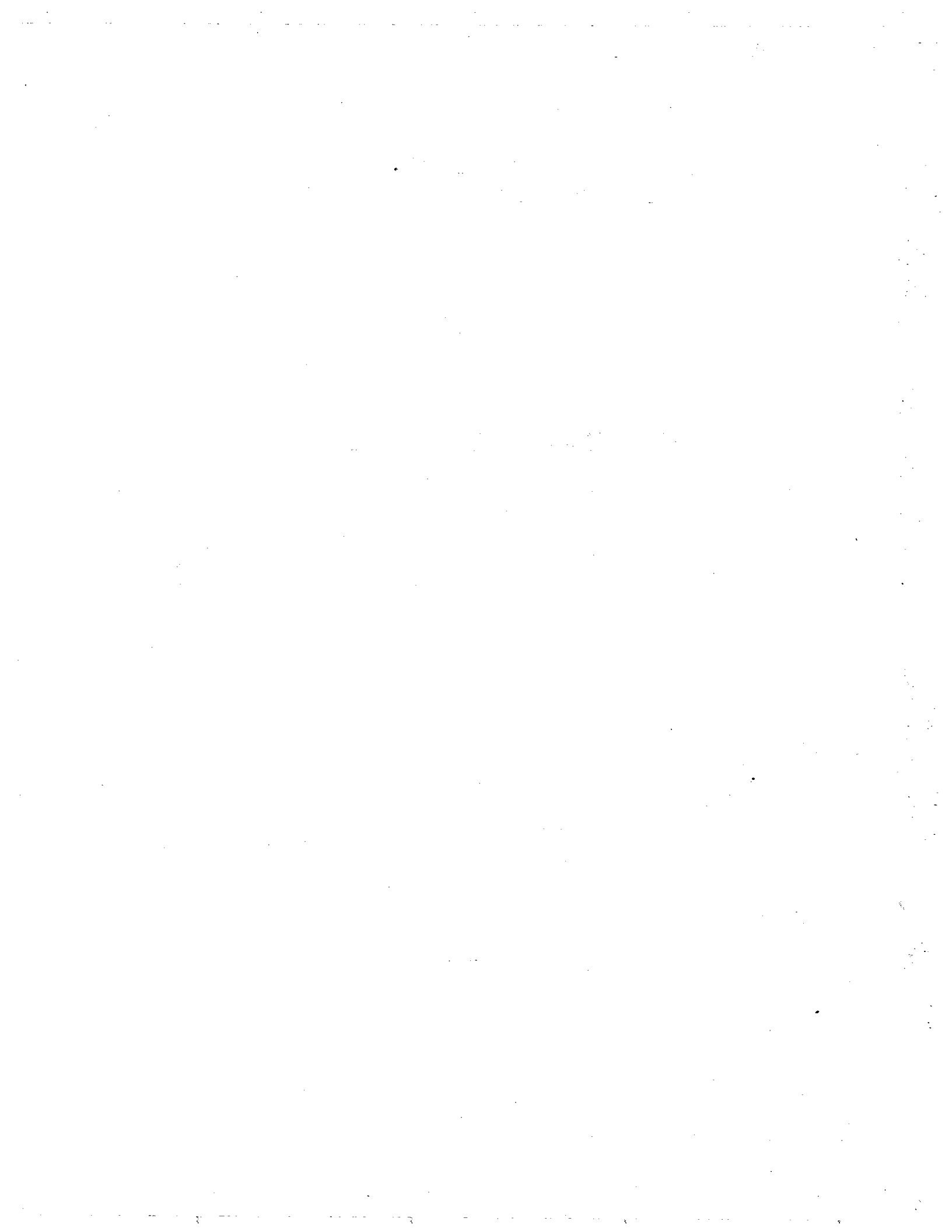


MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
METEOROLOGICAL PAPERS, VOL. I, NO. 3

THERMODYNAMICS APPLIED TO  
AIR MASS ANALYSIS

BY  
C.-G. ROSSBY

CAMBRIDGE, MASSACHUSETTS  
1932



## INTRODUCTION

SINCE the beginning of 1929 systematic work has been carried on at the Massachusetts Institute of Technology to develop practical methods for the identification and characterization of air masses with the aid of upper air soundings of pressure, temperature and humidity. A brief report on this work was published in October 1930<sup>1</sup>. It was then shown that by plotting against each other two meteorological elements, which under certain well defined conditions are recognized as conservative, namely, specific humidity and potential temperature, curves are obtained which, in winter time, to a high extent remain unchanged and characteristic of the individual air masses. In view of this property the curves were named "invariant curves." Since the invariance is restricted to the winter season, but the curves always may be advantageously used to determine the vertical structure and life history of air masses, they shall, in the following, be referred to as "characteristic curves."

The report also stated that by means of characteristic curves a new method had been created of indicating certain differences in stability between the principal American air masses. These differences may be expressed in terms of the variation with elevation of specific entropy, and it was therefore decided to continue the investigation and to include in it a study of the equivalent-potential temperature, which, in an easily comprehensible form, measures the specific entropy of moist air. The excellent results obtained by Robitzsch<sup>2</sup> through the introduction of equivalent-potential temperature into practical meteorological work lent additional support to this decision.

Through the courtesy of the Chief of the Weather Bureau, Professor C. F. Marvin, we obtained photostat copies of the meteorograph records for the periods October 1, 1929 to March 31, 1930 and July 1 to August 31, 1930 from the five Weather Bureau kite stations, Ellendale, Broken Arrow, Groesbeck, Royal Center, and Due West. In addition, the Navy Department placed at our disposal upper air data from the naval air stations at San Diego, Seattle and Pensacola for parts of the periods referred to above. In all about thirteen hundred soundings were obtained for the investigation. To supplement these records with observations from New England, the Meteorological Department of the Massachusetts Institute of Technology is now (since November 16, 1931) making regular airplane soundings at Boston and the material thus collected will be included in the final analysis.

To facilitate the discussion of these data, tables for the calculation of the equivalent-potential temperature were prepared. Since potential temperatures and specific humidities (or mixing ratios) have to be calculated for the construction of characteristic curves, it was decided to express the equivalent-potential temperature in terms of these two arguments.

Everybody connected with the meteorological course as instructor, assistant or graduate student during the winter 1930-1931 has taken some part in the calculation of the tables for the equivalent-potential temperature or in the evaluation of the kite

<sup>1</sup> Earl, K. and Turner, T. A., A Graphical Means of Identifying Air masses, *Massachusetts Institute of Technology Professional Notes, No. 4*, Cambridge, Mass., 1930.

