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# THE ELECTRICAL FIELD INDUCED BY OCEAN CURRENTS AND WAVES, WITH APPLICATIONS TO THE METHOD OF TOWED ELECTRODES

BY

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# PREFACE

THIS paper originated in an informal conference between the authors at Woods Hole in September, 1951. A preliminary draft, entitled "A manual of examples of the electrical effects of different types of water motion in the ocean," was circulated in May, 1952. Since then the paper has been revised and extended to its present form.

One of us (M. S. L. H.) is indebted to the Commonwealth Fund, New York, for a Fellowship which enabled him to study at the Woods Hole Oceanographic Institution and the Scripps Institution of Oceanography during 1951 and 1952. The work of the other two (M. E. S. and H. S.) was supported from funds of the Office of Naval Research, Contract No. N6onr-27701 (NR-083-004).

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# INTRODUCTION

THE purpose of this paper is to discuss the nature of the electrical field induced in the ocean by particular types of velocity distribution. It is believed that these examples will be helpful in the interpretation of measurements by towed electrodes in the sea.

The electrical field induced by waves and tidal streams, originally predicted by Faraday (1832), was first measured experimentally by Young, Gerrard and Jevons (1920),* who used both moored and towed electrodes in their observations. Recently, the technique of towed electrodes has been developed by von Arx (1950, 1951) and others into a useful means of detecting water movements in the deep ocean. While the method has been increasingly used, the problem of interpreting the measurements in terms of water movements has become of great importance. Two of the present authors have made theoretical studies (Longuet-Higgins 1949, Stommel 1948) dealing with certain cases of velocity fields, and Malkus and Stern (1952) have proved some important integral theorems. There seems, however, to be a need for a more extended discussion of the principles underlying the method, and for the computation of additional illustrative examples. This is all the more desirable since some of the theoretical discussions published previously have been misleading.

A fundamental difficulty is that the electrical field at any point depends not only on the local water velocity, but also on the electrical current-density, which is determined to a greater or less extent by the whole velocity field. Thus it is generally impossible to deduce the velocity at the surface from measurements of the electrical field taken only at the surface. However, such measurements may provide very useful information when taken in conjunction with other observations; and, in some cases when the form of the streams is known (e.g., if it is known that they are mainly confined to a shallow surface layer), a more or less accurate estimate of the surface velocity can be made. In this paper we attempt to evaluate the degree of uncertainty involved in such estimates, by computing the electrical potential in specific cases and comparing it with the actual surface velocity.

We begin, in Chapter 1, by giving an account of the principle of induction, with simple circuit analogies to illustrate the relation between observations by stationary and moving electrodes, in moving water. The theory of towed electrodes is set out in detail. Next, in Chapter 2, we give a qualitative discussion of an important practical case — that of an infinitely extended stream of finite width in water of uniform depth. The influence of the conductivity of the bottom and of variations in water velocity with depth are roughly evaluated. In Chapter 3 we give the field equations necessary for an exact analysis, and discuss two-dimensional motion in particular. In Chapters 4, 5 and 6 we work out some particular solutions for the cases of a sinusoidal stream, and streams of rectangular and elliptical cross-section, respectively. The solution for the rectangular section is of special interest, since it can be used to build up the solution for a stream of arbitrary cross-section, to any degree of approximation. Chapter 7 deals with the field induced by sea waves, and of the effect of the magnetic field on them. Fluctuations of the velocity or of the magnetic field with time, and variations in the electrical conductivity of the oceans are considered in Chapter 8. The conclusions are summarised in Chapter 9.

Parts of the paper (for example Chapters 1 and 2) are mainly expository, and may serve as an introduction to the subject. For those not wishing to go into the mathematical details of the solutions, a general picture will be provided by Chapters 1, 2, 8 and 9.

* Tidally induced potentials had previously been noticed in submarine telegraph cables; for references see Longuet-Higgins (1949).

