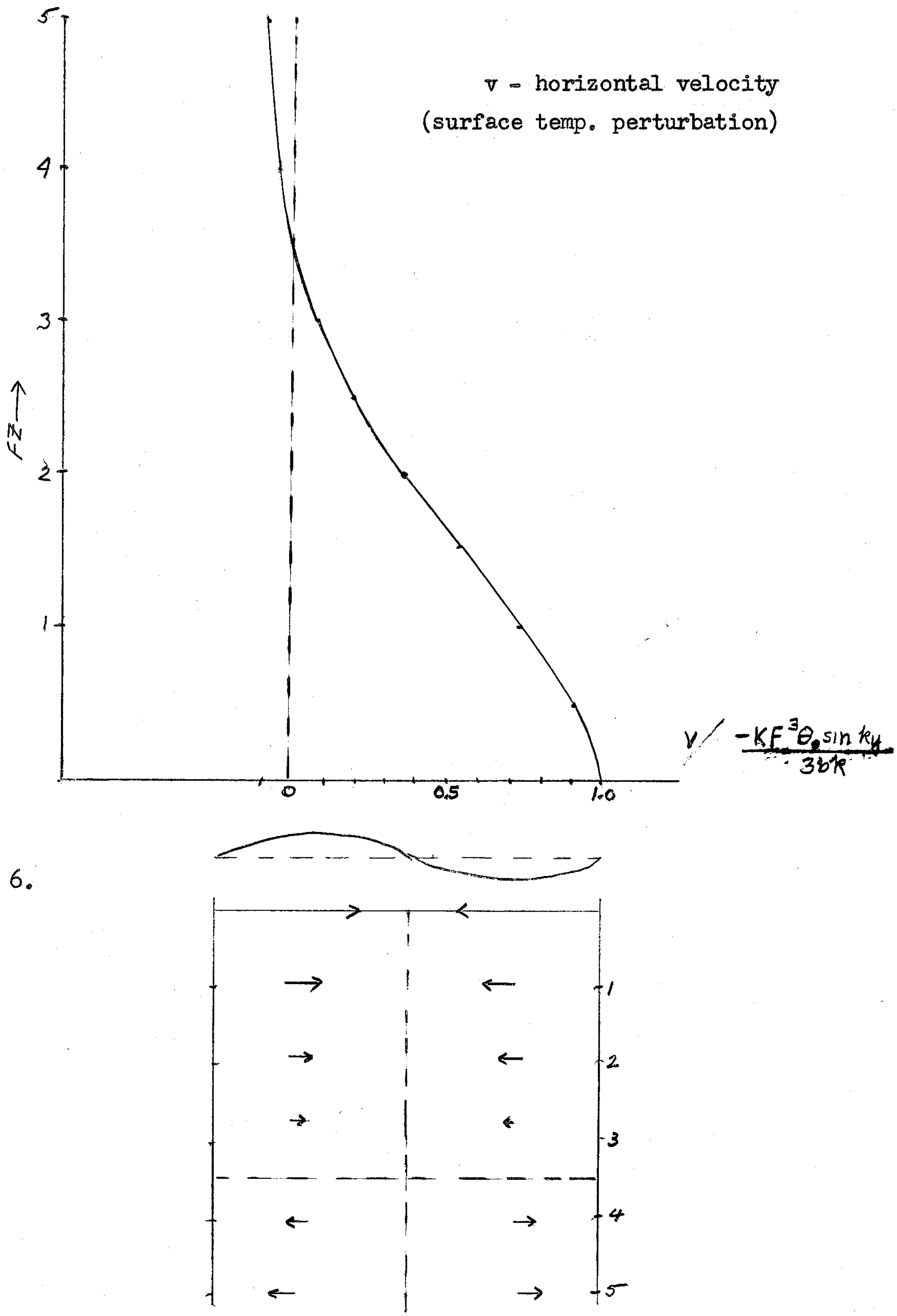

v - horizontal velocity
(surface temp. perturbation)
FZ →
$v / \frac{-KF^3\theta_0 \sin ky}{3b^2 k}$
0
0.5
1.0
1
2
3
4
5

Fig. 6.

much more rapidly with depth, passes through zero at Fz = 1.5 and then goes into a current moving in the opposite direction from that of the surface. In the temperature perturbation case the surface current extends to a depth Fz = 3.0, beneath which there is also a negative current extending to great depths.

The vertical velocity profiles also differ quite a bit in the two cases. Again for the winds stress perturbation case the velocity field is rather shallow, reaching a maximum at about Fz = 1.2 and then dropping off rapidly to zero at about Fz = 5. For case B, w grows more slowly with depth to a maximum at Fz = 2.5, then very gradually lessens in intensity, still retaining half the maximum amplitude at Fz = 5.

These general features are the most interesting thing about this problem; deep temperature perturbations and shallow velocity fields in the wind stress case with the opposite effect in the surface temperature perturbation case. We might expect these to show up in the rotational case as well but that is not clear.

Of course these results are not terribly unique and exciting because there are several other combinations of boundary conditions which give equally interesting results. It would be instructive to go through these also to gain a complete picture of the influence that the different boundary conditions play in the determination of the z dependence of the motions.

D. Concluding remarks

In this short paper we have considered only the easiest case, that of non-rotation. From this point the next step would be to attack the problems of uniform rotation and non-uniform rotation, comparing as above the two different cases. As Stommel and Veronis showed in their paper, the variation of the Coriolis parameter with latitude exerts an important influence upon the extent of the motions so that solutions to this problem with non-uniform rotation should be very interesting.

However, since these problems present many mathematical difficulties I have not considered them here.

A Simplified Model of Flow Over an Obstacle - with Application to the Atmosphere.

W. Blumen

Mass. Inst. of Tech.

I. Introduction

When a basic horizontal atmospheric flow is forced over an obstacle (small hills, mountains) it is observed that a system of stationary sinusoidal waves form downstream from the obstacle, called "lee waves". The wave amplitudes damp downstream, usually reach a maximum in the middle troposphere and gradually decay upward.

The basic character of these motions depends on:

1. the static stability of the air, i.e. the temperature lapse rate
2. the wind variation with height
3. the height and width of the obstable.

Whether or not lee wave formation will take place also depends upon these parameters. In this problem we assume that wave motion has occurred and investigate the modification of the flow.

The theory of mountain waves has only been treated with any success in two dimensions. This is basically a non-linear problem, since the static stability and velocity are invariably functions of height. However, linearized two dimensional theory has been able to describe the gross features of the wave pattern which forms in the lee of an infinite ridge (y direction). We shall investigate the results of a small scale laboratory model of the wave phenomenon (1). The purpose of this model experiment was to vary the significant parameters in order to simulate observed atmospheric wave patterns.

II. Model

A. Laboratory model

A channel 20 ft. long, 6 in. wide and 2 ft. high was used. An idealized cross section of the Sierra Nevada range was situated on the bottom. The obstacle was started from rest and moved along the bottom at a known rate of speed in a fluid mixture of water and salt which has a basic linear density profile in the vertical. Time exposures of streaks due to aluminum particles showed the type of motion which resulted. If the rate at which the obstacle moved was not large (~ 3 cm/sec) it was observed that a lee wave pattern developed downstream from two small sinusoidally shaped ridges contained within the whole Sierra Nevada system. It is this motion with which we shall be concerned.

B. Mathematical Model

The basic assumptions are:

1. A steady state ($\frac{}{t} = 0$)
2. two dimensional
3. viscous
4. incompressible flow
5. with constant stability
6. constant basic current (rate of speed of the obstacle)
7. flow over a small amplitude sinusoidal ridge
8. the earth's rotation is not considered.

The schematic model is shown in figure 1.

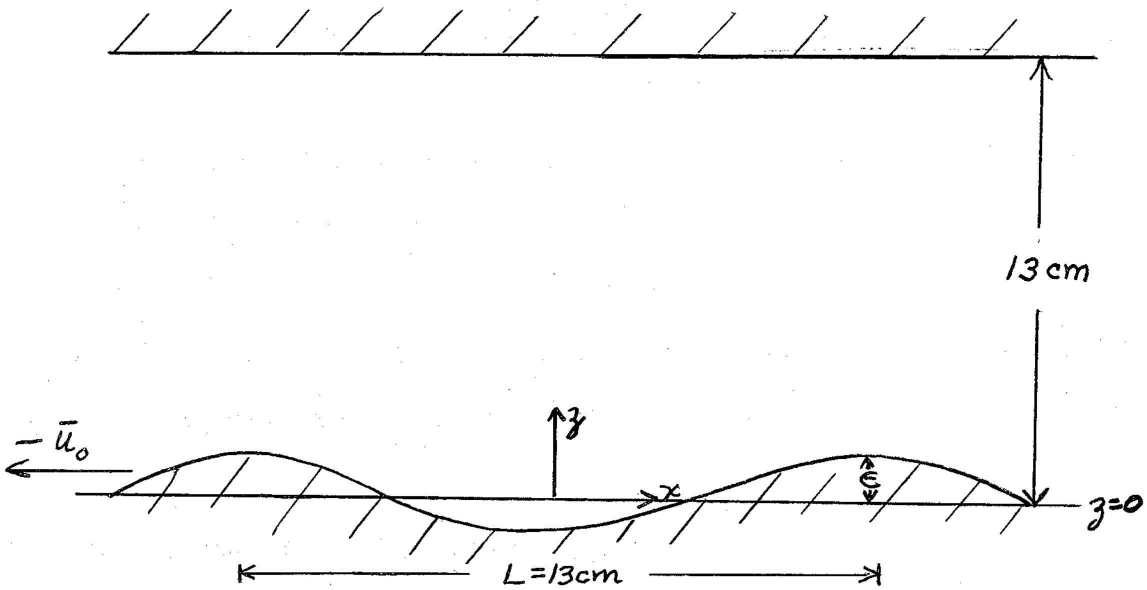


Fig. 1.

The coordinate system is stationary with respect to the mountain. Therefore to an observer moving with the mountain at a rate $-\bar{u}_o$ there is a wind velocity $+\bar{u}_o$.

The system of equations for this problem is:

1. $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

2. $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\nabla^2 w$$

3. $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0$$

4. $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ and the variables have the standard meaning.

Here we are interested in a disturbance and the modification of the mean flow pattern. We express the dependent variables as

5. $(u, w, p, \rho) = (\bar{u}, \bar{w}, \bar{p}, \bar{\rho}) + (u', w', p', \rho')$

where the bar indicates an average over one wave length of the sinusoidal obstacle. The obstacle is assumed to be infinitely sinusoidal and the problem reduces to studying the average motion as a function of height over one wave length. If (5) is substituted into the system (1-4) and this system is then averaged the mean equations result.

If the mean equations are subtracted from the total equations (1-4) the perturbation equations are determined. Since we have assumed that $\frac{\epsilon}{L} \leq 1$ (ϵ : height of the obstacle), we expand all the variables in terms of this small parameter, $\frac{\epsilon}{L} \equiv \epsilon'$. This says,

(6)
$$\bar{u} = \bar{u}_0 + \bar{u}_2 \epsilon'^2 + \dots$$
$$\bar{w} = \bar{w}_0 + \bar{w}_2 \epsilon'^2 + \dots$$

and similarly for the other mean quantities;

(7)
$$u' = u' \epsilon' + u'_2 \epsilon'^2 + \dots$$
$$w' = w'_1 \epsilon' + w'_2 \epsilon'^2 + \dots$$

and similarly for the other perturbation quantities.

When (6) and (7) are substituted into the mean and perturbation equations and the coefficients of the powers of ϵ' are equated to zero we get a system of mean and perturbation equations corresponding to each power of ϵ'. Since we wish to compare model and atmospheric flow it is desirable to non-dimensionalize the equations by putting

$$\bar{u} = \bar{u}'' \, \bar{u}_o, \; \bar{w} = \bar{w} \; \bar{u}_o, \; \bar{\rho} = \bar{\rho}'' \bar{\rho}_m \;, \; \bar{p} = \bar{p}'' \bar{\rho}_m \bar{u}_o^{\,2}$$

$$u' = u'' \, \bar{u}_o, \; w' = w \; \bar{u}_o, \; \rho' = \rho'' \bar{\rho}_m \;, \; p' = p'' \bar{\rho}_m \bar{u}_o^{\,2}$$

$x = x''L$, $z = z''L$, where $\bar{u}_o$ is the constant basic velocity, L is the wave length of the obstacle and the height of the fluid and $\bar{\rho}_m$ is a characteristic mean density. The first two systems of mean equations are (dropping the double primes)

$$\epsilon^o : \; \bar{u}_o = 1 = \text{constant}$$

$$\bar{w}_o = 0$$

$$(8) \qquad 0 = -\frac{1}{\rho_o}\frac{\partial \bar{p}_o}{\partial z} - \frac{L}{\bar{u}_o^{\,2}} g$$

$$\epsilon^2 : \; \frac{\partial}{\partial z}(\overline{u_1 w_1}) = \frac{1}{R}\frac{\partial^2 \bar{u}_2}{\partial z^2}$$

$$(9) \qquad \frac{\partial (\overline{w_1 w_1})}{\partial z} = -\frac{1}{\rho_o}\frac{\partial \bar{p}_2}{\partial z} + \frac{1}{R}\frac{\partial^2 \bar{w}_2}{\partial z^2}$$

$\bar{w}_2 \dfrac{\partial \rho_o}{\partial z} + \dfrac{\partial}{\partial z}(\overline{w_1 \rho_1}) = 0$, where $R = \dfrac{\bar{u}_o L}{\nu}$ is the Reynolds number.

The first order perturbation equations are

$$\epsilon' : \; \frac{\partial u_1}{\partial x} = -\frac{1}{\rho_o}\frac{\partial p_1}{\partial x} + \frac{1}{R}\nabla_H^2 u_1$$

$$(10) \qquad \frac{\partial w_1}{\partial x} = -\frac{1}{\rho_o}\frac{\partial p_1}{\partial z} - \frac{L}{\bar{u}_o^2}\frac{\rho_1}{\bar{\rho}_o} g + \frac{1}{R}\nabla_H^2 w_1$$

$$\frac{\partial \rho_1}{\partial x} + w_1 \frac{\partial \bar{\rho}_o}{\partial z} = 0$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0$$

If the first order perturbations (u_1, w_1) can be determined, the first modification to the mean field can be found from (9) by

$$\bar{u}_2 = R \int_1^z \overline{u_1 w_1}\, dz$$

Then, $\bar{u} = 1 + \bar{u}_2 \epsilon^2$.

The second equation of motion in (10) can be written

$$\frac{\partial w'}{\partial x} = -\left(\frac{\partial}{\partial z} - \frac{1}{\bar{\rho}_0}\frac{\partial \bar{\rho}_0}{\partial z}\right)\left(\frac{p_1}{\bar{\rho}_0}\right) - \frac{L}{\bar{u}_0^2}\frac{\rho_1}{\bar{\rho}_0} g + \frac{1}{R}\nabla_H^2 w_1 .$$

$$\frac{\partial}{\partial z} - \frac{1}{\bar{\rho}_0}\frac{\partial \bar{\rho}_0}{\partial z} \sim \frac{1}{L} - \frac{1}{L}\frac{\Delta \bar{\rho}_0}{\bar{\rho}_0} \sim \frac{1}{L} - \frac{1}{L}\left(\frac{1}{10}\right).$$

Therefore we neglect the term $\frac{1}{\bar{\rho}_0}\frac{\partial \bar{\rho}_0}{\partial z}$ in the equation of motion but not in the thermodynamic equation. Then (10) reduces to the Boussinesq system of equations. If all the dependent variables but w_1, are eliminated we get the fourth order differential equation

$$(11)\quad \frac{\partial^2 w}{\partial z^2} + (S^2 - K^2)w = -\frac{i}{KR}\left(\frac{\partial^4}{\partial z^4} - 2K^2\frac{\partial^2}{\partial z^2} + K^4\right)w = 0,$$

where $-S^2 \equiv \frac{L}{\bar{u}_0^2}\frac{1}{\bar{\rho}_0}\frac{\partial \bar{\rho}_0}{\partial z} g.$

Here we have assumed $w_1 = w(z)e^{iKx}$, where K is the non-dimensional wave number.

Consider the inviscid equation $\frac{\partial^2 w}{\partial z^2} + (S^2 - K^2)w = 0$. Similarity between the model and atmosphere implies that S^2 and K^2 be equal in both.

In the model $S^2 \approx 10^2$. If we demand that $S^2 \approx 10^2$ in the atmosphere, then the model describes slow motion (~ 3 m/sec) over a small sinusoidal ridge (~ 1 km length; ~ 100 m high). Also,

$K^2 = 2\pi$ in both model and atmosphere. In order for the Reynolds number (R) to be the same in both systems requires an eddy viscosity coefficient $\nu \sim 10^4$ cm^2/sec. This will not be stipulated here since it implies a one-to-one correspondence between molecular friction in the model and turbulent friction in the atmosphere.

When $w(z) \sim e^{mz}$ is substituted into (11) and the resulting quartic equation in m is solved, we get the four roots

$$m_1 = \frac{1}{2}\frac{S^4/KR}{\sqrt{S^2-K^2}} - \sqrt{S^2-K^2}\, i$$

$$m_2 = -\frac{1}{2}\frac{S^4/KR}{\sqrt{S^2-K^2}} + \sqrt{S^2-K^2}\, i$$

$$m_3 \approx \sqrt{\frac{KR}{2}}\,(1+i)$$

$$m_4 \approx -\sqrt{\frac{KR}{2}}\,(1+i)$$

The general solution of (11) is

$$w_1(x,z) = C_1 e^{(m_1 z + iKx)} + C_2 e^{(m_2 z + iKx)} + $$

$$+ C_3 e^{(m_3 z + iKx)} + C_4 e^{(m_4 z + iKx)}$$

The solutions corresponding to the roots m_1 and m_2 are similar to the inviscid solutions but modified by a small exponential factor; the remaining roots correspond to boundary layer solutions.

III. Boundary Conditions

In the model experiment it was observed that the top boundary remained undisturbed. Therefore at the upper surface we set $w_1 = 0$. Since a free surface cannot support a tangential stress, we have $\frac{\partial u'}{\partial z} = \frac{\partial^2 w_1}{\partial z^2} = 0$, using the continuity equation. At the lower boundary $w = 0$ in viscous flow. The fourth boundary condition must introduce the effect of the obstacle. It was observed that the boundary layer was not large, so that at a small distance (δ) above the lower boundary the flow was practically unaffected by friction. Then the flow just above the obstacle essentially follows the contour of the lower boundary.

The four boundary conditions are

$$w_1 = \frac{\partial^2 w_1}{\partial z^2} = 0 \quad \text{at} \quad z = 1$$

$$w_1 = 0 \quad \text{at} \quad z = \mathcal{J}$$

$$w_1 = iKe^{iKx} \quad \text{at} \quad z = \mathcal{J} + \delta$$

$$\text{where} \quad \mathcal{J} = \text{Re}\; e^{iKx}$$

The first order solution which satisfied the boundary conditions has not been determined. It is hoped that the modified mean velocity profile agrees with the observational evidence of alternate layers of high and low speed.

Reference

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On the Propagation of Shallow Waves in a Viscous Fluid in a Rotating System

by

R.L. Duty

Abstract

We study the effect of viscous damping on the propagation of shallow waves in a rotating system for the case of a fluid of constant depth H. Travelling waves exist provided the wavelengths are sufficiently large, the exact magnitude depending on the viscosity ν and depth H of the fluid, as well as the Coriolis frequency f (= 2 ω). In all cases the intuitive result that viscosity damps the motion is supported mathematically.

Some interesting boundary layer results are found for the flow near an oscillating wall. Comparison is made between the completely inviscid results and those for small viscosity.

1. Introduction

Lamb (1879) discusses the effect of viscosity in dissipating energy associated with shallow waves. In a non-rotating system it is possible to derive a single equation for the wave surface ζ (x, y, t) free of fluid velocities u and v. In a rotating system, which we are considering, the fluid velocities u and v are coupled, with the result that two equations are obtained relating u to ζ. By seeking solutions in the form of travelling waves we are led to an algebraic relation connecting the physical variables. Conditions for the propagation of these waves are explicitly determined together with damping factors. A "forbidden wavelength band", which would be difficult to predict by a purely physical argument, is found for which waves do not propagate, while both longer and shorter wavelengths are propagated.

2. The Governing Equations

As is customary, we shall neglect the substantial derivative $\frac{Dw}{Dt}$ in the z component momentum equation, and neglect the convective terms in the x and y component equations, to obtain the viscous shallow wave equations. The effect of viscosity on the vertical component of the motion is also neglected which is in agreement with the neglected term $\frac{\partial w}{\partial t}$. The viscosity is retained in the x and y component equations, which leads to the viscous system,

$$
\begin{aligned}
&\text{Mz)} \qquad p = -\rho g(z - \zeta) \\
&\text{Mx)} \qquad u_t - fv + g\zeta_x = \nu\nabla^2 u \\
&\text{My)} \qquad v_t + fu + g\zeta_y = \nu\nabla^2 v \\
&\text{(c)} \qquad \zeta_t + (Hu)_x + (Hv)_y = 0
\end{aligned}
\tag{1}
$$

Seeking wave solutions of (1) we may write the time dependence of u, v, and ζ in the form

$$
\begin{aligned}
u &= e^{i\sigma t}\, u(x,y) \\
v &= e^{i\sigma t}\, v(x,y) \\
\zeta &= e^{i\sigma t}\, \zeta(x,y)
\end{aligned}
\tag{2}
$$

where it will be understood hereafter that u, v, and ζ have spatial dependence only. Having made the substitution (2) into (1) we arrive at the equations which relate other fluid velocities u and v to the wave surface ζ,

$$
\begin{aligned}
i\sigma u - fv + g\zeta_x &= \nu\nabla^2 u \\
i\sigma v + fu + g\zeta_y &= \nu\nabla^2 v \\
i\sigma\zeta + (Hu)_x + (Hv)_y &= 0
\end{aligned}
\tag{3}
$$

The first equation of (3) may be solved for the v-velocity component, which, when substituted into the second equation, leads to an expression between u and ζ alone,

$$\nu^2\nabla^4 u - 2i\sigma\nu\nabla^2 u + (f^2 - \sigma^2)u = g\left[\nu\nabla^2\zeta_x - i\sigma\zeta_x - f\zeta_y\right] \tag{4}$$

while $$v = \frac{1}{f}\left[i\sigma u + g\zeta_x - \nu\nabla^2 u\right] \tag{5}$$

Let us seek plane wave solutions of (4), in particular

$$\begin{aligned} \zeta &= A\, e^{-i\ell x} \\ u &= B\, e^{-i\ell x} \end{aligned} \tag{6}$$

which are waves travelling in the x-direction. We will satisfy the equation of continuity (the third equation of (3)), with solutions of the form (6) for the particular case of constant depth, provided we choose

$$A = \frac{\ell HB}{\sigma} \tag{7}$$

Having placed H equal to a constant (which we still denote by H, for convenience), we may substitute (7), together with the expression (6) for u and v, into the connecting relation (4). The result of this sequence of operations is an algebraic quartic equation in ℓ, or an algebraic cubic equation in σ. We are interested in the cubic equation which is

$$\sigma^3 - 2i\nu\ell^2\sigma^2 - (gH\ell^2 + \nu^2\ell^4 + f^2)\sigma + ig\nu H\ell^4 = 0 \tag{8}$$

4. Solutions to the σ equation

We have reduced the original problem to the simpler one of solving (8) for σ as a function of the other variables. In order to apply the ordinary theory of cubic equations it would be convenient if (8) had only real coefficients of σ. With this in view we make the transformation

$$\sigma = -i\gamma \tag{9}$$

In terms of the transformed variable γ, we have

$$\gamma^3 + 2\nu\ell^2\gamma^2 + (gH\ell^2 + \nu^2\ell^4 + f^2)\gamma + g\nu H\ell^4 = 0 \qquad (10)$$

which has real, positive, coefficients for real wave number ℓ. The theory for determining the nature of the roots of (10) is well developed and may be found in any elementary book on the theory of equations, e.g. Lovitt. The nature of the roots depend on the sign of the discriminant Δ. For the equation (10),

$$\Delta = -4\left[gH\ell^2 + f^2 - \frac{\nu^2\ell^4}{3}\right]^3 - \frac{1}{27}\rho^2\ell^4\left[9gH\ell^2 - 2\nu^2\ell^4 - 18f^2\right]^2 \qquad (11)$$

For future reference we now tabulate the well-known results,

Case 1: $\Delta > 0$

The roots γ_i are all real and distinct, hence the roots σ_i of (8) are all purely imaginary. In terms of the surface $\zeta(x,t)$ this implies that waves do not advance, but are of the form (where γ_i are all real)

$$e^{\gamma_i t} \cos \ell x \quad (\text{or } e^{\gamma_i t} \sin \ell x) \qquad (12)$$

On physical grounds one might expect that the γ_i should all be negative in order that the energy be dissipated. We shall later show this to be correct on mathematical grounds.

