

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
METEOROLOGICAL PAPERS, VOL. I, NO. 4

A GENERALIZATION OF THE THEORY OF  
THE MIXING LENGTH WITH APPLICATIONS TO  
ATMOSPHERIC AND OCEANIC TURBULENCE

BY  
C.-G. ROSSBY

*Published with the aid of a grant from the Woods Hole Oceanographic Institution*

CAMBRIDGE, MASSACHUSETTS  
1932



#### ACKNOWLEDGMENT

In the preparation of the present paper Mr. C. L. Pekeris has rendered valuable assistance, particularly in connection with the systematic search of the literature bearing on atmospheric turbulence. I am also indebted to my colleague, Prof. R. H. Smith, for various critical remarks and for numerous discussions of the principle of similarity in hydrodynamics.



# A GENERALIZATION OF THE THEORY OF THE MIXING LENGTH WITH APPLICATIONS TO OCEANIC AND ATMOSPHERIC TURBULENCE

By C.-G. ROSSBY

## I. INTRODUCTION

It has been shown by L. Prandtl<sup>1</sup> that the fictitious tangential stresses first introduced by Osborne Reynolds to account for the effect of superimposed turbulence on the mean motion of a fluid, in the case of straight, two-dimensional flow may be expressed in the form

$$(1) \quad \tau_x = \rho l^2 \frac{dU}{dz} \cdot \left| \frac{dU}{dz} \right|,$$

where  $\rho$  is the density and  $\frac{dU}{dz}$  the rate of shearing of the mean motion, assumed to be directed along the  $x$ -axis. In the above formula,  $l$ , the "Mischungsweg" or molar free path, is a characteristic length of the state of turbulence and corresponds roughly to the mean free path in case of molecular motion. Prandtl and his collaborators<sup>2</sup> have succeeded to integrate the equations of motion in special cases, making use of (1). The integrals found agree well with observations.

The mixing length is not a constant, as in the molecular theory, but depends, in a fashion not yet fully known, upon the mean velocity distribution and upon the distance from and the character of the external boundaries. In the case of heterogeneous fluids moving in a gravitational field it is obvious that the value of  $l$  must depend also upon the stability of the medium under consideration.

Assuming in each point the existence of a certain fundamental turbulent disturbance restricted to the immediate vicinity of this point, v. Kármán<sup>3</sup> has been able to derive an independent proof for (1) by requiring that the patterns of these fundamental disturbances be kinematically similar, leaving only the length scale and the velocity scale in each disturbance undetermined. Furthermore, v. Kármán obtains an expression for  $l$  of the form

$$(2) \quad l = k \cdot \sqrt{\frac{\frac{dU}{dz}}{\frac{d^2U}{dz^2}}}$$

<sup>1</sup> L. Prandtl: Bericht über Untersuchungen zur ausgebildeten Turbulenz, *Zeitschrift für angewandte Mathematik und Mechanik*, Band 5, 1925, p. 136.

<sup>2</sup> W. Tollmien: Berechnung turbulenter Ausbreitungsvorgänge, *Zeitschrift für angewandte Mathematik und Mechanik*, Band 6, 1926, p. 468.

<sup>3</sup> Th. v. Kármán: Mechanische Ähnlichkeit und Turbulenz, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1930, Heft 1, p. 58.

in which  $k$  is a non-dimensional constant. From a combination of (1) and (2) it follows that the tangential stress may be calculated from the formula

$$(3) \quad \tau_x = \rho k^2 \cdot \frac{\left| \frac{dU}{dz} \right|^3 \frac{dU}{dz}}{\left( \frac{d^2 U}{dz^2} \right)^2}$$

With the aid of this expression for the stress  $\nu$ , Kármán has determined the theoretical velocity distribution in turbulent flow through circular pipes and derived a result which well agrees with the measured distribution. The constant  $k$  was found to have a value between 0.36 and 0.38.

In the study of large scale atmospheric and oceanic movements all but two of the fictitious tangential stresses introduced by Reynolds may be disregarded. Assuming the positive  $z$ -axis to be directed upward, the two significant stresses are

$$(4a) \quad \tau_x = -\rho \overline{u'w'}, \quad \tau_y = -\rho \overline{v'w'}.$$

The bar represents a mean, for instance the arithmetic mean over a certain unit time  $T$ , so that

$$(4b) \quad \bar{f} = \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} f dt$$

The dashes indicate deviations from the mean. Thus  $u'$ ,  $v'$ ,  $w'$  represent the components of the deviation of the instantaneous velocity vector from the mean vector in the point under consideration.

It is generally assumed that these deviations may be represented in (4a) by expressions of the type

$$(5) \quad u' = \pm \lambda_x \frac{dU}{dz}, \quad v' = \pm \lambda_y \frac{dV}{dz}$$

The positive sign is associated with negative  $w'$ -values and vice versa.  $U$  and  $V$  stand for the horizontal components of the mean motion. Thus the formulae for the stresses  $\tau_x$  and  $\tau_y$  reduce to

$$(6) \quad \tau_x = \rho \overline{\lambda_x w'} \cdot \frac{dU}{dz}, \quad \tau_y = \rho \overline{\lambda_y w'} \cdot \frac{dV}{dz}.$$

The coefficients  $\rho \overline{\lambda_x w'}$  and  $\rho \overline{\lambda_y w'}$  are normally, for lack of conclusive evidence to the contrary, assumed to be equal.<sup>4</sup> Under these conditions the fictitious turbulent stress per horizontal unit area is parallel to the shearing vector and may be written

$$(7) \quad \tau_x + i \tau_y = A \left[ \frac{dU}{dz} + i \frac{dV}{dz} \right], \quad A = \rho \overline{\lambda w'}, \quad (i = \sqrt{-1})$$

<sup>4</sup> Certain data presented by L. F. Richardson in "Some Measurements of Atmospheric Turbulence," *Philosophical Transactions of the Royal Society of London*, Series A, Vol. 221, indicate a marked lack of isotropy in the eddy viscosity. Theoretical studies of the vertical distribution of the wind in the anisotropic case have been undertaken by S. Sakakibara in *The Geophysical Magazine*, Tokio, Vol. I, 130, 1930, and by Y. Isimaru in *Memoirs of the Imperial Marine Observatory*, Kobe, Japan, Vol. IV, No. 1, 1930. See also F. Möller: Austausch und Wind, *Meteorologische Zeitschrift*, Heft 2, 1931.

where  $A$  is the "Austausch" coefficient in Wilhelm Schmidt's terminology or "eddy viscosity" according to G. I. Taylor.<sup>5</sup> From the preceding discussion it is evident that  $A$  must depend upon the mean velocity distribution, upon stability and upon the distance from and character of the ground. It is assumed also that the same coefficient  $A$  enters into the expressions determining the eddy diffusion of heat, moisture and atmospheric or oceanic suspensions. Thus  $A$  is also a coefficient of "eddy-conductivity" and of "eddy-diffusivity."

The geophysical turbulence problem is really a twofold one. One phase concerns the determination of the diffusion of various atmospheric or oceanic properties when the state of turbulence ( $A$ ) is known. This problem has been greatly advanced through investigations made by, among others, V. W. Ekman, G. I. Taylor, Hesselberg and Sverdrup and W. Schmidt.<sup>6</sup> The second phase of the turbulence problem consists in the development of a rational theory for the eddy-viscosity coefficient  $A$ .

In a general, qualitative way the variations of  $A$  with distance from the ground (ocean surface), with wind velocity (drift velocity) and with stability are known, but no theory has been presented which enables us to calculate  $A$  from these factors. The principal result of significance is a criterion developed by L. F. Richardson<sup>7</sup> which determines the conditions under which atmospheric turbulence will increase or decrease; however, this criterion is incomplete since it applies to volumes so large that diffusion of eddies through the boundaries may be neglected, and since it does not in a satisfactory way take into account the dissipation of eddy energy by molecular action. Even if these points should be remedied,<sup>8</sup> it is obvious that as long as the connection between eddy energy and the eddy-viscosity coefficient  $A$  remains obscure, Richardson's criterion will be of limited value in the determination of  $A$ .