# 1. GENERAL PRINCIPLES

## 1.1 Faraday's Law of Induction

Suppose that a simple closed circuit $C$ moves in the presence of a magnetic field $\mathbf{H}$ (see Figure 1). By Faraday's law of induction an electromotive force (e.m.f.) is induced in $C$ equal to minus the

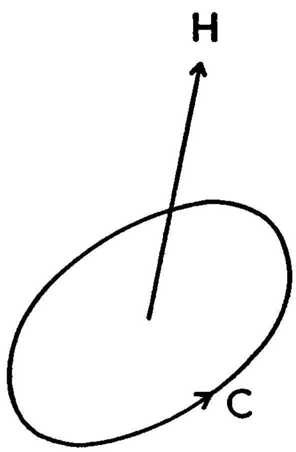


Figure 1

rate of change of magnetic flux through $C$. This change of flux may be due to either of two causes: a variation of the magnetic field with time, or the movement of the circuit. It will be supposed in the following that $\mathbf{H}$ is constant (whether this is permissible in the case of the Earth's magnetic field will be considered later). Then the rate of change of flux through $C$ is due solely to the motion of $C$ in the field, and is equal to the rate at which

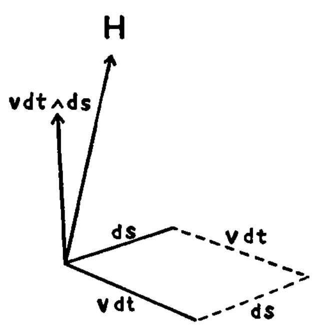


Figure 2

elements of the circuit are, on the whole, cutting lines of magnetic flux. Now an element ds of the circuit, moving with velocity $\mathbf{v}$, in a short time dt sweeps out an area whose projection in the direction of $\mathbf{H}$ is $(\mathrm{v}dt \wedge ds) \cdot \mathbf{H}$ (see Figure 2). This equals $-\mathbf{v} \wedge \mathbf{H} \cdot ds\, dt$, so that the rate of cutting of lines of magnetic flux by the circuit element ds is $-\mathbf{v} \wedge \mathbf{H} \cdot ds$, and the total e.m.f. $E$ developed in the circuit $C$ is the integral of $\mathbf{v} \wedge \mathbf{H} \cdot ds$ taken round $C$:

$$E = \int_C \mathbf{v} \wedge \mathbf{H} \cdot ds \qquad 1.1.1$$

Thus an alternative expression of Faraday's law is to say that at each point in space the motion induces an e.m.f. per unit distance represented by the vector $\mathbf{v} \wedge \mathbf{H}$. This vector is perpendicular to $\mathbf{v}$ and $\mathbf{H}$, and is proportional to both $\mathbf{v}$, $\mathbf{H}$ and the sine of the angle between them.

In the case of a simple wire circuit $C$ the e.m.f. induced in the wire would cause a current $I$ to flow such that

$$E = RI, \qquad 1.1.2$$

where $R$ is the resistance of the wire. In the case of continuous media we suppose that there is an electrical current-density $\mathbf{i}$ such that the circulation of $\rho\mathbf{i}$ round any circuit $C$ just equals the circulation of the e.m.f. $\mathbf{v} \wedge \mathbf{H}$ ($\rho$ being the electrical resistivity). In other words the circulation of $(\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i})$ round $C$ vanishes.

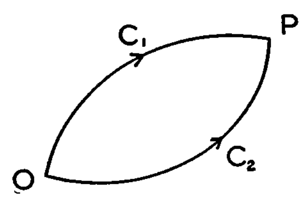


Figure 3

It follows that we may define an electrical potential $\phi$, at each point $P$ in space, as the integral of $(\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i})$ along any path from a fixed point $O$ to the point $P$:

$$\phi\,(P) = \int_O^P (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \cdot ds. \qquad 1.1.3$$

