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THE LAYER OF FRICTIONAL INFLUENCE IN WIND AND OCEAN CURRENTS

BY

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CONTENTS

I. INTRODUCTION

The purpose of the present paper is to analyze, in a reasonably comprehensive fashion, the principal factors controlling the mean state of turbulence and hence the mean velocity distribution in wind and ocean currents near the surface. The plan of the investigation is theoretical but efforts have been made to check each major step or result through an analysis of available measurements. The comparison of theory and observations is made difficult by the fact that in most cases measurements have been arranged without the aid of a working hypothesis concerning the dynamics of the effect studied; thus information is often lacking concerning parameters essential to the interpretation of the data.

A complete description of the effect of turbulence upon the mean motion is possible when we know, in each point of the fluid, the six components of Reynolds' symmetric stress tensor as functions of time. If, in addition, a complete description of the effect of turbulence on the distribution of various suspensions and solutes or upon the temperature distribution in the medium is desired another field tensor must be determined, the *Austausch* tensor.[1*]

For many meteorological and oceanographical purposes it is sufficient to consider individual cases where the mean motion is stationary, horizontal and independent of the horizontal coordinates. In so far as its effects upon the mean motion are concerned, the state of turbulence in any given point may then be described with the aid of a single quantity, the eddy-viscosity coefficient, which, however, varies greatly from point to point along the vertical and from case to case. Making the further assumption that the horizontal gradients of the various extraneous fields under consideration (for example temperature and specific humidity in air, salinity in water) are negligibly small, one component only of the *Austausch* tensor need to be taken into consideration. This component is identical with Wilhelm Schmidt's *Austausch*, which may or may not have the same value as the eddy-viscosity coefficient. Knowing the *Austausch* coefficient in each point of the fluid we are in a position to determine the rate of vertical transfer of various field properties from their vertical gradients.

We know from numerous investigations in the last thirty years that the great variations of the eddy-viscosity are controlled by certain external parameters; the most important of these are, in the case of the atmosphere, gradient wind velocity (horizontal pressure gradient), character (roughness) of the ground, vertical temperature gradient (stability), and latitude. In oceanic drift currents the corresponding factors are: surface wind velocity (determining the frictional drag and also the roughness of the sea surface), internal vertical stability, and latitude. Turbulence in bottom currents is, of course, influenced by the character of the ocean bottom. Furthermore, as in air, the recent history of the body of water under consideration may be of significance, i.e. the state of turbulence may not adjust itself instantaneously to the prevailing values of the controlling external parameters but may show some lag.

The present investigation aims at the development of a quantitative theory for the eddy-viscosity coefficient in terms of the controlling parameters listed above. The eddy-viscosity is never measured directly but always obtained by indirect means from ob-

* References are indicated by superior figures and are listed at the end of the paper.

served velocity distributions. Some of the earlier methods used in this calculation are entirely unsatisfactory or have been applied, incorrectly, to mean velocity distributions obtained from heterogeneous data, giving eddy-viscosity values which are entirely fictitious. However, even the best methods of computing the eddy-viscosity coefficient give values which may be correct only to the order of magnitude. For this reason it seems preferable to check the theoretical results by a study of the observed velocity distribution for different values of the controlling parameters rather than by a study of generally rather uncertain eddy-viscosity values obtained indirectly from the same velocity profiles.

In the development of a rational theory for the eddy-viscosity and its interaction with the mean motion, liberal use of a fairly complicated but inexact mathematical apparatus is unavoidable. The physical principles upon which the theoretical discussion is based cannot always be said to have a rigid foundation; they are largely intuitive. It is hardly appropriate to erect an exact mathematical structure upon a weak physical foundation.

Our immediate object must be to indicate the general character of the relationships connecting the mean motion and hence the eddy-viscosity with the controlling external parameters and to serve as a guide in the organization of the proper measurements and in their interpretation. Under these circumstances the use of intuitive methods becomes permissible.

The introduction of the non-dimensional Reynold's number has served as a guiding principle and brought order and clarity to the field of aerodynamics. In the study of turbulent wind and ocean currents no such guiding principle exists. The writers believe that the ideas suggested below eventually will tend to remove this serious difficulty.

The present investigation forms a continuation of certain studies published by the first named author in 1932.[2] Frequent use will be made of the results obtained in this paper; for the sake of brevity, it will be referred to under the symbol A. Thus A 163 refers to formula 163 in A.

II. ADIABATIC ATMOSPHERE

1. Completion of Solution for Adiabatic Atmosphere

Two theoretical problems were treated in A. The first referred to a permanent drift current maintained by a steady wind in the surface layer of an unlimited homogeneous ocean. The second concerned the wind distribution maintained in a homogeneous (adiabatic) atmosphere by a constant horizontal pressure gradient. These two problems were solved with the aid of von Kármán's theory for the mixing length.[3] In each case it was found that the eddy-viscosity coefficient (η, referred to as A in A) decreases with increasing distance (z) from the boundary according to the law $\eta \propto (h-z)^2$. The distance h at which η disappears depends in a definite fashion upon the given external parameters. The mixing length (l) also decreases with increasing distance from the boundary and is proportional to $h-z$.

The corrugations of the ground and of the ocean surface must set an upper limit for the magnitude of the mixing length at the boundary and this upper limit must reduce to the dimensions of the molecular free path in the case of an absolutely smooth surface. It is obvious, therefore, that the solutions obtained in A and leading to increasing values of the mixing length with approach towards the boundary must be supplemented with solutions applicable in the immediate vicinity of the surface.

The solution of the atmospheric problem treated in A was incomplete also in another respect; the angle between surface wind and gradient wind remained undetermined. The analogous oceanic problem, treating the modifications set up in a steady gradient (slope) current due to frictional drag at the bottom, is of fundamental importance in Ekman's current theory.[4] In each case the angle between the surface (bottom) velocity vector and the gradient velocity vector controls the transversal circulation (net transport across the isobaric surfaces). In Ekman's theory the transversal component of the bottom current represents the sole agent through which convergence or divergence of surface water can be offset. Since the permanent atmospheric wind system is of such a character as to maintain pronounced convergence in certain ocean regions, divergence in others, the significance of this angle from the point of view of the theory of ocean currents becomes apparent.

The first step in our investigation will consist in an effort to supply the missing elements which have been referred to above. Later other parameters such as stability, will be considered.