Case 2: $\Delta = 0$

The roots γ_i are all real. Two of the roots are equal. Again, the γ_i are all negative.

Case 3: $\Delta < 0$

One of the roots (say γ_3) is real. Two of the roots are complex conjugates of each other, i.e. $\gamma_2 = \bar{\gamma}_1$. Let us for future reference write

$$\gamma_1 = a + ib \, , \quad \gamma_2 = a - ib$$

which gives solutions of the plane wave-type for $\zeta(x,t)$

$$\zeta(x,t) = e^{at} \cos(bt - \ell x) \quad \left[\text{or } e^{at} \sin(bt - \ell x)\right] \tag{13}$$

Again, it will be shown a is negative, corresponding to damping. It will also be shown γ_3 is negative.

5. Physical Nature of the Solutions.

The sign of the discriminant, Δ, determines the type of wave solutions expected. In order to display the dependence of Δ on the physical variables more explicitly it is useful to expand the Δ of (11). Doing this, we find

$$\begin{aligned}\Delta = {} & \nu^2(g^2H^2 - 4\nu^2f^2)\ell^8 + 4gH(5\nu^2f^2 - g^2H^2)\ell^6 - \\ & - 4f^2(2\nu^2f^2 + 3g^2H^2)\ell^4 - \\ & - 12gHf^4\ell^2 - 4f^6\end{aligned} \tag{14}$$

Several interesting results are immediately evident from (14). Let us first consider waves which have sufficiently long wavelengths so that we may write approximately,

$$\Delta \approx -4f^6 \tag{15}$$

which is, of course, an equality for $\ell = 0$. This result implies that (according to case 3) waves of a sufficiently long wavelength are travelling waves which are damped by an exponential time factor, regardless of the other physical variables. (We have not yet proved damping.) If in fact we retained terms to order ℓ^4 in the expression (15) for long wavelengths, we would still find travelling waves since Δ is negative. Let us now consider the propagation of waves of short wavelength for which ℓ is large. If in particular we approximate Δ by the highest order term in ℓ for large ℓ we have

$$\Delta \approx \nu^2(g^2H^2 - 4\nu^2f^2)\ell^8 \qquad (16)$$

If waves of very short wavelengths are to be propagated, it is again necessary for Δ to be negative, which implies the inequality on the physical situation,

$$2\nu f > gH \qquad (17)$$

We have shown that waves of very long wavelength are always propagated, and if the inequality (17) is satisfied, waves of very short wavelength will be propagated also. If the physical situation is such that

$$2\nu f < gH \qquad (18)$$

then the discriminant will be negative and short wavelength waves will not be propagated.

A rather interesting situation may occur if both very long- and very short-wavelengths are present, i.e. assume (17) is satisfied. We may for convenience re-write (17) as an equality,

$$4\nu^2f^2 = g^2H^2 + \epsilon^2 \qquad (19)$$

where ϵ^2 is real and positive (non-zero). We shall now investigate the possibility of existence of a band of wavelengths $\Gamma(\ell)$ (short, but limited above and below) which are not propagated.

Let us consider waves for which a reasonable approximation to Δ is given by retaining the two highest powers of ℓ, i.e.

$$\Delta \approx \nu^2(g^2H^2 - 4\nu^2 f^2)\ell^8 + 4gH(5\nu^2 f^2 - g^2H^2)\ell^6 \tag{20}$$

If we substitute (19) into (20) we may express the discriminant,

$$\Delta \approx \ell^6 \left[-\nu^2\epsilon^2\ell^2 + 4gH(\nu^2 f^2 + \epsilon^2)\right]$$

the condition that waves not be propagated is satisfied for wave-lengths

$$\ell^2 < \frac{4gHf^2}{\epsilon^2}$$

By considering a physical situation in which ℓ is sufficiently large that (20) is reasonably valid, while ϵ^2 is small, we see $\Gamma(\ell)$ is bounded above for large ℓ. As we consider longer wavelengths, i.e. $\ell \to 0$, the approximation (20) ceases to be valid, and from the exact expression (14) we see waves of sufficiently long wavelength are always propagated. Thus we have shown that viscosity allows the possibility of existence of a set of non-propagating wavelengths $\Gamma(\ell)$, bounded between propagating waves.

6. Damping by Viscosity

We shall now prove the physically obvious result that viscosity damps the motion. It is convenient to consider cases 1 and 2 (see result 12) together, case 3 separately (see result 13).

Cases 1 and 2: Solutions are of the form

$$e^{\gamma_i t}\cos \ell x \qquad \left[\text{or } e^{\gamma_i t}\sin \ell x\right] \tag{12 bis}$$

where the γ_i are real. We wish to show that $\gamma_i (i = 1,2,3)$ are negative. We are given that γ_i are solutions of the cubic equation (10), which has all real, positive coefficients. It follows immediately by relating the γ_i to the coefficients, which are all positive, that the $\gamma_i (i = 1,2,3)$ are negative if they are real.

Case 3: Solutions are of the form

$$\begin{array}{ll} e^{\gamma_3 t} \cos \ell x & \left[\text{or } e^{\gamma_3 t} \sin \ell x\right] \\ e^{at} \cos(\pm\, bt - \ell x) & \left[\text{or } e^{at} \sin(\pm\, bt - \ell x)\right] \end{array} \qquad (13\text{ bis})$$

where $\gamma_1 = a + ib$, $\quad \gamma_2 = a - ib$

We wish to show that $a < 0$ and $\gamma_3 < 0$. Again we refer to the cubic equation (10). For real ℓ the constant term in (10) is positive, hence we prove γ_3 is negative. That a is negative follows by assuming solutions, one real, two complex, and relating the coefficients of (10) to these solutions. In particular if we obtain an equation for a alone we find it is a cubic equation with all real positive coefficients. The real solution (s) to such an equation is (are) negative which proves $a < 0$.

Thus, in all cases, viscosity damps the motion.

7. Solution for waves of long wavelength

We may formally write the solution of the σ-equation (8) by the standard method of solving a cubic equation. This will involve complicated algebraic quantities. We may, by seeking a solution for small ℓ, hope to simplify the answer while still retaining the essential characteristics. With this purpose in view, let us seek the solution of (8) for small ℓ by neglecting terms of order ℓ^4. With this approximation, the σ-equation becomes

$$\sigma^3 - 2i\nu\ell^2\sigma^2 - (gH\ell^2 + f^2)\sigma = 0$$

which, provided $\sigma \neq 0$, may be written

$$\sigma^2 - 2i\nu\ell^2\sigma - (gH\ell^2 + f^2) = 0 \tag{21}$$

Thus we see the explicit wave solutions are of the form

$$e^{-\nu\ell^2 t}\cos(\pm\, bt - \ell x) \qquad \left[\text{or } e^{-\nu\ell^2 t}\sin(\pm\, bt - \ell x)\right] \tag{22}$$

where

$$b = \sqrt{gH\ell^2 + f^2 - \nu^2\ell^4}$$

The solution (22) exhibits explicitly that (within our approximation) the shorter waves are damped more strongly than the larger ones.

8. Infinite half-plane with periodically oscillating boundary

Let us consider the infinite half-plane $x \geq 0$ with the boundary $x = 0$ oscillating periodically, i.e.

$$u\Big|_{x=0} = U e^{i\sigma t} \tag{23}$$

where σ is real.

The no slip condition at the boundary is

$$v\Big|_{x=0} = 0 \tag{24}$$

We now wish to investigate the boundary layer, and coastal wave effect which may be compared with the inviscid result. Let us non-dimensionalize the equation (8) as follows

$$\sigma = \sqrt{\frac{g}{H}}\,\bar{\sigma} \qquad \ell = \frac{1}{H}\bar{\ell}$$

$$f = \sqrt{\frac{g}{H}}\,\bar{f}$$

where the barred quantities are non-dimensional. In terms of these variables, (8) becomes

$$\bar{\ell}^4 - \frac{\bar{\sigma}(2iN\bar{\sigma}+1)\,\bar{\ell}^2}{N(i-N\bar{\sigma})} + \frac{\bar{\sigma}(\bar{\sigma}^2-\bar{f}^2)}{N(i-N\bar{\sigma})} = 0 \tag{25}$$

where

$$N = \frac{\nu}{H\sqrt{gH}}$$

Hereafter we shall drop the bars, and it will be understood the quantities are non-dimensionalized as given above. The equation (25) may be solved for ℓ^2 to give

$$\ell^2 = \frac{(2iN\sigma+1)\sigma \pm \sqrt{\sigma\,(\sigma - 4N^2\sigma f^2 + 4iNf^2)}}{2N(i-N\sigma)} \tag{26}$$

It is clear that (26) may be solved for ℓ as a function of σ. This representation involves several radicals. As we are primarily interested in small viscosity, we may initially assume N is small, and immediately obtain simplifications. In particular we shall assume the following approximation,

$$\sqrt{\alpha^2 + N\beta} = \alpha\left(1 + \frac{N\beta}{2\alpha^2}\right) \tag{27}$$

With this simplification, (26) may be written for small N,

$$\ell_+^2 = \left(\frac{\sigma}{N}\right)\left[\left(\sigma + \frac{f^2}{\sigma}\right) - i\right] \tag{28a}$$

$$\ell_-^2 = (\sigma^2 - f^2) \tag{28b}$$

And hence we may write for the roots of (26),

$$\ell_{1,2} = \pm\left(\frac{\sigma}{N}\right)^{\frac{1}{2}} \left\{ \left[\frac{\sqrt{(\sigma + f^2/\sigma)^2 + 1} + (\sigma + f^2/\sigma)}{2}\right]^{\frac{1}{2}} - i\left[\frac{\sqrt{(\sigma + f^2/\sigma)^2 + 1} - (\sigma + f^2/\sigma)}{2}\right]^{\frac{1}{2}} \right\} \tag{29}$$

$$\ell_{3,4} = \pm\sqrt{\sigma^2 - f^2}$$

As we are now ready to write the solution, we refer to (6) and (7) in non-dimensional form, which is

$$\zeta = \sum_{j=1}^{4} R_j \, e^{-i\ell_j x} \, e^{i\sigma t} \tag{30}$$

$$u = \sum_{j=1}^{4} \frac{\sigma R_j}{\ell_j} \, e^{-i\ell_j x} \, e^{i\sigma t} \tag{31}$$

We must choose R_2 and R_4 identically zero in order that the height of the surface be finite as $x \rightarrow \infty$. Then we may write (30)(31) as

$$\begin{aligned} \zeta &= \left[R_1 e^{-i\ell_1 x} + R_3 e^{-i\ell_3 x}\right] e^{i\sigma t} \\ u &= \left[\frac{\sigma R_1}{\ell_1} e^{-i\ell_1 x} + \frac{\sigma R_3}{\ell_3} e^{-i\ell_3 x}\right] e^{i\sigma t} \end{aligned} \tag{32}$$

The constants R_1 and R_3 are to be determined by the boundary conditions (23) and (24). We have from before, in non-dimensional form,

$$v = \frac{1}{f}\left[i\sigma u + \zeta_x - N\nabla^2 u\right] \tag{5 bis}$$

The boundary conditions then imply

$$\begin{aligned} R_1 &= \frac{U\ell_1}{\sigma} \frac{\left[\sigma N(\ell_3^2 - 2\ell_1^2) + i(2\ell_1^2 - \ell_3^2 - \sigma^2)\right]}{\left[\sigma N(\ell_3^2 - \ell_1^2) + i(\ell_1^2 - \ell_3^2)\right]} \\ R_2 &= \frac{U\ell_3}{\sigma} \frac{\left[\sigma N\ell_1^2 + i(\sigma^2 - \ell_1^2)\right]}{\left[\sigma N(\ell_3^2 - \ell_1^2) + i(\ell_1^2 - \ell_3^2)\right]} \end{aligned} \tag{33}$$

where ℓ_1 and ℓ_3 are defined by (29), and U is the non-dimensional boundary amplitude corresponding to that given in (23).

We will now study these results. From (28a) we see the ℓ_+^2 quantity depends on viscosity while in (28b), viscosity does not enter explicitly. It is to be remembered of course that small N was assumed in deriving these expressions. Let us first interpret (28b).

For a completely inviscid analysis we find that

$$\ell^2 = (\sigma^2 - f^2) \qquad (34)$$

while for small viscosity,

$$\ell_-^2 = (\sigma^2 - f^2) \qquad \text{(28b bis)}$$

which is identical. We thus see we have recovered the inviscid solution for the case of small viscosity, but we have an additional solution (28a) which now requires consideration.

From (28a) we see for small viscosity

$$\ell_1 = 0(\tfrac{\sigma}{N})^{\frac{1}{2}}$$

which then implies a boundary layer exists near the wall, and its thickness is of the order $(\frac{N}{\sigma})^{\frac{1}{2}}$. The physical quantities u, v, and ζ may be calculated from (5) and (32). For all these quantities we observe that one term modifies the solution significantly near $x = 0$, while the other term is important for $x > (\frac{N}{\sigma})^{\frac{1}{2}}$.

9. Examples and Numerical Results

Let us first seek an example in which the forbidden wave-length band of Part 5 may exist. Recall the desired physical situation is one such that

$$2\nu f > gH \qquad (17\text{ bis})$$

where these are dimensional quantities. It is natural to first consider the ocean for which table 1 applies.

Quantity	Homogeneous	Reduced gravity (internal)
ν	$(10)^7$ cm^2/sec	$(10)^7$ cm^2/sec
g	$(10)^3$ cm/sec^2	2 cm/sec^2
f	$(10)^{-4}$ sec	$(10)^{-4}$ sec
H	$4(10)^5$ cm	$4(10)^4$ cm

Table 1.

In the case of shallow waves on a homogeneous ocean, the inequality (17) is greatly violated by a factor of $(10)^5$. If we consider the ocean as a two-density model and consider internal waves, we introduce a reduced gravity g' which is defined

$$g' = g\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)$$

The upper layer extends one-fifth of the total depth below the surface. The values for the reduced gravity case are given in table 1. The inequality (17) is violated by a factor of $(10)^1$ for this case. This factor is not sufficiently large to preclude the possibility of (17) being satisfied due to the fact that ν is not necessarily correct to one part in ten. Thus we must admit the possibility of the forbidden band, although unfortunately we cannot guarantee its existence. We may experimentally set up the forbidden band situation by filling a large tank with two fluids of very nearly equal density, and generating waves. The difficulties associated with such an experiment would be technical ones.

In the oscillating boundary problem of Part 8 we observed a boundary layer having thickness of order $\left(\frac{N}{\sigma}\right)^{\frac{1}{2}}$. For $\sigma^2 < f^2$ we have from the inviscid results a coastal wave having thickness of order $(f^2 - \sigma^2)^{-\frac{1}{2}}$. We may ask for what value of σ will the boundary layer be of comparable thickness with the coastal wave, i.e. we require

$$\left(\frac{N}{\sigma}\right)^{\frac{1}{2}} = (f^2 - \sigma^2)^{-\frac{1}{2}}$$

hence

$$\sigma = \frac{1}{2}\left[-\frac{1}{N} \pm \sqrt{\left(\frac{1}{N}\right)^2 + 4f^2}\right]$$

For the ocean there are two cases of interest, shallow waves extending to the surface, and internal reduced gravity waves. For the first case, $N = (10)^{-2}$, $f = (10)^{-3}$, where f is non-dimensional, corresponding to the value in table 1. Due to the relative magnitude of (1/N) and f we may approximate the square root, as was done in (27) to get

$$\sigma = Nf^2 = 10^{-8}$$

or in dimensional terms

$$\sigma = (10)^{-9}/\text{sec}$$

This means the boundary layer thickness will be comparable with the coastal wave thickness if the period of oscillation of the boundary is 60 years.

A more interesting case is that of internal waves for which $N = 10$, $f = 10^{-2}$, where f is non-dimensional, corresponding to the value in table 1. For this case

$$\sigma = 10^{-3}$$

or in dimensional terms

$$\sigma = 10^{-4}/\text{sec.}$$

This means the period of oscillation should be slightly less than one day. A period of this magnitude may be realized physically if we simulate the effect of a pressure front moving up the coast by the oscillating boundary. There is still a further result which may be of interest.

Suppose we are experimentally able to measure the net effect of the viscous boundary layer and coastal wave at some distance from the shore, i.e. we evaluate the quantity

$$A \exp\left[-(f^2 - \sigma^2)^{\frac{1}{2}}x\right] + B \exp\left[-\left(\frac{\sigma}{N}\right)^{\frac{1}{2}}x\right]$$

by experiment, where A and B are known functions of σ and f, the viscosity being assumed the unknown quantity. Using the previous relations we may evaluate N, the macroscopic viscosity.

Acknowledgements

I wish to thank Dr. Louis Howard (MIT) who has offered much valuable advice during the preparation of this manuscript. Thanks are also due to Dr. Willem Malkus, who encouraged me as he did all the summer fellows; to the Fellowship Committee for their financial support; to Dr. W.H.Reid who originally acquainted me with the program of summer study.

Convection of Water Maintained by Cooling from Below

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Introduction

Numerous papers have been written on convection of liquid maintained by heating from below and cooling from above. As a deviation from this pattern, the process of convection by cooling from below and warming from above was investigated.

Since water has the property of being most dense at 4°C it was an ideal fluid to set up a convecting motion by cooling from below. Experiments were performed in which convection was observed when a vessel of water was cooled on the bottom at 0°C. An attempt at a mathematical analysis of the process was far from being successful.

I. The Experiment

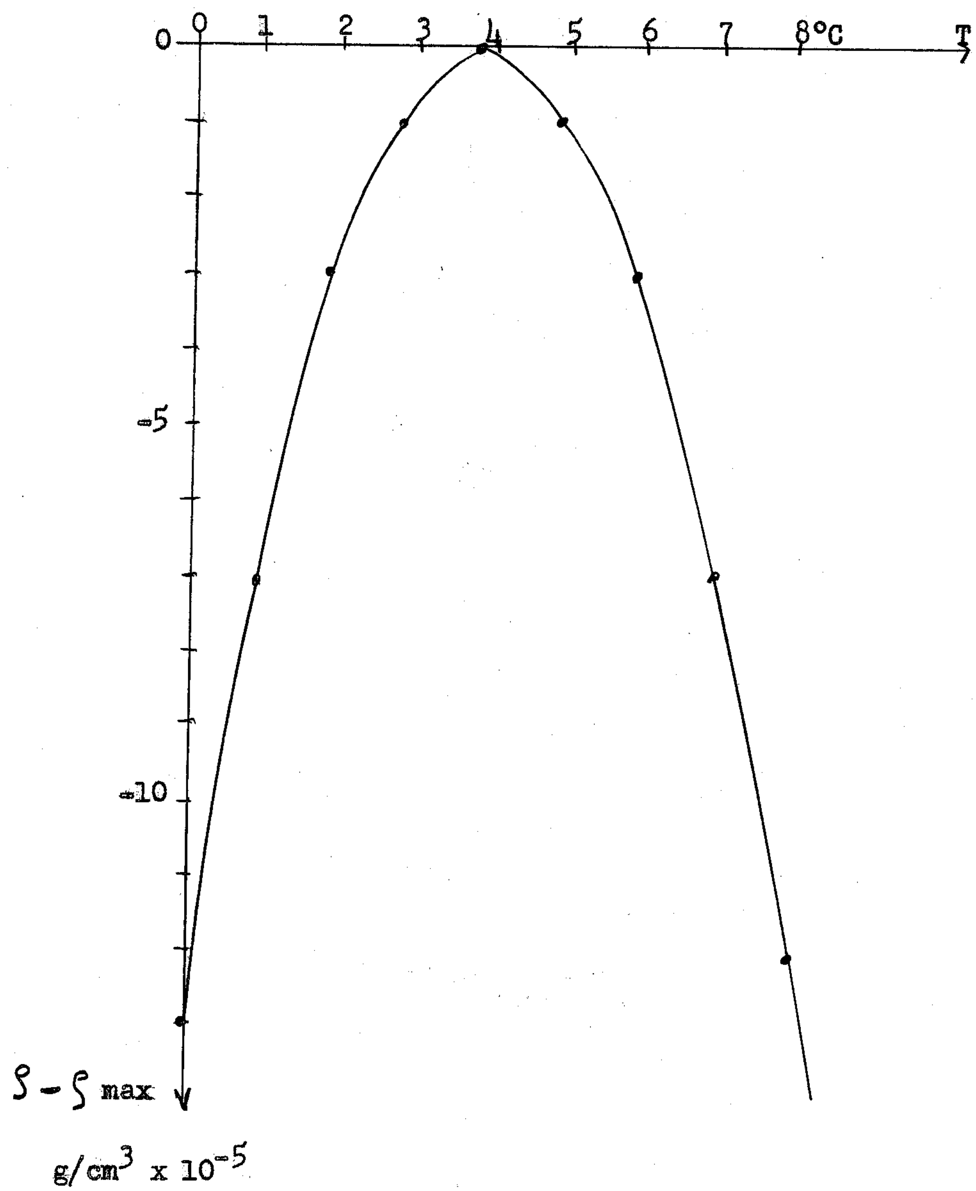
I - A. Apparatus and Equipment

Figure 1 shows the density - temperature profile of water. From this it can be inferred that if water in a vessel was cooled at 0°C on the bottom and kept at a warm temperature, say 16°C, on the surface, some sort of motion should occur. For somewhere in the vessel there should be a layer of water at 4°C. And as 0°C water is more buoyant than 4°C water, the colder water should tend to rise.