The path of integration is arbitrary; for if $C_1$ and $C_2$ are any two paths from $O$ to $P$ (see Figure 3) the integral round the closed circuit consisting of $C_1$ and $-C_2$ is zero; the integral along $C_2$ therefore equals the integral along $C_1$. From 1.1.3 we have also

$$\nabla\phi = \mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i} \cdot \qquad 1.1.4$$

The potential $\phi$ is the same as would be measured by a stationary potentiometer connected between $O$ and $P$. For suppose this to consist of an external wire circuit $C'$ (see Figure 4) which contains a cell of voltage $E'$ so adjusted as to reduce the current in $C'$ to zero. If $C$ is any circuit from $O$

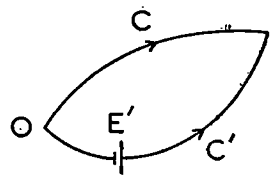


FIGURE 4

to $P$ through the fluid, the total e.m.f. developed in the circuit $C$ and $-C'$ is

$$\int_{C-C'} \mathbf{v} \wedge \mathbf{H} \cdot \mathbf{ds} - E', \qquad 1.1.5$$

and this must equal

$$\int_{C-C'} \rho \mathbf{i} \cdot \mathbf{ds}. \qquad 1.1.6$$

But on $C'$ both $\mathbf{v}$ and $\mathbf{i}$ vanish. Hence on equating 1.1.5 and 1.1.6 we have

$$E' = \int_C (\mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i}) \cdot \mathbf{ds} = \phi(P). \qquad 1.1.7$$

It is essential to this argument that the potentiometer be stationary. If on the other hand the potentiometer circuit is in motion with velocity $\mathbf{v}'$, then the contribution to 1.1.5 from $C'$ does not vanish and we find

$$E' = \phi(P) - \int_{C'} \mathbf{v}' \wedge \mathbf{H} \cdot \mathbf{ds}. \qquad 1.1.8$$

By taking $P$ sufficiently close to $O$ we might obtain a measure of the potential gradient at $O$. For then we have for the stationary electrodes

$$E' = \phi(P) - \phi(O) = \nabla \phi \cdot \overline{OP} \qquad 1.1.9$$

(to first order in $\overline{OP}$). For the moving electrodes we should obtain

$$E' = (\nabla \phi - \mathbf{v}' \wedge \mathbf{H}) \cdot \overline{OP}, \qquad 1.1.10$$

so that the apparent gradient measured would be

$$\nabla \phi - \mathbf{v}' \wedge \mathbf{H}. \qquad 1.1.11$$

It should be noticed, however, that $\mathbf{v}' \wedge \mathbf{H}$ is not necessarily the gradient of a potential function as was incorrectly assumed by von Arx in Section 3 of his 1950 paper.

## 1.2 EXAMPLES

(a) Imagine a conducting rod $AB$ of length $L$ moving horizontally at right angles to itself, with velocity $V$, say, and let there be a uniform magnetic field of strength directed vertically upwards (see Figure 5). Then an e.m.f. will be induced in $AB$, of magnitude $VH$ per unit distance and directed from left to right as one faces in the direction of motion. Suppose the ends of the rod are connected to a stationary potentiometer $P$ by a rectangular circuit $BCDA$, as in Figure 5. The sides $BC$ and $DA$ of the rectangle are not cutting any lines of magnetic flux, and $CD$ is stationary. Since no electric current flows, the voltage measured by $P$ is simple $VHL$.

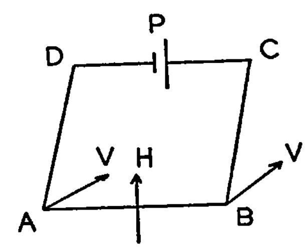


FIGURE 5

(b) Suppose now that the potentiometer, instead of being stationary, moves horizontally with the same velocity as $AB$. The e.m.f. induced in $CD$ will just balance that induced in $AB$, and $P$ will record zero voltage.