<sup>2</sup> Robitzsch, M., Die Verwertung der durch aerologische Versuche gewonnenen Feuchtigkeitsdaten zur Diagnose der jeweiligen atmosphärischen Zustände, *Die Arbeiten des Preussischen Aeronautischen Observatoriums bei Lindenberg*, XVI. Band, Wissenschaftliche Abhandlungen, Heft C, Braunschweig, 1928.

and airplane records. The final analysis of the upper air data, including a study of the properties of the principal American air masses, is being undertaken by Dr. H. C. Willett. The object of the present communication is primarily to develop certain advantageous *methods of attack* of air mass problems and only incidentally to present results concerning the structure of American air masses. Specifically, the purpose is threefold:

1. To present simple, consistent definitions of the equivalent-potential temperature and the equivalent temperature and to discuss theoretically their most important properties as well as the general properties of the characteristic curve.
2. To illustrate by examples the use of characteristic curves and equivalent-potential temperature diagrams in the analysis of air masses.
3. To present tables and diagrams for the equivalent-potential temperature in terms of potential temperature and mixing ratio.

For the sake of those readers who are unwilling to wade through lengthy mathematical derivations before arriving at some practical conclusions, mathematical theory and applications are kept apart in the following presentation. Thus the third section contains a non-mathematical discussion of selected characteristic curves in conjunction with a study of the corresponding synoptic charts. Section one is devoted to theoretical definitions. In this section not all the results are new, but the definitions given are more rigid and consistent than those usually presented. Section two is devoted to the theoretical solution of a problem of atmospheric convection with the aid of equivalent-potential temperatures. Section four consists entirely of tables, diagrams and plates.

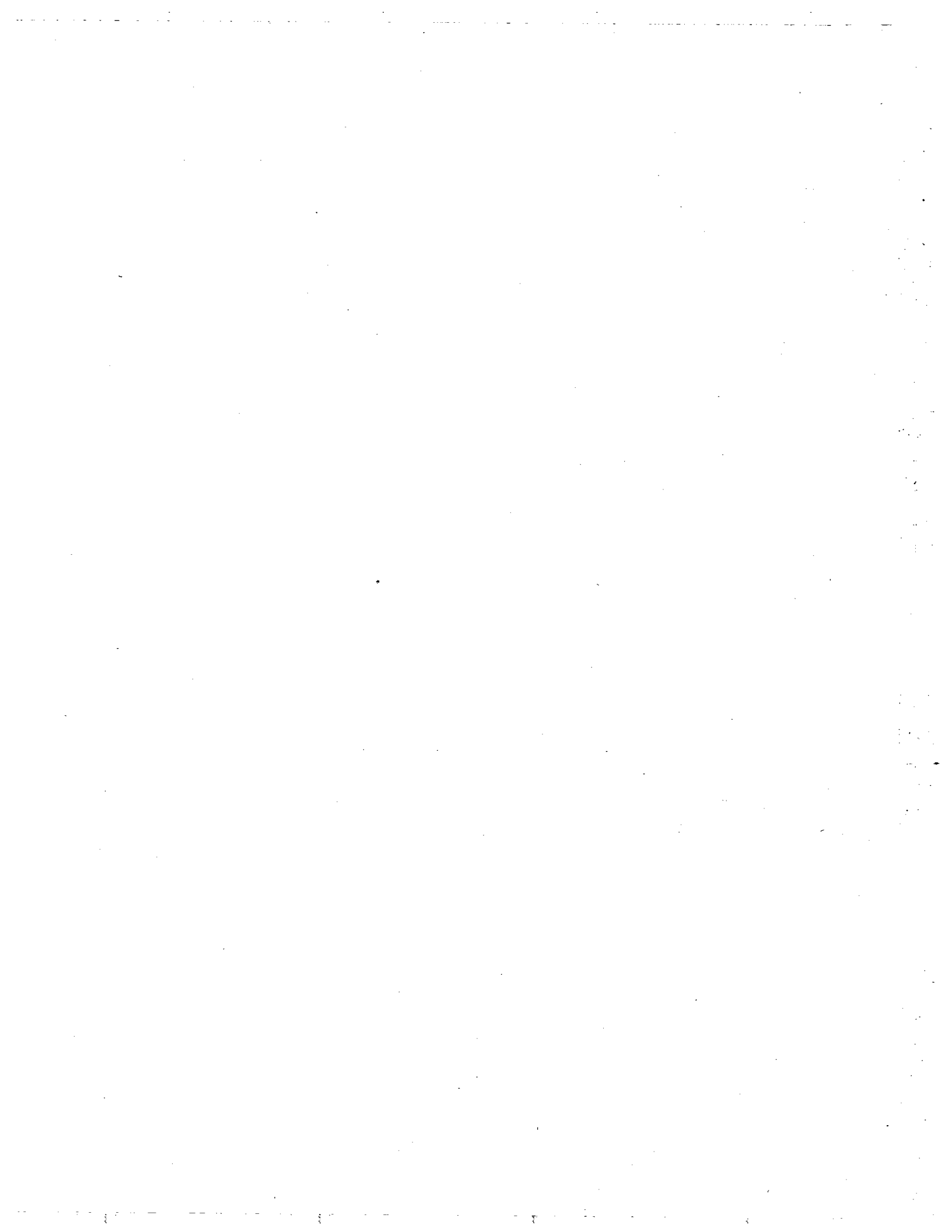
I am obliged to Lt. T. J. Raftery, U.S.N., and to Lt. W. M. Graesser, U.S.N., who calculated part of the data discussed in section three. Their results were originally presented in an (unpublished) thesis on file at the Massachusetts Institute of Technology.

CAMBRIDGE, MASSACHUSETTS

*December, 1931*

## LIST OF SYMBOLS

$P$	standard pressure (1000 mb.)
$p_d$	partial pressure of dry air
$e$	partial pressure of water vapor
$e_m$	pressure of saturated water vapor
$f = \frac{e}{e_m}$	relative humidity
$p$	total air pressure
$\rho_d$	density of dry air
$\rho_w$	density of water vapor
$\rho$	total air density
$T$	absolute temperature in centigrades
$t$	temperature in Celsius
$w$	water vapor content in grams per gram of dry air
$W$	water vapor content in grams per kilogram of dry air
$q$	specific humidity in grams per gram of moist air
$m_d$	molecular weight of dry air
$m_w$	molecular weight of water vapor
$\frac{m_w}{m_d} = 0.622$	
$m$	molecular weight of moist air
$R$	universal gas constant
$k = \frac{R}{m_d c_p} = 0.288$	
$c_p$	specific heat of dry air under constant pressure (0.241 cal/gr)
$c_v$	specific heat of dry air by constant volume
$\frac{c_p}{c_v} = 1.40$	
$L$	heat of condensation of water vapor in calories per gram (606.5 - 0.695t)
$\theta$	potential temperature
$\theta_d$	partial potential temperature (dry air alone considered)
$\theta_E$	equivalent-potential temperature
$T_E$	equivalent temperature
$s$	specific entropy of air (moist or dry)
$S$	total entropy of system (moist or dry)
$g$	acceleration of gravity
log	natural logarithm