For the experiment, the container used was a cylindrical shell of lucite with $\frac{1}{4}$ inch thick walls, an outside diameter of

Figure 1

Variation of density with temperature



12 inches and a height of about 10 inches. One end of the lucite cylinder was machined smooth so it sat flush upon a plate of aluminum $\frac{1}{4}$ inch thick, 13 inches wide and 16 inches long. To render the vessel water-tight, stop cock grease was applied to the contact area between the lucite shell and aluminum plate.

Figure 2 shows the arrangement for the first few experiments. The height of the water was about 10 cm. The container was made to rest on a block of ice large enough so that the plate was contained by the ice.

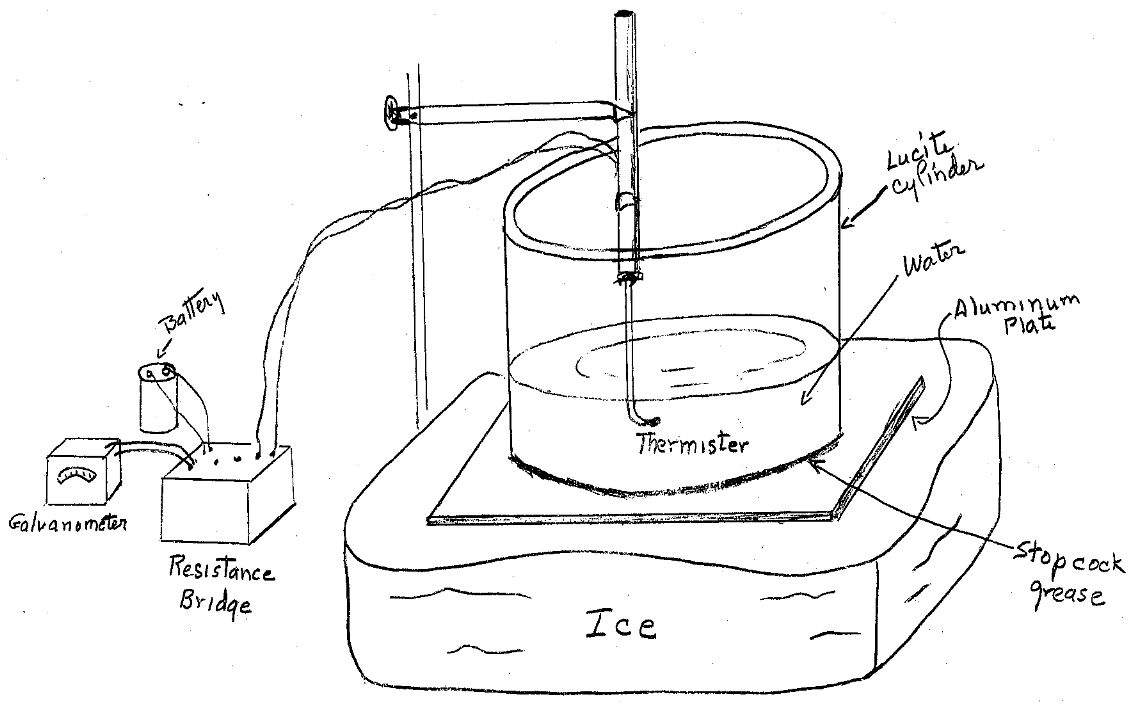


FIG. 2

To measure the temperature of water at various levels, a thermister connected to a bridge circuit was used. The thermistor was attached to a movable rod with a graduated scale so that the depth of the thermistor from the surface of water could be measured.

The bridge circuit consisted of a potentiometer with external galvanometer and battery. At each reading the resistance of the thermistor was measured by the bridge circuit and then the reading was converted to temperature by a calibration scale.

A particle of $AgNO_3$ crystal dropped into the water proved to be an excellent indicator for the moving water. As ordinary tap water was used, there were sufficient amounts of various chlorides present to form AgCl. The fine particles of AgCl were excellent tracers when a beam of light was focused into the vessel at right angles to the line of observation.

The arrangement of placing the vessel upon a block of ice was unsatisfactory for prolonged observations. The uneven melting of ice warped the aluminum plate, which brought about leakage of water at the contact interface of the shell and plate. For a more stable set up, salt water and ice combination was used to cool the bottom. The arrangement is shown in Figure 3. Salt water was readily available in the form of sea water which was piped into the laboratory from nearby Vineyard Sound. The best measurement of the

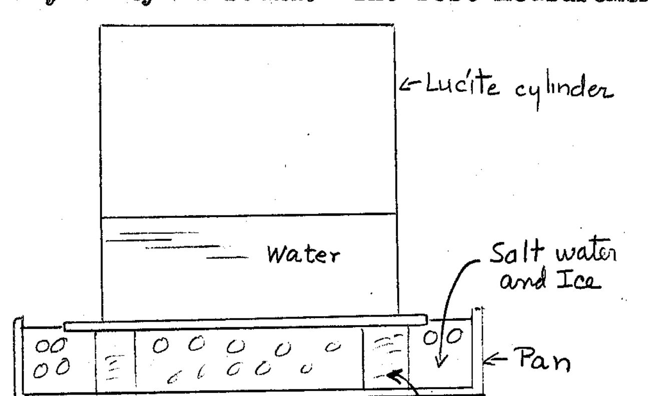


Fig. 3

saltwater-ice combination in contact with the plate indicated a steady -0.2°C.

I - B. Procedure

It was found that if water at room temperature was poured into the vessel to a height of 10 cm and the bottom cooled at 0°C, it took about 6 hours before any convective motion was noticed. To eliminate the long waiting period, water cooled to about 5°C was poured into the vessel to a height of 5 cm. A circular piece of cardboard was placed upon the surface. Water at room temperature was then poured gently upon the cardboard and allowed to run off the edges of the cardboard. In this way roughly water of two layers was obtained. After the motion of water resulting from pouring was allowed to settle, temperature measurement was taken. The temperature profile was not linear, but more that of a parabola. (Figure 4.)

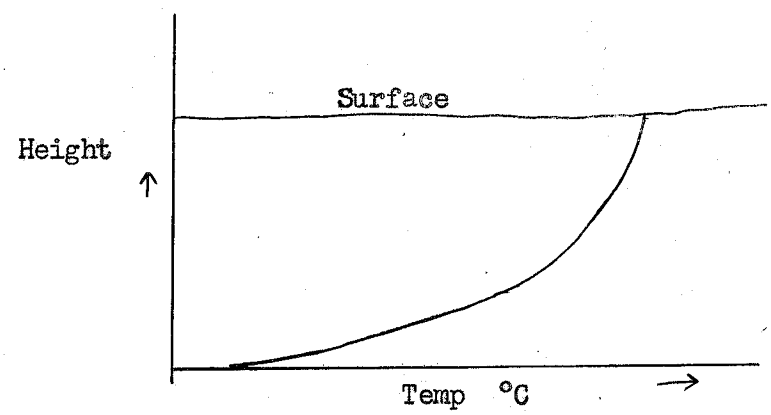


Fig. 4

After the equipment and apparatus were set up and the bridge circuit connected, the temperature profile of the water was taken every hour. As two minutes between readings had to be allowed for the thermistor to be in isothermal equilibrium with the surrounding water, one profile took about 20 minutes.

The data were plotted on a graph of temperature vs. depth. A number of these successive graphs gave a history of the evolving temperature profile. Besides the temperature, the densities of the water corresponding to the various temperatures were also plotted.

I - C. Observations and Data

There were some interesting observations connected with convective motion.

At the onset of convection, the height of the upward motion of the water was 1 cm. This seemed to be the minimum height of convection. The upward motion occurred along ridges about 7 cm long and curved. Figure 5 shows the type of motion.

Fig. 5

Upward Convection at Onset

When the motion had developed to a height of 2.5 cm, it showed an irregular cell-like pattern. A top view of the motion is shown in Figure 6. The lines that seemed to indicate the boundaries of cells were regions of upward motion. In general the average diameter of these irregular patterns was about 5.0 cm.

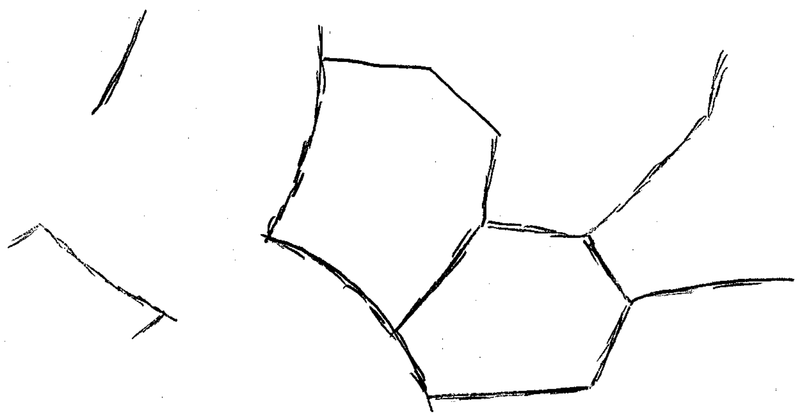


Fig. 6

Top View of Convecting Water
Broken Lines Suggesting Cells

When the upward motion reached a height of 7 cm, there was a change in the pattern. There were lines indicative of boundaries of irregular cells. The average diameter of these cells was 13 cm. However, this time the upward motions occurred in what seemed to be the centers of the cells and the downward motion occurred along the boundaries.

At the onset of the convective motion, the motion was similar to a roll. At the second stage of convection, the upward motion occurred in thin walls that suggested cell boundaries. At the third stage the upward motion was concentrated in thin streaks while other parts of the water were relatively undisturbed. (Figure 7).

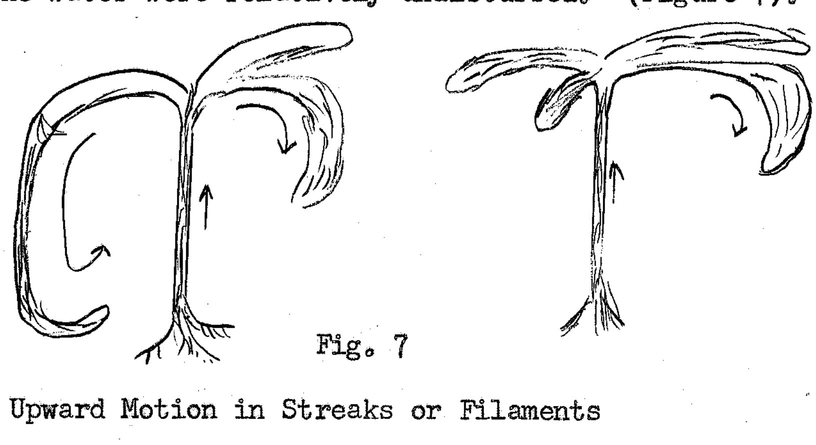


Fig. 7

Upward Motion in Streaks or Filaments

The data from two of the experiments have been included in this report. Graphs resulting from the data were also included.

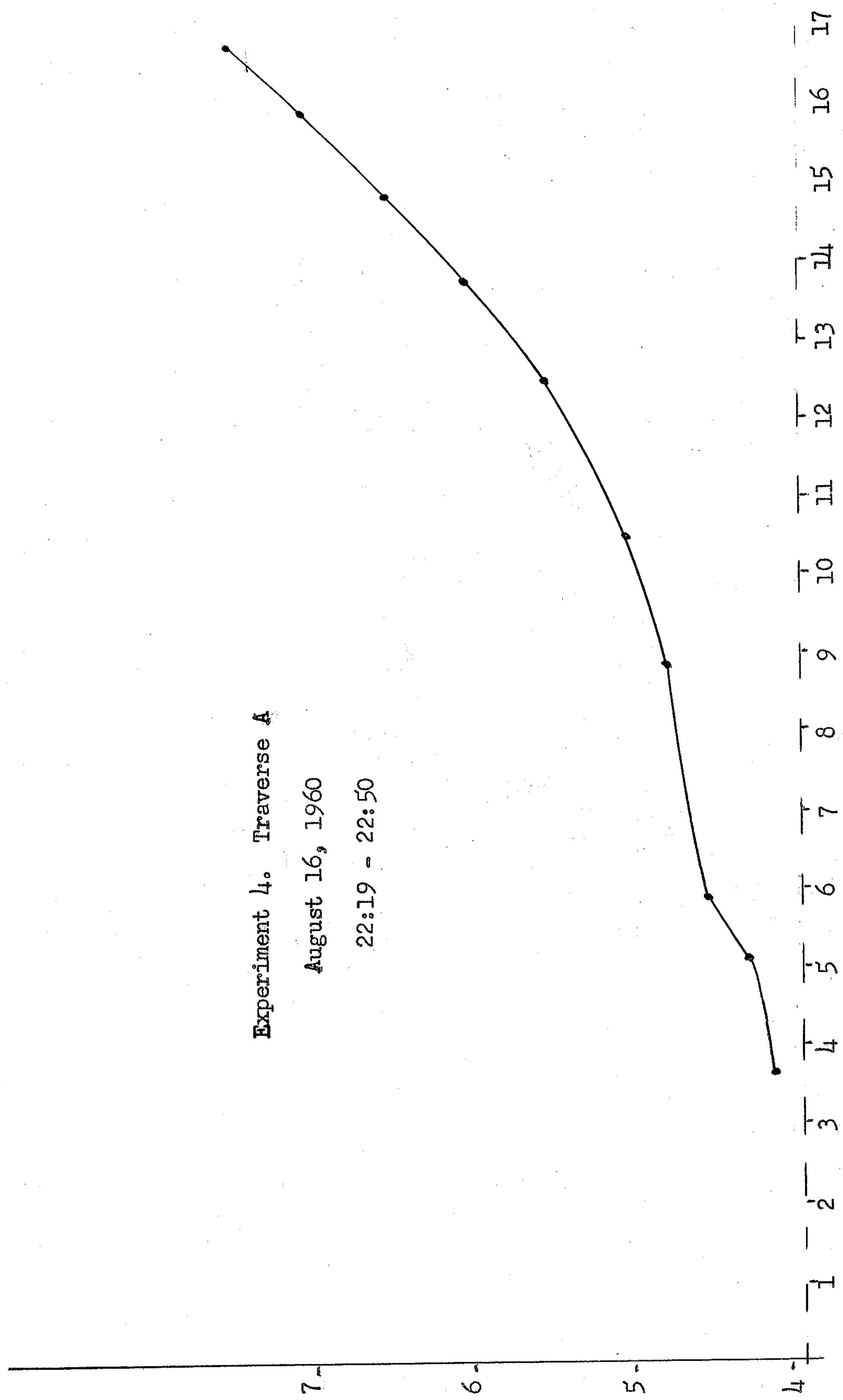
Experiment 4.

Date: August 16, 1960
Height of water column 10.05 cm
Vessel set on block of ice.

Traverse A.

Depth Scale	Thermistor Reading	Temperature	Time
4.1 in (.18 in from bottom)	2562 ohms	3.7°C	p.m.
4.25	2402	5.1	10:19 p.m.
4.5	2301	5.9	10:21
4.75	1995	8.8	10:23
5.0	1857	10.5	10:25
5.5	1678	12.5	
6.0	1570	13.8	10:30
6.5	1496	14.8	10:32
7.0	1427	15.9	10:34
7.5	1367	16.8	10:36
4.65	2322	5.7	10:41
4.55	2385	5.2	
4.40	2450	4.7	10:45

At 10:25 Height of convection 0.5 in
10:27 0.8 in

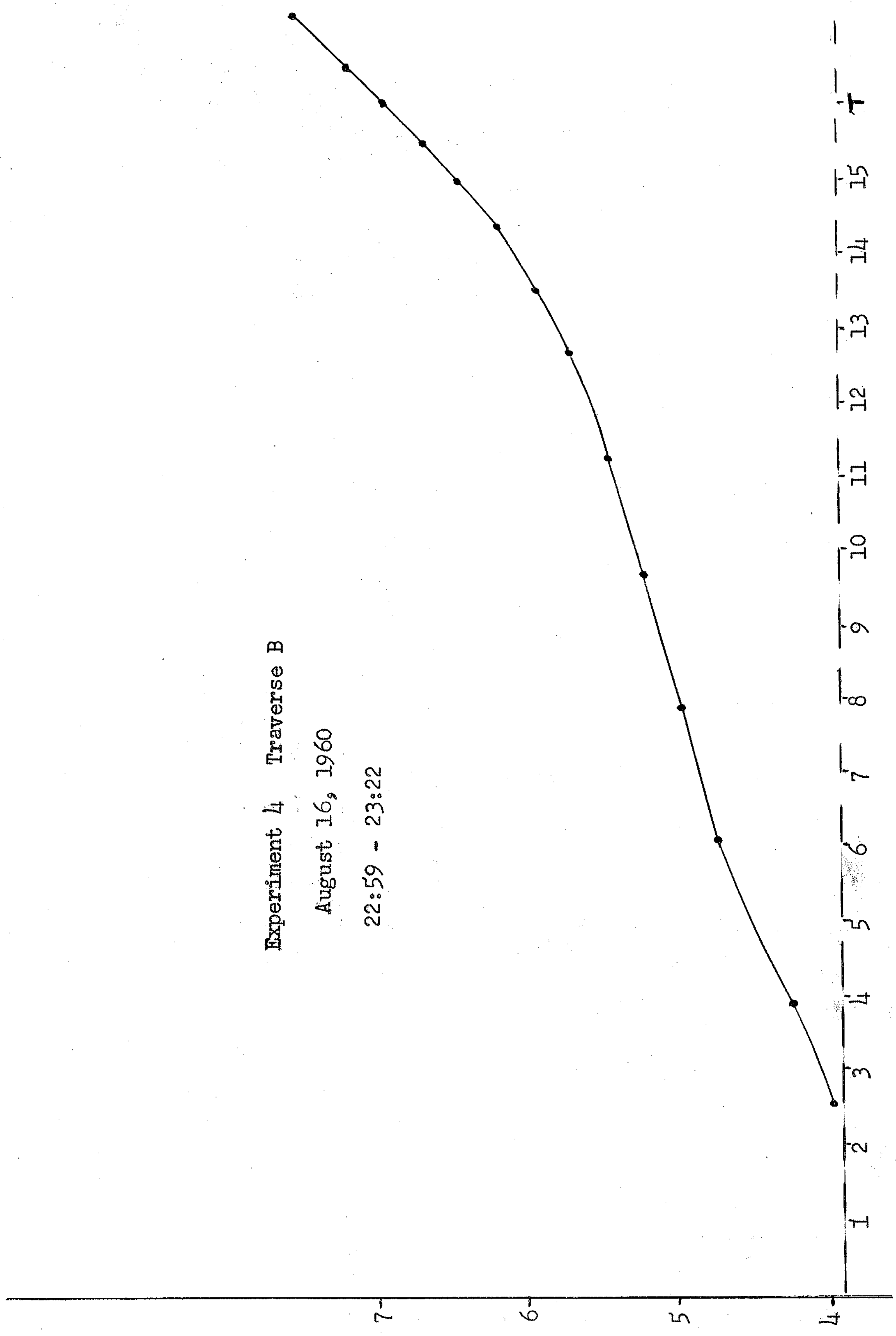

Experiment 4. Traverse A
August 16, 1960
22:19 - 22:50
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
4
5
6
7

Experiment 4 (cont')

	Depth Scale	Thermistor Reading	Temperature	Time
Traverse B.				
	4.0 inches	2712 ohms	2.6 °C	
	4.25	2542	3.9	
	4.50	2435	4.7	10:59 p.m.
	4.75	2285	6.1	11:00
	5.0	2086	7.9	11:02
	5.25	1911	9.7	11:04
	5.50	1774	11.3	11:06
	5.75	1667	12.7	11:07
	6.00	1600	13.5	11:09
	6.25	1531	14.4	11:10
	6.50	1487	15.0	11:12
	6.75	1447	15.5	11:14
	7.00	1413	16.0	11:15
	7.25	1383	16.5	11:18
	7.50	1355	17.1	11:20
	7.6	1335	17.2	11:22

At 11:17 convection height 1.3 in.

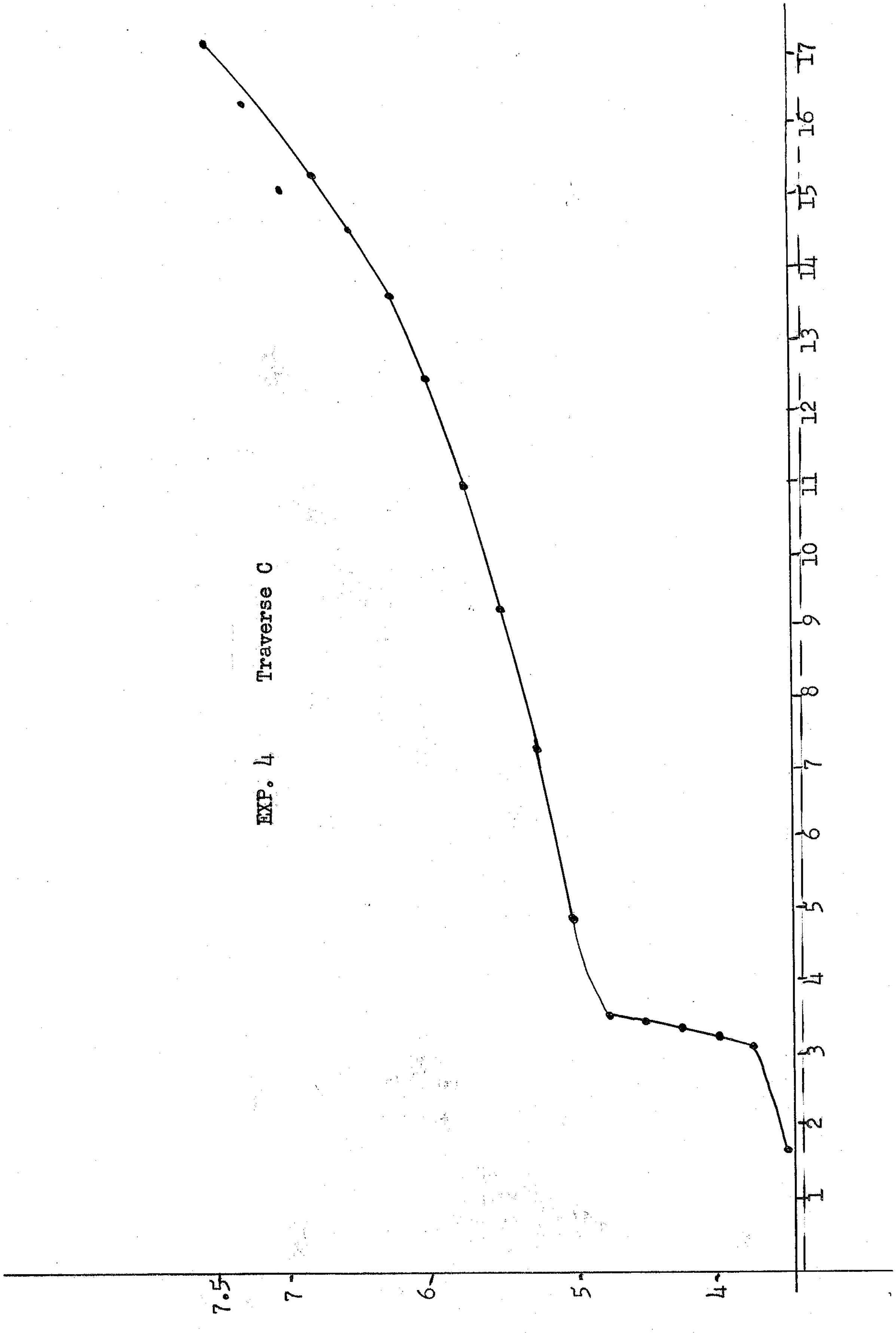
	Depth Scale	Thermistor Reading	Temperature	Time
Traverse C.				
	7.5	1347	17.1	11:53
	7.25	1393	16.3	11:54
	7.0	1423	15.1	11:55
	6.75	1468	15.3	11:59
	6.5	1521	14.5	12:00
	6.25	1588	13.6	12:02
	6.00	1675	12.5	12:05
	5.75	1800	11.0	12:08
	5.5	1967	9.2	12:10
	5.25	2153	7.3	12:11


Experiment 4 Traverse B
August 16, 1960
22:59 - 23:22
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
4
5
6
7

Experiment 4 (cont')

	Depth Scale	Thermistor Reading	Temperature	Time
Traverse C (cont')				
	5.00 inches	2424 ohms	4.9 °C	12:15 p.m.
	4.5	2598	3.5	
	4.75	2597	3.5	12:18
	4.25	2629	3.3	12:20
	4.0	2590	3.5	
	3.75	2646	3.1	
	3.60	2844	1.7	

Motion height 1.7 in at 11:56


EXP. 4 Traverse C
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
4
5
6
7
7.5

Experiment 7.