(c) Now let the potentiometer be stationary as in example (a), but suppose that the ends of the rod are made to move along two parallel conducting rails in electrical contact with one another as in Figure 6. If the resistance $R_1$ of the external path is low compared with $R$, the system will be

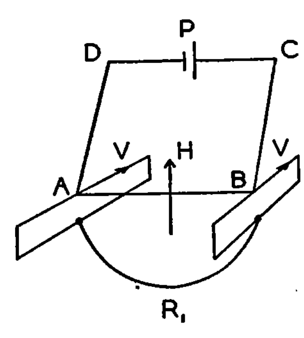


FIGURE 6

short-circuited: a current $I$ will flow in $AB$ (returning through $R_1$) which will reduce the voltage between $A$ and $B$ to zero. The additional voltage produced by the current must therefore be $-VHL$. If the ratio $R_1/R$ is at first large and then decreases to zero the voltage recorded by $P$

will gradually fall from its maximum value $VHL$ to zero.

(d) Assume the same arrangement as in (c) but with the potentiometer in motion as in (b). The rate at which the circuit $ABCD$ is cutting lines of magnetic force is, on the whole, zero. But a current $I$ still flows in $AB$. Thus when $R_1/R$ is small the potentiometer records a voltage $-VHL$, the negative of the voltage measured in example (a). Again, if $R_1/R$ gradually decreases to zero, the voltage will gradually increase (in absolute magnitude) from $O$ to $-VHL$.

(e) In examples (a) to (d) the moving rod may be replaced by a uniform stream of water moving in a long channel of rectangular cross-section and width $L$; the external path may be replaced by the walls of the channel (see Figure 7). If the walls of

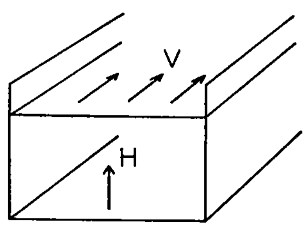


FIGURE 7

the channel are nonconducting, the potential difference between the sides of the channel will be $VHL$, if measured by a stationary potentiometer, or zero if measured by a potentiometer moving with the fluid. If on the other hand the walls of the channel are perfectly conducting the stationary potentiometer will measure zero voltage, and the moving potentiometer $-VHL$.

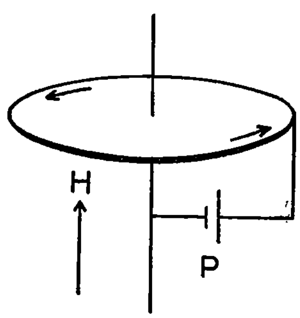


FIGURE 8

(f) *The Faraday disc.* Suppose a conducting circular disc spins about a vertical axis, in the presence of a vertical magnetic field (see Figure 8). The vector $\mathbf{v} \wedge \mathbf{H}$ is now directed radially outwards from the centre of the disc. However, because of the circular symmetry, no electrical current will normally be able to flow. Let a stationary

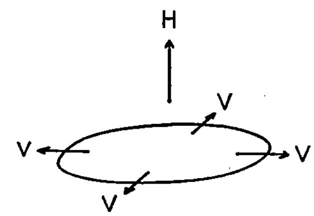


FIGURE 9

potentiometer be connected between the centre of the disc and its circumference (where there must be a brush or sliding contact). If $a$ is the radius and $\omega$ the angular velocity of the disc, a voltage $E'$, given by

$$E' = \int_0^a \omega r \, H \, dr = \tfrac{1}{2} \omega a^2 H, \qquad 1.2.1$$

will be recorded. This voltage could be entirely short-circuited by electrical contact between the rotating disc and a parallel, stationary disc of high conductivity. If the potentiometer were made to rotate with the moving disc, but without any connection to the stationary disc, the voltage measured would again be zero; but if the external connections were made, the measured voltage would be $-\frac{1}{2} \omega a^2 H$.

(g) In the previous example the disc may be replaced by a whirlpool or vortex, with circular symmetry about a vertical axis. If the vortex extends uniformly in the vertical direction there will be a potential difference (with regard to stationary electrodes) between points at different distances from the axis; but if the current is only very shallow in relation to its radius, and if the water below is stationary, much of this voltage may be short-circuited; this would be expected, for example, in the case of a shallow Ekman current.