## CHAPTER I

### THEORY OF THE EQUIVALENT-POTENTIAL TEMPERATURE AND THE EQUIVALENT-POTENTIAL TEMPERATURE DIAGRAM

IN THE atmosphere a small, initially unsaturated element is given, containing one gram of dry air and  $w$  grams of water vapor. This element is permitted to expand under gradually decreasing pressure. Due to mechanical cooling, part of the water vapor will condense. It is assumed that the liquid water drops out of the element immediately upon condensation. Throughout the process, no heat is added, and no heat leaves the element except as contained in the precipitating water; this latter amount of heat is supposed to be negligibly small. On account of the prevalence of subcooled water in the atmosphere we may assume that the condensation takes place in the form of liquid water even below the freezing point; thus the heat of fusion is not available to raise the temperature of the air. Hence the process outlined follows the "pseudo-adiabatic" rain stage curve.

We then define the equivalent-potential temperature as the ultimate potential temperature assumed by the element when all its water vapor has been removed. This value of the potential temperature is approached asymptotically as the pressure in the element decreases toward zero.

#### WATER VAPOR CONTENT

To express our definition analytically we must first review the definitions of some auxiliary quantities. The water vapor content or mixing ratio ( $w$ ) is expressed in terms of the number of grams of water vapor associated with one gram of *dry* air. Assuming the perfect gas law to hold for dry air and water vapor alike, we have, by definition,

$$(1) \quad w = \frac{\rho_w}{\rho_d} = \frac{m_w}{m_d} \cdot \frac{e}{p_d} = 0.622 \cdot \frac{e_m}{p_d} \cdot f$$

or

$$(2) \quad w = w_m \cdot f, \quad \left( f = \frac{e}{e_m} \right)$$

$w_m$  signifying the maximum water vapor content possible at the given pressure and temperature,  $f$  being the relative humidity. The quantity  $w$  is closely related to, but not the same as, the specific humidity  $q$ , which measures the amount of vapor associated with one gram of *moist* air. Thus,

$$(3) \quad q = \frac{\rho_w}{\rho_w + \rho_d} = \frac{w}{1 + w}.$$

The value of  $w$  seldom exceeds 0.02 (20 grams of water per kilogram of dry air). Considering the inaccuracy of our humidity readings, we are therefore generally permitted to neglect  $w$  in the denominator. Hence approximately,

$$(4) \quad q = w.$$

## POTENTIAL AND PARTIAL POTENTIAL TEMPERATURE

The potential temperature  $\theta$  is defined by Poisson's equation

$$(5) \quad \frac{T}{\theta} = \left( \frac{p}{P} \right)^k \quad \left( k = \frac{R}{m_d c_p} \right)$$

\*

The exponent  $k$  should be corrected for the presence of water vapor but the correction is insignificant. It is possible and, for our purpose, preferable to introduce another potential temperature  $\theta_d$ , which we may name the partial potential temperature since it is calculated from the partial pressure of dry air. It is defined by

$$(6) \quad \frac{T}{\theta_d} = \left( \frac{p_d}{P} \right)^k$$

Since

$$(7) \quad p = p_d + e$$

it follows that

$$(8) \quad \frac{T}{\theta} = \left( \frac{p_d + e}{P} \right)^k$$

or

$$(9) \quad \frac{T}{\theta} = \left( \frac{p_d}{P} \right)^k \left( 1 + \frac{e}{p_d} \right)^k$$

If this last equation is combined with (1) and (6), one obtains

$$(10) \quad \frac{T}{\theta} = \frac{T}{\theta_d} \left( 1 + \frac{m_d w}{m_w} \right)^k$$

or

$$(11) \quad \theta_d = \theta \left( 1 + \frac{m_d w}{m_w} \right)^k,$$

which latter equation for all practical purposes may be written

$$(12) \quad \theta_d = \theta \left( 1 + k \frac{m_d w}{m_w} \right) = \theta \left( 1 + \frac{R}{m_w c_p} w \right).$$

As long as the element remains unsaturated  $\theta$  and  $w$ , and therefore also  $\theta_d$ , will remain unaffected by adiabatic changes of state. To determine  $\theta_d$  from  $\theta$  it is necessary to add a correction term

$$(13) \quad \Delta \theta_d = \theta \left[ \left( 1 + \frac{m_d w}{m_w} \right)^k - 1 \right]$$

This correction term is tabulated in Table B. When the potential temperature is determined graphically from the adiabatic chart, the partial potential temperature may be obtained directly by entering the chart with the partial pressure  $p_d$  and the temperature  $T$  as arguments.

\* The error thus introduced is about as follows: Dry air and moist air with  $w = 10^{-2}$  both at 500 mb have potential temperatures differing by about 0.15 C. This is only a tenth of the partial potential temperature correction in Table B.



## TEMPERATURE AT CONDENSATION LEVEL

The temperature at which an adiabatically expanded air particle reaches the condensation point must be determined in order to permit the calculation of the equivalent-potential temperature. Initially the element has the partial pressure  $p_d$ , the temperature  $T$  and the moisture content  $w$ . At the condensation level the corresponding quantities are  $p_{od}$ ,  $T_o$  and  $w$ , the moisture content  $w$  and the partial potential temperature  $\theta_d$  remaining unchanged up to this point. From Poisson's equation it then follows that

$$(14) \quad \frac{p_{od}^k}{T_o} = \frac{p_d^k}{T} = \frac{P^k}{\theta_d}$$

At the condensation level,

$$(15) \quad w = \frac{m_w}{m_d} \cdot \frac{e_{om}}{p_{od}}$$

where  $e_{om}$  represents the saturation vapor pressure at the temperature  $T_o$ . Thus

$$(16) \quad p_{od} = \frac{m_w}{m_d} \cdot \frac{e_{om}}{w}$$

and

$$(17) \quad \frac{\left(\frac{m_w \cdot e_{om}}{m_d \cdot w}\right)^k}{T_o} = \frac{P^k}{\theta_d}$$

or

$$(18) \quad \left(\frac{m_w}{m_d P}\right)^k \cdot \frac{e_{om}^k}{T_o} = \frac{w^k}{\theta_d}$$

The left side of this equation we have plotted as a function of  $T_o$ . For  $e_{om}$  the values given by Robitzsch<sup>3</sup> were used (Table A). Since these values are given in millimeters instead of millibars and since  $P$  is equal to 1000 mb, equation (18) may be written

$$(19) \quad \left(\frac{4}{3} \frac{m_w}{m_d}\right)^k \cdot \frac{e_{om}^k}{T_o} = \frac{W^k}{\theta_d}, \quad (e_{om} \text{ in mm Hg})$$

where  $W$  represents the number of grams of water vapor associated with one *kilogram* of dry air. Equation (19) may then be solved graphically. For the sake of convenience both members may be multiplied by  $10^k$ ,  $100^k$  or  $1000^k$  and the values of the quantities  $N^k$  taken from a set of tables calculated by T. H. Doerr.<sup>4</sup>

<sup>3</sup> Robitzsch, op. cit. in fn. 2.