Date: August 25, 1960

Height of water column 3.9 in

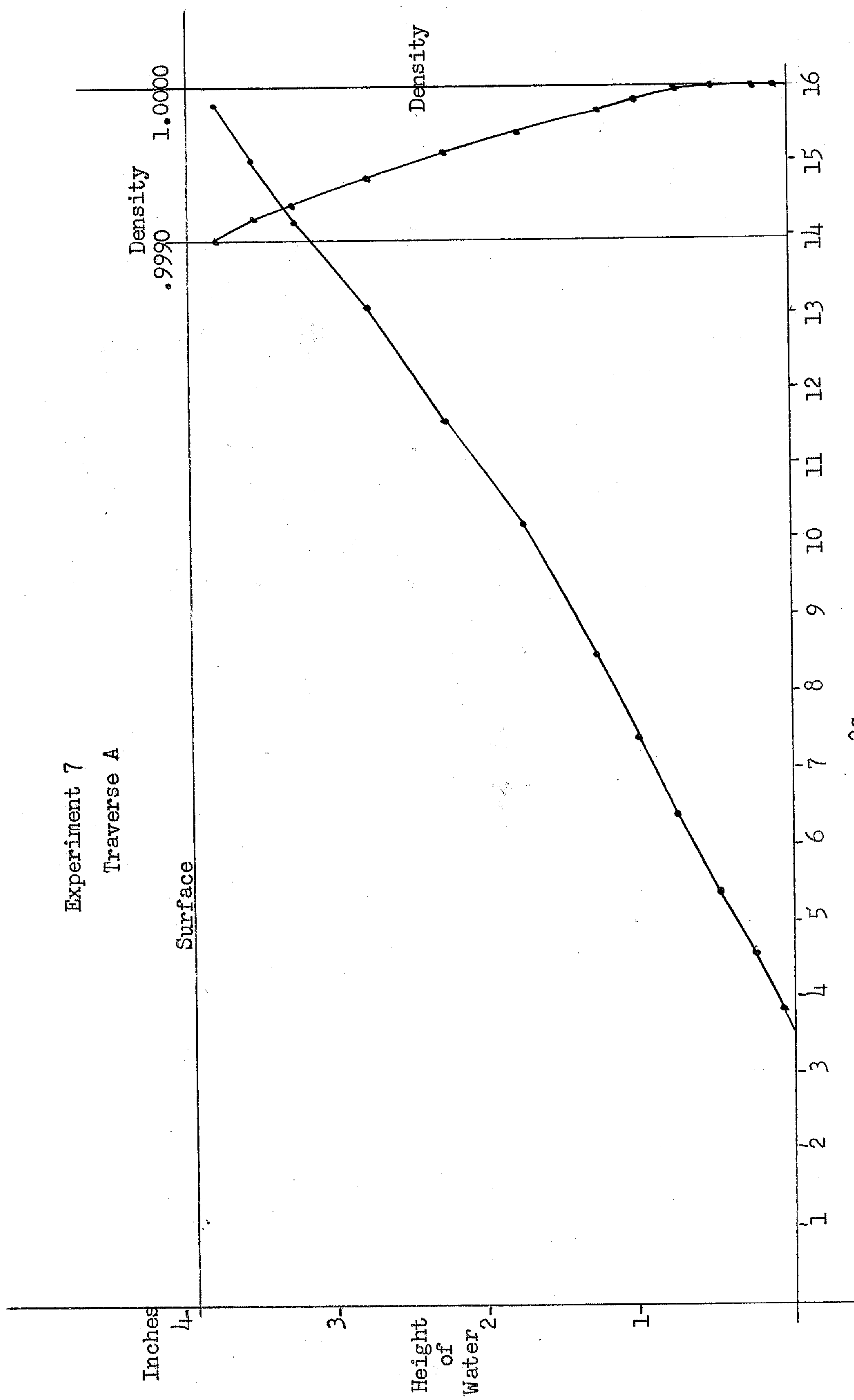
Vessel cooled by salt water-ice combination at -0.2°C.

Room temp. 23.5 °C.

Depth: from bottom	Thermistor Reading	Temperature	Time
Traverse A.			
.1 in	2560 ohms	3.9°C	
.25	2460	4.6	
.50	2361	5.4	21:38
.75	2253	6.4	
1.0	2139	7.5	
1.25	2030	8.5	
1.75	1868	10.3	
2.25	1740	11.6	
2.75	1630	13.1	
3.25	1534	14.3	
3.5	1483	15.0	
3.75	1430	15.8	21:53

Height of motion .5" at 21:38
.8" at 21:42
1.1" at 21:54

Traverse B.			
3.75	1394	16.2	
3.5	1515	14.6	22:00
2.75	1615	13.3	22:02
2.25	1728	11.9	
1.75	1896	9.5	22:06
1.25	2190	6.9	
1.0	2340	5.5	
0.75	2470	4.4	


Experiment 7
Traverse A
Surface
Density
.9990
1.0000
Density
°C
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
Inches
Height of Water
1 2 3 4

Experiment 7 (cont')

Traverse B (cont')

Depth: from bottom	Thermistor Reading	Temperature	Time
.5 inches	2575 ohms	3.6 °C	
.25	2575	3.6	
.15	2650	3.1	
.1	2705	2.7	22:25

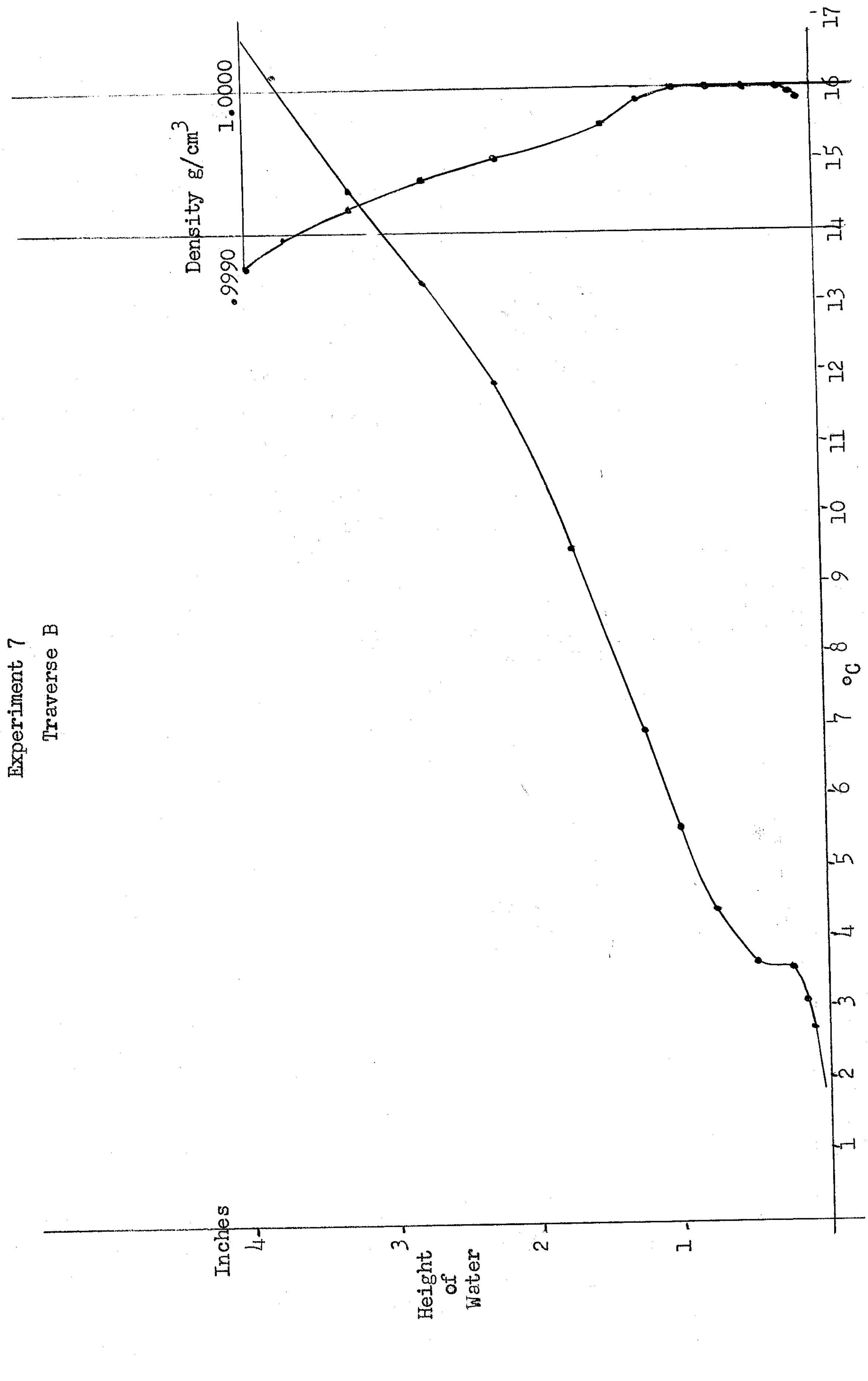
Experiment 7

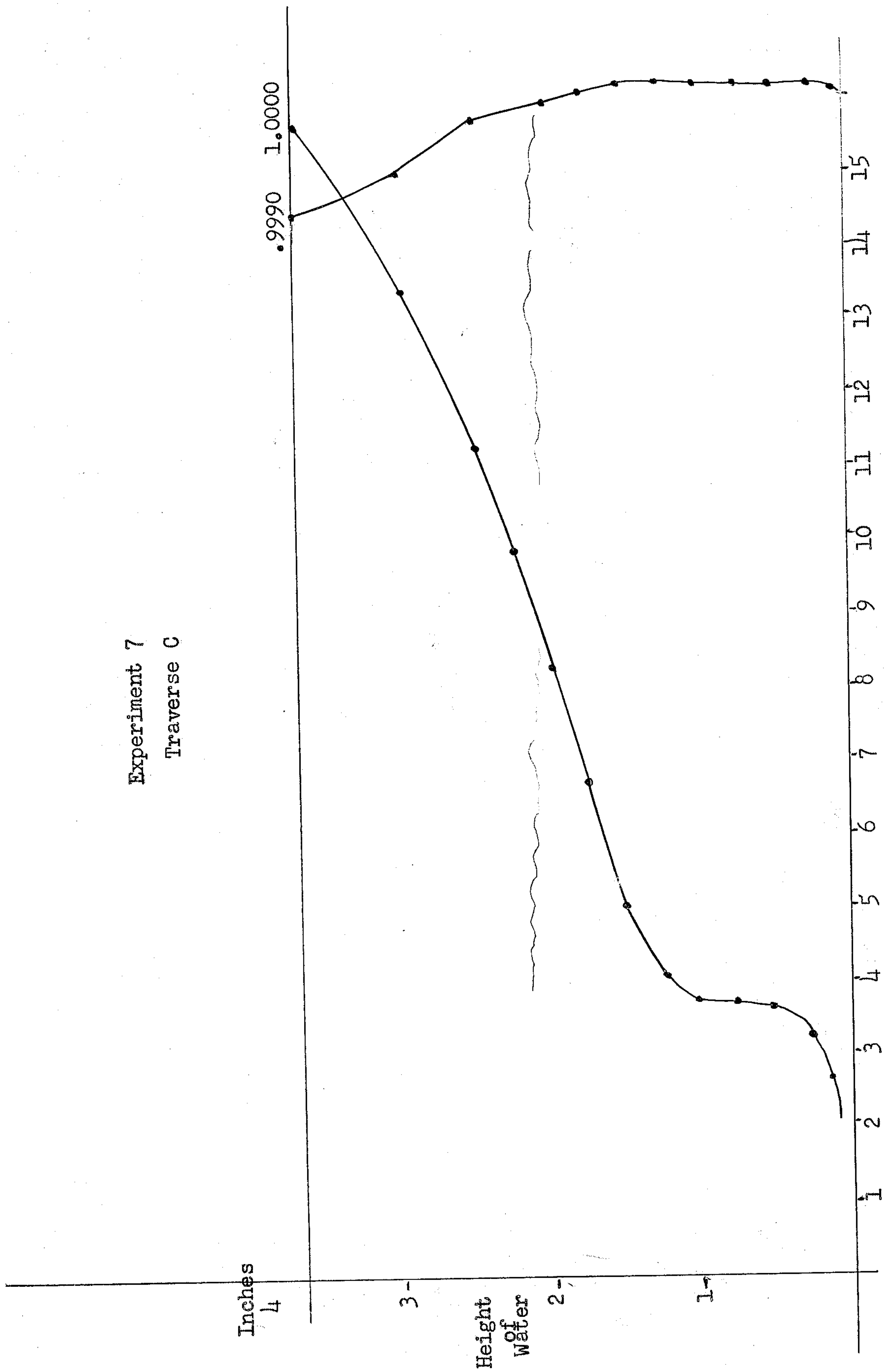
Motion 1.5 in high at 22:14

Traverse C.

.1 in	2708	2.7°C	23:03
.25	2614	3.3	
.5	2588	3.5	
.75	2580	3.6	
1.00	2580	3.6	23:09
1.25	2525	4.0	
1.5	2410 ± 10	5.0	
1.75	2210 ± 10	6.7	
2.00	2045	8.3	
2.25	1910	9.8	
2.5	1785	11.2	
3.0	1615	13.3	
3.7	1463	15.4	

Motion 2.1 in high 23:24
4.5 in across


Experiment 7
Traverse B
Density g/cm3
.9990
1.0000
°C
Inches
Height of Water
1
2
3
4
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17


Experiment 7
Traverse C
.9990
1.0000
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
°C
Inches
4
3
2
1
Height of Water

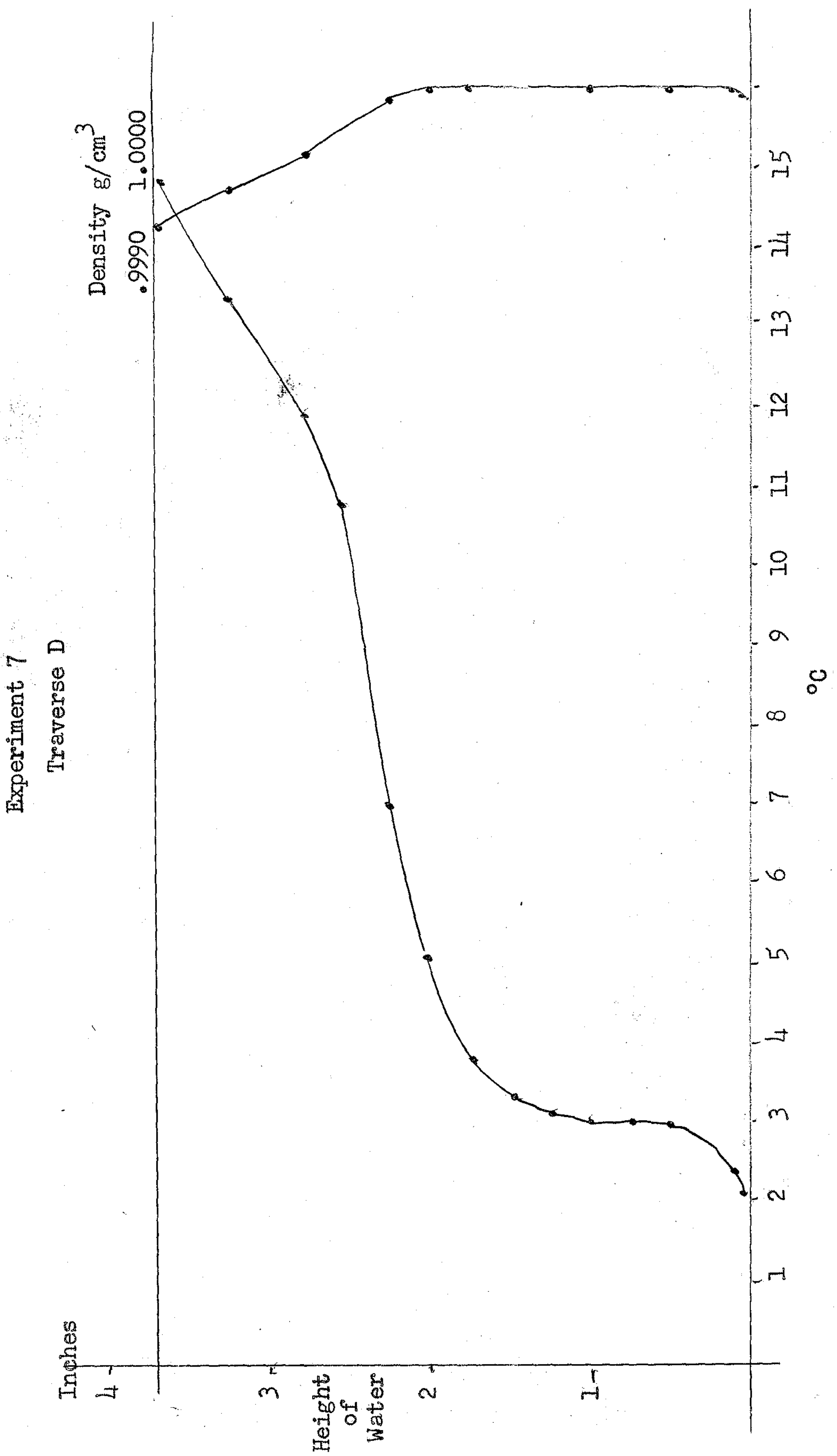
Experiment 4.

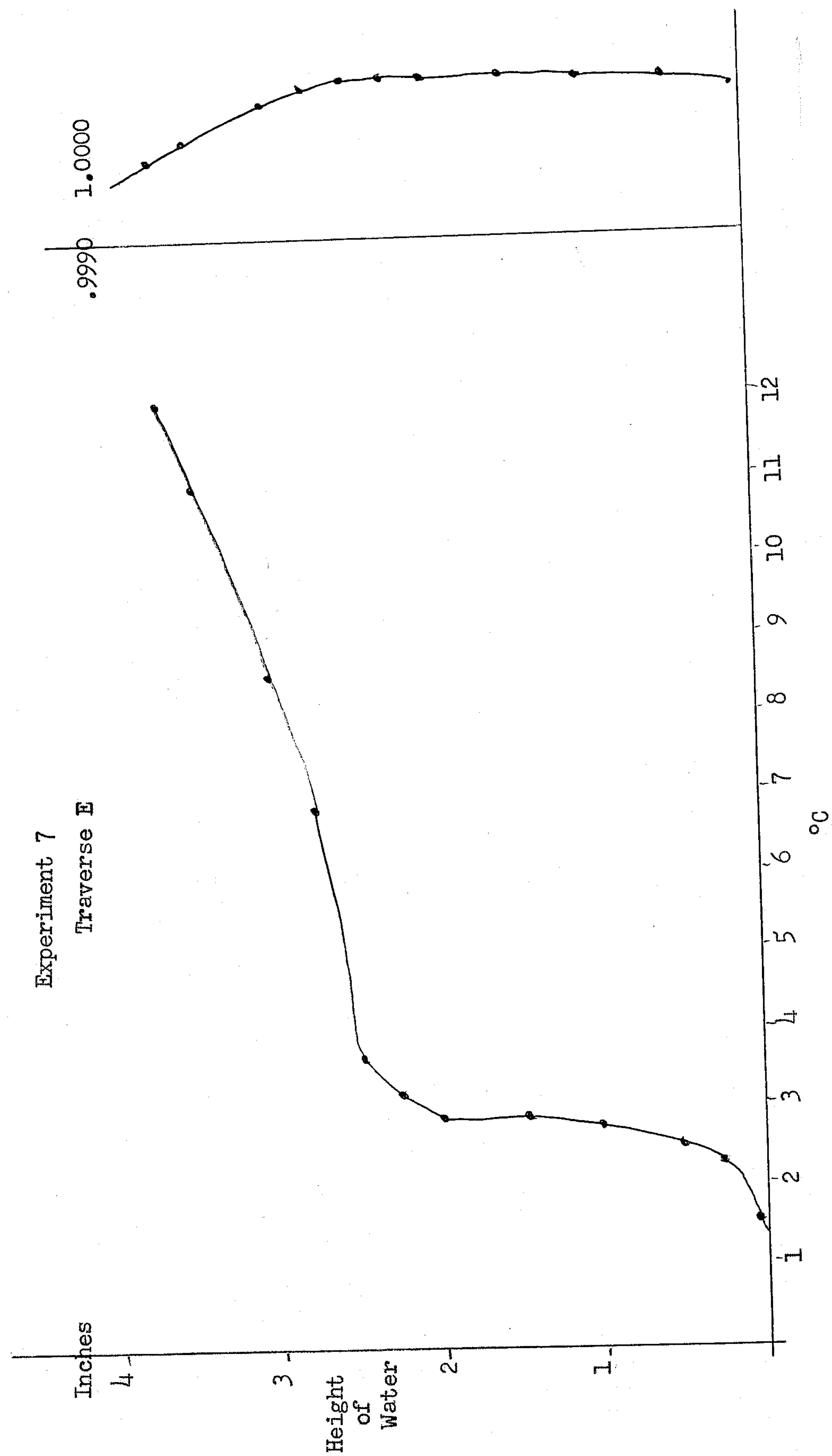
Depth from bottom	Thermistor Reading	Temperature	Time
Traverse D.			
3.7 in	1493 ohms	14.8 °C	
3.25	1614	13.3	24:09
2.75	1815	11.9	
2.25	2183	7.0	
2.00	2400	5.1	
1.75	2560	3.8	
1.50	2625	3.3	
1.25	2645	3.1	
1.00	2655	3.0	
.75	2655	3.0	
.5	2655	3.0	
.25	2610	3.4°	
.15	2630	3.3	
.10	2730	2.4	
.05	2780	2.1°	

Height of motion 2.6 in at 24:23

Traverse E.			
.05	2860	1.6°	1:16 Aug.26,1960
.25	2750	2.3	
.50	2730	2.5	
1.00	2690	2.8	
1.50	2680	2.9	
2.00	2680	2.9	
2.25	2675	3.2	
2.5	2570	3.7	
2.75	2210	6.8	
3.0	2030	8.5	
3.5	1810	10.9	
3.7	1725	11.9	1:26

Height of motion 3 in at 1:27


Experiment 7
Traverse D
Density g/cm3
.9990 1.0000
Inches
Height of Water
°C
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
1 2 3 4


1.0000
.9990
Experiment 7
Traverse E
°C
1 2 3 4 5 6 7 8 9 10 11 12
Inches
4
3
2
1
Height of Water

I - D A Few Results

When the heat flux through the water was compared to the height of the convective motion, a somewhat linear relation was found. The quantities are tabulated below while the graph is given in Figure 8:

	Heat Flux ($\frac{cal}{sec\ cm^2}$)	Height (cm)
Exp.4	3.6×10^{-3}	1.27
"	3.8×10^{-3}	3.3
"	4.0×10^{-3}	4.8
Exp.7	2.1×10^{-3}	2.03
	2.8×10^{-3}	3.0
	3.16×10^{-3}	5.33
	3.6×10^{-3}	7.6

But the appearance was deceptive. A better linear relation was found when a quantity $\frac{2 g \alpha \beta^2}{\nu K} h^4$ resembling the Rayleigh number was plotted against the heat flux. The symbols are:

$g = 980 \frac{cm}{sec^2}$

α = coefficient of expansion of water near 4°C,

β = mean temperature gradient

ν = kinematic viscosity

K = thermal conductivity of water

h = height of convective motion.