(h) Imagine a circular wire circuit expanding radially in all directions in the presence of a magnetic field parallel to its axis of symmetry (Figure 9). The induced e.m.f. will be everywhere along the circuit. There will be no hindrance to the flow of electric current, which will therefore reduce the measured potential to zero.

(i) The comparable case for a continuous medium would be that of flow outwards from a line source, or of a symmetrical upwelling (see Figure 10). The electrical current-density would circulate freely round the axis, and the measured potential gradient would be zero.

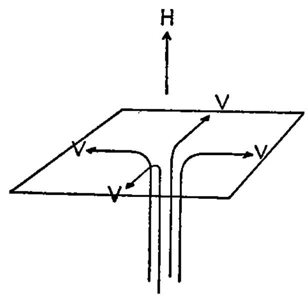


FIGURE 10

## 1.3 The Method of Towed Electrodes

Suppose two electrodes $A$ and $B$ are towed in line behind a ship (Figure 11). Let the velocity of the water be $\mathbf{v}$, assumed to be the same for both the ship and the electrodes. Then the direction of the vector $\overline{AB}$ joining the electrodes may be expected to be the same as that of the velocity $\mathbf{v}_1$ of the ship relative to the velocity of the water; this is given by

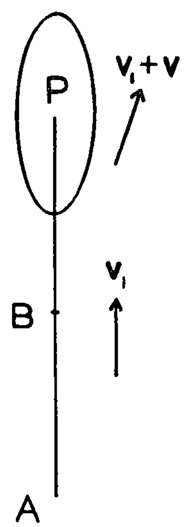


FIGURE 11

$$\mathbf{v}_1 = \mathbf{v}_p + \mathbf{v}_w \qquad 1.3.1$$

where $\mathbf{v}_p$ is the velocity due to the ship's propellors (which may be estimated) and $\mathbf{v}_w$ the additional velocity due to the wind (windage). The absolute velocity of the ship is

$$\mathbf{v} + \mathbf{v}_1 \qquad 1.3.2$$

which, if a steady state has been reached, will also be the absolute velocity of the electrodes. If the electrodes were stationary the voltage between them would be simply that due to the potential gradient in the direction of the electrodes, i.e.,

$$\nabla \phi \cdot \overline{AB}. \qquad 1.3.3$$

However, since the system is in motion with velocity $\mathbf{v} + \mathbf{v}_1$, the actual voltage recorded is, by 1.1.10,

$$[\nabla \phi - (\mathbf{v} + \mathbf{v}_1)] \cdot \overline{AB}. \qquad 1.3.4$$

But since $\overline{AB}$ is parallel to $\mathbf{v}_1$ we have

$$\mathbf{v}_1 \wedge \mathbf{H} \cdot \overline{AB} = 0, \qquad 1.3.5$$

that is to say, the electrode-line itself is cutting no lines of magnetic flux by virtue of being towed through the water. Thus the voltage measured is

$$[\nabla \phi - \mathbf{v} \wedge \mathbf{H}] \cdot \overline{AB}, \qquad 1.3.6$$

which is independent of the towing-velocity $\mathbf{v}_1$ (except in so far as this determines the direction of $\overline{AB}$). Since

$$\nabla \phi = \mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i} \qquad 1.3.7$$

the above may also be written

$$-\rho \mathbf{i} \cdot \overline{AB}. \qquad 1.3.8$$

*The electrodes therefore measure, in effect, the component of the electrical current-density* $-\rho\mathbf{i}$ *in the direction of the electrode-line* $\overline{AB}$.

Now in many cases the potential gradient is completely short-circuited, as for example in Section 1.2 (b), (d), (e), (h) and (i). We then have

$$\nabla \phi = 0 \qquad 1.3.9$$

and so

$$\rho \mathbf{i} = \mathbf{v} \wedge \mathbf{H}. \qquad 1.3.10$$

The voltage recorded is therefore

$$-\mathbf{v} \wedge \mathbf{H} \cdot \overline{AB}, \qquad 1.3.11$$

and is a direct measure of the component of velocity at right-angles to the electrode-line.