In his paper on "Meteorologische Anwendung der Strömungslehre" Prandtl[5] has given a simple solution for the wind distribution in the vicinity of the ground, that is, in the region where the theory of A ceases to apply. Prandtl's solution may be summarized thus:

We consider a boundary layer H which is so thin that its motion is controlled entirely by the frictional drag on its two horizontal boundaries, i.e. it is assumed that volume forces (horizontal pressure gradient, deflecting force) may be neglected. The frictional drag per unit horizontal area will then be constant in intensity and direction from the ground up to the level H. The wind direction must also be constant within the layer H and must coincide with the direction of the frictional drag. The assumption that all volume forces may be neglected represents a severe restriction; its significance will be discussed in a later section (II, 4).

Within the layer H Prandtl assumes the mixing length (l) to vary directly with the distance (z) from the boundary. At the ground, l has a finite value which, as a first approximation, may be assumed to be proportional to the average height (ϵ) of the roughness elements of the ground. The variation of l within the layer H may then be expressed through the relation

(1) $$l = k_0(z + z_0),$$

where the non-dimensional constant k_0 according to Prandtl and von Kármán has the value 0.38. The constant z_0 is, according to the statement above, proportional to ϵ,

(2) $$z_0 = s\epsilon.$$

The value of the factor s is not definitely known. It probably depends upon the shape and distribution of the roughness elements, but Prandtl suggests a tentative value for s of about 1/30.

With these assumptions the expression for the frictional stress (τ) may be written in the form

(3) $$\tau = \rho l^2\left(\frac{dW}{dz}\right)^2 = \rho k_0^2(z + z_0)^2\left(\frac{dW}{dz}\right)^2,$$

where $\frac{dW}{dz}$ represents the shear of the mean motion and ρ represents the density. In the particular case under consideration τ is constant ($\tau = \tau_0$) and the density change with elevation negligible. The preceding equation may therefore be integrated and gives

$$\frac{dW}{dz} = \frac{1}{k_0\,(z + z_0)}\sqrt{\frac{\tau_0}{\rho}}, \quad W = \frac{1}{k_0}\sqrt{\frac{\tau_0}{\rho}}\ln\frac{z + z_0}{z_0}. \tag{4}$$

Turning this formula around we obtain

$$\tau_0 = \rho\,\frac{k_0^2 W^2}{\left(\ln\frac{z + z_0}{z_0}\right)^2}. \tag{5}$$

From a comparison of Prandtl's expression for the stress (3) with the one commonly used in meteorology it is seen that the eddy-viscosity must be given by

$$\eta = \rho l^2 \left|\frac{dW}{dz}\right|. \tag{6}$$

In the present case, then,

$$\eta = \rho k_0 (z + z_0)\sqrt{\frac{\tau_0}{\rho}} = \rho k_0^2 (z + z_0)\,\frac{W_a}{\ln\frac{z_a + z_0}{z_0}}, \tag{7}$$

where W_a represents the wind velocity at the anemometer level (z_a). This important result shows that in a homogeneous atmosphere and in the immediate vicinity of the ground, the eddy-viscosity is a linear function of the distance from the ground and, at a fixed elevation, is directly proportional to the wind-velocity. We shall now present some observational evidence in support of the theory outlined above.

Prandtl's formula (5) for the stress agrees with an empirical formula obtained by G. I. Taylor[6] from a study of Dobson's pilot balloon data from Salisbury Plain. Taylor found

$$\tau_0 = \rho\gamma^2 W_a^2.^{*} \tag{8}$$

The two formulae agree if

$$\gamma^2 = \frac{k_0^2}{\left(\ln\frac{z_a + z_0}{z_0}\right)^2}. \tag{9}$$

This formula shows how Taylor's coefficient γ^2 varies with the elevation of the anemometer and with the roughness of the ground. Taylor's values for γ^2 range between 22×10^{-4} and 32×10^{-4}, W_a being measured 30 m. above the ground.

Mildner[7] has recently published a few determinations of η from hodographs of vertical wind distributions observed near Leipzig. The hodographs were constructed from individual pilot balloon runs or from means of homogeneous data, and the η-values were then computed by Solberg's method.[8] From the mean of 28 balloon runs made during one October day Mildner obtains the following vertical variation of η.

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* In A the symbol κ was used instead of γ^2.

TABLE 1

Height (z) in m.	80	135	190	240	295	405	460	510
η in c.g.s. units	125	270	310	500	246	117	70	70

The hodograph for the corresponding wind distribution indicates a velocity of 9.3 m.p.s. at an elevation of 35 m. above the ground. It will be shown below that the appropriate value of the roughness parameter z_0 for a smooth lawn lies in the vicinity of 0.5 cm. Mildner's data were collected at an airport for which this z_0-value would seem to be appropriate. The objection might be made that the eddy-viscosity (i.e. the stirring) depends upon the roughness along the trajectory of the air current as well as upon the roughness at the point of observation and that for this reason the value of z_0 should be chosen so as to be characteristic of the entire trajectory. To this objection two comments may be added: first, that we believe that the adjustment of the state of turbulence to the prevailing external parameters is fairly rapid,* and second, that the parameter z_0 occurs in the logarithmic part of expression (7) for η due solely to the substitution of the wind velocity at anemometer level for the stress. There can hardly be any doubt that in the relation between stress and wind velocity near the ground the appropriate z_0-value is the one characteristic of the point and moment of observation.

Inserting the above values of W_a and z_0 in (7) we obtain

$$\eta = 0.02(z + z_0), \tag{10}$$

in excellent agreement with Mildner's values for the first 240 m. above the ground. The rapid decrease in the observed η-values beyond 240 m. indicate very plainly that the validity of the preceding theory is restricted to a layer in the vicinity of the ground. Even if we were to take into account the gradual decrease of the stress with elevation by introducing a variable τ in (3), it would be found that the assumption of a mixing length proportional to the distance from the ground leads to impossible results beyond a certain elevation.

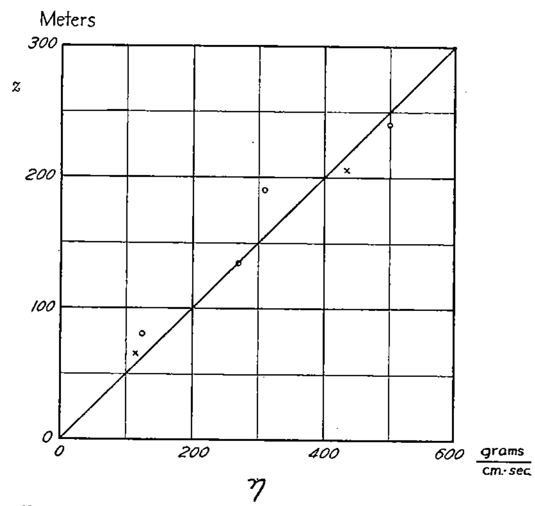

FIG. 1.—Vertical variation of eddy-viscosity (based on Mildner's data).