<sup>4</sup> Doerr, T. N., Tables of 0.288th Powers, *Quarterly Journal of the Royal Meteorological Society*, Vol. 47, pp. 196-202, London, 1921.

## THE EQUIVALENT-POTENTIAL TEMPERATURE

It is now easy to determine the equivalent-potential temperature. The characteristic differential equation of the pseudo-adiabatic rain stage<sup>5</sup> is, in our notations,

$$(20) \quad c_p \frac{dT}{T} - \frac{R}{m_d} \frac{dp_d}{p_d} + d \left( \frac{Lw}{T} \right) = 0.$$

We may at once integrate the above equation. Thus

$$(21) \quad c_p \log T - \frac{R}{m_d} \log p_d + \frac{Lw}{T} = C. \quad (C = \text{constant.})$$

From the definition of the partial potential temperature (6) we find

$$(22) \quad c_p \log \theta_d - c_p \log T = \frac{R}{m_d} \log P - \frac{R}{m_d} \log p_d.$$

The characteristic equation may therefore be written

$$(23) \quad c_p \log \theta_d - \frac{R}{m_d} \log P + \frac{Lw}{T} = C.$$

As the pressure decreases towards zero and the water drops out, the potential temperature of the element asymptotically approaches the equivalent-potential temperature. Thus

$$(24) \quad c_p \log \theta_E - \frac{R}{m_d} \log P = C$$

and, consequently,

$$(25) \quad c_p \log \theta_E = c_p \log \theta_d + \frac{Lw}{T}$$

or

$$(26) \quad \theta_E = \theta_d e^{\frac{Lw}{c_p T}} \quad (e = \text{base of natural logarithms})$$

for any point on the pseudo-adiabatic curve. Obviously  $\theta_E$  is constant along a pseudo-adiabatic line.

If the element is unsaturated, having a partial potential temperature  $\theta_d$  and a vapor content  $w$ , its equivalent-potential temperature is evidently obtained from

$$(27) \quad \theta_E = \theta_d e^{\frac{L_o w}{c_p T_o}},$$

where  $L_o$ , the heat of condensation, is taken at the temperature  $T_o$ . The latter quantity represents the temperature at the condensation level; it depends upon  $w$  and  $\theta_d$  and may be calculated by means of (19). Thus, to obtain the equivalent-potential temperature

<sup>5</sup> Humphreys, W. J., *Physics of the Air*, Second Edition, p. 257, McGraw-Hill Book Company, New York and London, 1929. The equation given by Humphreys differs slightly from the exact form derived by J. Fjeldstad, but the discrepancy between the two is small. Humphreys' formula has the advantage of greater simplicity. Fjeldstad's equation is given in *Geofysiske Publikasjoner*, Vol. III, No. 13, Oslo, 1925.

from the partial potential temperature a correction term  $\Delta\theta_E$  has to be added which is given by

$$(28) \quad \Delta\theta_E = \theta_d \left( e^{\frac{L_o w}{c_p T_o}} - 1 \right).$$

$\Delta\theta_E$  is tabulated in Table C. It was mentioned above that  $L_o$ , the heat of condensation, varies with the temperature  $T_o$ . This variation<sup>6</sup> may be expressed in the form

$$(29) \quad L_o = 606.5 - 0.695 (T_o - 273).$$

In calculating  $\Delta\theta_E$  for different values of  $w$  and  $\theta_d$  we combined (19) and (29) and determined graphically the value of  $\frac{L_o}{c_p T_o}$  for different values of  $w$  and  $\theta_d$ .

### EQUIVALENT TEMPERATURE

The equivalent temperature  $T_E$  we shall here define as the temperature which a particle of air would assume if it were lifted pseudo-adiabatically until all its moisture had been removed and then were brought back dry-adiabatically to its original dry air pressure. From this definition it follows that

$$(30) \quad \frac{T_E}{\theta_E} = \left( \frac{p_d}{P} \right)^k = \frac{T}{\theta_d}$$

or

$$(31) \quad T_E = T \cdot \frac{\theta_E}{\theta_d}$$

Thus the equivalent temperature stands in the same relation to the equivalent-potential temperature as the ordinary temperature to the partial potential temperature. The product of  $c_p$  in the difference between the equivalent temperature and the actual temperature,

$$(32) \quad \Delta T_E = T \left( e^{\frac{L_o w}{c_p T_o}} - 1 \right),$$

may in a way be said to measure the latent heat of the water vapor originally present in the air. The quantity  $\Delta T_E$  we shall call the equivalent temperature difference. It is well to notice that the equivalent temperature difference defined above is a function not only of the temperature and the water vapor content but also of the pressure; analytically this is apparent from the fact that a change in pressure affects a change in the temperature  $T_o$  at which condensation is reached.

It is readily seen that the equivalent temperature difference may be written in the form

$$(33) \quad \Delta T_E = \frac{T}{\theta_d} \cdot \Delta\theta_E$$

and calculated from Table C. Frequently the moisture content of the air, and therefore the exponent in (32) is so small that it is permissible to write

$$(34) \quad \Delta T_E = \frac{T}{T_o} \frac{L_o w}{c_p}.$$

<sup>6</sup> Robitzsch, op. cit. in fn. 2.

The expressions for the equivalent temperature difference given in (33) and (34) suffer from the disadvantage that they depend upon three variables,  $w$ ,  $T$  and  $p_d$  (or  $f$ , the relative humidity). However, the ratio  $\frac{T}{T_0}$  will be close to unity if the relative humidity of the air particle is high. Then (34) reduces to

$$(35) \quad \Delta T_E = \frac{L_0 w}{c_p}, \quad \left( \because T_E = T + \frac{L_0 w}{c_p} \right)$$

which expression, if the slight variation of the heat of condensation ( $L_0$ ) with temperature be disregarded, depends on one variable only, namely, the water vapor content  $w$ . The approximate expression (35) is identical with the definition given by Robitzsch<sup>7</sup> for the equivalent temperature difference. The advantage of the strict definitions of equivalent-potential temperature and equivalent temperature set forth in the preceding discussion may be found in the fact that they are based upon well-defined processes by means of which either of these temperatures may be realized.