Although this case may be regarded as normal for currents in the deep ocean, it is essential to consider under what conditions $\nabla \phi$ may be expected to vanish, and what errors will be introduced when these conditions are not exactly satisfied. This will be done in later Sections.

It will be convenient to consider here one or two errors that may be introduced when the conditions are different from those assumed.

*Variation of water velocity with depth.* Suppose the water velocity varies with depth in such a way that the effective water velocity for the ship is different from the velocity $\mathbf{v}$ for the electrodes. Suppose the water velocity for the ship is $\mathbf{v} + \mathbf{v}_2$, so that the absolute velocity of the ship and of the electrodes is $\mathbf{v} + \mathbf{v}_1 + \mathbf{v}_2$. The vector $\overline{AB}$ may be expected to be parallel to $\mathbf{v}_1 + \mathbf{v}_2$ which is the velocity of the ship relative to the water velocity for the electrodes. Hence all the previous equations are valid if only $\mathbf{v}_1$ is replaced by $\mathbf{v}_1 + \mathbf{v}_2$. The direction of $\overline{AB}$ will be slightly altered; but the measured voltage will still depend only on the velocity $\mathbf{v}$ and the current-density $\mathbf{i}$ in the neighbourhood of the electrodes.

*Deflection of the electrode-line.* Suppose that the electrode-line $\overline{AB}$, instead of being parallel to $\mathbf{v}_1 + \mathbf{v}_2$, makes an angle with this direction (in a horizontal plane). Such an effect might be produced, for example, by a wind $\mathbf{W}$ acting at right-angles to the electrode-line (see Figure 12). Then

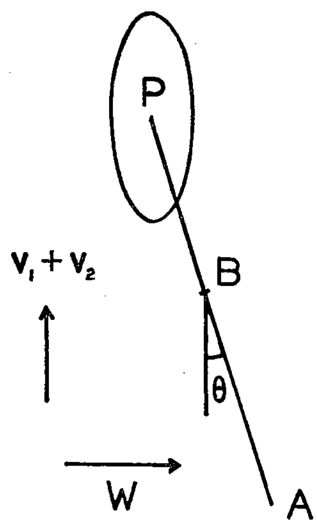


FIGURE 12

the direction of the measured component of potential gradient will again be slightly different; and there will also be a contribution from the term

$$-(\mathbf{v}_2 + \mathbf{v}_1) \wedge \mathbf{H} \cdot \overline{AB}. \qquad 1.3.12$$

The magnitude of this term is

$$V_0 H_z \cdot \overline{AB} \sin\theta, \qquad 1.3.13$$

where $V_0$ is the absolute value of $\mathbf{v}_1 + \mathbf{v}_2$ and $H_z$ is the vertical component of $\mathbf{H}$. The voltage acts in the same sense as that which would be produced by a water velocity

$$V_0 \tan\theta \qquad 1.3.14$$

flowing across the electrodes in the same sense as the deflecting wind.

Similarly if the cable droops, so that $A$ is lower than $B$ by an amount $AB \sin\theta'$, a signal will be recorded from the horizontal component of field $H_x$, and will be of magnitude $V_0 H_x \sin\theta'$.

*General Case.* We shall now discuss the general case where the water velocity varies horizontally and is not necessarily steady. The following analysis might apply, for example, to small eddies or to waves where length was not large compared with the electrode-separation $AB$.