Mildner determined additional η-values from two individual balloon runs of the group discussed above. Of these the balloon run at 0915 gave quite low η-values, probably due to the stabilizing influence of remnants of an early morning temperature inversion. The pilot balloon run made at noon gave the following η-values:

TABLE 2

Height (z) in m.	65	205	360	510
η in c.g.s. units	115	435	225	50

The two first η-values of this table as well as the four first η-values of Table 1 are plotted in Fig. 1 together with the theoretical line (10).

* Compare II, 3.

Mildner states that the lapse rate on the day of his observations was in the vicinity of 0.65°C per 100 m. It will be seen below that the degree of stability indicated by this lapse rate is quite effective in suppressing turbulence. However, the air reaching the airport at which Mildner's observations were made had just before passed over the city of Leipzig. It seems fairly certain that the highly increased stirring resulting from the passage of the wind over a large city must have brought about complete mixing and the establishment of an adiabatic lapse rate in the lowest two hundred meters, as indicated by the surprisingly high level at which the maximum value of η was found and the abnormally rapid decrease of η above this level.

The ratio between the wind velocities at two levels z_1 and z_2 near the ground is given by

$$\frac{W_2}{W_1} = \frac{\ln \dfrac{z_2 + z_0}{z_0}}{\ln \dfrac{z_1 + z_0}{z_0}} = \frac{\log (z_2 + z_0) - \log z_0}{\log (z_1 + z_0) - \log z_0}. \tag{11a}$$

At all levels except those a few centimeters above the ground, z_0 is negligible compared with z_1 and z_2. Thus the above ratio reduces to the form

$$\frac{W_2}{W_1} = \frac{\log z_2 - \log z_0}{\log z_1 - \log z_0} \tag{11b}$$

which is well suited for determinations of z_0. If anemometer readings are available from three levels we may form the ratio

$$\frac{W_3 - W_2}{W_2 - W_1} = \frac{\log z_3 - \log z_2}{\log z_2 - \log z_1}, \tag{12}$$

which is independent of both stress and roughness parameter and depends solely upon the three anemometer levels. Equation (12) is particularly useful in attempts to verify the theory.

Hellmann's wind velocity measurements in Nauen and in Potsdam offer an excellent opportunity for a check of the preceding theoretical discussion and for a determination of z_0. Observations were collected for a period of about four years at three (later five) levels over a level grass-field or lawn northwest of the radio station in Nauen. In Hellmann's final report,[9] no subdivision of the material according to wind velocity is made and for this reason the data are less suitable for our purposes. In an earlier paper,[10] Hellmann presents a survey of preliminary results obtained from measurements during the period December 10, 1912 to November 20, 1913. These data will be used in the following analysis.

Hellmann's anemometers were mounted at three levels, 2 m., 16 m., and 32 m., above the ground. From the entire observational material for this first year, Hellmann selects two groups, referred to as days with light wind and days with strong wind. Days with light wind are defined by the condition that the 24-hour mean of the wind velocity 2 m.

above the ground must be ≤ 2 m.p.s. During days with strong wind the same mean must be ≥ 5 m.p.s. Table 3 is a reproduction of Hellmann's Table 4 and gives ratios and differences between the wind readings at the three different levels for light and for strong wind. It is reasonable to assume that the lapse rate is practically dry-adiabatic between 0900 and 1500 on windy days. In Fig. 2 we have plotted Hellmann's mean values of $\frac{W_{16}}{W_2}$ and $\frac{W_{32}}{W_2}\left(=\frac{W_{32}}{W_{16}}\cdot\frac{W_{16}}{W_2}\right)$ for the period 0900–1500 for the entire year against log z. Corresponding ratios for the summer half year and for the winter half year have been entered on the same diagram. The points fall very nearly on a straight line, the slope of which indicates a z_0-value of 0.55 cm. Assuming Prandtl's value for $s(=1/30)$ to be correct, this corresponds to a roughness of about 16 cm., which would seem to be a reasonable value for level, grass covered fields.

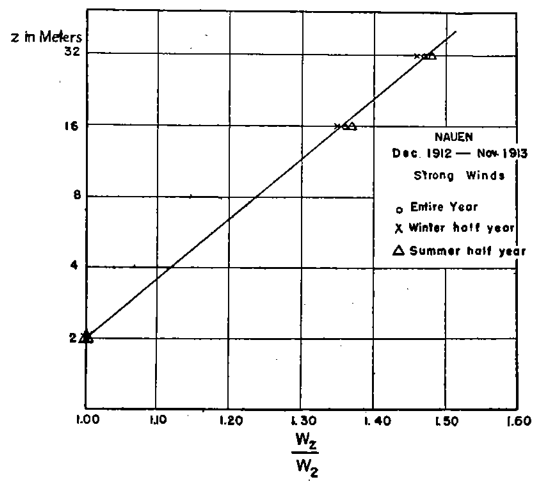

FIG. 2.—Vertical variation of strong winds with elevation (from Hellmann's data).

During the autumn of 1918 Hellmann[11] measured the wind velocity at five different levels above a smooth lawn near Potsdam. The grass was cut repeatedly within the area (100 m.^{2}) where the anemometers were mounted, 5, 25, 50, 100 and 200 cm. above the ground. The anemometers had a cup diameter of 4 cm. In view of the strong wind gradient next to the ground and the relatively large diameter of the cups it is to be expected

TABLE 3

RATIOS AND DIFFERENCES OF WIND VELOCITIES AT DIFFERENT LEVELS DURING DAYS WITH LIGHT WIND AND DAYS WITH STRONG WIND (HELLMANN).

C stands for winter half year, W for summer half year and Y for entire year.