TABLE 1  
 $w = 10 g^r/kg$      $t = +16^\circ C = 289^\circ A$     ( $e_m = 18.174$  mb.)

$p_d$	$f\%$	$\theta_d$	$\theta_E$	$T_0$	$\Delta_1 = T \left( e^{\frac{L_0 w}{c_p T_0}} - 1 \right)$	$\Delta_2 = \frac{T}{T_0} \frac{L_0 w}{c_p}$	$\Delta_3 = \frac{Lw}{c_p}$
1000	88	289.0	315.1	287	26.1	25.0	25.2
900	80	297.9	325.0	285	26.3	25.2	25.2
800	71	308.2	336.6	283	26.6	25.5	25.2
700	62	320.3	350.1	280	26.9	25.7	25.2

TABLE 2  
 $w = 10 g^r/kg$      $t = +26^\circ C = 299^\circ A$     ( $e_m = 33.604$  mb.)

1000	48	299.0	326.3	284	27.3	26.1	25.2
900	43	308.2	336.6	282	27.6	26.3	25.2
800	38	318.8	348.5	280	27.8	26.6	25.2
700	33	331.3	362.6	278	28.2	26.9	25.2

TABLE 3  
 $w = 5 g^r/kg$      $t = +26^\circ C = 299^\circ A$     ( $e_m = 33.604$  mb.)

1000	24	299.0	313.1	272	14.1	13.8	12.6
900	22	308.2	322.9	271	14.3	14.0	12.6
800	19	318.8	334.2	269	14.4	14.1	12.6
700	17	331.3	347.4	267	14.5	14.2	12.6

In the calculation of  $\Delta_3$  it was assumed that  $L = 606.5$ .

<sup>7</sup> Robitzsch, op. cit. in fn. 2.

It is easy to show that the variation of  $\Delta T_E$  with pressure is slight and generally of the same order of magnitude as the errors due to inaccuracies in the reading of the relative humidity and therefore in the calculated value of the water vapor content  $w$ . As an illustration some equivalent temperature differences have been calculated and reproduced in tables 1, 2, 3. There is a slight increase of the equivalent temperature difference with decrease of pressure, but in all cases the variation caused by a drop in pressure from 1000 mb to 700 mb is less than four per cent of the total equivalent temperature difference.

The variation of the equivalent temperature difference with temperature can be seen from a comparison of table 1 with table 2, for corresponding pressures. Other quantities being equal, the equivalent temperature difference increases with increasing temperature, but the variation in  $\Delta T_E$  corresponding to a temperature increase of  $10^\circ$  is in all tabulated cases less than five per cent of the total value of  $\Delta T_E$ . Thus also the variation with temperature may generally be disregarded.

It follows from the preceding discussion that the variation of the equivalent temperature difference with temperature and pressure is such that this difference, *ceteris paribus*, is greater the lower the relative humidity. To understand this variation we must first consider the temperature at the condensation level. This temperature is determined by (19), which equation after substitution of  $p_d$  and  $T$  for  $\theta_d$  may be written

$$(19b) \quad C \cdot \frac{e_{om}^k}{T_o} = \frac{w^k \cdot p_d^k}{T},$$

$C$  denoting a constant. The left side of this equation increases steadily with  $T_o$ . Thus, for a constant water vapor content  $w$ , the condensation temperature  $T_o$  increases with increasing initial pressure ( $p_d$ ) of the element and decreases with increasing initial temperature  $T$ .

Now consider the following simplified process. A particle containing one gram of dry air and  $w$  grams of water vapor is expanded adiabatically from  $(p_d, T)$  to the condensation point  $(p_{od}, T_o)$ . A definite, very small fraction of the moisture content,  $\Delta w$ , is then allowed to condense, precipitate and raise the temperature of the element by the amount  $\Delta T_o = \frac{L_o \Delta w}{c_p}$ . Thereupon the element is brought back, dry-adiabatically, to the

original pressure  $p_d$ . To obtain the equivalent temperature difference  $\Delta T_E$  for this condensation process it is necessary to calculate the amount of energy gained in the two adiabatic processes involved. This amount is obviously equal to

$$-c_v (T - T_o)$$

for the expansion and

$$c_v \left[ (T + \Delta T_E) - (T_o + \Delta T_o) \right]$$

for the compression. From Poisson's equation it follows that these two quantities may be written

$$-c_v T_o \left[ \left( \frac{p_d}{p_{od}} \right)^k - 1 \right]$$

and

$$c_v (T_o + \Delta T_o) \left[ \left( \frac{p_d}{p_{od}} \right)^k - 1 \right].$$

The algebraic sum of these two quantities,

$$c_v \Delta T_o \left[ \left( \frac{p_d}{p_{od}} \right)^k - 1 \right],$$

represents a gain and is stored up by the element as heat. By definition, the equivalent temperature difference is equal to the sum of the temperature increase due to condensation,

$$\Delta T_o = \frac{L_o \Delta w}{c_p},$$

and the temperature increase brought about mechanically,

$$\Delta T_o \left[ \left( \frac{p_d}{p_{od}} \right)^k - 1 \right].$$

Thus

$$\Delta T_E = \Delta T_o \left( \frac{p_d}{p_{od}} \right)^k = \frac{T}{T_o} \cdot \Delta T_o$$

Now, if the initial pressure is kept constant while the initial temperature is raised, the temperature and therefore also the pressure at the condensation level decreases ( $w$  is all the time constant). Thus the net gain of work increases and the equivalent temperature difference increases. In the second place, assume the initial temperature  $T$  to remain constant, while the initial pressure is decreased. The condensation temperature  $T_o$  again decreases. It is easily seen that the work of compression as well as the work of expansion increases. The equivalent temperature difference increases as may be seen immediately

from the expression  $\Delta T_E = \frac{T}{T_o} \cdot \Delta T_o$ .

Thus the cause of the variation of the equivalent temperature difference with relative humidity lies in the fact that this difference, as here defined, measures not only the latent heat of the water vapor present but also a certain amount of work done by the surroundings on the element. This amount of work increases with decreasing relative humidity.

#### EQUIVALENT-POTENTIAL TEMPERATURE DIAGRAMS

A graphical representation of the equivalent-potential temperature is given in Plate I. The water vapor content is plotted along the horizontal axis. The partial potential temperature is set off, logarithmically, on the vertical axis. Since the latter quantity is connected with the specific entropy of dry air by the formula

$$(36) \quad s = c_p \log \theta_d$$

linear distances on the vertical axis represent differences in specific entropy.

The lines of constant equivalent-potential temperature ( $\theta_E$ -lines) have been drawn with two degree intervals. The incomplete section at the bottom of the diagram represents saturated air at pressures higher than those normally occurring at sea level ( $>1100$  mb).