If, in Figure 11, the ship and electrodes were stationary, the measured e.m.f. between them would be

$$\phi(B) - \phi(A) = \int_A^B (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \cdot d\mathbf{s}. \qquad 1.3.15$$

The motion of the cables induces an additional e.m.f.

$$-\int_{AP+PB} \mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s}, \qquad 1.3.16$$

the above integral being taken along the electrode cables. But between $B$ and $P$ the two cables follow the same paths though they are opposite in direction. Therefore 1.3.16. reduces to

$$-\int_{AB} \mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s} \qquad 1.3.17$$

and the measured e.m.f. $E'$ is

$$E' = \int_A^B (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \cdot d\mathbf{s} - \int_A^B \mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s}. \qquad 1.3.18$$

Now in general $\mathbf{v}'$ is different from $\mathbf{v}$, but if, as the cable is trailed through the water, it has no transverse motion relative to the water we shall have

$$\mathbf{v}' = \mathbf{v} + \mathbf{v}'' \qquad 1.3.19$$

where $\mathbf{v}''$ is parallel to $d\mathbf{s}$ and so

$$\mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s} = \mathbf{v} \wedge \mathbf{H} \cdot d\mathbf{s}, \qquad 1.3.20$$

since $\mathbf{v}'' \wedge \mathbf{H} \cdot d\mathbf{s}$ vanishes, or in other words the component of $\mathbf{v}'' \wedge \mathbf{H}$ along the cable contributes nothing to the rate at which it is cutting lines of magnetic flux. From 1.3.4. and 1.3.6. we have

$$E' = -\int_A^B \rho\mathbf{i} \cdot d\mathbf{s} \qquad 1.3.21$$

the path of integration being along the cable. In the case when all the induced e.m.f. is short-circuited we have

$$\rho\mathbf{i} = \mathbf{v} \wedge \mathbf{H} \qquad 1.3.22$$

and so

$$E' = -\int_A^B \mathbf{v} \wedge \mathbf{H} \cdot \mathbf{ds} \qquad 1.3.23$$

If the cable is deflected by wind, as in Figure 12, or if the path of the cable is curved, as in Figure 13, the cable may have a small transverse component relative to the water. In place of equation 1.3.19. we have, in general

$$\mathbf{v}' = \mathbf{v} + \mathbf{v}'' + \mathbf{v}''', \qquad 1.3.24$$

where $\mathbf{v}'''$ is perpendicular to ds. The additional e.m.f. due to $\mathbf{v}'''$ is given by

$$-\int_A^B \mathbf{v}''' \wedge \mathbf{H} \cdot \mathbf{ds}. \qquad 1.3.25$$

The extent of this "side-slipping" might be measured by towing the electrodes in a full circle, first clockwise and then anti-clockwise.

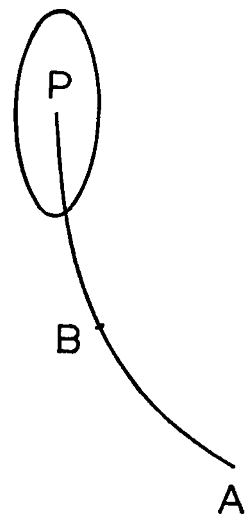


FIGURE 13

## 2. A LONG, STRAIGHT STREAM: QUALITATIVE TREATMENT

WE shall now consider some simple ideal cases of ocean currents, and investigate by rough physical reasoning the effect of the conductivity of the sea bed, and other factors, on the electrical potential. The methods used will mostly be approximate; some exact solutions are given later, in Chapters 4–6.

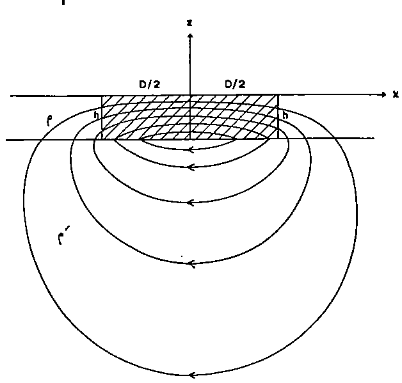


FIGURE 14. The electrical current-density due to $H_z$ (sketch only).