It is evident that v. Kármán's theory, if at all applicable to atmospheric and oceanic phenomena, eliminates part of the difficulties enumerated above since it leads to an expression for the stress (3) which in a supposedly complete fashion takes into account the effect of the mean velocity distribution.

Through a comparison of (3) and (6) it is seen that

$$(8) \quad A = \rho k^2 \cdot \frac{\left| \frac{dU}{dz} \right|^3}{\left( \frac{d^2U}{dz^2} \right)^2}$$

In its original form v. Kármán's theory was restricted to the case of two-dimensional flow. Thus, with a view towards geophysical applications it becomes necessary to generalize the original theory so as to include motion of the atmospheric and oceanic type, that is, a mean motion in parallel horizontal planes, constant within each plane,

<sup>5</sup> G. I. Taylor: On Eddy-Motion in the Atmosphere, *Philosophical Transactions*, Series A, Vol. 215, p. 22.

W. Schmidt has summarized his numerous investigations in "Der Massenaustausch in freier Luft und verwandte Erscheinungen," *Probleme der Kosmischen Physik*, VII, Hamburg, 1925.

<sup>6</sup> In addition to the references given in fn. 5 see V. W. Ekman: On the Influence of the Earth's Rotation on Ocean Currents, *Arkiv f. Mat. Astr. o. Fys*, 2, Nr. 11, Stockholm, 1905, and Th. Hesselberg and H. U. Sverdrup: Die Reibung in der Atmosphäre, *Veröffentlichungen des Geophysikalischen Instituts der Universität Leipzig*, Heft 10, 1915.

<sup>7</sup> L. F. Richardson: The Supply of Energy from and to Atmospheric Eddies, *Proceedings of the Royal Society*, Series A, Vol. 97, No. A686.

<sup>8</sup> C.-G. Rossby: The Vertical Distribution of Atmospheric Eddy Energy, *Monthly Weather Review*, Vol. 54, No. 8.

but varying in azimuth and intensity from plane to plane. For the sake of convenience this particular type of motion will here be referred to as planar motion. The generalization will be carried out in section II.

It will be found that v. Kármán's condition of kinematic similarity can be satisfied only by a certain general type of planar motion. This permissible type is characterized by the fact that the endpoints of all the shearing vectors,

$$\frac{dU}{dz} + i \frac{dV}{dz},$$

form a logarithmic spiral in the  $xy$ -plane. We may refer to motion of this kind as logarithmic planar motion.

It is a well-established fact that the vertical distributions of the wind and of the drift in pure ocean drift currents belong to this general type. The result obtained would therefore seem to encourage the establishment of a theory of geophysical turbulence upon the basis created by v. Kármán.

It should be stressed that the justification for and the physical meaning of the condition of kinematic similarity remain somewhat obscure. This is emphasized by v. Kármán, who admits that the principal justification so far must be sought in the agreement between the theoretical results and observed facts. No exhaustive analysis of this question will be undertaken since it would require complete knowledge of the fundamental turbulent disturbance.

The generalization of v. Kármán's condition of similarity presented below is purely formal and is therefore neither more nor less acceptable than the original condition. It would seem then that even a partial agreement between conclusions drawn from this generalized theory and observed atmospheric and oceanic facts should add to the strength of the original theory.

In this paper no attempt will be made to calculate the effect of stability upon the molar free path.<sup>9</sup> In a homogeneous medium any fluid element may take the place of any other element; in a heterogeneous medium it must be assumed, either that each individual fluid element after passing through the fundamental turbulent disturbance returns to its original layer or else that mixing between strata of different densities takes place. If it be assumed, as is customary in meteorology and oceanography, that the mechanism which is responsible for the transport of momentum is responsible also for the turbulent diffusion of other properties, then the second of these alternatives must be accepted. Motion of this type, however, is not yet accessible to rigid mathematical treatment.

The expression derived by v. Kármán for the stress is not valid in the immediate vicinity of boundaries. The same restriction obviously applies to the generalized theory, which therefore must be supplemented with suitable, heuristic expressions for the stresses within the boundary layers.

<sup>9</sup> This problem has recently been attacked by L. Prandtl in "Meteorologische Anwendung der Strömungslehre," *Beiträge zur Physik der freien Atmosphäre*, Bjerknes-Festband (XIX. Band), p. 188.

## II. CONDITION OF KINEMATIC SIMILARITY

An incompressible, homogeneous liquid is maintained in a state of planar motion normal to the  $z$ -axis.

From the previously given definition of this type of motion, it follows that the velocity components  $U$  and  $V$  are functions of  $z$  only. By impressing suitable translations on the system, one may reduce the velocity at any given level to zero. Counting the  $z$ -coordinate from this particular level one has

$$(9) \quad U + iV = (U' + iV')z + \frac{U'' + iV''}{2} \cdot z^2 + \dots$$

The complex coefficients in this series may be written in plane polar coordinates,

$$(10) \quad U' + iV' = C' e^{i\psi'}, U'' + iV'' = C'' e^{i\psi''}, \dots$$

Thus:

$$(11) \quad U + iV = C' e^{i\psi'} \cdot z + \frac{1}{2} C'' e^{i\psi''} \cdot z^2 + \dots$$

A variation in the orientation of the  $x$ -axis has the effect of changing the angles  $\psi'$ ,  $\psi''$  etc., by a constant amount.

Now let a slight disturbance with the components  $f_x(xyz)$ ,  $f_y(xyz)$  and  $f_z(xyz)$  be superimposed upon the mean motion defined in (11). The three components  $f_x$ ,  $f_y$ , and  $f_z$  shall differ from zero only in the vicinity of  $x = y = z = 0$ , that is, the disturbance is limited to the immediate surroundings of this point. On account of the incompressibility of the fluid, the functions introduced above are not independent but connected mutually through the equation of continuity,

$$(12) \quad \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = 0.$$

The total flow is given by the two expressions,

$$(13) \quad U_t + iV_t = C' e^{i\psi'} z + \frac{1}{2} C'' e^{i\psi''} z^2 + \dots + f_x + i f_y$$

and

$$(14) \quad W_t = f_z.$$

The subscript  $t$  stands for total (velocity).

It is assumed that the disturbance defined above remains stationary, at least for a short period of time. During that time it is carried along bodily with the mean motion of the fluid. However, it may be shown that this assumption is not essential to the following argument. Let such a pattern of flow exist at every point.

We shall now require that the patterns of flow defined by (13) and (14) be kinematically similar with a length and velocity scale that may vary from point to point. Since the mean motion varies in azimuth from one level to the next, it is evident that we must permit a variation in azimuth of our coordinate system in order to bring out the kinematic similarity.

It is possible to determine by simple methods the requirements for this kinematic similarity.

We compare conditions at two levels,  $L_1$  and  $L_2$ . In the vicinity of two points  $P_1$  and  $P_2$ , one at each level, the small disturbances  $D_1$  and  $D_2$  are superimposed upon the mean motion.  $D_1$  differs from zero only in the immediate vicinity of  $P_1$  and  $D_2$  is zero everywhere except in the neighborhood of  $P_2$ . If the patterns of flow in the vicinity of these two levels are to be kinematically similar when referred to  $P_1$  and  $P_2$ , it follows that the mean motions at these two levels must be kinematically similar.

We introduce a coordinate system  $(x, y, z)$  with its origin in  $P_2$  and another system  $(\xi, \eta, \zeta)$  with its origin at  $P_1$ . If the length scale of the two patterns be  $l$  and the velocity scale be  $\frac{A}{l}$ , we have, for corresponding points,

$$(15) \quad x = l\xi, y = l\eta, z = l\zeta.$$

Furthermore,

$$(16) \quad f_x = \frac{A}{l} f_\xi, f_y = \frac{A}{l} f_\eta, f_z = \frac{A}{l} f_\zeta.$$

The symbol  $A$ , introduced in conformity with v. Kármán's notation, should not be confused with the "Austausch"-coefficient  $A$ .

The mean velocity in the vicinity of  $P_2$  is given by  $U_2 + iV_2$ , where  $U_2$  and  $V_2$  are the components along the  $x$ - and  $y$ -axes. The same velocity, resolved at  $P_2$  along the  $\xi$ - and  $\eta$ -directions is  $U_{21} + iV_{21}$ . Then

$$(17) \quad U_2 + iV_2 = (U_{21} + iV_{21}) e^{-i\varphi},$$

$\varphi$  being the angle which the  $x$ -axis forms with the  $\xi$ -axis (counted positive from  $\xi$  toward  $\eta$ ).