In the Rayleigh number β to the first power is used, while β^2 was more appropriate to this problem because the density of water

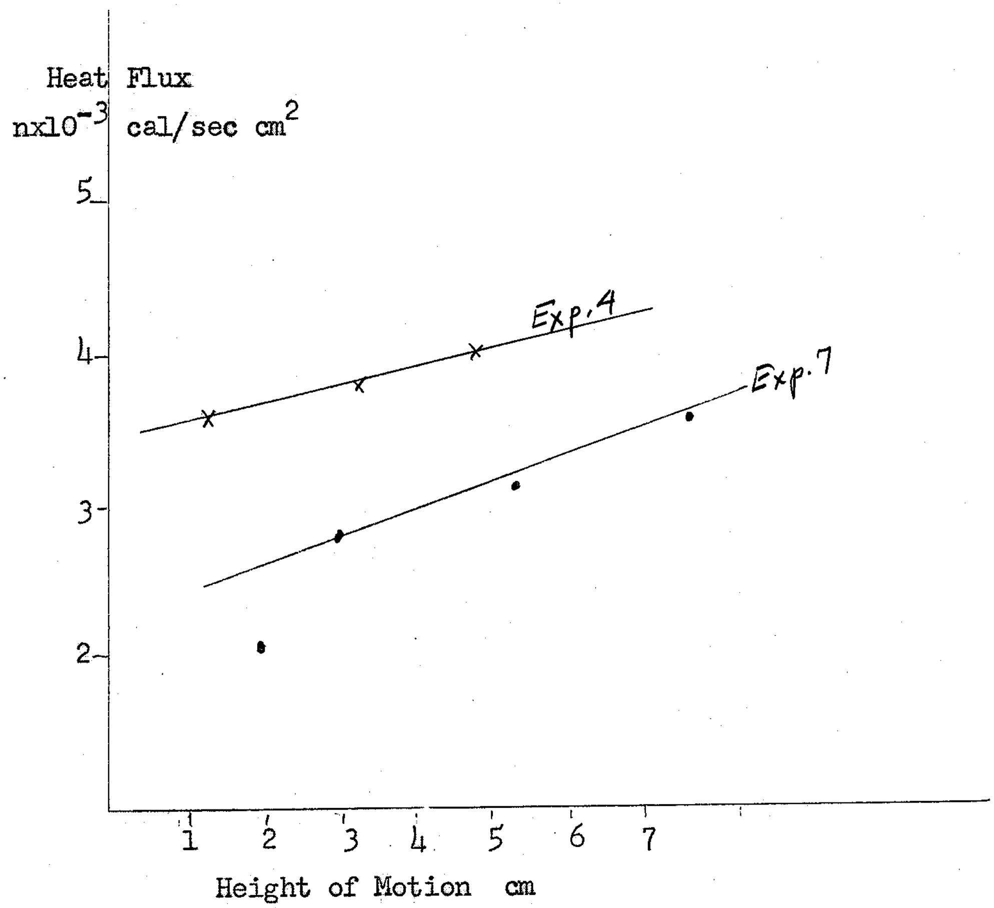


Figure 8

near 4°C is a parabolic function of temperature. The results are:

	Heat Flux (cal/sec cm^2)	$\frac{2g\alpha\beta^2h^4}{\nu K}$
Exp.4	3.6×10^{-3}	8,400
	3.8×10^{-3}	56,700
	4.0×10^{-3}	142,000
Exp.7	2.1×10^{-3}	21,500
	2.8×10^{-3}	47,300
	3.16×10^{-3}	149,000
	3.6×10^{-3}	300,000

The results were plotted (Figure 9). The first point of Experiment 7 probably represented a different regime from the other points.

As a minimum the quantity $\frac{2g\alpha\beta^2 h^4}{\nu K}$ turned out to be 5250

II. Mathematical Analysis

The attempted mathematical analysis followed closely the arguments presented in a paper by Anne Pellew and R.V. Southwell on convective motion in a fluid heated from below (1940).

II - A The Governing Equations

In rectangular coordinates with the origin of the coordinate system at the midpoint of a layer of water, and with the x-y plane horizontal and z-direction being vertically upward, the equations of motion are

$$\rho \frac{D}{Dt}(u,v,w) = \rho(X,Y,Z) - \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\bar{p} + \nu\rho\nabla^2(u,v,w) \quad (1)$$

u,v,w are the components of velocity; X,Y,Z are the components of body force; ρ is the density of water; ν the viscosity.

In this problem, we have

$$X = 0, \quad Y = 0, \quad Z = -g. \qquad (2)$$

The equation of continuity is

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad (3)$$

As was shown in Figure 1 of this paper, the density profile of water near 4°C resembles a parabola. Hence we assume that density ρ is

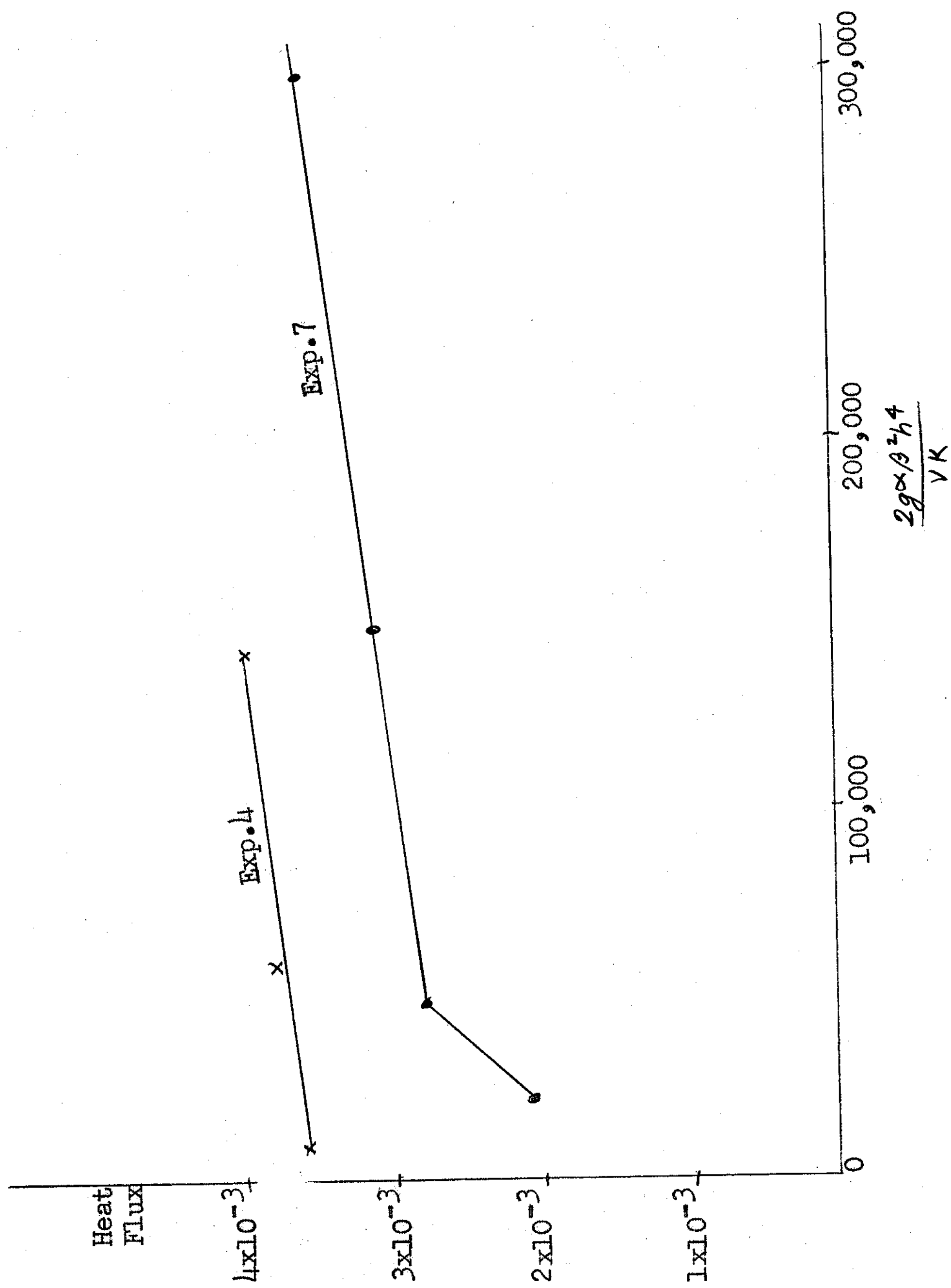

Heat Flux
4x10^{-3}
3x10^{-3}
2x10^{-3}
1x10^{-3}
0
100,000
200,000
300,000
Exp.4
Exp.7
$\frac{2g\alpha\beta^2h^4}{\nu K}$

Figure 9

represented by

$$\rho = \rho_0 \left(1 - \alpha \left[\bar{\theta} - T_0\right]^2\right) \tag{4}$$

where ρ_0 is the density at 4°C, or 1 g/cm^2 in the c.g.s. system, $\bar{\theta}$ is the actual temperature in °C, $T_0 = 4$°C, $\alpha = 7.8 \times 10^{-6}$(°C)$^{-2}$.

The equation of conduction of heat

$$\frac{D\bar{\theta}}{Dt} = k \nabla^2 \bar{\theta} \tag{5}$$

where K is the diffusivity for the temperature.

In the initial state where convection has not yet begun, u,v,w vanish severally. From the first three equations as given in (1), it is seen that $\bar{p} = p_0$ is independent of x and y, and we have only

$$0 = -g\rho - \frac{\partial p_0}{\partial z} \tag{6}$$

p_0 denotes the initial (steady) pressure at (x,y,z). Then equation (3) is satisfied identically.

The temperature is also held steady in the initial case and is independent of x and y. Then as

$$\frac{D\theta}{Dt} = 0$$

we have

$$0 = k \frac{\partial^2 \bar{\theta}}{\partial z^2} \tag{7}$$

and the temperature gradient will be constant. In the experiment, we did not obtain this idealized situation, but this can be brought about if we were patient enough to wait six hours before convection began.

We may express the initial temperature θ_o in the form

$$\theta_o = \Theta_o + \beta z \tag{8}$$

where β is the steady temperature gradient, and Θ_o is the temperature of the midpoint of the layer of water.

As density is given by

$$\rho = \rho_o\left[1 - \alpha_1(\theta_o - T_o)^2\right]$$

we have upon substitution

$$\rho = \rho_o\left[1 - \alpha_1(\beta z - T_o + \Theta_o)^2\right] \tag{9}$$

Also

$$\frac{\partial p_o}{\partial z} = -g\rho_o\left[1 - \alpha_1(\beta z - T_o + \Theta_o)^2\right]$$

We assume that u, v, w, the velocities in the convective motion, to be sufficiently small that their squares and products can be neglected. This is justified as observations showed that it took several minutes for the tracers to travel what seemed to be a cycle. Then we may replace $\frac{D}{Dt}$ by $\frac{\partial}{\partial t}$.

We designate the temperature as

$$\bar{\theta} = \theta_o + T = \Theta_o + \beta z + T \tag{10}$$

where T is the deviation from the initial temperature profile. We assume T to be small so that second order terms can be neglected. Upon substituting (10) into (5), we have

$$-\beta\frac{Dz}{Dt} = \left[\frac{\partial}{\partial t} - k\nabla^2\right]T$$

or

$$-\beta\omega = \left[\frac{\partial}{\partial t} - k\nabla^2\right]T \qquad (11)$$

Density also can be expressed with T, so that

$$\rho = \rho_0\left[1-\alpha_1(\beta z + T - T_0 + \Theta_0)^2\right].$$

Further we can express $\bar{p} = p_o + p$ where p_o is pressure in the initial state and is a function of z only, while p is the perturbation. We had

$$\frac{\partial p_0}{\partial z} = -g\rho$$

but ρ here is only in the unperturbed state, so that we must express ρ as $\rho = \rho_0\left[1-\alpha_1(\beta z - T_0 + \Theta_0)^2\right]$

Then

$$\frac{\partial p_0}{\partial z} = -g\rho_0\left[1-\alpha_1(\beta z - T_0 + \Theta_0)^2\right].$$

If we substitute these into equation (1) and let $\frac{D}{dt}$ become $\frac{\partial}{\partial t}$, we obtain

$$\rho\frac{\partial\vec{u}}{\partial t} = -g\rho_0\alpha_1\left\{-2T(\beta z - T_0 + \Theta_0)\right\} - \nabla p + \nu\rho\nabla^2\vec{u}$$

in which second order terms in T have been neglected. We may rewrite

$$\frac{\partial\vec{u}}{\partial t} = 2g\alpha_1 T(\beta z - T_0 + \Theta_0)\vec{K} - \frac{1}{\rho}\nabla p + \nu\rho\nabla^2\vec{u} \qquad (12)$$

We may replace equation (3) by

$$\nabla\cdot\vec{u} = 0 \qquad (13)$$

By combining u and v equations of (12) with (13) we obtain

$$\left[\frac{\partial}{\partial t} - \nu\nabla^2\right]\frac{\partial\omega}{\partial z} = \frac{1}{\rho}\nabla_1^2 p \qquad (14)$$

where $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Then eliminating p between (14) and the w-equation of (12), we have

$$\left[\frac{\partial}{\partial t} - \nu \nabla^2\right] \nabla^2 w = 2 g \alpha \beta z \nabla_1^2 T \tag{15}$$

If we eliminate w between (15) and (11) we have

$$\left[\frac{\partial}{\partial t} - \nu \nabla^2\right]\left[\frac{\partial}{\partial t} - \kappa \nabla^2\right] \nabla^2 T + 2 g \alpha \beta^2 z \nabla_1^2 T = 0 \tag{16}$$

As a first step, we assume that T has the form $f(x,y)\ F(z)\ \Phi(t)$. $\Phi(t)$ is an exponential function while $\nabla_1^2 f \propto f$. From this we can impose the condition

$$h^2 \nabla_1^2 T + a^2 T = 0$$

where h is half the depth of a convecting cell while a is an undetermined characteristic number.

II-B. Boundary Conditions

The boundary conditions on the cell walls and horizontal surfaces must be expressed in terms of T, as we have only equation (16) to work with. A similar equation in terms of w is not obtainable.

First, we consider the walls of the cell. The walls were surfaces of symmetry. Then we have

$$\frac{\partial w}{\partial n} = 0 \quad \text{and} \quad \frac{\partial T}{\partial n} = 0$$

where n stands for the normal outward along the wall.

The horizontal surfaces can be free or rigid. If the surface is free,

$$w = \frac{\partial^2}{\partial z^2} w = T = 0$$

If the surface is rigid

$$w = \frac{\partial}{\partial z} w = T = 0$$

For the free surface, consider first

$$-\beta w = \left[\frac{\partial}{\partial t} - k \nabla^2 T\right]$$

and assume that a maintained convective motion was obtained. Then $\frac{\partial}{\partial t} = 0$. For free surface $w = 0$, and $k\nabla^2 T = 0$.

But as $\nabla^2 T = \left(\frac{\partial^2}{\partial z^2} + \nabla_1^2\right)T$

and $$h^2\nabla_1^2 T + a^2 T = 0$$

$$\nabla^2 T = \left(\frac{\partial^2}{\partial z^2} - \frac{a^2}{h^2}\right)T = 0$$

But $T = 0$, and we have

$$\frac{\partial^2}{\partial z^2} T = 0$$

If we operate $\frac{\partial^2}{\partial z^2}$ on $-\beta w = -k\nabla^2 T$ and remember that $\frac{\partial^2}{\partial z^2} w = 0$ on a free surface, we obtain

$$\frac{\partial^4}{\partial z^4} T = 0.$$

For a free surface, we have then

$$T = \frac{\partial^2}{\partial z^2} T = \frac{\partial^4}{\partial z^4} T = 0.$$

For a rigid surface, $w = \frac{\partial}{\partial z} w = T = 0$. We also have $\frac{\partial^2}{\partial z^2} T = 0$ by similar arguments. If we operate $\frac{\partial}{\partial z}$ on $-\beta w = -k\nabla^2 T$, we have

$$-\beta\frac{\partial}{\partial z} w = 0 = -k\frac{\partial}{\partial z}\nabla^2 T = -k\left(\frac{\partial^3}{\partial z^3} - \frac{a^2}{h^2}\frac{\partial}{\partial z}\right)T$$

Then $\left[\frac{\partial^3}{\partial z^3} - \frac{a^2}{h^2}\frac{\partial}{\partial z}\right]T = 0$ For a rigid surface we have

$$T = \frac{\partial^2}{\partial z^2} T = \left(\frac{\partial^3}{\partial z^3} - \frac{a^2}{h^2}\frac{\partial}{\partial z}\right)T = 0$$

To simplify calculations we substitute $\frac{z}{h} = \zeta$ and $D \equiv \frac{\partial}{\partial \zeta}$.

This transforms the equations into

$$[D^2 - a^2]^3 T - \frac{2g\alpha\beta^2}{\nu K} h^5 a^2 \zeta T = 0$$

for the steady maintained state of $\frac{\partial}{\partial t} = 0$, and for the boundaries

$$T = D^2 T = D^4 T = 0 \quad \text{at} \quad \zeta = \pm 1$$

for the free surfaces and

$$T = D^2 T = (D^3 - a^2 D)T = 0 \text{ at } \zeta = \pm 1$$

for the rigid surfaces.

Various attempts were made to solve these equations but all ended in failure. A more careful study will probably bring about success, as these equations do not seem to be too formidable.

III. Conclusion

Experimental approach to this problem gave a few results while the analytical approach merely set forth the problem.

This problem is not one of mere curiosity. For the question of penetrative convection has far reaching implications in geophysical phenomena, such as convection to the tropopause, solar convection and convection in the core of the earth.

Stability of Salt Fingers

Recent investigations of the stability of a gravitationally stable system consisting of hot salty water over cold fresh water lead to the following results. Because the diffusivity of salt (K_S) is much smaller than the temperature diffusivity (K_T), an instability in the form of falling salt fingers occurs in which the salt in the upper part of the fluid falls in densely packed fingers. The phenomena of this instability is explained as follows: if a salty hot particle is displaced downward it loses heat but not salt since $K_T >> K_S$. It then becomes heavier than the surrounding fluid and continues to fall. A steady state solution for these fingers has been obtained and certain experimental observations indicate that these "fingers" themselves may be unstable. It is our purpose to investigate the salt finger stability.

The simplified model whose stability we will investigate is one suggested by W. Malkus. (Fig. 1)

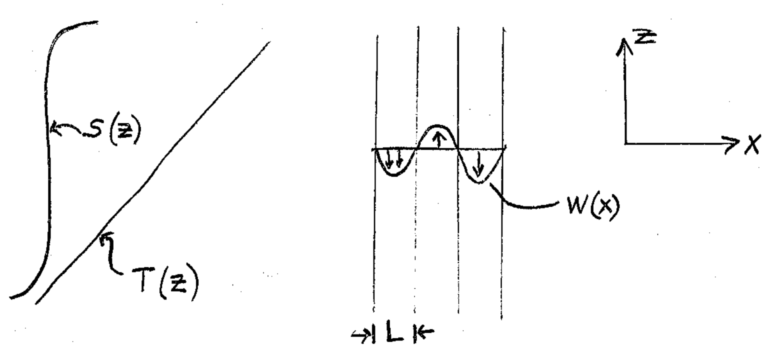


Fig. 1.

In this two dimensional model the vertical velocity is harmonic in the x direction; the temperature is a harmonic function of x and a linear function of z and the salinity is a harmonic function of x only. This approximation is imagined valid except in the regions near the end of the salt fingers or at their beginning. It is this harmonic mean flow whose stability we wish to investigate.

We will specialize even further. In the laboratory situation the salt amplitude ΔS, and the salt diffusivity are very small. We will therefore investigate the situation where $\Delta S \to 0$ and $K_S \to 0$ but when $\frac{\Delta S}{K_S} \to$ constant value to preserve the physics.

The perturbation equations we shall use are the following; for small disturbances:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \qquad (1)$$

$$u'_t + \bar{\omega} u'_z = -\frac{1}{\rho_0} p'_x + \nu \Delta u' \qquad (2)$$

$$w'_t + \bar{w} w'_z + w \bar{w}_x = -\frac{1}{\rho_0} p'_z + \nu \Delta w' - g\{\gamma s' - \alpha T'\} \qquad (3)$$

$$\left\{\frac{\partial}{\partial t} - K_T \Delta\right\} T' = -\bar{w} T'_z - w' \bar{T}_z - u' \bar{T}_x \qquad (4)$$

$$\left\{\frac{\partial}{\partial t} - K_S \Delta\right\} s' = -\bar{w} s'_z - u' \bar{s}_x \qquad (5)$$

since we take

$$\rho' = \rho_0 \{\gamma s' - \alpha T'\}.$$

We have used the fact that the mean salinity gradient in the z direction is zero. In these equations the primed quantities are perturbations and the barred quantities the steady flow quantities. We now wish to examine whether the perturbations on the mean field grow with time or are damped.