In this diagram *an unsaturated element of air undergoing adiabatic expansion will be represented by a characteristic point ( $w, \theta_d$ ), which will remain stationary until the level of condensation is reached.* If the further expansion of the particle takes place pseudo-adiabatically, its characteristic point will move upward along the line of constant equivalent-potential temperature ( $\theta_E$ -line) through the starting point. If the element is again compressed, its characteristic point will remain stationary at the last point reached during the preceding expansion, since no liquid water is available for evaporation and the compression therefore takes place dry-adiabatically.

### CHARACTERISTIC CURVES

Temperature, pressure and humidity data obtained through aerological soundings in a given vertical air column may easily be converted into partial potential temperatures and water vapor contents by means of formulae (1) and (6). It should be noticed that this conversion can be made without a preceding evaluation of the altitudes corresponding to the various pressure levels. If the partial potential temperatures and water vapor contents for this particular sounding are plotted against each other on the equivalent-potential temperature diagram, one obtains the *characteristic curve* for this air column. Below some of the general properties of these curves are derived.

A thin horizontal stratum, within which the moisture content varies from  $w$  to  $w+dw$  and the potential temperature from  $\theta_d$  to  $\theta_d+d\theta_d$  may be represented on the equivalent-potential temperature diagram by a short straight line element. The layer may be compressed or expanded or it may be stretched or contracted laterally so that its vertical depth changes, but as long as these various changes take place adiabatically and no condensation occurs, the water vapor content and the partial potential temperature of each individual point will obviously remain unchanged. Thus the characteristic curve of the given infinitesimal layer will remain the same even though the latter may be stretched until its thickness becomes a fraction of what it originally was. This result holds also for finite vertical air columns consisting of a great number of superimposed very thin strata, each undergoing a different change of the type described above. Thus we find the following law:

I. *The characteristic curve for a vertical air column of finite height remains invariant with respect to arbitrary adiabatic deformations provided no condensation (or evaporation) takes place and provided no new strata are introduced or others made to disappear during the deformation of the original air column.*

*and if there is  
no mixing*

It is obvious that the invariance of the characteristic curve makes it an exceedingly useful tool in following the displacement of individual air masses across the synoptic chart. There are other elements which may be used for the same purpose (for instance, atmospheric suspensions of various kinds) but the *two quantities selected above ( $\theta_d$  and  $w$ ) are the only ones which result in a characteristic curve of definite dynamic significance.* A similar method of identification, based on the construction of so-called ST-diagrams, has been in use in physical oceanography for years. Salinities and temperatures for corresponding depths are plotted against each other. In the diagram thus obtained, salinity obviously takes the place occupied by the moisture content in the atmospheric characteristic curve, since both indicate the percentual composition of the medium which is being studied. In the oceanographic problem there is no compressibility to consider; therefore the ordinary temperature may be used in place of the potential temperature. With the aid

of ST-diagrams Helland-Hansen<sup>8</sup> has been successful in establishing the Mediterranean origin of certain strata in the Atlantic, west of Gibraltar.

Next it may be asked how the characteristic curve varies from one part to another within a given air mass. As long as the air mass is strictly homogeneous, horizontally, the characteristic curve will obviously remain the same from one vertical air column to another. It was pointed out above that the characteristic curve for a given stratum is independent of its thickness. It follows that the characteristic curve will remain the same throughout the air mass even though the individual strata may vary in thickness from vertical to vertical (station to station), *provided* the succession of strata is exactly the same along each vertical. This property of the characteristic curves may also be stated in the following manner:

II. *The characteristic curves for a given air mass will have overlapping parts which are identical provided the surfaces of constant partial potential temperature and the surfaces of constant water vapor content coincide throughout the air mass.*

#### STRATIFICATION

It is well known that if a given air layer of finite thickness is thoroughly stirred and if the amount of moisture present is so small that no condensation takes place, then the layer will gradually assume an adiabatic temperature lapse rate while its water vapor content per unit mass ( $w$ ) approaches a constant value. This development results from the effect of turbulence in carrying water vapor and potential temperature in the direction of the gradient of these elements. The above result may be expressed in the following law.

III. *The characteristic curve for a thoroughly stirred unsaturated layer limited above and below by planes through which no transport of heat or water vapor takes place, reduces to a point.*

A stable air layer characterized by a number of inversions and a rapid decrease of water vapor content upward, obviously has an elongated characteristic curve. To obtain a picture of the stratification of a given air column from its characteristic curve one may interpolate, from the original data, the pressures, temperatures, and relative humidities for 0.5, 1, 2, 3 . . . km. elevation above sea level and convert these values into water vapor contents ( $w$ ) and partial potential temperatures ( $\theta_a$ ). It is then possible to mark, on the characteristic curve, the points corresponding to the above standard elevations. These points shall be referred to below as *standard characteristic points*. The preceding discussion may be summarized in the following law:

IV. *The more crowded the standard characteristic points, the more homogeneous (stirred) the air mass; the further apart these points are, the more pronounced is the stratification.*

It is necessary to consider the effect of stirring also in those cases when condensation occurs. If the layer under consideration is limited above and below by planes through which no transport of water vapor or realized heat takes place, stirring will first bring about condensation in a narrow layer next to the top. Within this layer a water vapor

gradient upward ( $\frac{dw}{dz} < 0$ ) is established which, because of the turbulent stirring, gradu-

ally spreads downward into the unsaturated strata. Thus an upward flow of water vapor and therefore also of latent heat is established.

<sup>8</sup> Helland-Hansen, B. and Nansen, F., The Eastern North Atlantic, *Geofysiske Publikasjoner*, Vol. IV, No. 2, Oslo, 1925. For additional references see Defant, A., *Dynamische Ozeanographie*, p. 95, *Einführung in die Geophysik*, III, Springer, Berlin, 1929.



The condensation and precipitation of liquid water at the upper boundary is attended by a liberation of latent heat and therefore also by an increase of the partial potential temperature. Thus a downward gradient of the partial potential temperature and consequently a downward flow of realized heat is established. In the particular case under discussion complete equilibrium is not reached until so much liquid water has left the system that the water vapor barely reaches saturation at the top and  $w$  and  $\theta_d$  have constant values throughout the whole column.

However, long before this equilibrium is reached a practically steady state develops in which the upward flow of latent heat is approximately equal to and balanced by the turbulent downward transport of realized heat. Per unit area and time the upward flow

of latent heat has the value  $-cL \frac{dq}{dz}$ ,  $c$  representing the "eddy conductivity" for water

vapor of the air layer under discussion and  $q$  as before the specific humidity. This value is obtained by multiplying the well-known expression for the transport of water vapor<sup>9</sup> in the latent heat of water vapor (considered constant). The transport of latent heat is, under the conditions here assumed, constant from the lower boundary of the layer under consideration and up to the condensation level.