If

$$(18) \quad U_{21} + iV_{21} = C_2' e^{i\psi_2'} \cdot \zeta + \frac{1}{2} C_2'' e^{i\psi_2''} \cdot \zeta^2 + \dots$$

we have

$$(19) \quad U_2 + iV_2 = C_2' e^{i(\psi_2' - \varphi)} \zeta + \frac{1}{2} C_2'' e^{i(\psi_2'' - \varphi)} \zeta^2 + \dots$$

In  $P_1$  the mean motion is given by

$$(20) \quad U_1 + iV_1 = C_1' e^{i\psi_1'} \zeta + \frac{1}{2} C_1'' e^{i\psi_1''} \cdot \zeta^2 + \dots$$

Thus the angles  $\psi_2'$ ,  $\psi_1'$ ,  $\psi_2''$ ,  $\psi_1''$  etc., are now measured from the same fixed direction.

In view of the required kinematic similarity of the two patterns when referred to the  $x, y, z$ -system and the  $\xi, \eta, \zeta$ -system respectively, we must have, in each corresponding pair of points,

$$(21) \quad \boxed{U_2 + iV_2 = \frac{A}{l} (U_1 + iV_1)}$$

It follows that

$$(22a) \quad \frac{d}{dz} (U_2 + iV_2)_{z=0} = \frac{A}{l^2} \frac{d}{d\xi} (U_1 + iV_1)_{\xi=0}$$

$$(22b) \quad \frac{d^2}{dz^2} (U_2 + iV_2)_{z=0} = \frac{A}{l^3} \frac{d^2}{d\xi^2} (U_1 + iV_1)_{\xi=0}$$

etc.

Through (15) and (16) we have insured kinematic similarity of the two disturbances. Through (22a, b, etc.) we have insured kinematic similarity of the mean motion at the two levels, the length and velocity scales being the same as those of the disturbances. Thus it is obvious that the total motion will be kinematically similar in the two points. This may be verified through direct substitution in Helmholtz' equations for the transfer of vorticity.<sup>10</sup>

If we introduce the values for  $U_2 + iV_2$  and  $U_1 + iV_1$  from (19) and (20) we find, from (22a),

$$(23) \quad C_2' e^{i(\psi_2' - \varphi)} = \frac{A}{l^2} C_1' e^{i\psi_1'}$$

or

I.

$$\frac{A}{l^2} = \frac{C_2'}{C_1'}$$

and

II.

$$\varphi = \psi_2' - \psi_1'$$

Since the orientation of the  $\xi$ -axis is arbitrary, it is permissible to assume that  $\psi_1' = 0$ . Thus, going from one level to another, we find that the patterns of flow are rotated through an angle which is equal to the rotation of the shearing vector,  $\frac{d}{dz} (U + iV)$ .

It is now possible to form an expression for the tangential stress in the vicinity of  $P_2$ . Its components, resolved along the  $x$ - and  $y$ -axes, are  $\tau_{x_2}$  and  $\tau_{y_2}$ . Resolving the same stress with respect to the fixed directions  $\xi$  and  $\eta$ , one obtains

$$(24) \quad \tau_{\xi 2} + i\tau_{\eta 2} = e^{i\varphi} \cdot (\tau_{x_2} + i\tau_{y_2}).$$

But, according to (4a),

$$(25) \quad \begin{aligned} \tau_{x_2} + i\tau_{y_2} &= -\rho (\overline{f_x f_x} + i\overline{f_y f_x}) \\ &= -\rho \frac{A^2}{l^2} (\overline{f_\xi f_\xi} + i\overline{f_\eta f_\xi}) = \frac{A^2}{l^2} (\tau_{\xi 1} + i\tau_{\eta 1}) \end{aligned}$$

<sup>10</sup> Horace Lamb: Hydrodynamics, fourth edition, Cambridge University Press, 1916, p. 198.

Thus, making use of I and II, we find

III.

$$\tau_x + i\tau_y = \rho l^2 C'^2 \cdot e^{i(\psi' + \sigma)}$$

In the above formula unnecessary subscripts have been dropped.  $\sigma$  is a constant angle, which may or may not be equal to zero. A constant factor has been absorbed in  $l$ , which as yet remains undetermined. The above expression indicates that the stress forms a constant angle  $\sigma$  with the direction of the shearing vector. The stress is proportional to the square of the rate of shearing ( $C'$ ) and to the square of the scale factor  $l$ , the mixing length. If  $\sigma$  is zero and the discussion be applied to two-dimensional motion, the above formula is identical with the one derived by Prandtl.

If we now, in formulating our condition of kinematic similarity, take into account also terms of second order in  $\zeta$  in the mean motion, we find, from (19), (20), and (22b), the following relation:

$$(26) \quad C_2'' e^{i(\psi_2'' - \varphi)} = \frac{A}{\beta^3} C_1'' e^{i\psi_1''}$$

Thus,

$$(27) \quad \frac{A}{\beta^3} = \frac{C_2''}{C_1''},$$

or, if we make use of I,

IV.

$$l = k \frac{C'}{C''},$$

where  $k$  is a proportionality factor. This expression, identical with the one obtained by v. Kármán, determines the mixing length in terms of the mean velocity distribution.

From (26) and II follows that

$$(28) \quad \varphi = \psi_2'' - \psi_1'' = \psi_2' - \psi_1'$$

or

V.

$$\psi'' - \psi' = \text{constant.}$$

Thus a new condition has now been impressed upon the mean motion, restricting the permissible planar motions to the particular type in which the shearing vector and its rate of change normal to the plane of motion always form a constant angle. This condition may also be formulated in another way. From the definitions

$$(29) \quad \frac{d^2}{d\zeta^2} (U + iV) = C'' e^{i\psi''}$$

and

$$(30) \quad \frac{d}{dz}(U + iV) = C' e^{i\psi'}$$

we obtain

$$(31) \quad \frac{d}{dz}(C' e^{i\psi'}) = \left( \frac{dC'}{dz} + iC' \frac{d\psi'}{dz} \right) e^{i\psi'} = C'' e^{i\psi''}$$

If we introduce a constant  $\alpha$  defined by

$$(32) \quad \tan(\psi'' - \psi') = \frac{1}{\alpha},$$

we find, from (31) and (32),

$$(33) \quad C' \frac{d\psi'}{dz} = \frac{1}{\alpha} \frac{dC'}{dz}$$

or, after integration,

$$(34) \quad C' = K e^{\alpha\psi'}$$

Thus, if projected on one plane and set off from the same point with  $\psi'$  as azimuth, the endpoints of the shearing vectors form a logarithmic spiral. This particular type of planar motion shall be referred to as logarithmic planar motion. In the theory for pure drift currents developed by Ekman, planar motion of this particular type is obtained. In Ekman's theory the angle  $\psi'' - \psi'$  has the value  $\frac{\pi}{4}$ .

Making use of (34) and IV we can write the expression for the stress in a different form. From (33) and (31) it follows that

$$(35) \quad (C'')^2 = \left( \frac{dC'}{dz} \right)^2 + C'^2 \left( \frac{d\psi'}{dz} \right)^2 = C'^2 \left( \frac{d\psi'}{dz} \right)^2 (1 + \alpha^2).$$

Thus

$$(36) \quad \frac{C'^4}{C''^2} = \frac{C'^2}{1 + \alpha^2} \left( \frac{dz}{d\psi'} \right)^2 = \frac{K^2}{1 + \alpha^2} \left( \frac{dz}{d\psi'} \right)^2 e^{2\alpha\psi'}$$

and, finally,

$$\text{VI.} \quad \boxed{\tau_x + i\tau_y = \frac{\rho k^2 K^2}{1 + \alpha^2} \left( \frac{dz}{d\psi} \right)^2 e^{2\alpha\psi + i(\psi + \sigma)}}$$

In VI we have dropped some unnecessary dashes and subscripts. It appears that the mean motion is now determined with exception for the rate at which the shearing vector rotates with elevation,  $\frac{d\psi}{dz}$ . A comparison of (34), (35) and IV shows that the mixing length is inversely proportional to the rate of rotation of the shearing vector.