	16m.:2m.			32m.:16m.			16m.–2m.			32m.–16m.			
Hour	C	W	Y	C	W	Y	C	W	Y	C	W	Y	
0– 3	2.07	2.32	2.23	1.28	1.42	1.37	1.44	1.27	1.33	0.78	0.94	0.88	
3– 6	2.03	2.22	2.16	1.31	1.48	1.41	1.15	1.05	1.09	0.70	0.91	0.84	
6– 9	1.75	1.40	1.50	1.30	1.15	1.20	1.04	0.63	0.76	0.63	0.33	0.46	
9–12	1.26	1.26	1.26	1.05	1.06	1.06	0.55	0.57	0.57	0.14	0.18	0.16	
12–15	1.26	1.27	1.27	1.06	1.07	1.07	0.55	0.69	0.64	0.15	0.22	0.20	Light Wind
15–18	1.68	1.40	1.46	1.16	1.12	1.12	0.91	0.87	0.89	0.36	0.37	0.35	
18–21	1.95	1.91	1.91	1.29	1.25	1.27	1.26	1.37	1.32	0.75	0.71	0.74	
21–24	1.91	2.19	2.07	1.30	1.33	1.32	1.26	1.56	1.45	0.80	0.94	0.89	
Mean	1.73	1.75	1.73	1.22	1.24	1.23	1.02	1.00	1.01	0.54	0.58	0.56	
0– 3	1.45	1.51	1.47	1.12	1.15	1.13	2.44	2.26	2.38	0.94	0.99	0.95	
3– 6	1.43	1.46	1.44	1.12	1.12	1.12	2.40	2.29	2.37	0.93	0.97	0.95	
6– 9	1.41	1.38	1.40	1.11	1.09	1.10	2.45	2.46	2.43	0.93	0.83	0.88	
9–12	1.36	1.37	1.36	1.08	1.08	1.08	2.60	2.68	2.63	0.74	0.80	0.76	
12–15	1.34	1.37	1.36	1.08	1.08	1.08	2.61	2.73	2.66	0.77	0.81	0.78	Strong Wind
15–18	1.40	1.38	1.40	1.09	1.08	1.09	2.48	2.66	2.55	0.75	0.81	0.84	
18–21	1.45	1.45	1.45	1.12	1.12	1.12	2.50	2.38	2.47	0.97	0.93	0.96	
21–24	1.46	1.49	1.47	1.11	1.13	1.11	2.47	2.21	2.40	0.87	0.91	0.87	
Mean	1.41	1.43	1.42	1.10	1.11	1.10	2.49	2.46	2.48	0.86	0.88	0.87	

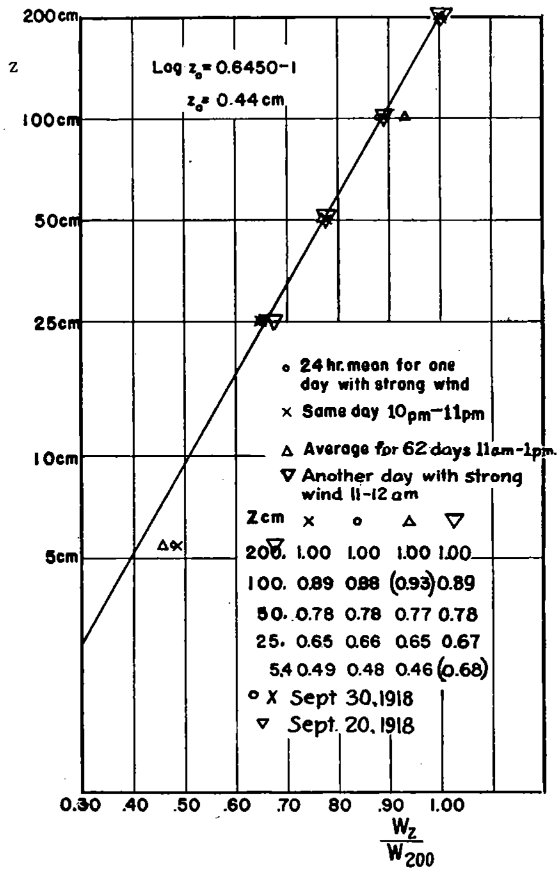

FIG. 3.—Vertical variation of wind velocity in the immediate vicinity of the ground (from Hellmann's data).

that the lowest anemometer should show too high wind velocities. Table 1 in Hellmann's paper[11] gives the diurnal variation of the wind velocity at the different levels based on an observation period of 62 days. The table indicates fairly steady conditions between 1100 and 1300. In the fall, in the middle of the day, the temperature lapse rate should be fairly close to the dry-adiabatic; thus the corresponding velocity distribution should be of the logarithmic type.

In Fig. 3 we have plotted the average values of the ratio W_z/W_{200} for the middle of the day (1100–1300) against log z. The anemometer at 5 cm. gives too high values for the wind ratio, the probable explanation of which has been mentioned above.* Also at 100 cm. the ratio seems to be abnormally high. This may be due to some slight error in the calibration of this particular anemometer for the prevailing light winds (about 3.4 m.p.s.) since the discrepancy does not appear in some readings reproduced by Hellmann and entered in our figure to illustrate the vertical wind gradient on days with strong winds. The z_0-value indicated by this figure is 0.44 cm., or somewhat less than the value obtained from Nauen.

Shaw[12] has published average values for the ratio of the wind velocity at various levels between 0.5 m. and 30 m. to the wind velocity at 10 m. These values, which are reproduced in Fig. 4, apply to open grass land. They are taken from "Annual Wind Summary 1916" in "British Meteorological and Magnetic Year Book, 1916." There is no information available concerning the time of day or the season to which these values apply. The slope of the line in Fig. 4 indicates a roughness parameter of 3.18 cm., which would seem to be a reasonable

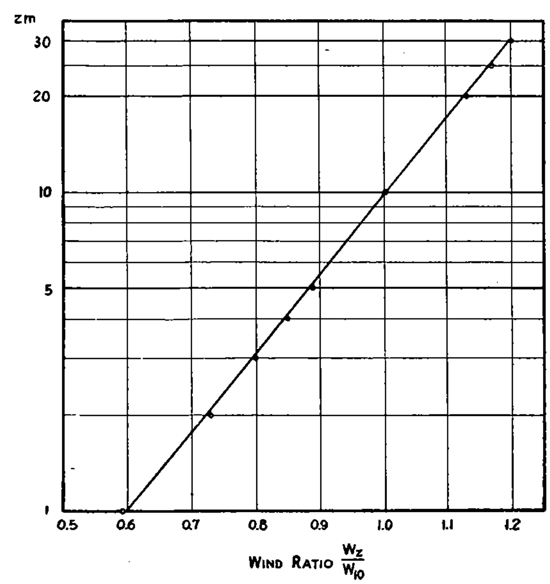

FIG. 4.—Vertical variation of wind velocity over open grass land (from Shaw's data).