The turbulent flow of realized heat is of a more complicated nature. On account of the release of latent heat above, the air column must have a stable stratification. The constant stirring of the layer then requires a steady supply of work which is continually being transformed into heat. This heat is added to the downward flow of heat emanating from the condensation level. Therefore, the heat transport varies somewhat from level to level and has its maximum below. A detailed analysis of this question is beyond the scope of the present paper; it has been treated by L. F. Richardson.<sup>10</sup> Equalizing the downward flow of realized heat at the condensation level with the upward flow of latent heat and using Richardson's expression for the eddy-flux of heat we find, just below the condensation level,

$$(37) \quad c \cdot c_p \frac{T}{\theta} \frac{d\theta}{dz} = -L \cdot c \cdot \frac{dq}{dz}$$

In this equation the two  $c$ 's introduced represent the eddy conductivities for heat and water vapor respectively. It is generally assumed that these coefficients are identical; they may then be dropped from the equation just given. If  $T$  is regarded as constant ( $T=T_0$ ) and if furthermore, the small differences between  $\theta$  and  $\theta_d$  and between  $w$  and  $q$  are disregarded we find, in the vicinity of the condensation level and, with a fair degree of accuracy, throughout the whole air column below,

$$\frac{d \log \theta_d}{dz} + \frac{L}{c_p T_0} \frac{dw}{dz} = 0$$

or

$$\theta_d e^{\frac{Lw}{c_p T_0}} = \text{constant} = \theta_E$$

<sup>9</sup> Richardson, L. F., *Weather Prediction by Numerical Process*, pp. 65-71, Cambridge University Press, 1922.

<sup>10</sup> Richardson, op. cit. in fn. 9, pp. 70-71.

V. The equivalent-potential temperature remains approximately constant also within the lower unsaturated layer, the water vapor content decreasing upward and the partial potential temperature increasing upward in such a fashion that the upward flow of latent heat and the downward flow of realized heat balance each other.

Verifications of this law will be found in chapter III.

We have thus found, on theoretical grounds, that thoroughly stirred strata, regardless of their degree of saturation, necessarily are characterized by a fairly constant equivalent potential temperature, *i.e.*, their characteristic curves must follow the  $\theta_E$ -lines in the equivalent-potential temperature diagram.

In case of a *saturated* air layer in pseudoadiabatic equilibrium it is easily shown that the standard characteristic points are distributed in a definite fashion along the characteristic curves (here coinciding with  $\theta_E$ -lines). That such must be the case follows from the fact that during a pseudoadiabatic expansion there is a definite relation between pressure, (partial) potential temperature and water vapor content. Combining this relation with the hydrostatic equation we may eliminate the pressure and insert instead the altitude.

The relation between pressure and partial potential temperature referred to above makes it possible for us to enter the isobaric lines corresponding to pseudoadiabatic equilibrium on the equivalent-potential temperature diagram. This has been done on the diagram reproduced in Plate I. At first the isotherms were calculated with the aid of (19). For each isotherm the value of the left member of this equation was determined. Then  $\theta_d$  was determined for different values of the moisture content ( $W$ ). Having entered the isotherms on the equivalent-potential temperature diagram, the isobars are easily plotted by means of Poisson's equation,

$$\theta_d = T \cdot \left( \frac{P}{p_d} \right)^k,$$

the factor  $\left( \frac{P}{p_d} \right)^k$  being constant along any isobar. As a check, certain isobars were re-calculated from

$$W = \frac{622}{p_d} e_m,$$

$e_m$  being a function of the temperature alone. Both isobars and isotherms are printed in red on the diagram.

To obtain the distribution of the standard characteristic points it is convenient to start from the expression for the variation in geopotential on the temperature-entropy diagram<sup>11</sup>, namely

$$(38) \quad g d z = c_p T \frac{d\theta}{\theta} - c_p d T$$

in which expression the partial potential temperature may be introduced from

$$(11) \quad \theta_d = \theta \left( 1 + \frac{m_d}{m_w} w \right)^k.$$

<sup>11</sup> Shaw, Sir Napier, Manual of Meteorology, Volume III, The Physical Processes of Weather, p. 298, Cambridge University Press, 1930.

Thus

$$(39) \quad g dz = c_p T \frac{d\theta_d}{\theta_d} - c_p dT - T \frac{R}{m_w} \frac{dw}{1 + \frac{m_d}{m_w} w},$$

the third term in the right number usually being small in comparison with the two first terms. To eliminate  $dT$  we may use the differential form of the equation for pseudo-adiabatic equilibrium

$$(40) \quad \frac{d\theta_d}{\theta_d} + d\left(\frac{Lw}{c_p T}\right) = 0.$$

Here  $T$  has been introduced instead of  $T_0$  since we are dealing with a saturated atmosphere. Assuming  $L$  and  $c_p$  to be constant, this gives

$$(41) \quad \frac{dT}{T} = \frac{dw}{w} + \frac{c_p T}{Lw} \frac{d\theta_d}{\theta_d}.$$

Therefore,

$$(42) \quad g dz = c_p T \left[ \frac{d\theta_d}{\theta_d} - \frac{c_p T}{Lw} \frac{d\theta_d}{\theta_d} - \frac{dw}{w} - \frac{R}{m_w c_p} \frac{dw}{1 + \frac{m_d}{m_w} w} \right]$$

or, after some substitutions,

$$(43) \quad g dz = c_p T \left[ \frac{d\theta_d}{\theta_d} \frac{\log \theta_E - \log \theta_d - 1}{\log \theta_E - \log \theta_d} - \frac{dw}{w} \frac{1 + \frac{m_d}{m_w} (1+k) w}{1 + \frac{m_d}{m_w} w} \right].$$

The air being saturated, one may eliminate  $T$  from the above formula by means of (27).

The above expression for the geopotential may be simplified if it is remembered that, very nearly,

$$(44) \quad \frac{1 + \frac{m_d}{m_w} (1+k) w}{1 + \frac{m_d}{m_w} w} \approx 1 + \frac{R}{m_w c_p} w \approx 1.$$

Thus, approximately,

$$(45) \quad g dz = c_p T \left[ \frac{d\theta_d}{\theta_d} \frac{\log \theta_E - \log \theta_d - 1}{\log \theta_E - \log \theta_d} - \frac{dw}{w} \right].$$

VI. If an air column is in pseudoadiabatic equilibrium its standard characteristic points are separated by definite intervals which may be once for all entered upon the equivalent-potential temperature diagram.

#### STABILITY

It is customary to indicate the stability of a given air layer by the behaviour of a small atmospheric element which has been lifted, adiabatically or pseudoadiabatically, from its equilibrium position, while the atmosphere as a whole remains at rest. In this way one is led to distinguish between three different degrees of stability.