The preceding discussion well illustrates the arbitrary character of the condition of kinematic similarity. This condition implies that the mean motion in the vicinity of two different levels must be kinematically similar. If we consider terms of the first order only, one of our scale factors,  $l$ , remains undetermined and we are led to Prandtl's expression for the stress. By considering terms of the second order in  $z$ , one obtains a formula for  $l$  in terms of the mean velocity distribution, but at the same time a certain restriction is impressed upon the mean motion. If terms of higher order are considered, additional restrictions upon the mean motion are introduced and cannot generally be fulfilled. Thus it would seem that the success of the method outlined above essentially depends upon the fact that the linear dimensions of the turbulent disturbances are so small that within their range the mean motion may be sufficiently accurately described by means of the two first terms in the development (9).

From the equations (4a), (5) and (6) it is seen that the constant  $k$  may be said to measure the correlation between  $u'$  (or  $v'$ ) and  $w'$ . When it is considered that the significant turbulent velocity fluctuations in pipes have a frequency of several hundred per second while the corresponding frequency in the atmosphere or in the ocean is in the vicinity of 15 per minute,<sup>11</sup> it is reasonable to expect that the correlation between  $u'$  and  $w'$  must be decidedly smaller in geophysical than in laboratory turbulence. The applications discussed below show that this is the case.

The circumstance that the fundamental turbulent disturbance in our case is three-dimensional and not two-dimensional as in the problem discussed by v. Kármán may also contribute to the explanation of the discrepancies between values of  $k$  obtained from laboratory experiments and from geophysical measurements. We may express this by saying that the value of  $k$  occurring in (VI) is a function of  $\alpha$ .

<sup>11</sup> Sir Napier Shaw: *Manual of Meteorology, Volume IV (Revised edition of Part IV)*, Cambridge University Press, 1931, p. 144.

### III. RESTRICTION IMPOSED UPON THE MEAN MOTION DUE TO THE CORIOLIS' FORCE OF THE EARTH'S ROTATION

In the preceding section we have derived an expression for the tangential stress of the form

$$(37) \quad \tau_x + i\tau_y = \frac{\rho k^2 K^2}{1 + \alpha^2} \left( \frac{dz}{d\psi} \right)^2 e^{2\alpha\psi + i(\psi + \sigma)}$$

When the above formula is applied to the study of geophysical phenomena it will be found that the deflecting force of the earth's rotation places an additional restriction on the types of logarithmic planar motion possible in the ocean or in the atmosphere. It is the purpose of this section to derive this restriction.

The horizontal deflecting force per unit mass ( $D_x + iD_y$ ) of the earth's rotation is proportional to the velocity relative to the surface of the earth and, on the northern hemisphere, directed ninety degrees to the right of the velocity vector. Thus

$$(38) \quad D_x + iD_y = f e^{-\frac{\pi}{2}i} (U + iV), \quad f = 2 \Omega \sin L$$

Here  $\Omega$  represents the angular velocity of the earth and  $L$  the latitude (positive on the northern hemisphere, negative on the southern hemisphere).

If the motion takes place under the influence of a *vertically and horizontally constant* horizontal pressure gradient  $\frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y}$ , we have

$$(39) \quad 0 = \rho f \cdot e^{-\frac{\pi}{2}i} (U + iV) - \left( \frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y} \right) + \frac{\rho k^2 K^2}{1 + \alpha^2} \frac{d}{dz} \left[ \left( \frac{dz}{d\psi} \right)^2 e^{2\alpha\psi + i(\psi + \sigma)} \right].$$

Differentiating this equation once with respect to  $z$  and making use of the fact that

$$(40) \quad \frac{d}{dz} (U + iV) = K e^{\alpha\psi + i\psi},$$

we find

$$(41) \quad 0 = \rho f K e^{\alpha\psi + i(\psi - \frac{\pi}{2})} + \frac{\rho k^2 K^2}{1 + \alpha^2} \frac{d^2}{dz^2} \left[ \left( \frac{dz}{d\psi} \right)^2 e^{2\alpha\psi + i(\psi + \sigma)} \right]$$

or

$$(42) \quad 0 = \frac{f(1 + \alpha^2)}{k^2 K} e^{\alpha\psi + i(\psi - \frac{\pi}{2} - \sigma)} + \frac{d^2}{dz^2} \left[ \left( \frac{dz}{d\psi} \right)^2 e^{2\alpha\psi + i\psi} \right].$$

The double differentiation indicated in the second term may easily be carried out and gives the result

$$(43) \quad \frac{d^2}{dz^2} \left[ \left( \frac{dz}{d\psi} \right)^2 e^{(2\alpha + i)\psi} \right] = \frac{2z'''' + 3(2\alpha + i)z''' + (2\alpha + i)^2 z''}{z'} e^{(2\alpha + i)\psi},$$

in which expression

$$(44) \quad z' = \frac{dz}{d\psi}, \quad z'' = \frac{d^2z}{d\psi^2}, \quad z''' = \frac{d^3z}{d\psi^3}.$$

Thus we find, from (42) and (43),

$$(45) \quad 0 = \frac{f(1+\alpha^2)}{k^2K} e^{-i(\frac{\pi}{2}+\sigma)+(\alpha+i)\psi} \cdot z' + [2z''' + 3(2\alpha+i)z'' + (2\alpha+i)^2z'] e^{(2\alpha+i)\psi}.$$

If we introduce the abbreviation

$$(46) \quad M + iN = e^{-i(\frac{\pi}{2}+\sigma)} \cdot \frac{f(1+\alpha^2)}{k^2K}, \quad (M \text{ and } N \text{ real}),$$

the complex equation (45) reduces to

$$(47) \quad 2z''' + 3(2\alpha+i)z'' + (2\alpha+i)^2z' + (M+iN)z'e^{-\alpha\psi} = 0.$$

This is a linear, second order differential equation for  $z'$ . Since  $z'$  necessarily must be real it is obvious that the preceding equation actually consists of two which must be satisfied simultaneously. This is not possible generally but may be achieved if the parameters have certain definite values.

The two equations are

$$(48) \quad 2z''' + 6\alpha z'' + [(4\alpha^2-1) + M e^{-\alpha\psi}] z' = 0.$$

$$(49) \quad 3z'' + [4\alpha + N e^{-\alpha\psi}] z' = 0.$$

By successive differentiations of the second of these two equations and substitution in the first, one obtains an expression of the form

$$(50) \quad t_1 e^{-2\alpha\psi} + t_2 e^{-\alpha\psi} + t_3 = 0$$

which must be *identically* satisfied.

In (50) the coefficients  $t_1$  and  $t_2$  and  $t_3$  depend upon the three parameters  $M$ ,  $N$ , and  $\alpha$  only. Assuming at first that  $\alpha \neq 0$  it is evident that equation (50) can be satisfied identically then and then only, when

$$(51) \quad t_1 = t_2 = t_3 = 0 \quad (\alpha \neq 0)$$

But  $t_1$  is equal to  $\frac{2N^2}{9}$ . Thus  $t_1$  will disappear only if  $N$  disappears. According to (46) this would occur when

$$(52) \quad \frac{\pi}{2} + \sigma = \pm n\pi \quad (n \text{ integer})$$

Thus

$$(53) \quad \sigma = -\frac{\pi}{2} \pm n\pi$$

which would imply that the stress everywhere is directed at right angles to the shearing vector. If this alternative be discarded as contrary to available evidence, our only way of satisfying (50) identically is to require

$$(54) \quad \alpha = 0$$

$$\text{and (55) } \quad (t_1 + t_2 + t_3)_{\alpha=0} = 0.$$

The second of these relations shows that one of the two parameters  $M$  and  $N$  may be expressed in terms of the other. We shall now inquire into the meaning of the condition  $\alpha = 0$ .