* At this lowest level it is necessary to use the exact relation (11a), so W_5/W_{200} is plotted against $z+z_0=5.4$ cm.

value for open grass land, broken by bushes, hedges and ditches, but it is also conceivable that the greater value of z_0 obtained from Shaw's ratios may be due to the inclusion of observations made in light winds under stable temperature conditions.

In September 1919, Wüst[13] made some measurements of the vertical wind distribution at an anchor station in the southeastern part of the Baltic. The sea increased from 1 to 3 on the international scale during the 24-hour period of the observations. Wüst gives some values for the vertical temperature gradient during approximately the same period (Wüst's Table 21, I). They indicate a superadiabatic lapse rate between 0.2 m. and 1.2 m. above sea level, but a slight inversion from 2 m. to 6 m. However, the air temperatures measured between 2 m. and 6 m. were, according to Wüst, influenced by radiation from the vessel, while the temperatures between 0.2 m. and 1.2 m. were measured from a dory and should be reliable. From these comments it would appear that there was no marked inversion present to produce strong wind shear and thus lead to artificially high values of z_0.

Wüst's wind values are given in Table 4. The graphical representation in Fig. 5 indicates a z_0-value of about 4 cm., reasonable in view of the state of the sea during the observation period.

TABLE 4

HEIGHT IN *m.*	*W* IN m.p.s.	$\frac{W}{W_1}$	z_0
6	5.24	1.56	3.98
2.5	4.32	1.28	3.89
1.0	3.37	1.00	—
0.2	2.84	0.84	(0.00)*

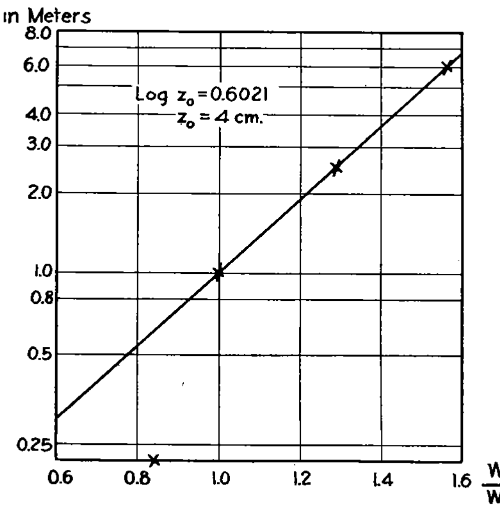

FIG. 5.—Vertical variation of the wind velocity above the sea surface (from Wüst's data).

The wind velocity at 0.2 m. seems very high. Wüst interprets this value as indicating slipping at the surface. It is probable that the strong wind gradient next to the surface may have contributed to the abnormally high value recorded by the anemometer.

The preceding discussion and illustrations should suffice to prove that the normal velocity distribution next to the surface is of the logarithmic type predicted by Prandtl's theory. Logarithmic formulae for the wind distribution near the ground are not new to meteorology; formulae of the type

$$W = a \log (z + b) + c \tag{13}$$

have been suggested by Hellmann[9] and Chapman.[14] However, this as well as other empirical formulae have had no theoretical foundation and their various constants have never been interpreted in terms of the significant physical parameters.

Establishing the validity of the logarithmic velocity distribution is by no means identical with proving the validity of Prandtl's entire theory. To do the latter we must verify the relation (5) between stress and wind velocity.† This is easily done for flow

* This point should have been plotted as indicated in the footnote on page 10, i.e. at the ordinate 0.24.

† The η-values determined by Mildner and plotted in our Fig. 1 furnish such a verification.

through tubes, where τ follows directly from the pressure gradient along the axis of the tube and thus may be regarded as given. On the other hand, to determine the frictional drag between the free atmosphere and the ground we must know the wind distribution within the entire frictional layer.

As a preliminary check we may calculate the coefficient γ^2 from (9) to see if the values so obtained agree with Taylor's empirical values. It was found, with the aid of Hellmann's observations, that a z_0-value of 0.5 cm. is fairly characteristic of a smooth grass field or lawn. If this value is inserted in (9) and if we assume a z_a-value of 30 m. in accordance with the conditions characteristic of Taylor's data, it follows that

$$\gamma^2 = 19 \cdot 10^{-4}.$$

This value is just below the lower limit of the range in κ^2 obtained by Taylor. If we insert the value of z_0 characteristic of open grass land, 3.2 cm., the result is

$$\gamma^2 = 31 \cdot 10^{-4}$$

close to the upper limit of Taylor's range.

Assuming a constant gradient wind U_g along the x-axis, the horizontal equations of steady motion may be written in the form

$$-\rho f i(U + iV - U_g) + \frac{\partial}{\partial z}(\tau_x + i\tau_y) = 0, \tag{14}$$

where U and V represent the two mean wind velocity components, τ_x and τ_y the components of stress, and ρ the density.

$$f = 2\Omega \sin L \tag{15}$$

is the Coriolis' parameter, with the dimension of an angular velocity. Ω is the angular velocity of the earth around its axis and L the latitude.

If we consider the real part of (14) it follows from an integration between the ground and the level h that

$$[\tau_x]_0^h = -f\int_0^h \rho V dz = -\rho f\int_0^h V dz. \tag{16}$$

We choose for h a level where constant gradient wind prevails. Then

$$\tau_{x0} = \rho f \int_0^h V dz. \tag{17}$$

Next to the ground, stress and wind must be parallel. Thus, if φ_s represents the angle between the surface wind and the isobars,

$$\tau_{x0} = \tau_0 \cos \varphi_s, \tag{18}$$

and

$$\tau_0 = \frac{\rho f}{\cos \varphi_s}\int_0^h V dz. \tag{19}$$

This formula is independent of any assumptions regarding the variation of η or τ with elevation and may confidently be used to calculate τ_0 whenever the gradient wind direction remains constant with elevation. Numerical examples of this method are found in (IV, 7).

Under certain conditions the determination of τ_0 from (19) may be simplified by the following procedure, which, to our knowledge, is new.

The vertical distribution of the wind is normally obtained with the aid of pilot balloons ascending at a constant rate. The displacement of the balloon normal to the isobars during the time interval t is

$$D = \int_0^t V dt, \tag{20a}$$

or, because of the constant rate of ascent (C),

$$D = \frac{1}{C}\int_0^{z_t} V dz. \tag{20b}$$

When the balloon reaches the gradient wind region it ceases to move normal to the isobars. Thus, if D now indicates the total displacement of the balloon normal to the isobars, we have

$$CD = \int_0^h V dz \tag{20c}$$

and consequently

$$\tau_0 = \frac{\rho f C D}{\cos \varphi_s}. \tag{21a}$$

The ascensional rate C is known and the distance D may be taken directly from the graphical representation of the horizontal trajectory of the balloon without evaluation of the wind velocities and wind directions. To do so it is necessary to determine the asymptote to the trajectory representing movement of the balloon along the isobars. The length of the perpendicular from the starting point to this line gives D. Unfortunately, it is often difficult to determine accurately the position of the asymptote.