Let us first examine the equations in the limit when ΔS

and Ks go to zero. As Δs goes to zero so do $\overline{w}$, $\overline{s}x$, $\overline{T}x$ since (from a calculation by Melvin Stern)

$$\overline{w} = -\frac{g\,\gamma\Delta s}{\nu}\left(\frac{L}{\pi}\right)^2$$

$$\overline{T}(x) \sim \frac{\overline{W}\,\overline{T}z}{K_T}\left(\frac{L}{\pi}\right)^2$$

$$\overline{S}(x) \sim \Delta S.$$

Consider equation (5). As Δ S and $K_s \to 0$ then each term in the equation except the local time derivative of S' is zero. Now if $\frac{\partial s'}{\partial t} = 0$ we get the trivial solution $s' = 0$. In order for this not to occur it must be necessary that the time scale should be on the order of $(K_s/a^2)^{-1}$. With that time scaling the salinity equation survives intact. The effect on the remaining dynamical equations is easily seen. In the temperature equation as K_s and $\Delta s \to 0$ we are left with the following

$$K_T\ \Delta T' = \overline{T}_z w' \qquad (6)$$

In the momentum equations we are left with

$$0 = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} + \nu\Delta w' - g\left\{\gamma\ s' - \alpha T'\right\} \qquad (7)$$

$$0 = -\frac{1}{\rho_o}\frac{\partial p'}{\partial x} + \nu\Delta\ u' \qquad (8)$$

The continuity equation remains unblemished.

If now we take the curl of the momentum equations twice and examine the vertical component we obtain:

$$g\ \frac{\partial^2}{\partial x^2}\left\{\gamma\ s' - \alpha T'\right. = \nu\Delta\Delta\ w' \qquad (9)$$

Our final equations are then equations (1), (5), (6) and (9).

In these equations, $\bar{w} = w_o \cos \frac{2\pi x}{a}$

$$\bar{s}_x = \frac{2\pi}{a} \delta \sin \frac{2\pi x}{a}$$

$$\bar{T}_z = \text{constant}$$

where here a = 2L.

Since the mean field is harmonic in x and is independent of z ant t we will search for solutions for the perturbation quantities of the following form.

$$\left.\begin{aligned} U' &= \sum_{n=1} Un \sin \frac{n\pi x}{a} \\ W' &= \sum_{n=1} Wn \cos \frac{n\pi x}{a} \\ S' &= \sum_{n=1} Sn \cos \frac{n\pi x}{a} \\ T' &= \sum_{n=1} Tn \cos \frac{n\pi x}{a} \end{aligned}\right\} e^{ikz-iwt}$$

Substituting these forms in the dynamical equations gives the following results. From the equation of continuity

$$\boxed{Un = -\frac{i\,k\,a\,w_n}{n\pi}} \qquad (10)$$

From the thermal equation:

$$-K_T\left\{\frac{n^2\pi^2}{a^2} + k^2\right\} Tn = \bar{T}_z\, Wn$$

or with $l_n^{\,2} \equiv \frac{n^2\pi^2}{a^2} + k^2$

$$\boxed{Tn = \left\{-\frac{\bar{T}z}{ln^2}\right\}\frac{Wn}{K_T}} \qquad (11)$$

From the momentum equation

$$- g \frac{n^2 \pi^2}{a^2} \left\{ \gamma Sn - \alpha Tn \right\} = \nu \left\{ ln \right\}^4 Wn$$

or

$$\boxed{Wn = - \frac{g n^2 \pi^2}{\nu l^4 a^2} \left\{ \gamma Sn - \alpha Tn \right\}} \qquad (12)$$

The more complicated salinity equation is the following:

$$\boxed{\begin{aligned} \sum_n Sn \cos \frac{n \pi x}{a} \left\{ - iw + K_s ln^2 \right\} &= - W_o \cos \frac{2 \pi x}{a} ik \sum Sn \cos \frac{n \pi x}{a} \\ &\quad - \sum Un \sin \frac{n \pi x}{a} \delta \frac{2\pi}{a} \sin \frac{2 \pi x}{a} \end{aligned}} \qquad (13)$$

We remember the following trigonometric identities:

$$\cos \frac{2 \pi x}{a} \cos \frac{n \pi x}{a} = \frac{1}{2} \left\{ \cos (n+2) \frac{\pi x}{a} + \cos (n-2) \frac{\pi x}{a} \right\} \qquad (14a)$$

$$\sin \frac{2 \pi x}{a} \sin \frac{n \pi x}{a} = \frac{1}{2} \left\{ \cos (n-2) \frac{\pi x}{a} - \cos (n+2) \frac{\pi x}{a} \right\} \qquad (14b).$$

It is possible using the continuity and momentum equations to write Un in terms of Sn. We obtain the relationship

$$Un = \frac{ik \gamma gn \pi}{\nu l^4 a} \left(\frac{1}{1 + \frac{\alpha gn^2 \pi^2 \bar{T}_z}{\nu l^6 a^2 K_T}} \right) \delta n \qquad (15)$$

If we substitute in turn equations (14a and b) and equation (15) into equation (13) we finally obtain an equation in just the salinity, i.e.

$$\sum Sn \cos \frac{n\pi x}{a}\left\{-iw + K_s l_n^2\right\} = -\frac{W_o ik}{z}\sum Sn\left(\cos(n+2)\frac{\pi x}{a} + \cos(n-2)\frac{\pi x}{a}\right)$$

$$-\sum_n \frac{\pi^2 \delta ik\sigma gn}{\nu l_n^{4a}\left\{1+R_n\right\}} Sn \left[\cos(n-2)\frac{\pi x}{a} - \cos(n+2)\frac{\pi x}{a}\right] \tag{16}$$

where $Rn \quad \dfrac{\alpha\, gn^2 \pi^2 \bar{T}_z}{\nu l_n^6 a^2 K_T}$

We first observe that if we start our sum at $n = 0$ and assume $S_o = 0$ then all the even modes in n are zero and we may limit our investigation to n odd.

Taking advantage of the orthogonality of the cosine we see that equation (16) gives us an infinite number of linear equations, homogeneous, in the Sn's. If they are to have a solution the determinant of the coefficients must vanish. This will essentially determine the parameter w as a function of other internal and external parameters in the problem. The above mention determinant is infinite in size, however, and is therefore intractable. What we shall do is to consider lower order determinants associated with the existence of just the lower harmonics and hypothesize that the neglect of the higher harmonics will not affect the calculation of w very seriously. In particular we will consider the subharmonic and the first harmonic above the mean (i.e. the equations for S_1 and S_3) and take Sn, $(n > 3) = 0$.

Equation (16) then gives us the following:

$$0 = S_1\left\{-i\omega + K_s\ell_1^2 + \frac{W_0 ik}{2} + \frac{\pi^2 ik\gamma g\delta}{a^2\nu\ell_1^4\{1+R_1\}}\right\} + S_3\left\{\frac{\pi^2\delta ikg3}{a^2\nu\ell_3^4\{1+R_3\}} + \frac{W_0 ik}{2}\right\}$$

$$S_3\left\{-i\omega + K_s\ell_3^2\right\} = -\frac{W_0 ik}{2}S_1 + \frac{S_1\pi^2\delta ik\gamma g}{a^2\nu\ell_1^4(1+R_1)}$$

The determinant of the coefficients is

$$0 = \begin{vmatrix} -i\omega + K_s\ell_1^2 + \frac{W_0 ik}{2} + \frac{\pi^2 ik\gamma g\delta}{a^2\nu\ell_1^4(1+R_1)} & \frac{W_0 ik}{2} + \frac{3\pi^2\delta(ik)\gamma g}{a^2\nu\ell_1^4(1+R_3)} \\ \frac{W_0 ik}{2} - \frac{\pi^2\delta ik\gamma g}{a^2\nu\ell_1^4(1+R_1)} & -i\omega + K_s\ell_3^2 \end{vmatrix}$$

leading to the following complex characteristic equation for w.

$$0 = \omega^2 + \omega\left\{iK_s\left[\ell_1^2+\ell_3^2\right] - \frac{W_0 k}{2} - \frac{k\pi^2\gamma g\delta}{a^2\nu\ell_1^4(1+R_1)}\right\} - K_s\ell_1^2\ell_3^2 - \frac{ikW_0K_s\ell_3^2}{2} - \frac{\pi^2 ik\gamma g\delta K_s\ell_3^2}{a^2\nu\ell_1^4(1+R_1)}$$

$$-\frac{W_0 k3\pi^2\delta k\gamma g}{2a^2\nu\ell_3^4(1+R_3)} - \frac{W_0^2k^2}{4} + \frac{3\pi^4\delta^2k^2\sigma^2g^2}{a^4\nu^2\ell_1^4\ell_3^4(1+R_1)(1+R_3)} - \frac{\pi^2\delta k^2W_0\gamma g}{2a^2\nu\ell_1^4\{1+R_1\}} \qquad (17)$$

To solve this complex equation we may write $w = \alpha + i\beta$. Now with a choice of time dependence the case of stability corresponds to $\beta < 0$. If we break equation (17) into its real and imaginary parts we obtain the following two real equations:

<u>Real part</u>

$$\alpha^2 - \beta^2 - \beta\left\{K_s\ell_1^2 + K_s\ell_3^2\right\} - \alpha\left\{\frac{W_0 k}{2} + \frac{\pi^2 k\gamma g\delta}{a^2\nu\ell_1^4(1+R_1)}\right\} - K_s^2\ell_1^2\ell_3^2$$

$$-\frac{W_0 3\pi^2\delta k^2\gamma g}{2a^2\nu\ell_3^4(1+R_3)} - \frac{W_0^2k^2}{4} + \frac{3\pi^4\delta^2k^2\gamma^2g^2}{a^4\nu^2\ell_1^4\ell_3^4(1+R_1)(1+R_3)} - \frac{\pi^2\delta k^2W_0\gamma g}{2a^2\nu\ell_1^4\{1+R_1\}} = 0 \qquad (18)$$

Imaginary part

$$2\alpha\beta+\alpha\left\{K_s\ell_1^2+K_s\ell_3^2\right\}-\beta\left\{\frac{W_0k}{2}+\frac{k\pi^2\gamma g\delta}{a^2\nu\ell_1^4(1+R_1)}\right\}$$

$$-\frac{kW_0K_s\ell_3^2}{2}-\frac{\pi^2k\gamma g\delta K_s\ell_3^2}{a^2\nu\ell_1^4(1+R_1)}=0 \tag{19}$$

If αd = 0 equation (19) tells us that

$$\boxed{\beta = -K_s\ell_3^2} \tag{20}$$

That is to say, that the case for marginal stability does not correspond to w = 0 since for the non-oscillatory modes the perturbations are damped. Rather the marginally stable case corresponding to $\beta = 0$ is from equation (19) for a frequency α of the magnitude

$$\alpha = \frac{\dfrac{W_0k}{2}+\dfrac{\pi^2\gamma kg\delta}{a^2\nu\ell_1^4(1+R_1)}}{\left(1+\dfrac{\ell_1^2}{\ell_3^2}\right)} \equiv \alpha_0 \tag{21}$$

In general:

$$\beta = \frac{K_s\ell_3^2\left\{\dfrac{W_0k}{2}+\dfrac{\pi^2k\gamma g\delta}{a^2\nu\ell_1^4(1+R_1)}-\alpha\left(1+\dfrac{\ell_1^2}{\ell_3^2}\right)\right\}}{2\alpha-\dfrac{W_0k}{2}-\dfrac{k\pi^2\gamma g\delta}{a^2\nu\ell_1^4(1+R_1)}} \tag{22}$$

What are the conditions for instability i.e. for β to be greater than zero. If $\alpha < 0$ then equation (22) tells us that $\beta < 0$, if $\alpha = 0$; $\beta < 0$ by equation (20). If $\alpha > 0$ the following conditions must hold if β is to be positive.

It is necessary that:

$$\frac{W_0 k}{2} + \frac{\pi^2 \kappa \gamma g \delta}{a^2 \nu \ell_1^4 (1+R_1)} > \alpha \left(1 + \frac{\ell_1^2}{\ell_3^2}\right)$$

$$2\alpha > \frac{W_0 k}{2} + \frac{\pi^2 \kappa \gamma g \delta}{a^2 \nu \ell_1^4 (1+R_1)}$$

or

$$2\alpha > \frac{W_0 k}{2} + F > \alpha \left(1 + \frac{\ell_1^2}{\ell_3^2}\right)$$

and by equation (20)

$$2\alpha > \alpha_0 \left(1 + \frac{\ell_1^2}{\ell_3^2}\right) > \alpha \left(1 + \frac{\ell_1^2}{\ell_3^2}\right)$$

i.e. $\alpha_0 > \alpha$. That is there is a lower limit on the frequency for $\beta > 0$ and an upper limit. α_0, the upper limit on the frequency gives us a lower limit on the period of these overstable waves. From observational data if the vertical wavelength is 5 cm and the salt finger width L is .3 cm for a concentration $\delta = 10^{-3}$ we get:

$T_0 = \frac{2\pi}{\alpha_0} = 216$ sec which is in fair agreement with experiment.

In conclusion we may remark that the instability associated with the salt fingers seems to be of the overstable variety with a long period. What remains to be done is to take α_0 from equation (21) and use equation (18) to determine the minimum salt concentrations needed for instability and to determine which vertical wave numbers are most unstable.

Joseph Pedlosky

Notation

a = coefficient of thermal expansion

α = real part of circular frequency

β = imaginary part of circular frequency

γ = coefficient of volumetric expansion due to dissolved salt.

ν = binematic viscosity

δ = salinity amplitude

g = acceleration due to gravity

K_T = coefficient of thermal diffusion

K_S = coefficient of saline diffusion

k = vertical wave number

p = pressure

ρ = density

a = 2 L

L = salt finger width

W = vertical velocity

u = horizontal velocity

S' = salt perturbation

T' = temperature perturbation

$R_n = \dfrac{\alpha g n^2 \pi^2 \bar{T}_z}{\nu \ell_n^6 a^2 K_T}$

$\ell_n^2 = \dfrac{n^2\pi^2}{a^2} + k^2$

The quantities with primes are perturbation quantities. Mean quantities have a subscript $_o$ or are barred.

Stabilities of Thermally Stratified Shear Flow

I. Introduction. The title of this lecture describes the situation under consideration adequately. The motivation for considering this complex situation, being somewhat exceptional, warrants, however, a few additional words.

In the past few years Dr. W.V.R. Malkus has been able to predict the profiles for fully turbulent shear flow and fully turbulent heat flow in a channel using the following hypothesis: 'In turbulent flows the mean fields approach (but do not go beyond) the limits for marginal inviscid instability.' Whether the limit is approached from the stable or from the unstable direction is not specified; in fact the procedure differs in different cases. More properly, the above hypothesis may be considered the basis for a procedure where inequalities for mean fields are obtained from stability considerations. In shear flow we have as a necessary and sufficient condition for inviscid stability that $\frac{\partial^2 w}{\partial y^2}$ not change sign (w = dimensionless velocity in x-direction and y is vertical coordinate) - it will suffice to consider it as being always positive. In heat flow our condition for instability is $\beta > 0$ (where β is the negative vertical temperature gradient); $\beta < 0$ being the condition for stability.

Now for any positive function which can be represented by a finite Fourier expansion, it has been shown that

$$I = \sum_{n=-n_o}^{n_o} I_n \, e^{in\phi} \xrightarrow[n_o \to \infty]{\text{asymptotically}} F^2 \, \frac{\cos^2(n_o + 1)\,\Theta}{\cos^2\Theta} \qquad (1)$$

for a symmetric geometry where Θ is the vertical coordinate and ranges from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and F^2 is a function of n_o. Asserting that $\beta > 0$ in full turbulent heat flow we obtain

$$\frac{K\beta}{H} = F^2 \frac{\cos^2(n_o+1)\,\Theta}{\cos^2\Theta} \qquad (2)$$

and by integrating we obtain

$$\frac{K(T_m - T(0))}{H} \qquad \frac{F^2 g_1^{\,2}}{2} \left(\tan\Theta + O\left(\frac{1}{n_o}\right)\right) \qquad (3)$$

For shear flow $\frac{\partial^2 U}{\partial z^2} > 0$ was asserted and the following profile was obtained

$$\frac{U_{max} - U}{U_\tau} = \frac{F^2 g_1^{\,2}}{2} \left(\ln\left(\frac{1}{\cos\Theta}\right) + O\left(\frac{1}{n_o}\right)\right) \qquad (4)$$

Both these profiles appear to adhere closely to experimental results although in the case of turbulent convection experimental data is hardly abundant.

The success of the above procedure in predicting profiles, however, may be entirely fortuitous since, to be sure, we are not working within a deductive framework. Our doubts might be mitigated by actually deducing in rigorous fashion the correctness or wrongness of the above procedure, but past experience indicates that such a task will be far from elementary. We will therefore proceed with a stopgap procedure. We will attempt to determine the limit for marginal inviscid instability for a more complicated situation (i.e. thermally stratified shear flow) and utilize the above procedure to obtain some asymptotic relation for the mean profiles. Hopefully, we may then obtain empirical checks for the profile predicted. If the experiments substantiate the results predicted, we may believe that there exists a strong basis for our procedure in physical fact, and with this as motivation proceed to interpret our procedure physically and to seek proofs. If the experiments fail to substantiate the predictions, we must then check the use of the procedure in this case and perhaps re-evaluate the procedure itself.

The above, of course, is only an idealized outline of what we would like to do. Thus far, only limited inroads have been made on the problem of finding the limits of marginal inviscid instability; and most of these results have, it must be admitted, been previously established by Lees and Lin - sometimes, however, by different methods.

II. As is standard in stability we shall begin with the equations of motion, solve for disturbances, linearize, and dimensionalize. It was felt, that some good might come from following the procedure for a rather general situation. Only in choosing characteristic measures, suitable for situations where inertial forces are of importance have we significantly restricted ourselves.

The following is the notation to be used; noughts refer to mean stream quantities (the notation chosen is that used by Prof. Howard essentially).

Dimensional Quantities	Dimensionless Quantities	Characteristic Measure
Positional Coordinates		
x^*	x	l
y^*	y	l
Since		
t^*	t	$l/\bar{u}_o^*$
Velocity components in x and y directions		
$u^* = \bar{u}^* + u^{*\prime}$	$u = w(y)+f(y)e^{i\alpha(x-ct)}$	$\bar{u}_o^*$
$v^* = \bar{v}^* + v^{*\prime}$	$v = \alpha\phi(y)e^{i\alpha(x-ct)}$	$\bar{u}_o^*$
Components of strain tensor		
$\epsilon^*_{xx} = \bar{\epsilon}^*_{xx} + \epsilon^{*\prime}_{xx}$	$\epsilon_{xx} = \bar{\epsilon}_{xx} + \epsilon'_{xx}$	$\bar{u}_o^*/l$
$\epsilon^*_{xy} =$ -----		
$\epsilon^*_{yy} =$ ------		
Components of stress tensor		
$\sigma_{xx} = \bar{\sigma}^*_{xx} + \sigma^{*\prime}_{xx}$	$\sigma_{xx} = \bar{\sigma}_{xx} + \sigma'_{xx}$	$\bar{p}_o^*$
$\sigma^*_{xy} =$ -----		
$\sigma^*_{yy} =$ ------		
Density of the gas		
$\rho^* = \bar{\rho}^* + \rho^{*\prime}$	$\rho(y) + r(y)e^{i\alpha(x-ct)}$	$\bar{\rho}_o^*$
Pressure of the gas		
$p^* = \bar{p}^* + p^{*\prime}$	$p(y) + \pi(y)e^{i\alpha(x-ct)}$	$\bar{p}_o^*$
Temperature of the gas		
$T^* = \bar{T}^* + T^{*\prime}$	$T(y) + \Theta(y)e^{i\alpha(x-ct)}$	$\bar{T}_o^*$

Dimensional Quantities	Dimensionless Quantities	Characteristic Measure
Coefficients of viscosity of the gas		
$\mu^* = \bar{\mu}^* + \mu^{*\prime}$	$\mu(y) + m(y)e^{i\alpha(x-ct)}$	$\bar{\mu}_o^*$
$\lambda^* = \bar{\lambda}^* + \lambda^{*\prime}$	$\lambda(y) + \ell(y)e^{i\alpha(x-ct)}$	$\bar{\mu}_o^*$
Thermal conductivity		
$K^* = \bar{K}^* + K^{*\prime}$	$\frac{1}{\sigma}\mu(y) + \frac{1}{\sigma_o}K(y)e^{i\alpha(x-ct)}$	$c_p\bar{\mu}_o^*$
Wave number of disturbance		
$\alpha^* = 2\pi/\lambda^*$	$\alpha = 2\pi/\lambda$	ℓ^{-1}
Phase velocity of disturbance		
c^*	c	$\bar{u}_o^*$
Specific heat at constant volume		
c_v	1	c_v
Specific heat at constant pressure		
c_p	γ	c_v
Gas constant per gram		
R^*	$\gamma - 1$	c_v
Acceleration due to gravity		
g	$\frac{1}{F^2}$	$\bar{u}_o^{*2}/\ell$
Froude number	$F = \frac{\bar{u}_o^*}{\sqrt{g\ell}}$	
Reynolds number	$R = \bar{\rho}_o^*\bar{u}_o^*\ell/\bar{\mu}_o^*$	
Mach number	$M = \bar{u}_o^*/\sqrt{\gamma R^*\bar{T}_o^*}$	
Prandtl number	$\sigma = c_p\bar{\mu}^*/\bar{K}^*$	

The general equations in two dimensions for a fluid are as follows:

a) motion

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\rho^*} \left(\frac{\partial \sigma_{xx}^*}{\partial x^*} + \frac{\partial \sigma_{xy}^*}{\partial y^*} \right) \tag{1}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = \frac{1}{\rho^*} \left(\frac{\partial \sigma_{xy}^*}{\partial x^*} + \frac{\partial \sigma_{yy}^*}{\partial y^*} \right) \tag{2}$$

b) continuity

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x^*} (\rho^* u^*) + \frac{\partial}{\partial x^*} (\rho^* v^*) = 0 \tag{3}$$

c) energy

$$\rho^* c_v \left\{ \frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial x^*} \right\}$$

$$= (\sigma_{xx}^* \epsilon_{xx}^* + 2 \sigma_{xy}^* \epsilon_{xy}^* + \sigma_{yy}^* \epsilon_{yy}^*)$$

$$+ \frac{\partial}{\partial x^*} \left(K^* \frac{\partial T^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*} \left(K^* \frac{\partial T^*}{\partial y^*}\right) \tag{4}$$

d) state

$$p^* = \rho^* R^* T^* \tag{5}$$

and,

$$\epsilon_{xx}^* = \frac{\partial u^*}{\partial x^*}, \quad \epsilon_{xy}^* = \frac{1}{2}\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right), \quad \epsilon_{yy} = \frac{\partial v^*}{\partial y^*} \tag{8}$$

$$\left.\begin{aligned}
\sigma_{xx}^* &= -p^* + 2\mu^* \frac{\partial u^*}{\partial x^*} + \frac{2}{3}(\lambda^* - \mu^*)\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right) \\
\sigma_{xy}^* &= \mu^* \left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right) \\
\sigma_{yy}^* &= -p^* + 2\mu^* \frac{\partial v^*}{\partial y^*} + \frac{2}{3} (\lambda^* - \mu^*)\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right)
\end{aligned}\right\} \tag{9}$$