Generally, in the case of logarithmic planar motion, one has

$$(56) \quad \frac{dU}{dz} + i \frac{dV}{dz} = K e^{(\alpha+i)\psi}$$

and, therefore, in accordance with (54),

$$(57) \quad \frac{dU}{dz} + i \frac{dV}{dz} = K e^{i\psi}.$$

*Thus the shearing vector has everywhere the same absolute value but changes direction from level to level. The endpoints of this vector now form a circle. Since*

$$(32) \quad \tan(\psi'' - \psi') = \frac{1}{\alpha}$$

it is evident that

$$(58) \quad \psi'' = \psi' \pm \frac{\pi}{2}$$

and thus *the shearing vector and its vectorial rate of change with elevation must be directed at right angles to each other.*

It has been pointed out previously that the corresponding angle in Ekman's theory is  $\frac{\pi}{4}$ , or only one-half of the value derived from v. Kármán's theory.

The stress reduces to the form

VII.

$$\tau_x + i\tau_y = \rho k^2 K^2 \left( \frac{dz}{d\psi} \right)^2 e^{i(\psi + \sigma)}.$$

#### IV. APPLICATION TO PURE DRIFT CURRENTS

A wind of uniform velocity is blowing over a very large ocean surface and exerts a stress  $\tau_0$  per unit area. The nature of the resulting drift current is sought. The water is homogeneous and its surface is level. Thus there are no horizontal pressure gradients to consider.

In this case it is advantageous to place the origin in the ocean surface with the  $z$ -axis directed vertically downward. The  $x$ -axis points in the direction of the force exerted by the wind and the  $y$ -axis is directed  $90^\circ$  to the right from the  $x$ -axis. Then

$$(59) \quad \tau_x + i\tau_y = \rho k^2 K^2 \left( \frac{dz}{d\psi} \right)^2 e^{i(\psi + \sigma)}$$

represents the stress exerted at the depth  $z$  by the water below this level on the water above.

In view of the particular orientation of the coordinate system, the deflecting force has the value

$$(60) \quad D_x + iD_y = \rho f e^{\frac{\pi}{2}i} (u + iv).$$

In this formula  $u + iv$  represents the drift velocity vector.

Since there are no horizontal pressure gradients to consider, the equations of motion may be condensed into one,

$$(61) \quad \rho f e^{\frac{\pi}{2}i} (u + iv) + \rho k^2 K^2 \frac{d}{dz} \left[ \left( \frac{dz}{d\psi} \right)^2 e^{i(\psi + \sigma)} \right] = 0.$$

Differentiation gives

$$(62) \quad \frac{f}{k^2 K} e^{(\frac{\pi}{2} - \sigma)i + i\psi} + \frac{d^2}{dz^2} \left[ \left( \frac{dz}{d\psi} \right)^2 e^{i\psi} \right] = 0,$$

or, after completion of the differentiation in the second term,

$$(63) \quad 2z''' + 3iz'' - z' + \frac{f}{k^2 K} e^{(\frac{\pi}{2} - \sigma)i} \cdot z' = 0.$$

We must therefore have, simultaneously,

$$(64) \quad 2z''' = z' [1 - q \sin \sigma]$$

$$(65) \quad 3z'' = -q \cos \sigma \cdot z'$$

$$(66) \quad q = \frac{f}{k^2 K}$$

These equations can be fulfilled only if

$$(67) \quad \frac{q^2 \cos^2 \sigma}{9} = \frac{1}{2} (1 - q \sin \sigma)$$

Then

$$(68) \quad z = R \cdot e^{-\frac{q \cos \sigma \cdot \psi}{3}} + S,$$

where  $R$  and  $S$  are constants of integration. We know from experience that  $\psi$  increases with  $z$ . When  $\psi$  approaches infinity,  $z$  approaches a certain limiting depth  $b$ ,

$$(69) \quad b = S$$

There is also a solution corresponding to a  $\psi$  which decreases with depth, but this solution must be discarded since it leads to values of the drift velocity which increase downward.

At sea level,  $\psi$  has the value  $\psi_0$ . Thus

$$(70) \quad z = b \left[ 1 - e^{-\frac{q \cos \sigma}{3} (\psi_0 - \psi)} \right].$$

The drift velocity at any level is obtained from

$$(71) \quad u + iv = u_s + iv_s + \int_0^z \frac{d}{dz} (u + iv) dz,$$

in which formula  $u_s + iv_s$  represents the surface drift. We have

$$(72) \quad \frac{d}{dz} (u + iv) = K e^{i\psi}$$

and from (70),

$$(73) \quad \frac{dz}{d\psi} = b \frac{q \cos \sigma}{3} e^{-\frac{q \cos \sigma}{3} (\psi_0 - \psi)}.$$

Making use of (72) and (73) we find

$$(74) \quad u + iv = u_s + iv_s + b \cdot \frac{q \cos \sigma}{3} \cdot K \int_{\psi_0}^{\psi} e^{i\psi} + \frac{q \cos \sigma}{3} (\psi_0 - \psi) d\psi.$$

The integration indicated in (74) is easily performed and yields the result

$$(75) \quad u + iv = u_s + iv_s + b K \cos \beta \cdot e^{i\beta} \left[ e^{i\psi_0} - e^{i\psi} + \cotg \beta (\psi_0 - \psi) \right],$$

in which  $\beta$  is defined by

$$(76) \quad \cotg \beta = \frac{q \cos \sigma}{3}.$$

At the depth  $b$  the drift velocity vanishes. Thus

$$(77) \quad u_s + iv_s = -b K \cos \beta \cdot e^{i(\psi_0 + \beta)},$$

and therefore, finally

$$(78) \quad u + iv = (u_s + iv_s) e^{(\cotg \beta - i)(\psi_0 - \psi)}.$$

Thus the drift velocity vector itself forms a logarithmic spiral.

We must now determine the surface drift  $u_s + i v_s$ . The stress at the depth  $z$  is, according to (59) and (73), given by

$$(79) \quad \tau_x + i \tau_y = \rho k^2 K^2 b^2 \cotg^2 \beta \cdot e^{2 \cotg \beta (\psi_0 - \psi)} + i (\psi + \sigma)$$

At the surface,  $z = 0$ , this value reduces to

$$(80) \quad (\tau_x + i \tau_y)_{z=0} = \rho k^2 K^2 b^2 \cotg^2 \beta \cdot e^{i (\psi_0 + \sigma)}$$

This force must be equal and opposite to the wind force  $\tau_o$ . Thus

$$(81a) \quad \pi = \psi_0 + \sigma$$

$$(81b) \quad \tau_o = \rho k^2 K^2 b^2 \cotg^2 \beta.$$

Inserting (81a) in (77) we find, for the surface drift,

$$(82) \quad u_s + i v_s = b K \cos \beta e^{i (\beta - \sigma)}$$

Thus

$$(83) \quad w_s = \sqrt{u_s^2 + v_s^2} = b K \cos \beta$$

and

$$(84) \quad \tau_o = \frac{\rho k^2 w_s^2}{\sin^2 \beta}.$$

The unknown quantities  $\sigma$ ,  $\beta$  and  $q$  are connected through the two equations

$$(67) \quad \frac{q^2 \cos^2 \sigma}{9} = \frac{1}{2} (1 - q \sin \sigma)$$

$$(76) \quad \frac{q \cos \sigma}{3} = \cotg \beta.$$

In addition we have, by definition,

$$(85) \quad q = \frac{f}{k^2 K}.$$

If we know the angle  $\sigma$  which the stress forms with the shearing vector, the values of  $q$  and thus also of  $K$  follow from (67) and (85). By substitution in (81b) we may then determine  $b$ , the depth of the drift current, as a function of the surface stress.