(1) *Absolute Instability*. The temperature lapse rate is greater than the dry-adiabatic ( $\frac{g}{c_p}$ ). In this case, regardless of the moisture content of the air but assuming that viscous forces may be neglected, any small impulse is sufficient to set a particle of air into motion at an increasing speed away from its original equilibrium position.

(2) *Conditional Instability*. The temperature lapse rate is less than the dry-adiabatic and higher than the condensation- or pseudoadiabatic lapse rate. In this case the atmosphere is stable with respect to impulses tending to displace air particles downward. If the relative humidity is high a sufficiently strong impulse upward may set an air particle in flight away from its original position. The magnitude of the impulse required decreases with increasing relative humidity and varies also with the values of the actual and condensation-adiabatic lapse rates at the point under consideration. Conditional instability is of no significance in case of very dry air since then the impulses required to upset the equilibrium are very great.

(3) *Absolute Stability*. The temperature lapse rate is less than the condensation adiabatic (pseudoadiabatic).

In the second and third cases the potential temperature increases with elevation, in the first case it decreases upward. It is, of course, possible to tell from the characteristic curve if the potential temperature (strictly speaking the partial potential temperature) decreases or increases with elevation, but it is not always possible to say whether the atmosphere is conditionally unstable or absolutely stable.

It is, however, feasible to discuss from another point of view the stability of a given air column by means of its characteristic curve. Assuming a normal vertical distribution of the water vapor content, that is, a decrease of  $w$  with altitude, we may distinguish between the three following cases.

1. The characteristic curve slopes down towards the lower left corner of the equivalent-potential temperature diagram. In this case the partial potential temperature decreases with elevation. This case represents *absolute instability* as described above.

2. The characteristic curve ascends to the left but its slope is less than that of the pseudoadiabatic lines; thus *the equivalent potential temperature ( $\theta_E$ ) decreases with elevation*. In this case a layer of *finite* depth, lifted by convection adiabatically, and eventually pseudoadiabatically towards higher levels, must gradually become absolutely unstable. The process is illustrated in figure 1. We may term this case *convective instability*. It is possible to estimate, from the equivalent-potential temperature diagram, the decrease in pressure required to bring about absolute instability. Convective instability may or may not be accompanied by conditional instability.

Figure 1 is part of an equivalent-potential temperature diagram on which has been plotted part of the characteristic curve for Broken Arrow on December 13, 1929. The layer to be analyzed extends from the 925 mb. to the 797 mb. isobaric surface (these values refer to the partial pressures of the dry air). The atmosphere is saturated from the base up to the 880 mb. level. At the top (797 mb.) the relative humidity is about 72 per cent. The whole layer is characterized by *convective instability* since the equivalent-potential temperature drops from about 334.5 at the lower boundary to about 227.5 at the upper boundary.

The top point of the layer falls on the 740 mb. isobar in the diagram. This indicates that the partial pressure of the dry air must be reduced adiabatically to 740 mb. in order to bring about condensation at the upper boundary of the layer under consideration.

This may be accomplished by a bodily lifting of the whole layer. Such lifting would reduce the pressures at the lower boundary and at the intermediate point by approximately the same amount as at the top, namely 57 mb. (The last statement is strictly true only with reference to the total pressure—dry air pressure plus partial pressure of water vapor—and then only when the removal of mass by precipitation may be neglected.) Since the lower part of the layer is saturated, the lifting results in a displacement of the corresponding characteristic points along the pseudoadiabatic lines to the proper pressures, 823 mb. and 868 mb. The characteristic point for the top surface remains stationary.

In its new position the layer is saturated throughout. It is easy to see that the difference in partial potential temperature between the two boundaries has decreased.

Additional bodily and pseudoadiabatic lifting will displace the characteristic curve of the layer in the manner indicated by the various positions on the diagram. Finally, when the top surface reaches the 300 mb. level, the partial potential temperature of the

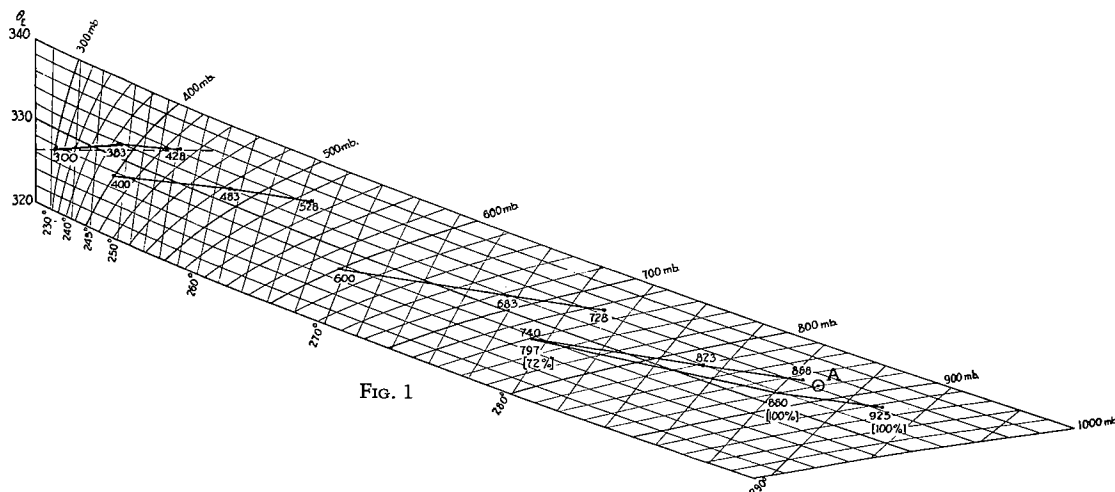


FIG. 1

upper boundary is lower than that of the intermediate point and about the same as that of the lower boundary, that is, even if its saturated state be disregarded, the layer is now absolutely unstable. The broken line on the diagram represents a constant partial potential temperature line.

A *saturated* layer is absolutely unstable whenever the equivalent-potential temperature decreases with elevation. This is easily shown graphically. At first it may be pointed out that the actual pressure in any point on a characteristic curve always must be equal to or higher than the value of pressure on the printed isobar going through the point in question, since the latter indicates the pressure at the level to which the point must be lifted in order to reach condensation. In the case of saturation, actual pressures and those indicated on the equivalent-potential temperature diagram are identical.

If we now return to the lower part of the layer analyzed in figure 1, it is easily seen that a small particle of air from the lower boundary (925 mb.), if displaced pseudoadiabatically to the 880 mb. level (point A), will have a higher partial potential temperature than the one prevailing in the surrounding atmosphere at this level. This is evident from the fact that the characteristic point of the surroundings (marked 880) and