From equations (1) to (9) we get the equations for the disturbances (dropping second order terms)

$$\frac{\partial u^{*\prime}}{\partial t^*} + \left(u^{*\prime}\frac{\partial \bar{u}^*}{\partial x^*} + v^{*\prime}\frac{\partial \bar{u}^*}{\partial y^*}\right) + \left(\bar{u}^*\frac{\partial u^{*\prime}}{\partial x^*} + \bar{v}^*\frac{\partial u^{*\prime}}{\partial y^*}\right)$$

$$= \frac{1}{\bar{\rho}^*}\left(\frac{\partial \sigma_{xx}^{*\prime}}{\partial x^*} + \frac{\partial \sigma_{xy}^{*\prime}}{\partial y^*}\right) + \frac{\rho^{*\prime}}{\bar{\rho}^{*2}}\left(\frac{\partial \bar{\sigma}_{xx}^*}{\partial x^*} + \frac{\partial \bar{\sigma}_{xy}^*}{\partial y^*}\right) \tag{10}$$

$$\frac{\partial v^{*\prime}}{\partial t^*} + \left(u^{*\prime}\frac{\partial \bar{v}^*}{\partial x^*} + v^{*\prime}\frac{\partial \bar{v}^*}{\partial y^*}\right) + \left(\bar{u}^*\frac{\partial v^{*\prime}}{\partial x^*} + \bar{v}^*\frac{\partial u^{*\prime}}{\partial y^*}\right)$$

$$= \frac{1}{\bar{\rho}^*}\left(\frac{\partial \sigma_{xy}^{*\prime}}{\partial x^*} + \frac{\partial \sigma_{yy}^{*\prime}}{\partial y^*}\right) - \frac{\rho^{*\prime}}{\bar{\rho}^{*2}}\left(\frac{\partial \bar{\sigma}_{xy}^*}{\partial x^*} + \frac{\partial \bar{\sigma}_{yy}^*}{\partial y^*}\right) \tag{11}$$

$$\frac{\partial \rho^{*\prime}}{\partial t^*} + \frac{\partial}{\partial x^*}\left(\rho^{*\prime}\bar{u}^* + \bar{\rho}^* u^{*\prime}\right) + \frac{\partial}{\partial y^*}\left(\rho^{*\prime}\bar{v}^* + \rho^* v^{*\prime}\right) = 0 \tag{12}$$

$$c_v\bar{\rho}^*\left(\frac{\partial T^{*\prime}}{\partial t^*} + \bar{u}^*\frac{\partial T^{*\prime}}{\partial x^*} + \bar{v}^*\frac{\partial T^{*\prime}}{\partial y^*} + u^{*\prime}\frac{\partial \bar{T}^*}{\partial x^*} + v^{*\prime}\frac{\partial \bar{T}^*}{\partial y^*}\right)$$

$$+ c_v\rho^{*\prime}\left(\frac{\partial \bar{T}^*}{\partial t^*} + \bar{u}^*\frac{\partial \bar{T}^*}{\partial x^*} + \bar{v}^*\frac{\partial \bar{T}}{\partial y^*}\right)$$

$$= \frac{\partial}{\partial x^*}\left(\bar{u}^*\frac{\partial T^{*\prime}}{\partial x^*} + K^{*\prime}\frac{\partial \bar{T}^*}{\partial x^*}\right)$$

$$+ \left(\bar{\sigma}_{xx}^*\,\epsilon_{xx}^{*\prime} + 2\bar{\sigma}_{xy}^*\,\epsilon_{xy}^{*\prime} + \bar{\sigma}_{yy}^*\,\epsilon_{yy}^{*\prime}\right)$$

$$+ \left(\sigma_{xx}^{*\prime}\,\bar{\epsilon}_{xx}^* + 2\sigma_{xy}^{*\prime}\,\bar{\epsilon}_{xy}^* + \sigma_{yy}^{*\prime}\,\epsilon_{yy}\right) \tag{13}$$

$$\frac{p^{*\prime}}{\bar{p}^*} = \frac{\rho^{*\prime}}{\bar{\rho}^*} + \frac{T^{*\prime}}{\bar{T}^*} \tag{14}$$

and,

$$\epsilon_{xx}^{*'} = \frac{\partial u^{*'}}{\partial x^*}, \quad \epsilon_{xy}^{*'} = \frac{1}{2}\left(\frac{\partial u^{*'}}{\partial y^*} + \frac{\partial v^{*'}}{\partial x^*} \right), \quad \epsilon_{yy}^{*'} = \frac{\partial v^{*'}}{\partial y^*} \tag{15}$$

$$\left.\begin{aligned}
\sigma_{xx}^{*'} &= -p^{*'} + 2\left(\bar{\mu}^* \frac{\partial u^{*'}}{\partial x^*} + \mu^{*'} \frac{\partial \bar{u}^*}{\partial x^*}\right) \\
&\quad + \frac{2}{3}\left\{(\lambda^* - \mu^*)\left(\frac{\partial u^{*'}}{\partial x^*} + \frac{\partial v^{*'}}{\partial y^*}\right)\right. \\
&\quad \left. + (\lambda^{*'} - \mu^{*'})\left(\frac{\partial \bar{u}^*}{\partial x^*} + \frac{\partial \bar{v}^*}{\partial y^*}\right)\right\} \\
\sigma_{xy}^{*'} &= \mu^* \left(\frac{\partial u^{*'}}{\partial y^*} + \frac{\partial v^{*'}}{\partial x^*}\right) + \mu^{*'}\left(\frac{\partial \bar{u}^*}{\partial y^*} + \frac{\partial \bar{v}^*}{\partial x^*}\right) \\
\sigma_{yy}^{*'} &= -p^{*'} + 2\left(\bar{\mu}^* \frac{\partial v^{*'}}{\partial y^*} + \mu^{*'} \frac{\partial \bar{v}^*}{\partial y^*}\right) \\
&\quad + \frac{2}{3}\left\{(\bar{\lambda}^* - \bar{\mu}^*)\left(\frac{\partial u^{*'}}{\partial x^*} + \frac{\partial v^{*'}}{\partial y^*}\right)\right. \\
&\quad \left. + (\lambda^{*'} - \mu^{*'})\left(\frac{\partial \bar{u}^*}{\partial x^*} + \frac{\partial \bar{v}^*}{\partial y^*}\right)\right\}
\end{aligned}\right\} \tag{16}$$

In forming our final linearized dimensionless equations we will restrict ourselves to almost parallel flows for which

$$\bar{v}^* << \bar{u}^* \tag{17}$$

and for any property of the flow $\bar{Q}^*$

$$\frac{\partial \bar{Q}^*}{\partial x^*} << \frac{\partial \bar{Q}^*}{\partial y^*} \tag{18}$$

Also, for parallel flows our differential equations should not contain explicit reference to x^* and t^*. Hence we will seek solutions of the form

$$Q^{*'} = q^*(y^*)\, e^{+\alpha^* i(x^* - c^* t^*)} \tag{19}$$

We obtain from equations (11) - (19), restricting ourselves to flows with uniform free stream velocity $\vec{u}_o^*$, temperature $\overline{T}_o^*$, etc., the following linearized, dimensionless equations for small disturbances on parallel flows:

$$\alpha\rho\left\{i(w-c)f + w'\phi\right\} = -\frac{i\alpha\pi}{\gamma M^2}$$
$$+\frac{\mu}{R}\left\{f'' + \alpha^2(i\phi' - 2f)\right\}$$
$$+\frac{2}{3}\frac{\lambda-\mu}{R}\alpha^2\left\{-f + i\phi'\right\}$$
$$+\frac{1}{R}\left\{mw'' + m'w' + \mu'(f' + i\alpha^2\phi)\right\} \tag{20}$$

$$\alpha^2\rho\left\{i(w-c)\phi\right\} = \frac{-\pi'}{\gamma M^2} - \frac{r}{F^2} + \frac{\mu\alpha}{R}\left\{2\phi'' - if' - \alpha^2\phi\right\}$$
$$+\frac{2\alpha}{3}\frac{\lambda-\mu}{R}\left\{\phi'' + if'\right\}$$
$$+\frac{\alpha}{R}\left\{imw' + 2\mu'\phi' + \frac{2}{3}(\lambda-\mu)(\phi' + if)\right\} \tag{21}$$

$$i(w-c)r + \rho(\phi' + if) + \rho'\phi = 0 \tag{22}$$

$$\alpha\rho\left\{i(w-c)\theta + T'\phi\right. = -\alpha(\gamma-1)\rho T(\phi' + if)$$
$$+\frac{\gamma}{R\sigma_o}\left\{\mu(\theta'' - \alpha^2\theta) - (mT')' + \mu'\theta'\right\}$$
$$+\gamma\frac{(\gamma-1)}{R}M^2\left\{mw'^2 + 2\mu w'(f' + i\alpha^2\phi)\right\} \tag{23}$$

$$\frac{\pi}{p} = \frac{r}{\rho} + \frac{\theta}{T} \tag{24}$$

and for the mean profile we have

$$p = \rho T \tag{25}$$

Furthermore, we will introduce the hydrostatic equation

$$p' = \frac{\gamma M^2}{F^2} \rho \tag{26}$$

for pressure (a valid step for most problems of interest).

Anyone, wishing to work on a stability problem may now use equations (20) - (26) dropping terms as the problem may permit.

Since we are interested in inviscid conditions, we let $R \rightarrow \infty$ obtaining

$$\alpha \rho \left\{ i(w-c)f + w' \phi \right\} = - \frac{i\alpha \pi}{\gamma M^2} \tag{27}$$

$$\alpha^2 \rho \left\{ i(w-c) \phi \right\} = - \frac{\pi'}{\gamma M^2} - \frac{r}{F^2} \tag{28}$$

$$i(w-c)r + \rho(\phi' + if) + \rho' \phi = 0 \tag{29}$$

$$\alpha \rho \left\{ i(w-c)\Theta + T' \phi \right\} = -\alpha(\gamma-1) \rho T(\phi' + if) \tag{30}$$

$$\frac{\pi}{P} = \frac{r}{\rho} + \frac{\Theta}{T} \tag{31}$$

Unfortunately, even these equations comprise a very difficult problem; and therefore we will restrict ourselves to special cases, taking care to ascertain the physical meaning of our restrictions.

One case of special interest to aerodynamicists is that of infinite Froude number which has been extensively investigated by Lees and Lin. This situation is one in which inertial forces dominate completely over gravitational forces and is closely approximated in high speed flows.

The equations for $\begin{matrix} R \rightarrow \infty \\ F^2 \rightarrow \infty \end{matrix}$ are as follows

$$\rho\left\{i(w-c)f + w'\phi\right\} = -\frac{\pi}{\gamma M^2} \tag{32}$$

$$\rho\left\{i\alpha^2(w-c)\phi\right\} = -\frac{\pi'}{\gamma M^2} \tag{33}$$

$$i(w-c)r + \rho(\phi' + if) + \rho'\phi = 0 \tag{34}$$

$$\rho\left\{i(w-c)\Theta + T'\phi\right\} = -(\gamma-1)\rho T(\phi' + if) \tag{35}$$

$$\frac{\pi}{P} = \frac{r}{\rho} + \frac{\Theta}{T} \tag{36}$$

and

$$p = \rho T \tag{37a}$$

$$p' = \rho' T + \rho T' = 0 \tag{37b}$$

for mean profiles

Elimination of f, π, r, and Θ from equations (32) - (36) gives

$$f = i\,\frac{T\phi' - M^2 W'(w-c)\phi}{T - (w-c)^2 M^2} \tag{38}$$

and

$$\alpha^2\rho(w-c)\phi = \frac{d}{dy}\left\{\frac{(w-c)\phi' - w'\phi}{T - (w-c)^2 M^2}\right\} \tag{39}$$

Putting equation (39) into self adjoint form we get

$$\frac{d}{dy}\left\{\frac{1}{T - (w-c)^2 M^2}\,\phi'\right\}$$

$$-\left\{\frac{1}{w-c}\,\frac{d}{dy}\left\{\frac{w'}{T - M^2(w-c)^2}\right\} + \frac{\alpha^2}{T}\right\}\phi = 0 \tag{39a}$$

Note that if F^2 didn't go to ∞ we would have in eqn's. (39) an additional term

$$\frac{1}{F^2}\left\{\frac{(\gamma-1)\rho T\phi' - \rho\gamma M^2(w-c)w'\phi + \rho T'\phi}{i(w-c)T}\right\}$$

$$+ i\left\{\frac{(\gamma-1)\rho - \rho(w-c)^2\gamma M^2}{(w-c)}\right\}\left\{\frac{T\phi' - M^2 w'(w-c)\phi}{T-(w-c)^2M^2}\right\}$$

However, equation (39) is sufficiently complicated in itself; although necessary and sufficient conditions for subsonic disturbances in the case of $R \to \infty$, $F^2 \to \infty$ have been obtained by Lees and Lin their procedure is extremely long and complicated, and no attempt will be made to duplicate their work here. Their result is, fortunately, rather simple: <u>a necessary and sufficient condition for a subsonic disturbance to exist is that</u>

$$\underline{\frac{d}{dy}\frac{W'}{T}} \quad \underline{\text{change sign in the y-interval}} \tag{40}$$

The condition $F^2 \to \infty$ is most definitely a very unlikely one in geophysical problems; however, it would not seem impossible to establish an empirical situation where a temperature gradient is maintained, F^2 is very large, and the boundary layer is very small. For instance, $F^2 \sim 100$ for a 50 km/hr flow through a channel about 1 ft high.

In such a case, applying Dr. Malkus' procedure, we would have

$$\frac{d}{dy}\left(\frac{W'}{T}\right) > 0$$

Hence

$$\frac{d}{d\Theta}\left(\frac{\frac{dw}{d\Theta}}{T}\right) = F^2 g^2 \frac{\cos^2(n_o+1)\Theta}{\cos^2\Theta} \tag{41}$$

and

$$\frac{\frac{dw}{d\Theta}}{T} = \frac{F^2g^2}{2}\left(\tan\Theta + O\left(\frac{1}{n_o}\right)\right) \tag{42}$$

$$+ \text{ const. of integration}$$

where $F^2g^2/2$ are functions only of n_o

Of considerably greater geophysical significance is the case $M^2 \to 0$, $R \to \infty$.

In letting $M^2 \to 0$ we do not mean that either $\bar{u}_o^* \to 0$ (in which case F^2 and R also approach zero) nor that $\gamma \bar{T}_o^* \to 0$ which is thermodynamically impossible. What we are doing is considering the solutions of our equations as power series in M^2 and considering only the zero order terms.

Our equations for this case are

$$\alpha \rho \left\{ i(w-c)f + w' \phi \right\} = \frac{i\alpha \Pi}{\gamma M^2} \tag{43}$$

$$\alpha^2 \rho \left\{ i(w-c)\, \phi \right\} = - \frac{\Pi'}{\gamma M^2} - \frac{r}{F^2} \tag{44}$$

$$i(w-c)r + \rho\,(\phi' + if) + \rho' \phi = 0 \tag{45}$$

$$\alpha \rho \left\{ i(w-c)\Theta + T' \phi \right\} = -\alpha(\gamma - 1)\rho T(\phi' + if) \tag{46}$$

$$0 = \frac{r}{\rho} + \frac{\Theta}{T} \tag{47}$$

$$\begin{aligned} P &= \rho T \\ P' &= \rho' T + \rho T' = 0 \end{aligned} \tag{48}$$

Again, eliminating Π, r, f, and Θ we obtain (49) $f = -i\phi'$ as would be expected for uncompressible flow.

$$r = \frac{i\rho' \phi}{(w-c)} \tag{50}$$

and

$$\alpha^2 \rho(w-c)\phi = \frac{d}{dy}\left\{\rho\left[(w-c)\phi' - w'\phi\right]\right\} - \frac{1}{F^2}\frac{\rho'\phi}{(w-c)} \tag{51}$$

For ρ = const. equation (51) reduces to

$$(W-c)\left\{\phi'' - \alpha^2\phi\right\} - W''\phi = 0 \qquad (52)$$

which is the inviscid **Orr-Sommerfeld** equation.

Placing eqn. (51) in self-adjoint form we get

$$\frac{d}{dy}\left\{\rho\phi'\right\} - \left\{\frac{1}{(W-c)}\frac{d}{dy}\left\{\rho W'\right\} + \rho\alpha^2 + \frac{1}{F^2}\frac{\rho'}{(W-c)^2}\right\}\phi$$

$$= 0 \qquad (53)$$

With this simplified equation we may readily verify Lees' and Lin's result that $\frac{d}{dy}\left(\frac{W'}{T}\right)$ changing sign is a necessary condition for instability when $F^2 \rightarrow \infty$

Rewriting (53) in terms of T we get

$$\frac{\alpha^2(W-c)\phi}{T} = \frac{d}{dy}\left\{\frac{1}{T}\left[(W-c)\phi' - W'\phi\right]\right\} + \frac{1}{F^2}\frac{T'\phi}{(W-c)} \qquad (54)$$

or

$$\frac{d}{dy}\left\{\frac{1}{T}\phi'\right\} - \left\{\frac{\frac{d}{dy}\left(\frac{W'}{T}\right)}{(W-c)} + \frac{\alpha^2}{T} + \frac{1}{F^2}\frac{T'}{T^2(W-c)^2}\right\}\phi = 0 \qquad (55)$$

Letting $F^2 \rightarrow \infty$ we get

$$\frac{d}{dy}\left\{\frac{1}{T}\phi'\right\} - \left\{\frac{\frac{d}{dy}\left(\frac{W'}{T}\right)}{(W-c)} + \frac{\alpha^2}{T}\right\}\phi = 0 \qquad (56)$$

Multiply equation (56) by $\bar{\phi}$ and integrate

$$\int_{y_1}^{y_2}\left\{\bar{\phi}\frac{d}{dy}\left\{\frac{1}{T}\phi'\right\} - \left\{\frac{\frac{d}{dx}\left(\frac{W'}{T}\right)}{(W-c)} + \frac{\alpha^2}{T}\right\}|\phi|^2\right\}dy = 0 \qquad (57)$$

Integrating by parts, and using the fact that $\phi(y_1) = \phi(y_2) = 0$

we obtain

$$\int_{y_1}^{y_2} - \frac{1}{T}|\phi|^2 - \left\{ \frac{\frac{d}{dy}\left(\frac{W'}{T}\right)}{(w-c)} + \frac{\alpha^2}{T} \right\} |\phi|^2 = 0 \qquad (58)$$

Separating real and imaginary parts

$$\int_{y_1}^{y_2} \frac{|\phi'|^2}{T} + \frac{\alpha^2}{T}|\phi|^2 + \frac{(w - c_n)}{|w - c|^2} \frac{d}{dy}\left(\frac{W'}{T}\right) |\phi|^2 = 0 \qquad (59)$$

and

$$ci \int_{y_1}^{y_2} \frac{\frac{d}{dy}\left(\frac{W'}{T}\right)}{|w-c|^2} dy = 0 \qquad (60)$$

From equation (60) we see that a necessary condition for instability is that $\frac{d}{dy}\left(\frac{W'}{T}\right)$

change sign in the interval.

Rewriting equation (51) we get

$$(w-c)(\phi'' - \alpha^2\phi) - w''\phi + \frac{\rho'}{\rho}\left\{(w-c)\phi' - w'\phi\right\} - \frac{1}{F^2} \frac{\rho'}{\rho} \frac{\phi}{w-c} = 0 \qquad (61)$$

In the situation where the Boussinesq approximation holds $\frac{\rho'}{\rho}$ is set equal to zero everywhere except where it interacts with gravity. In this case equation (61) becomes

$$(w-c)(\phi'' - \alpha^2\phi) - w''\phi - \frac{1}{F^2}\frac{\rho'}{\rho}\frac{\phi}{w-c} = 0 \qquad (62)$$

Note, now, that if we rewrite

$\frac{1}{F^2}\frac{\rho'}{\rho}$ as $\frac{g\,l^2}{\bar{u}_o^*}\frac{\bar{\rho}^{*\prime}}{\bar{\rho}^*}$ we get a Richardson number. This

is in keeping with Batchelor's demonstration that the Richardson number is the only parameter involved in this situation.

Assuming $c_i \neq 0$ equation (62) becomes

$$\phi'' - \alpha^2 \phi - \frac{w''}{(w-c)}\phi - \frac{1}{F^2}\frac{\rho'}{\rho}\frac{\phi}{(w-c)^2} = 0 \tag{63}$$

Multiply equation (63) by $\bar{\phi}$ and integrate

$$\int_{y_1}^{y_2} \bar{\phi}\phi'' - \alpha^2 |\phi|^2 - \left\{ \frac{w''}{w-c} + \frac{1}{F^2}\frac{\rho'}{\rho}\frac{1}{(w-c)^2} \right\} |\phi|^2$$

$$= \int_{y_1}^{y_2} - |\phi'|^2 - \alpha^2 |\phi|^2 - \frac{1}{w-c}\left\{ w'' + \frac{1}{F^2}\frac{\rho'}{\rho}\frac{1}{w-c} \right\} |\phi|^2$$

$$= 0 \tag{64}$$

where we have integrated by parts and utilized the condition $\phi(y_i) = \phi(y_2) = 0$.

Separating real and imaginary parts we get

$$\int_{y_1}^{y_2} |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{|\phi|^2}{|w-c|^2}\left\{ \left[(w-c_r)w'' + \frac{1}{F^2}\frac{\rho'}{\rho}\frac{(w-c_n)^2 - c_i^2}{|w-c|^2} \right] \right\} = 0 \tag{65}$$

and

$$c_i \int_{y_1}^{y_2} \frac{|\phi|^2}{|w-c|^2}\left\{ w'' + \frac{2\rho'}{F^2\rho}\frac{(w-c_r)}{|w-c|^2} \right\} = 0 \tag{66}$$

If $c_i \neq 0$, $w'' + \frac{2\rho'}{F^2\rho}\frac{(w-c_r)}{|w-c|^2}$ must change sign in the interval; also, if ϕ is not to be identically equal to zero

$$(w-c_r)w'' + \frac{1}{F^2}\frac{\rho'}{\rho}\frac{(w-c_r)^2 - c_i^2}{(w-c_r)^2 + c_i^2}$$ must for some region be negative.

For a neutral disturbance to exist i.e. $c_i = 0$

$$(w-c)w'' + \frac{1}{F^2}\frac{\rho'}{\rho}\frac{(w-c)^2}{|w-c|^2} = (w-c)w'' + \frac{1}{F^2}\frac{\rho'}{\rho}$$ must be negative in

some region.