We assume at first that the stress is parallel to the vector of shear, or

$$(86) \quad \sigma = 0.$$

It then follows from (67) that

$$(87) \quad q = \frac{3}{\sqrt{2}}$$

and from (85) that

$$(88) \quad K = \frac{f\sqrt{2}}{3k^2}.$$

From (76) we find

$$(89) \quad \tan \beta = \sqrt{2}, \quad \beta = 54^\circ 44'$$

It follows from (82) that the surface drift is directed  $54^\circ 44'$  to the right of the wind force. We may calculate the surface drift velocity from (84), making use of the  $\beta$ -value given in (89).

$$(90) \quad \tau_o = \frac{3}{2} \rho k^2 w_s^2.$$

To determine the depth of the drift current we employ (83) which gives us

$$(91) \quad w_s = \frac{bK}{\sqrt{3}}$$

or, with the aid of (88),

$$(92) \quad b = \frac{3\sqrt{3}}{\sqrt{2}} \frac{k^2}{f} w_s = \frac{3k}{f} \sqrt{\frac{\tau_o}{\rho}}, \quad \tau_o = \rho \frac{f^2 b^2}{9k^2}.$$

The formulae (88), (89), (90) and (92) completely determine the drift current, assuming  $\sigma = 0$ . In order to evaluate these expressions numerically it is, of course, necessary to know the constant  $k$ . In v. Kármán's original, two-dimensional theory  $k = 0.38$ , but it has already been pointed out that we must not expect the same value to hold also in the case of planar motion.

One objection to the solution just given may be based on the fact that it leads to a deflection of the surface current relative to the direction of the wind stress of  $54^\circ 44'$ . Assuming surface wind and surface stress to be parallel, this angle seems somewhat too large. Ekman's theory leads to the value  $\frac{\pi}{4}$ , which apparently agrees better with observations, although P. H. Gallé<sup>12</sup> has found, from observations in the region between  $10^\circ$ - $20^\circ$  southerly latitude, an average deflection of  $49^\circ$ .

If we discard the assumption that the stress is everywhere directed along the shearing vector and assume instead that there is a constant angle between the two, then  $\sigma$  may be so determined as to give a surface drift deflection of  $45^\circ$ . The calculation is carried out below.

Going back to the expression (82) for the surface drift vector, we now require

$$(93) \quad \beta - \sigma = \frac{\pi}{4}.$$

To determine  $\beta$  and  $q$  one may introduce the symbols

$$(94) \quad x = q \cos \sigma, \quad y = q \sin \sigma.$$

<sup>12</sup> See O. Krümmel: Handbuch der Ozeanographie, Band II, Stuttgart, 1911, p. 456.

Thus (67) transforms into

$$(95) \quad x^2 = \frac{9}{2}(1-y)$$

and (76) may be written

$$(96) \quad x = 3 \cotg \beta$$

From (93) we find

$$(97) \quad \cotg \beta = \frac{x-y}{x+y}$$

and thus, combining (96) and (97),

$$(98) \quad \frac{x}{3} = \frac{x-y}{x+y} \quad \text{or} \quad \frac{x(3-x)}{3+x} = y.$$

The value of  $y$  obtained in (98) may be inserted in (95). This gives, after reductions,

$$(99) \quad (2x-3)(x^2+9) = 0.$$

The only real solution is

$$(100) \quad x = \frac{3}{2}, y = \frac{1}{2}$$

or

$$(101) \quad \tan \sigma = \frac{1}{3}, \tan \beta = 2.$$

It follows that

$$(102) \quad \sigma = 18^\circ 26', \beta = 63^\circ 26', q = \frac{1}{2}\sqrt{10}.$$

Thus

$$(103) \quad K = \frac{2f}{k^2\sqrt{10}}, \tau_o = \frac{\rho}{10} \frac{f^2 b^2}{k^2} = \frac{5}{4} \rho k^2 w_s^2.$$

The two alternative solutions derived above are reproduced graphically in Fig. 1. The spirals represent the endpoints of all the vectors  $u+iv$  projected on the plane of the paper and set off from one point. It should be remembered that  $w_s$  (the surface drift velocity) and  $b$  have different values in the two solutions, even though the external parameters  $\tau_o$ ,  $\rho$  and  $f$  are the same. Indicating the values corresponding to  $\sigma = 0$  by a dash, we find the ratios

$$(104) \quad \frac{w_s}{w_s'} = \sqrt{\frac{6}{5}}, \quad \frac{b}{b'} = \sqrt{\frac{10}{9}},$$

provided  $k$  is the same in both solutions.

In the diagram the scale has been chosen in such a fashion that  $w_s' = 1$ . Thus all velocities obtained from the spirals are expressed in fractions of the corresponding surface

drift for  $\sigma = 0$ . In order to compare the flow at different levels, we have set off, on the curve  $\sigma = 0$ , the points corresponding to  $\frac{z}{b'} = 0.1, 0.2, 0.3$ , etc. On the other curve we have marked the points

$$(105) \quad \frac{z}{b} = 0.1 \sqrt{\frac{9}{10}}, \quad 0.2 \sqrt{\frac{9}{10}}, \quad 0.3 \sqrt{\frac{9}{10}}, \text{ etc.}$$

so that corresponding points on the two curves refer to the same absolute rather than to the same relative depth.

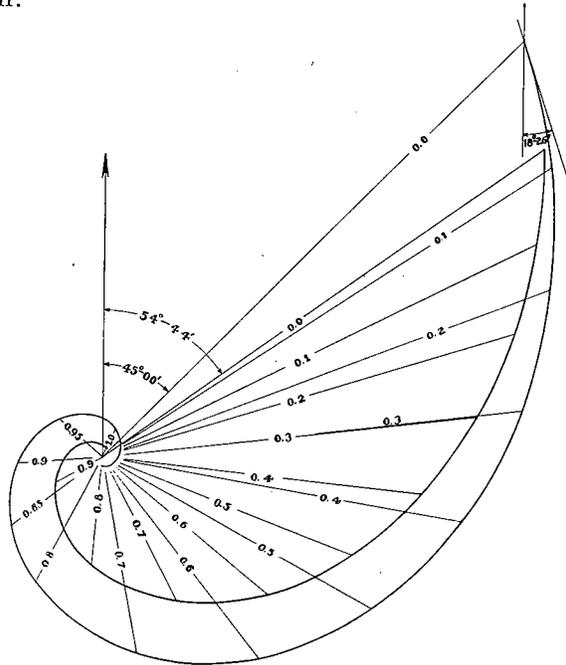


FIG. 1

The first spiral ( $\sigma=0$ ) is, according to (68), (78) and (89), given in parameter form by

$$(106a) \quad \frac{w'}{w_s'} e^{i(\varphi-\varphi_s')} = e^{(\frac{1}{\sqrt{2}}-i)(\psi_0-\psi)} \quad (\varphi_s' = 54^\circ 44')$$

$$(106b) \quad \frac{z}{b'} = 1 - e^{\frac{\psi_0-\psi}{\sqrt{2}}}$$

In this system  $\frac{w'}{w_s'}$  represents the drift velocity at the depth  $z$  in percentage of the surface drift. The angles  $\varphi'$  and  $\varphi_s'$  indicate the relative orientation of these two drifts.

The second spiral is given by a similar system, obtained from (68), (78) and (101).

$$(107a) \quad \frac{w}{w_s} e^{i(\varphi-\varphi_s)} = e^{(\frac{1}{2}-i)(\psi_0-\psi)} \quad [\varphi_s = 45^\circ]$$

$$(107b) \quad \frac{z}{b} = 1 - e^{\frac{\psi_0-\psi}{2}}$$

As mentioned before, it is desirable to compare velocities at the same absolute depths. For this purpose one may transform (107a) into

$$(108a) \quad \frac{w}{w_s'} e^{z(\varphi-\varphi_s)} = \sqrt{\frac{6}{5}} e^{(\frac{1}{2}-z)(\psi_0-\psi)}$$

and (107b) into

$$(108b) \quad \frac{z}{b'} = \sqrt{\frac{10}{9}} \left[ 1 - e^{\frac{\psi_0-\psi}{2}} \right]$$

The drift currents analyzed in this paper differ from Ekman's in three particular respects.<sup>13</sup> In the first place, according to the present theory the quadratic relation between stress and surface drift is independent of latitude, whereas Ekman finds that the surface drift produced by a given stress must be inversely proportional to the square root of the sine for the latitude. This would mean that the surface drift corresponding to a given stress reaches very high values in the vicinity of the equator. Since the stress is never measured directly, the customary way of verifying this particular result is to compare drift velocity and surface wind velocity and to assume a definite relation between stress and surface wind. The particular empirical relation employed by Ekman for the latter purpose assumes that the stress is independent of the latitude but proportional to the square of the wind velocity  $W_o$ ,

$$(109) \quad \tau_o = 32 \cdot 10^{-7} \cdot W_o^2.$$

The second important deviation from Ekman's theory is to be found in the relation between the surface drift velocity and the depth of the drift current  $b$ . This relation is here given by (83) and (85) which, if combined, give

$$(110) \quad b = \frac{k^2 q}{\cos \beta} \cdot \frac{w_s}{f}.$$

Thus the depth of the drift current is inversely proportional to the sine of the latitude and proportional to the surface drift. Ekman's corresponding, semi-empirical formula, takes the form

$$(111) \quad D = 600 w_s,$$

where  $D$  is the "depth of frictional influence" and fairly closely corresponds to  $b$ . Observations indicate a pronounced increase in the depth  $b$  (or  $D$ ) towards the equator<sup>14</sup>, but this may perhaps partly be ascribed to the permanent character of the trade winds as compared with winds of higher latitudes.