Under the auspices of the Deutsche Seewarte in Hamburg a number of pilot balloon observations have been made in various parts of the Atlantic Ocean and the results published in *Aus dem Archiv der Deutschen Seewarte*.[15] In most cases these reports contain reproductions of the horizontal trajectories of the individual ascents. On each trajectory, marks indicate the successive positions of the balloon at the 1000 m. level, the 2000 m. level et cetera. The ascensional rate varied from one balloon run to the next but in each case the scale of the reproduction of the trajectory was chosen in such a fashion as to give a uniform velocity scale; thus, from the distance between two successive points on any balloon trajectory the mean horizontal velocity for the corresponding layer may be determined with the aid of a single velocity scale.

In view of the fact that the ascensional rate (C) is unknown we must modify (21a) so as to eliminate C. If we indicate by t_0 the time required for the balloon to rise 1000 m., the expression for the stress may be written

(21b)
$$\tau_0 = \frac{\rho f}{\cos \varphi_s} \cdot C \cdot t_0 \cdot \frac{D}{t_0} = 10^5 \cdot \frac{\rho f}{\cos \varphi_s} \cdot \frac{D}{t_0}.$$

D/t_0 has the dimensions of a velocity and may be determined by setting off the length of the perpendicular from the starting point to the asymptote on the velocity scale.

It is easily seen that an error of α percent in the assumed ascensional rate will produce an error in τ_0 as computed from (21a) or (21b) of $\alpha^2+2\alpha$ percent. It is therefore advisable to correct the values of C given by the ordinary formulae so as to include the effect of turbulence on the ascensional rate in the surface layer or else to restrict the application of the method outlined to trajectories obtained with the aid of two theodolites.

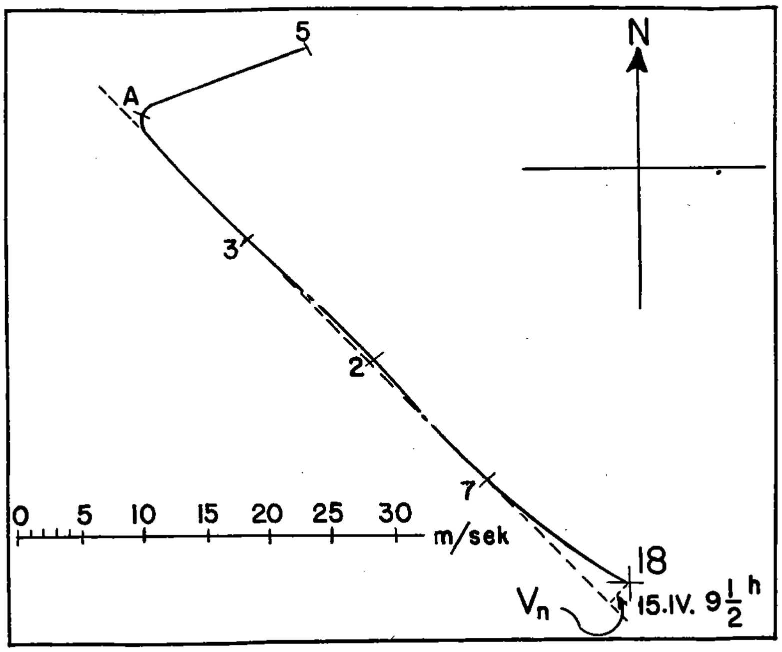

Fig. 6.—Determination of the surface wind force from pilot balloon trajectory. The trajectory is taken from "Aus dem Archiv der Deutschen Seewarte," Band 46, No. 2.

Fig. 6 is an illustration of the method suggested above for the determination of the surface stress. The broken line represents the gradient wind path of the balloon. The short line marked V_n represents the total displacement of the balloon normal to the isobars, here expressed as a velocity.

Averaging the results of such stress determinations from 13 selected pilot balloon trajectories published in "Aus dem Archiv der Deutschen Seewarte," we obtain

$$\frac{\tau_0}{\rho W_a^2} = \gamma^2 = 0.0013.$$

This value is unexpectedly low. In view of the difficulties associated with pilot balloon observations on board a moving steamer, too much weight should not be attached to this determination.

The velocity distribution discussed in the preceding pages applies to a boundary layer (H) within which the stress varies but little with elevation. The theory does not give us any information concerning the thickness of the boundary layer nor does it tell us the value of the stress. To determine these two quantities we must combine Prandtl's solution for the boundary layer with the solution derived in A for the free atmosphere. The latter theory was based upon the assumption that the atmosphere is subject to a vertically and horizontally constant pressure gradient and to a certain frictional drag parallel to the wind at the lower boundary, which we now take as the top of the boundary layer H. Measuring the vertical coordinate z from this level, the theoretical solution in A may be summarized in the formulae below.

The mixing length is given by

$$l = \frac{k(h - z)}{\sqrt{2}}, \tag{A 169}$$

where $z = h$ represents the level at which the frictional influence disappears and the actual wind becomes identical with the gradient wind. h then is the thickness of the layer to which the solution in A applies. The non-dimensional constant k was estimated to have a value of 0.065.

If τ_0 represents the intensity of the frictional drag at the lower boundary of the free atmosphere, then h and τ_0 are connected by the relation

$$\tau_0 = \rho \frac{f^2 h^2}{9k^2}, \tag{22}$$

which follows from a combination of (A 167) and (A 168).