None of the above conditions are conditions for the mean fields, independent of the phase velocities of the disturbance.

Dr. Bisshopp suggested the following change of variables (see also C.C.Lin, "Theory of Hydrodynamic Stability", p. 120).

Let $f = \frac{\phi}{w-c}$, $\phi = f(w-c)$

$$\phi' = f'(w-c) + w'f$$

$$\phi'' = f''(w-c) + 2f'w' + w''f$$

Substituting the above in equation (62) we get

$$f''(w-c) + 2f'w' - \alpha^2 f(w-c) - \frac{1}{F^2}\frac{\rho'}{\rho}\frac{f}{w-c} = 0 \qquad (67)$$

Multiplying equation (67) by $\bar{f}(w-c)$ and integrating yields

$$\int_{y_1}^{y_2} \bar{f}\, f''\, (w-c)^2 + 2\bar{f}\, f'\, (w-c)w' - \alpha^2 f\, \bar{f}(w-c)^2 - \frac{1}{F^2}\frac{\rho'}{\rho} f\, \bar{f} = 0 \qquad (68)$$

Integrating by parts

$$\int_{y_1}^{y_2} - f'\frac{d}{dy}\left(\bar{f}(w-c)^2\right) + 2f'\, \bar{f}(w-c)w' - \alpha^2 |f|^2 (w-c)^2 -$$

$$- \frac{1}{F^2}\frac{\rho'}{\rho}|f|^2 = 0 \qquad (69)$$

or

$$\int_{y_1}^{y_2} -|f'|^2 (w-c)^2 - \alpha^2 |f|^2 (w-c)^2 - \frac{1}{F}\frac{\rho'}{\rho}|f|^2 = 0 \qquad (70)$$

$$(w-c)^2 = (w-c_r - ic_i)^2$$
$$= (w-c_r)^2 - c_i^2 - 2i\, c_i (w-c_r)$$

Separating real and imaginary parts

$$\int_{y_1}^{y_2} |f'|^2 \left\{ (w-c_r)^2 - c_i^2 \right\} + \alpha^2 |f|^2 \left\{ (w-c_r)^2 - c_i^2 \right\} +$$
$$+ \frac{1}{F}\frac{\rho'}{\rho}|f|^2 = 0 \qquad (71)$$

and

$$c_i \int_{y_1}^{y_2} (w-c_r) \left\{ |f'|^2 + \alpha^2 |f|^2 \right\} \qquad (72)$$

Thus if $c_i \neq 0$ $(w-c_r)$ must change sign in the interval i.e. for parallel flow $w_{min} < c_r < w_{max}$.

If $c_i = 0$, then for ϕ to exist, we find from equation (71),

$$\frac{1}{F^2}\frac{\rho'}{\rho}\frac{1}{(w-c)^2} < -\alpha^2 \quad \text{for some region.}$$

Again our conditions are not independent of the phase velocity of the disturbance.

It is to be regretted, that at this point, no success has been achieved in finding general conditions on the mean profiles alone for stability and instability. Until such conditions are found it will be impossible to apply Dr. Malkus procedure to the problem of shear flow in a stratified fluid obeying the Boussinesq equations.

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Numerical Methods in Fluid Mechanics

The purpose of this lecture is to acquaint the reader with the advantages of numerical methods, which may be used to give more detail than a physical experiment may provide. They may be applied without the physical experiment, but are of greatest value when the results are compared between the two.

Consider, for example, the system of viscous equations,

$$\frac{d\vec{v}}{dt} + \frac{\nabla p}{\rho} = \nu \nabla^2 \vec{v} \qquad \nabla \cdot \vec{v} = 0 \tag{1}$$

Suppose we are given a three dimensional region R, boundary conditions of the viscous form, and initial conditions. The Eulerian approach to solving the problem would be to set up a grid in space, and keep track of the physical quantities at each grid point. These quantities would change in time, according to (1), but the grid would remain fixed.

A second approach (Lagrangian) would be to associate each grid point with a fluid particle. The particles would then move according to the Lagrangian equations, and their motion would be determined by the grid configuration, which determines the field.

A third approach would be to assume each particle carries its own properties, e.g. say it is repelled from the others by a force of $\frac{1}{r}$. This approach might be useful in determining a pressure field.

A fourth method is wave number space. We may break a scalar field into say its first 100 orthogonal projections on a transform space. The limitation of this method would be that the resolution is limited by the smallest eigenvector.

To elucidate the Eulerian approach, let us consider solving the problem of viscous, plane Couette flow. Suppose initially the fluid velocity profile is a straight line with velocity U_H on the upper plate, and velocity U_o in the opposite sense on the lower plate. We may now apply an arbitrary disturbance, and study if it grows, and exactly how the field changes. In other words we are going to solve the instability question by considering an initial value problem.

Let us set up an Eulerian grid. In order to avoid computational instability we must make the grid spacing small enough so that energy does not accumulate in any part of the grid system due to the spacing. That is, the grid Reynolds number, R_G, where

$$R_G = \frac{\left(\frac{\partial u}{\partial z}\right)(\Delta z)^2}{\nu}$$

must be of order one, while the experimental Reynolds number, is given by

$$R_E = \frac{(U_H - U_0)}{\nu} n(\Delta z) = n^2 R_G$$

Instability may arise for $R_E \approx 2000$. For $R_G = 1$, we must then choose the number of grid points across the channel, i.e. n, such that $n \geq 45$. We cannot of course consider an infinitely long channel, so we must assume, for example, a length of ten times the distance across, will be sufficient to approximate the infinite case. Doing this we see we need 45 x 20,250 grid points.

Let us now consider the time constant of the system. The longest time step must be such that a particle cannot go farther than one grid point. For illustrative purposes, say we wish to consider 450 time steps. We now will solve the equation

$$\frac{\partial}{\partial t}(\nabla^2\psi) = \nu\nabla^4\psi - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}(\nabla^2\psi) + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}(\nabla^2\psi) \tag{2}$$

subject to viscous boundary conditions, and an initial straight line velocity profile. The solution of (2) is a marching problem, i.e. given $\nabla^2\psi$, compute $\frac{\partial}{\partial t}(\nabla^2\psi)$, which gives $\nabla^2\psi$ at a later time.

We may now consider the amount of machine time required for the calculation of the time-evolved field. If we count the number of multiplications (a lengthy machine operation as contrasted with additions), we find 50 multiplications per point per time step. For all the points we have then $(10)^6$ mult./step. On the IBM 7090 the time required for the total calculation at any one step is 20 seconds. For 450 time steps, we would need $2\frac{1}{2}$ hours. The cost would be approximately \$1500. In addition we must allow between two to eight months for check-out. In order to condense the results into a form suitable for presentation, programs must be written involving statistics, correlations, etc., which may well double the time required.

This may appear very lengthy, and costly, but we must remember we have asked for a time-evolution of the flow which can give us considerable insight into the build-up of instability into turbulence.

submitted by R. L. Duty

A Number of Stationary Principles for High-order Eigenvalue Problems

Often in the formulation of problems of hydrodynamic stability one encounters a set of coupled ordinary differential equations of the form,

$$\begin{aligned} Lu &= \lambda\, l(z)v \\ Mv &= \mu\, m(z)u\,, \end{aligned} \tag{1}$$

with accompanying boundary conditions of the form,

$$U_i = 0\,, \quad V_j = 0\,, \quad i = 1, \ldots, n \quad j = 1, \ldots, m. \tag{2}$$

where

$$L \equiv p_o(z)\,\frac{d^n}{dz^n} + p_1(z)\,\frac{d^{n-1}}{dz^{n-1}} + \ldots + p_{n-1}(z)\,\frac{d}{dz} + p_n(z)$$

$$M \equiv q_o(z)\,\frac{d^m}{dz^m} + \ldots + q_m(z)$$

$$U_i = a_i^{(o)}\,u(z_o) + a_i^{(1)}\,u'(z_o) + \ldots + a_i^{(n-1)}u^{(n-1)}(z_o) +$$

$$+ b_i^{(o)}u(z_1) + b_i^{(1)}u'(z_1) + \ldots + b_i^{(n-1)}u^{(n-1)}(z_1)$$

$$V_j = c_j^{(o)}\,v(z_o) + \ldots + c_j^{(m-1)}v^{(m-1)}(z_o) +$$

$$+ d_j^{(o)}\,v(z_1) + \ldots + d_j^{(m-1)}v^{(m-1)}(z_1)$$

(The notation employed is essentially that of Ince, Ordinary Differential Equations).

We shall assume:

(1.), $p_o > 0$ and $q_o > 0$, $z_o \leq z \leq z_1$;

(2.), $l(z)$ and $m(z)$ do not change sign, $z_o < z < z_1$;

(3.), the reduced systems,

$$Lu = 0 \ , \quad U_i = 0$$

and

$$Mv = 0 \ , \quad V_i = 0 \ ,$$

admit only the trivial solutions,

$$u = v = 0 \ .$$

Under these conditions, equations (1) and (2) define a standard eigenvalue problem for which solutions exist only when the product $\lambda\mu$ is equal to one of a denumerable set of real eigenvalues, i.e. when

$$\lambda\mu = \Lambda_n \tag{3}$$

Accordingly, we write

$$\begin{aligned} L\, u_n &= \lambda_n u(z) v_n \ , \qquad U_i = 0 \\ M\, v_n &= {}_n m(z) u_n \ , \qquad V_j = 0 \ . \end{aligned} \tag{4}$$

(bear in mind that one of the two numbers, λ_n and μ_n, can be specified arbitrarily since only their product is determined).

Let us now construct a system of equations which has solutions $\bar{u}_m$ and $\bar{v}_m$ which are duals of u_n and v_n relative to some weighting functions $\rho(z)$ and $\sigma(z)$, i.e. for which

$$\left.\begin{aligned}(\bar{u}_m|\rho|v_n) &\equiv \int_{z_o}^{z_1} dz\,\rho(z)\,\bar{u}_m v_n = \delta_{mn}\\ (\bar{v}_m|\sigma|u_n) &\equiv \int_{z_o}^{z_1} dz\,\sigma(z)\,\bar{v}_m u_n = \delta_{mn}\end{aligned}\right\} \tag{5}$$

Such a system is usually called an adjoint system.

Consider the equations

$$\begin{aligned}\bar{L}\,\bar{u}_n &= \bar{\lambda}_n \bar{l}(z)\,\bar{v}_n\\ \bar{M}\,\bar{v}_n &= \bar{\mu}_n \bar{M}(z)\,\bar{u}_n\end{aligned} \tag{6}$$

with the boundary conditions

$$\bar{U}_i = 0 \text{ and } \bar{V}_j = 0 \tag{7}$$

If we now require

$$\left.\begin{aligned}(\bar{u}_m|L\,u_n) &= (u_n|\bar{L}\,\bar{u}_m)\\ \text{and}\quad (\bar{v}_m|M\,v_n) &= (v_n|\bar{M}\,\bar{v}_m),\end{aligned}\right\} \tag{8}$$

the operators, $\bar{L}$ and $\bar{M}$, and the boundary conditions, $\bar{U}_i$ and $\bar{V}_j$, are uniquely determined (cf. Ince Chap. IX). In virtue of equations (8), we have

$$\begin{aligned}\lambda_n(\bar{u}_m|l|v_n) &= \bar{\lambda}_m(u_n|\bar{l}|\bar{v}_m)\\ \mu_n(\bar{v}_m|M|u_n) &= \bar{\mu}_m(v_n|\bar{M}|\bar{u}_m),\end{aligned} \tag{9}$$

and thus

$$\lambda_n\mu_n(\bar{u}_m|l|v_n)(\bar{v}_m|M|u_n) = \bar{\lambda}_m\bar{\mu}_m(u_n|\bar{l}|\bar{v}_m)(v_n|\bar{M}|\bar{u}_m) \tag{10}$$

If we now let

$$\bar{l} = M \text{ and } \bar{M} = l , \tag{11}$$

we have

$$(\lambda_n \mu_n - \bar{\lambda}_m \bar{\mu}_m)(\bar{u}_m | l | v_n)(\bar{v}_m | M | u_n) = 0 , \tag{12}$$

and either

$$\left.\begin{aligned} &\lambda_n \mu_n = \bar{\lambda}_m \bar{\mu}_m \\ \text{or} \quad &(\bar{u}_m | l | v_n) = (\bar{v}_m | M | u_n) = 0 \end{aligned}\right\} \tag{13}$$

Let us presume the solutions of the original set of equations are numbered by the prescription

$$\lambda_n \mu_n < \lambda_{n+1} \mu_{n+1}$$

$$\lambda_{-1} \mu_{-1} < 0 < \lambda_0 \mu_0$$

The second set of equations has also a denumerable set of eigensolutions which we shall number by the prescription,

$$\bar{\lambda}_m \bar{\mu}_m = \lambda_m \mu_m \tag{14}$$

An even more convenient identification is

$$\bar{\mu}_m = \mu_m \quad \text{and} \quad \bar{\lambda}_m = \lambda_m , \tag{15}$$

for in this case the desirable normalizations,

$$(\bar{u}_m | l | v_n) = \delta_{mn} \quad \text{and} \quad (\bar{v}_m | M | u_n) = \delta_{mn} , \tag{16}$$

can be achieved simultaneously.

If the solutions u_m and v_m are complete sets, i.e. if there exist sequences of partial sums,

$$S_n = \sum_{j=-n}^{n} s_j(z') \; u_j(z)$$

and (17)

$$T_n = \sum_{j=-n}^{u} t_j(z') \; v_j(z) \quad ,$$

for which

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n = \delta(z'-z)$$

for all z and z' in (z_0, z_1), then the correspondence defined by equation (14) includes all solutions of the adjoint system. Suppose there is a non-trivial solution of equations (6) and (7) corresponding to $\bar{\Lambda}_* \neq \bar{\Lambda}_m$. Then from equation (10) we have

$$(\bar{u}_* | l | v_m) = (\bar{v}_* | M | u_m) = 0 \tag{18}$$

and thus

$$(\bar{u}_* | l | T_n) = (\bar{v}_* | M | S_n) = 0 \tag{19}$$

Presuming that the limit as $n \to \infty$ of the integral is the integral of the limit, we obtain

$$l(z') \, \bar{u}_*(z') = M(z') \, \bar{v}_*(z') = 0 \tag{20}$$

in contradiction with our original assumption as long as l and m do not vanish in any finite subinterval of (z_0, z_1).

Let us now use the adjoint system to construct stationary principles for the determination of $\lambda\mu$. With four differential equations (equations (4) and (6)) at our disposal, we find quite a variety of stationary principles, for we need not formulate the whole problem variationally; we can (if feasible) formally invert one or more of the four differential operators and treat the solution of the nonhomogeneous equation derived thereby as a subsidiary condition in a variational method based on the remaining equations. We shall now set forth a few of the possibilities.

1. Stationary principles containing no subsidiary conditions.

Consider

$$\lambda(\bar{u} | l\, v) = (\bar{u} | Lu) = \frac{1}{\mu}(\bar{u} | L \frac{1}{m} Mv)$$

or

$$\lambda\mu = \frac{(\bar{u} | L \frac{1}{m} Mv)}{(\bar{u} | l\, v)} = \frac{(\bar{M} \frac{1}{m} \bar{L}\, \bar{u} | v)}{(\bar{u} | l v)} \tag{21}$$

We assert that

$$\delta(\lambda\mu) \equiv \frac{\partial}{\partial \bar{u}}(\lambda\mu) \circ \partial\bar{u} + \frac{\partial}{\partial v}(\lambda\mu) \circ \partial v = 0$$

(i.e. $\delta\bar{u}$, δv independent of one another)

is a variational statement of equations (4) and (6). Proof of the assertion follows by direct computation of $\delta(\lambda\mu)$, from which we obtain

$$\begin{aligned} L \frac{1}{m} Mv &= \lambda\mu l\, v \\ \bar{M} \frac{1}{m} \bar{L}\, \bar{u} &= \lambda\mu\, l\, u . \end{aligned} \tag{22}$$

These are in fact one form the eliminants of equations (4) and (6) may take on.

Similarly one can show that

$$\lambda\mu = \frac{(\bar{v} | M \frac{1}{2} Lu)}{(\bar{v} | m u)} = \frac{(\bar{L} \frac{1}{2} \bar{M}\, \bar{v} | u)}{(\bar{v} m u)} \tag{23}$$

also provides a stationary principle.

The difficulty with the above formulations is that the trial functions must satisfy the boundary conditions:

$$U_i = \bar{U}_i = V_j = \bar{V}_j = 0 \tag{24}$$

and

$$(LU)_j = (\bar{L}\,\bar{U})_j = (M\,V)_i = (\bar{M}\,\bar{V})_i = 0$$

where $(LU)_j$ is the same function of LU as V_j is of V, etc. Such trial functions are likely to be rather unmanageable, so we shall see if a method which requires satisfaction of fewer boundary conditions can be formulated.

Consider

$$\lambda = \frac{(\bar{u}|Lu)}{(\bar{u}|\ell v)} \quad \text{and} \quad \mu = \frac{(\bar{v}|Mv)}{(\bar{v}|mu)} \tag{25}$$

We assert that

$$\begin{aligned} \delta(\lambda\mu) &= \lambda\,\delta\mu + \mu\,\delta\lambda \\ &= \lambda\left[\frac{\partial\mu}{\partial\bar{u}}\,\delta\bar{u} + \frac{\partial\mu}{\partial u}\,\delta u + \frac{\partial\mu}{\partial v}\,\delta v\right] \\ &\quad + \mu\left[\frac{\partial\lambda}{\partial\bar{v}}\,\delta\bar{v} + \frac{\partial\lambda}{\partial v}\,\delta v + \frac{\partial\lambda}{\partial u}\,\delta u\right] \\ &= 0 \end{aligned} \tag{26}$$

is a variational statement of the <u>original</u> set of equations, and that a similar principle with

$$\lambda = \frac{(\bar{u}|Lu)}{(\bar{v}|mu)} \quad \text{and} \quad \mu = \frac{(\bar{v}|Mv)}{(\bar{u}|\ell v)} \tag{27}$$

implies solution of the adjoint equations.

Straight-forward computations based on equation (25) give

$$
\begin{aligned}
Lu &= \lambda l v \\
Mv &= \mu m u \\
\bar{L}\,\bar{u} &= \lambda \frac{(\bar{u}\,|\,l v)}{(\bar{v}\,|\,l m u)}\, m \bar{v} \\
\bar{M}\,\bar{v} &= \mu \frac{(\bar{v}\,|\,m u)}{(\bar{u}\,|\,l v)}\, l \bar{u} \,,
\end{aligned} \qquad (28)
$$

while from equation (27) we obtain a reversed situation. Since the above method requires only satisfaction of the original boundary conditions,

$$U_i = \bar{U}_i = V_j = \bar{V}_j = 0 \,,$$

it may in some cases be easier to use.

2. Stationary principles with subsidiary equations.

There are four ways in which inversion of one operator can be incorporated - we shall carry out one; the others are equivalent in form.

Suppose given v, we solve

$$Lu = \lambda l(z)v \qquad (29)$$

with λ fixed ($\lambda = 1$ say) .

We now write

$$\mu = \frac{(\bar{v}\,|\,Mv)}{(\bar{v}\,|\,mu)}$$

and we assert

$$\delta\mu = \frac{\partial\mu}{\partial v}\delta v + \frac{\partial\mu}{\partial\bar{v}}\delta\bar{v} = 0 \,, \quad L\delta u = \lambda l \delta v \qquad (30)$$

is a variational statement of the problem.

Since $L\delta u = 0$ implies $\delta u = 0$ (assumption 3.) the particular variation, $\delta\bar{v}$ arbitrary y and $\delta v = 0$ yields the desired result

$$Mv = \mu mu \tag{31}$$

Note that since $\mu = \dfrac{(\bar{v}\,|\,Mv)}{(\bar{v}\,|\,mu)}$ contains $\bar{u}$ neither explicitly nor implicitly we obtain no information about it from this variation principle. In fact, we can easily find $\bar{u}$ either from

$$\begin{aligned} \bar{L}\,\bar{u} &= l(z)\,\bar{v} \\ \text{or} \qquad \bar{u} &= \frac{1}{\mu m}\,\bar{M}\,\bar{v} \end{aligned} \tag{32}$$

If $\bar{L}$ is difficult to invert, we shall of course prefer the second expression above.

There is yet another class of variation principles which can be constructed if we wish to solve two inhomogeneous equations.

Suppose, given v and $\bar{u}$ we solve

$$\begin{aligned} Lu &= l(z)v \qquad & \lambda &= 1 \\ \bar{M}\,\bar{v} &= l(z)\bar{u} \qquad & \bar{\mu} &= 1 \end{aligned} \qquad (\quad)$$

Then

$$\mu(\bar{v}\,|\,mu) = (v\,|\,l\bar{u})$$

$$(\bar{u}\,|\,lv) = \bar{\lambda}\,(u\,|\,m\,\bar{v})$$

$$\mu = \bar{\lambda} = \frac{(\bar{u}\,|\,lv)}{(u\,|\,m\,\bar{v})} \qquad ($$

We assert: $\delta\mu = \delta\bar{\lambda} = 0$, $L(u) = lv$, $\bar{M}\,\bar{v} = l\,\bar{u}$

implies solution of the problem.

Notice:

$$\left.\begin{array}{l} (\bar{u}, l\,\delta v) + (\delta\bar{u}, l\,v) = \mu\left[(\delta u, m\bar{v}) + (u, m\,\delta\bar{v})\right] \\ \quad\ \ \| \qquad\qquad \| \\ (\bar{u}, L\,\delta u) \quad (v, \bar{M}\,\delta\bar{v}) \\ \quad\ \ \| \qquad\qquad \| \\ (\delta u, \bar{L}\,\bar{u}) + (\delta\bar{v}, Mv) = \mu\left[(\delta u, m\bar{v}) + (\delta\bar{v}, mu)\right] \end{array}\right\} \qquad (3.\ )$$

Since by assumption 3 the Green's functions for L and $\bar{M}$ are unique, arbitrary variations, δv and $\delta\bar{u}$, are equivalent to arbitrary $\delta\bar{v}$ and δu ; and

$$\begin{aligned} \bar{L}\,\bar{u} &= \mu m\bar{v} = \lambda m\bar{v} \\ M\,v &= \lambda mu = \mu mu \end{aligned} \qquad (3\ )$$

as asserted

We may in some cases wish to solve

$$\begin{aligned} L\,u &= l(z)v \qquad\qquad (\lambda = 1) \\ \bar{L}\,\bar{u} &= m(z)\bar{v} \qquad\qquad (\bar{\lambda} = 1) \end{aligned} \qquad ($$

In this case a stationary principle follows from $\delta\mu = 0$ with

$$\mu = \frac{(\bar{v}|Mv)}{(\bar{v}|mu)} \qquad ($$

or

$$\bar{\mu} = \frac{(v|\bar{M}\,\bar{v})}{(v|l\bar{u})} \qquad ($$

(independent variations, δv and $\delta\bar{v}$, in either case).