A third and actually the most fundamental difference between the preceding theory and that of Ekman may be seen in the fact that the latter introduces four external parameters  $\tau$ ,  $\rho$ ,  $f$  and  $A$  (eddy-viscosity) or  $D$  (depth of frictional influence). When these four quantities are known, the drift current may be calculated in all its details. It was brought out in the introduction to this paper that  $A$ , the eddy-viscosity coefficient, is not a constant but varies with depth. For the sake of convenience, this fact is disregarded

<sup>13</sup> Recently a good summary of Ekman's theory has been presented by G. F. McEwen in *Bulletin of the National Research Council*, Number 85, Oceanography, Chapter 8, Washington, 1932.

<sup>14</sup> A. Defant: *Dynamische Ozeanographie, Einführung in die Geophysik*, III, Julius Springer, Berlin, 1929, p. 74.

in Ekman's original theory<sup>15</sup> and a mean value of  $A$  is introduced. The mean value obviously must vary with the stability of the stratification in some fashion which neither theory at present is capable of indicating, but if the discussion be restricted to the case of homogeneous water, there are only three independent external parameters,  $\tau$ ,  $\rho$  and  $f$ . It is the particular advantage of the theory presented above that the drift current may be completely determined from these three parameters. True, we have introduced two as yet unknown quantities,  $k$  and  $\sigma$ , but they are real constants and should be determined empirically once and for all. In a stratified ocean the stability will evidently affect the drift current by introducing changes in  $k$  and possibly in  $\sigma$ . It would therefore seem that we have succeeded in separating the influence of wind (stress) and stability and thus created at least a skeleton theory of oceanic turbulence.

We shall now attempt a preliminary, rough estimate of the constant  $k$  in order to be able to express our results numerically.

For this purpose we accept the second solution, based on the assumption that the surface drift forms an angle of  $45^\circ$  with the stress. We assume that Ekman's semi-empirical formula (111) reasonably well expresses the relation between  $b$  and  $w_s$  in middle latitudes where observations are most abundant.

From (103) we find

$$(112) \quad b = \frac{5}{\sqrt{2}} \frac{k^2}{f} \cdot w_s.$$

Now assume that  $k$  has the value 0.12. At latitude  $43^\circ$  one finds

$$(113) \quad f = 2\Omega \sin L = 10^{-4}$$

and therefore, at this particular latitude,

$$(114) \quad b = 509 w_s,$$

in fair agreement with Ekman's empirical formula. As a check we shall calculate the ratio between surface wind and surface drift for the same latitude, making use of the empirical relation (109). We set the density of sea water equal to one and find, from (103) and (109),

$$(115) \quad \frac{w_s}{W_o} = 1.33 \cdot 10^{-2}.$$

According to Ekman's theory<sup>16</sup> the same ratio should have the value

$$(116) \quad \frac{w_s}{W_o} = \frac{\lambda}{\sqrt{\sin L}},$$

where  $\lambda$  is a constant. This constant has been determined repeatedly and values have been obtained ranging from 0.0103 to 0.0160.<sup>17</sup> Assuming a mean value  $\lambda = 0.0135$  and inserting the proper value of  $L$  (about  $43^\circ$ ) in (116) we find

<sup>15</sup> Later various other assumptions regarding  $A$  have been made and the corresponding velocity distributions calculated. Thus Ekman has investigated the case of a stress proportional to the square of the rate of shear, and H. Solberg has treated the case of an eddy-viscosity coefficient of the form  $a(z+b)^2$ . See H. Solberg: Sur le frottement dans les couches basses de l'atmosphère, *Förh. Skand. Naturforskarmötet i Göteborg*, 57, p. 95.

<sup>16</sup> A. Defant: l. c. fn. 14, p. 74.

<sup>17</sup> A recent determination from current measurements on Finnish lightships has been made by E. Palmén: Zur Bestimmung des Triftstromes aus Terminbeobachtungen, *Journal du Conseil International pour l'Exploration de la Mer*, Vol. VI, No. 3, 1931.

$$(117) \quad \frac{1}{\sqrt{\sin L}} = 1.2$$

and thus, from Ekman's theory, for this particular latitude

$$(118) \quad \frac{w_s}{W_o} = 1.62 \cdot 10^{-2},$$

or somewhat in excess of the value found from the present theory (115). At the pole the Ekman ratio would have the value  $1.35 \cdot 10^{-2}$ . It should be emphasized that this ratio according to Ekman depends upon the latitude, since the surface drift produced by a given stress varies with latitude. The theory presented above implies that the drift is a function of the stress but not of the latitude. The influence of latitude will be noticed first when we consider the value of the mixing length next to the surface.

To determine the mixing length we shall make use of (IV), (73) and (70). The result is

$$(119a) \quad l = k \frac{C'}{C''} = k \left| \frac{dz}{d\psi} \right| = k \frac{q \cos \sigma}{3} (b-z) \begin{array}{l} \cong \frac{k(b-z)}{\sqrt{2}} \quad (\sigma = 0) \\ \cong \frac{k(b-z)}{2} \quad (\sigma = 18^\circ 26') \end{array}$$

Thus  $l$  has a maximum at the surface and decreases in a linear fashion with depth; it disappears at the lower boundary of the drift current. Since  $l$  measures the vertical displacements due to the eddying motion within the drift current, it is evident that its value must decrease with increasing stability. The stability is zero when the wind velocity reaches such a value that the stirring of the ocean water becomes complete. We may therefore expect to observe mixing lengths comparable to those indicated by (119) in strong winds; in light winds the observed values of  $l$  must be less. At the level  $z = 0$  the expression for  $l$  reduces to

$$(119b) \quad l_o = \frac{kb}{\sqrt{2}} = \frac{3k^2}{f\sqrt{2}} \sqrt{\frac{\tau_o}{\rho}} = 305 \cdot \frac{10^{-4}}{f} \sqrt{\frac{\tau_o}{\rho}} \quad (\sigma = 0)$$

It is seen that the maximum mixing length increases towards the equator as  $\frac{1}{\sin L}$ .

Knowing  $l$  we are in a position to calculate the eddy-viscosity coefficient, and find

$$(120a) \quad A = \rho l^2 C' \begin{array}{l} \cong \rho \frac{f(b-z)^2}{3\sqrt{2}} \quad (\sigma = 0) \\ \cong \rho \frac{f(b-z)^2}{2\sqrt{10}} \quad (\sigma = 18^\circ 26') \end{array}$$

Thus the viscosity rapidly decreases with increasing depth, a result which agrees well with observations by Sverdrup<sup>18</sup> and others. The maximum value of  $A$  occurs at the level  $z = 0$ . There we find, with the aid of the expressions (92) and (103) for  $b$ ,

<sup>18</sup> H. U. Sverdrup: The Wind-Drift of the Ice on the North-Siberian Shelf, *The Norwegian North Polar Expedition with the Maud*, Scientific Results, Vol. 4, No. 1, p. 27. See also E. Palmén, l. c. fn. 17, p. 397.

$$\begin{aligned}
 (120b) \quad A_{\max} &\cong \sqrt{4.5} \cdot k^2 \frac{\tau_o}{f} = 9.8 \cdot \frac{10^{-8}}{f} W_o^2 & (\sigma = 0) \\
 &\cong \sqrt{2.5} \cdot k^2 \frac{\tau_o}{f} = 7.3 \cdot \frac{10^{-8}}{f} W_o^2 & (\sigma = 18^\circ 26')
 \end{aligned}$$

Equation (120b) is not correct dimensionally, since it was obtained by means of Ekman's formula (109), in which the air density is suppressed. The maximum values obtained in (120b) for middle latitudes ( $f = 10^{-4}$ ) are about twice as large as certain characteristic mean values for the whole turbulent layer given by W. Schmidt.