The wind velocity at the lower boundary is

$$W_H = U_g\left(\cos \varphi_s - \frac{1}{\sqrt{2}} \sin \varphi_s\right), \tag{A 166}$$

where φ_s is the angle between the wind direction at the lower boundary (or at the ground) and the gradient wind direction. Between φ_s and h there exists a relation of the form

$$h = \frac{9k^2}{2} \frac{U_g}{f} \sin \varphi_s. \tag{A 167}$$

The wind distribution is characterized by a constant rate of shear K,

$$K = \left| \frac{dU}{dz} + i \frac{dV}{dz} \right| = \frac{\sqrt{2}f}{3k^2}. \tag{A 165}$$

If we indicate by ψ the direction of the shearing vector, so that

$$\frac{dU}{dz} + i \frac{dV}{dz} = Ke^{i\psi}, \tag{A 57}$$

then

(A 144) $$z = h\left(1 - e^{\frac{\psi-\psi_H}{\sqrt{2}}}\right) \quad (\psi_H = \varphi_s)$$

and the wind vector at any level is given by

(A 152) $$1 - \frac{U + iV}{U_g} = \left(1 - \frac{W_H}{U_g}\, e^{i\varphi_s}\right) e^{\left(i+\frac{1}{\sqrt{2}}\right)(\psi-\psi_H)}$$

or

(23) $$1 - \frac{U + iV}{U_g} = \left(1 - \frac{W_H}{U_g}\, e^{i\varphi_s}\right) \cdot \left(\frac{h - z}{h}\right)^{1+i\sqrt{2}}.$$

With the aid of (A 166) this may be written

(23b) $$\frac{U - U_g + iV}{U_g} = \sqrt{\frac{3}{2}}\, \sin \varphi_s \cdot e^{i(\pi-\beta+\varphi_s)} \cdot e^{\left(i+\frac{1}{\sqrt{2}}\right)(\psi-\psi_H)},$$

$$(\tan \beta = \sqrt{2},\ \beta = 54^\circ 44').$$

From (6) it is seen that the eddy-viscosity is given by

(24) $$\eta = \rho l^2 K = \frac{\rho f}{3\sqrt{2}}\,(h - z)^2$$

and hence

(A 180) $$\eta_{\max} = \frac{\rho f h^2}{3\sqrt{2}} = \frac{27k^4}{4\sqrt{2}} \cdot \frac{\rho U_g^2}{f}\, \sin^2\varphi_s.$$

In patching together the two solutions we shall require continuity in frictional drag, in mixing length, and in wind velocity and wind direction. These conditions ensure continuity in eddy-viscosity and in rate of shear. Denoting by H the thickness of the boundary layer within which Prandtl's solution obtains, the condition of continuity in the mixing length takes the form

(25) $$k_0(H + z_0) = \frac{kh}{\sqrt{2}}$$

or, in view of the smallness of z_0,

(26) $$H = 0.12h.$$

Since maximum eddy-viscosity occurs at the level H, (26) indicates that in a turbulent layer of a total depth of 1000 m., maximum stirring is reached 107 m. above the ground. Combining (A 167) with (26) one finds that

(27) $$H = 0.54k^2\, \frac{U_g}{f}\, \sin \varphi_s,$$

showing that the level of maximum turbulence is displaced upward with increasing gradient wind and with decreasing latitude.

From the required continuity in wind velocity and wind direction it follows, through combination of (4) and (A 166), that

$$\frac{1}{k_0}\sqrt{\frac{\tau_0}{\rho}}\ln\frac{H+z_0}{z_0} = U_g\left(\cos\varphi_s - \frac{1}{\sqrt{2}}\sin\varphi_s\right), \tag{28}$$

and, because of the continuity in frictional drag, substitution of (22) gives

$$\frac{fh}{3kk_0}\ln\frac{H+z_0}{z_0} = U_g\left(\cos\varphi_s - \frac{1}{\sqrt{2}}\sin\varphi_s\right). \tag{29}$$

In this formula h may be eliminated by means of (A 167). The result is

$$\frac{3k}{2k_0}\ln\frac{H+z_0}{z_0} = \cot\varphi_s - \frac{1}{\sqrt{2}}; \tag{30a}$$

and finally

$$\frac{H+z_0}{z_0} = e^{\frac{2k_0}{3k}\left(\cot\varphi_s - \frac{1}{\sqrt{2}}\right)}. \tag{30b}$$

With the aid of (25) and (A 167) this formula reduces to

$$N = \frac{U_g}{fz_0} = \frac{2\sqrt{2}k_0}{9k^3}\cdot\frac{1}{\sin\varphi_s}\cdot e^{\frac{2k_0}{3k}\left(\cot\varphi_s - \frac{1}{\sqrt{2}}\right)}. \tag{31a}$$

Equation (31a) provides us with an opportunity to calculate φ_s from the nondimensional number N, which depends upon external parameters only, namely gradient wind speed, latitude and roughness of the ground. Presumably the same relation may be used to calculate the angle between bottom current and gradient current from the roughness of the bottom, the intensity of the bottom current and the latitude. Inserting the proper numerical values in (31a) we find

$$N = \frac{U_g}{fz_0} = \frac{435}{\sin\varphi_s}\,e^{3.9\cot\varphi_s - 2.76} \tag{31b}$$

or

$$\log N = 1.694\cot\varphi_s - \log\sin\varphi_s + 1.441. \tag{31c}$$

This equation is represented graphically in Fig. 7 by the curve marked $u=1$. The angle between the surface wind and the gradient wind increases with roughness and with latitude; it decreases with increasing gradient wind. However, the variations in φ_s, especially with the last two parameters, are quite limited. To illustrate this point Table 5 was prepared in which N and φ_s are tabulated as functions of latitude and gradient wind velocity for $z_0 = 3.18$ cm., the value characteristic of open grass land. The table indicates that a range in latitude of 70° (20°–90°) produces a practically negligible variation (2°) in φ_s. For a given latitude, a range in U_g of from 5 m.p.s. to 40 m.p.s. corresponds to a range of 3° in φ_s.

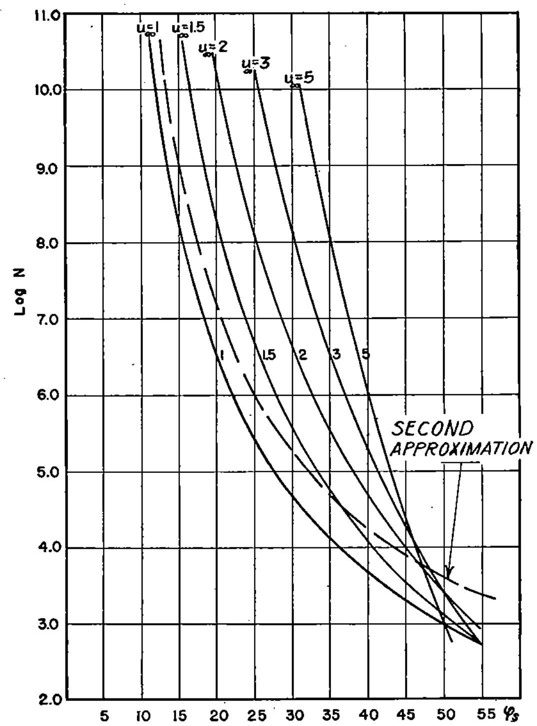

FIG. 7.—Relation between surface wind direction (φ_s) and the non-dimensional number $N(=U_g/fz_0)$.