The preceding theory leads to surface values of the mixing length and of the eddy viscosity which become infinite with approach to the equator. In reality the surface values of these quantities depend upon the immediate interaction between air and water and must therefore be functions of  $\tau_o$ , the wind stress, and of  $\epsilon$ , the vertical measure of the roughness of the sea surface. In a permanent wind blowing over a large ocean expanse,  $\epsilon$  is a function of  $\tau_o$ . Thus the surface values of the mixing length and of the eddy viscosity depend upon  $\tau_o$  alone.

We must therefore assume that there exists, next to the ocean surface, a certain transition layer within which  $l$  increases from its fixed surface value  $l_o$  to the theoretically determined maximum value (119b). It is furthermore assumed that  $l_o$  and  $\epsilon$  are identical. It is evident from the equation (119b) that the transition layer must increase in depth with decreasing latitude. However, excluding a belt next to the equator, we shall assume the transition layer to be everywhere so thin that within it variations of the stress with depth may be disregarded. This implies that all mass or volume forces (deflecting force, horizontal pressure gradients) are neglected.

Within the same layer it is assumed that  $l$  increases linearly with depth, ( $l = k_o z + \epsilon$ ). To the coefficient  $k_o$  we assign the value 0.38, obtained by v. Kármán for two-dimensional motion. The  $z$ -coordinate is now counted from the ocean surface and has the value  $H$  at the bottom of the transition layer.

Using Prandtl's original expression for the stress (1) and remembering that the stress is assumed to be constant and directed along the  $x$ -axis, we find

$$(121) \quad \frac{\tau_o}{\rho} = k_o^2 \left( z + \frac{\epsilon}{k_o} \right)^2 \left( \frac{du}{dz} \right)^2,$$

or, after integration,

$$(122a) \quad u_s = u_o - \frac{1}{k_o} \sqrt{\frac{\tau_o}{\rho}} \log \frac{k_o H + \epsilon}{\epsilon}.$$

In (122a)  $u_o$  and  $u_s$  represent the components of the velocity along the  $x$ -axis (direction of the wind stress). Between  $u_s$  and  $w_s$ , the total velocity at  $H$ , exists the relation

$$(122b) \quad u_s = \frac{w_s}{\sqrt{3}} = 0.577 w_s. \quad (\sigma = 0)$$

In view of our previous assumptions the  $y$ -component of the velocity remains constant throughout the transition layer and equals

$$(122c) \quad v = w_s \sqrt{\frac{2}{3}} = 0.816 w_s.$$

The mixing length must be continuous at  $z = H$ . Thus

$$(123) \quad k_o H + \epsilon = \frac{kb}{\sqrt{2}} = \frac{3k^2}{f\sqrt{2}} \sqrt{\frac{\tau_o}{\rho}}$$

From (122), (123) and (122c) we obtain

$$(124a) \quad w_o \cos \varphi_o = \frac{w_s}{\sqrt{3}} + \frac{1}{k_o} \sqrt{\frac{\tau_o}{\rho}} \log \left( \frac{3k^2}{f\epsilon\sqrt{2}} \sqrt{\frac{\tau_o}{\rho}} \right)$$

$$(124b) \quad w_o \sin \varphi_o = w_s \sqrt{\frac{2}{3}}$$

$\varphi_o$  represents the angle between the stress and the surface drift ( $w_o$ ). This result shows that the surface drift must reach very high values at low latitudes.

In order to estimate numerically  $w_o$  and  $\varphi_o$  we make use of (90) and (124).

$$(125a) \quad w_o \cos \varphi_o = \frac{w_s}{\sqrt{3}} \left[ 1 + \frac{3k}{\sqrt{2}k_o} \log \frac{3k^2}{f\epsilon\sqrt{2}} \sqrt{\frac{\tau_o}{\rho}} \right],$$

$$(125b) \quad w_o \sin \varphi_o = w_s \sqrt{\frac{2}{3}}$$

According to (109) we have,

$$(126) \quad \sqrt{\tau_o} = 0.18 \cdot 10^{-2} W_o \quad (W_o = \text{surface wind}).$$

Cornish<sup>19</sup> has given a rule, according to which the maximum roughness of the ocean surface is proportional to the wind velocity and given by

$$(127) \quad \epsilon = 0.37 W_o.$$

Krümmel finds that this formula gives too high  $\epsilon$ -values for light winds and too small  $\epsilon$ -values for winds between 15 and 30 m.p.s. It may however be accepted as reasonably accurate for our purposes. Assuming  $k = 0.12$  and making use of (126) and (127) we are in a position to evaluate (125) numerically. One finds

$$(128) \quad \tan \varphi_o = \frac{\sqrt{2}}{1 + 1.54 \cdot 10 \log \frac{1.5}{10^4 f}}$$

$$(129) \quad \frac{w_o}{w_s} = \sqrt{1 + 1.03 \cdot 10 \log \frac{1.5}{10^4 f} + 0.79 \left( 10 \log \frac{1.5}{10^4 f} \right)^2}$$

From (90), (126) and (129) we find, for the relation between surface drift and surface wind ( $W_o$ )

$$(130) \quad \frac{w_o}{W_o} = 1.225 \cdot 10^{-2} \sqrt{1 + 1.03 \cdot 10 \log \frac{1.5}{10^4 f} + 0.79 \left( 10 \log \frac{1.5}{10^4 f} \right)^2}$$

<sup>19</sup> Krümmel: l. c. fn. 12, Band II, p. 74.

To obtain the depth of the transition layer it is necessary to go back to (123), (126) and (127). The result is

$$(131) \quad \frac{H}{b} = 0.223 [1 - 0.673 \cdot 10^4 f]$$

and therefore, for the total depth of the drift current,

$$(132) \quad H + b = \frac{7.93}{10^4 f} [1 - 0.123 \cdot 10^4 f] W_0$$

In Table 1 we have calculated  $\varphi_0$ ,  $\frac{w_0}{w_s}$ ,  $\frac{w_0}{W_0}$ ,  $\frac{H}{b}$  and  $(H + b)$  as functions of the latitude.  $H + b$  was computed for a wind velocity of 10 m.p.s. For the sake of comparison we have also included two additional columns. One of them represents the ratio  $\frac{w_0}{W_0}$  as calculated from Ekman's formula

$$(133) \quad \frac{w_0}{W_0} = \frac{\lambda}{\sqrt{\sin L}} \quad (\lambda = 0.0135, L = \text{latitude}).$$

The last column gives  $D$ , the "depth of frictional influence," as a function of latitude. It is given by

$$(134) \quad D = \frac{7.5 W_0}{\sqrt{\sin L}} \cdot 20$$

The variation of  $(H + b)$  with latitude agrees fairly well with observations. Defant quotes values for the depth of the drift current of 50 m. in the Weddell Sea (lat. 65° S) and 150 m. in lat. 8° N. It is evident that the present formula leads to a more rapid variation with latitude than does Ekman's theory.

TABLE I  
Calculated characteristics of drift current maintained by a steady surface wind of 10 m.p.s.

$L$	$\varphi_0$	$\frac{w_0}{w_s}$	$\frac{w_0}{W_0} 10^2$	$\frac{H}{b}$	$H + b$ Meters	$\left(\frac{w_0}{W_0} 10^2\right)_{Ekman}$	$D$ Meters
15°	36.3	1.38	1.69	0.166	201	2.65	147
30°	43.6	1.18	1.45	0.114	99	1.91	106
45°	48.5	1.09	1.34	0.068	67	1.61	89
60°	51.8	1.04	1.27	0.034	53	1.45	81
75°	53.6	1.01	1.24	0.012	47	1.37	76
90°	54.2	1.01	1.23	0.004	45	1.35	75

<sup>20</sup> Krümmel: l. c. fn. 12, Band II, p. 545.