TABLE 5

L	$U_g=5$ m.p.s.	$U_g=10$	$U_g=20$	$U_g=40$
20°	$N=3.2\cdot10^6$ $\varphi_s=20°$	$6.3\cdot10^6$ 19°	$12.6\cdot10^6$ 18°	$25.2\cdot10^6$ 17°
30	2.2 21°	4.3 20°	8.6 19°	17.2 18°
40	1.7 21°	3.3 20°	6.7 19°	13.4 18°
50	1.4 22°	2.8 21°	5.6 20°	11.2 18°
60	1.3 22°	2.5 21°	5.0 20°	10.0 19°
70	1.2 22°	2.3 21°	4.6 20°	9.2 19°
80	1.1 22°	2.2 21°	4.4 20°	8.7 19°
90	1.1 22°	2.2 21°	4.3 20°	8.6 19°

It is seen that the effect of latitude on φ_s is negligible for latitudes greater than 30°, the range being only 1°. φ_s depends primarily on z_0 and secondarily on U_g, so for all practical purposes, except at low latitudes, it can be readily found from Fig. 8. This figure was computed from the relation $N=U_g/fz_0$ and Fig. 7, using a middle-latitude value of $f=10^{-4}$, corresponding to $L=43°27'$.

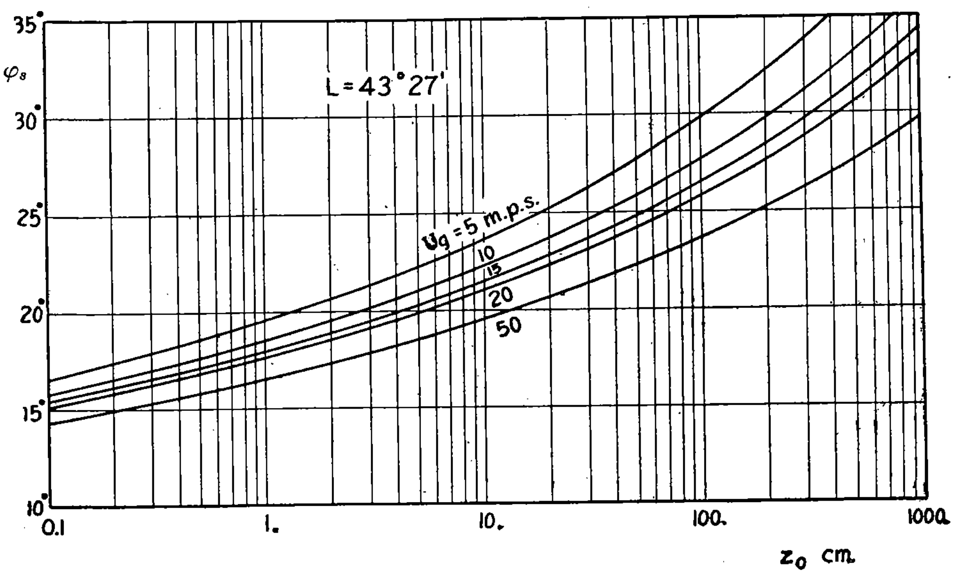

FIG. 8.—Relation between surface wind direction (φ_s) and roughness (z_0).

In discussing the fundamental equation (31c) it is important to keep in mind that it applies to an adiabatic (but not unstable) atmosphere and that it assumes gradient wind direction and gradient wind velocity to be constant with elevation. If the gradient wind varies along the vertical, shearing forces must exist at higher levels which are independent of the frictional layer next to the ground; under those conditions the actual wind may nowhere coincide with the gradient wind and the preceding solution evidently does not apply. The ideal frictional layer is found in initially slightly stable air currents which, under the influence of moderate to strong, steady gradient winds, gradually are stirred to a height corresponding approximately to the theoretical limit $H+h$. If the initial stability is very pronounced the stirring may be insufficient to establish an adiabatic lapse rate all the way up to $H+h$; if the atmosphere initially is in superadiabatic equilibrium, or if the condensation level is lower than $H+h$, convection currents will develop. In either case the final régime will differ from the one here analyzed.

Keeping these restrictions in mind it is easily seen from dimensional considerations that φ_s, which is a pure number, must be a function of U_g/fz_0, this being the only non-dimensional quantity which can be formed from the three external parameters U_g, f and z_0. It is well established that φ_s increases with increasing roughness (z_0). It follows that φ_s must increase with f and decrease with U_g.

Ekman[16] has analyzed Jeffreys' data on the wind variations over the North Sea[17] and states that there is no reliable correlation between φ_s and U_g, a result which is in good agreement with the theoretical data in table 5. Jeffreys' data are divided in four groups, representing different intensities of the gradient wind. The corresponding mean values of φ_s are given in Table 6.

TABLE 6

U_g in m.p.s.	0–8	8.5–12	12.5–18	>18
Number of observations	166	164	160	71
φ_s	15.7°	18.2°	17.2°	16.3°

In view of the fact that the data on which this table is based* are given in compass points (32 points = 360°), the numerical values of φ_s for the different groups may not be very accurate. It is nevertheless interesting to note that φ_s decreases slightly as U_g increases from moderate to strong winds, in good agreement with our theoretical conclusions. For very low values of U_g, φ_s is smaller. However, there is less reason to assume that the lapse rate is adiabatic and the gradient wind constant with elevation when U_g is small and the stirring slight. This point will be analyzed more closely in the next section. It is entirely reasonable to attribute most importance to the groups with moderate or strong winds in attempting to verify the theory.

Dobson's pilot balloon data,[18] which were used by Taylor to verify his theory of the turbulent layer, show a very similar trend. They are reproduced in Table 7.

TABLE 7

U_g in m.p.s.	4.6	9.1	15.6
Number of observations	16	58	23
φ_s	13.1°	21.5°	19.7°
$W_{30m.}/U_g$	0.72	0.65	0.61

Assuming z_0-values of between 3.2 cm. and 4 cm., corresponding to Shaw's data for open grass land and to Wüst's data for the Baltic, and assuming f-values of $1.12\cdot 10^{-4}$

* Jeffrey's tables I and II.