### 2.1 THE EFFECT OF A CONDUCTING SEA BED

Suppose that the water velocity is everywhere horizontal and parallel to the $y$-axis, and that its strength is independent of $y$. Let the depth of water be $h$, and let the resistivity of the water and of the "sea bed" be $\rho$ and $\rho'$, respectively. It will be assumed for the purpose of the discussion that both $\rho$ and $\rho'$ are uniform, though this may be far from true in the case of the actual ocean floor. However, it may certainly be assumed that $\rho << \rho'$.

Consider first the case when a rectangular block of water of width $D$ ($D >> h$) moves parallel to the $y$-axis with uniform velocity $V$, the remaining water being at rest (see Figure 14).

Assuming the magnetic field $\mathbf{H}$ to be uniform, we may consider separately the potentials induced by the three components $H_x$, $H_y$ and $H_z$. The magnetic component $H_y$, being parallel to the water velocity, induces no electrical field. In this and the following section we shall consider the field due to the vertical component $H_z$; the field due to $H_x$ will be considered in Section 2.3.

$H_z$ will induce a horizontal e.m.f. acting from left to right (see Figure 14), which will tend to cause an electric current to flow from left to right through the water, with a return current back through the sea bed. The system is analogous to the wire circuit shown in Figure 15, which consists of a cell of voltage $E$ and internal resistance $R_i$ connected in series to an external resistor $R_e$. The voltage $E$ corresponds to the total induced e.m.f. across the circuit, i.e. $VH_zD$; the internal resistance $R_i$ is of the order of $\rho D/h$; and, assuming that the electric current spreads downwards into the "sea bed" to a depth comparable with the width of the current, the external resistance is of the order of $\rho' D/D$, or $\rho'$. Thus

$$E \sim VH_zD; \quad R_i \sim \rho D/h; \quad R_e \sim \rho'. \qquad 2.1.1$$

The total current $I$ flowing round the circuit of Figure 15 is

$$I = \frac{E}{R_i + R_e} \qquad 2.1.2$$

and the potential difference between $A$ and $B$ is

$$\phi(A) - \phi(B) = I\,R_e = \frac{E}{1 + R_i/R_e} \qquad 2.1.3$$

Hence the potential difference between $A$ and $B$ in Figure 14 is given (to an order of magnitude) by

$$\phi(A) - \phi(B) \sim \frac{VH_zD}{1 + \rho D/\rho' h} \qquad 2.1.4$$

and the horizontal component of potential gradient is given by

$$\frac{\phi(A) - \phi(B)}{D} \sim \frac{VH_z}{1 + \rho D/\rho' h}. \qquad 2.1.5$$

Thus the effect of the conductivity of the sea bed depends upon the ratio

$$R_i/R_e \sim \rho D/\rho' h. \qquad 2.1.6$$

When this ratio is small, the conductivity of the sea bed has little effect and the sea bed can be regarded as an insulator; when it is large, the potential gradient is effectively short-circuited by the sea bed; and when it is of order unity the potential gradient depends critically on the sea-bed conductivity.

Let us consider some practical examples:

(a) *The English Channel* (see Longuet-Higgins, 1949). The resistivity of the water is nearly uniform, on account of the high degree of mixing, and has a mean value of about 25 ohm-cm. On the other hand the resistivity of the rocks beneath the sea bed is highly variable. In the uppermost sediments the resistivity can be expected to be nearly the same as that of the water; but at lower levels, the resistivity, if equal to that of crystalline rocks, may be as much as $10^6$ ohm-cm. Thus the ratio $\rho/\rho'$ may vary, so far as can be estimated, between 1 and $2{\cdot}5 \times 10^{-5}$. Now for most sections of the English Channel the ratio of the width $D$ to the mean depth $h$ is of the order of $10^3$. Hence the ratio $\rho D/\rho' h$ may lie anywhere between $10^3$ and 0·025. Thus the potential gradient due to tidal or other streams cannot be accurately predicted; it becomes more useful to find a value of $\rho'$ from the observed potential gradients.

Measurements by Cherry and Stovold (1946) on submarine telephone cables have shown that

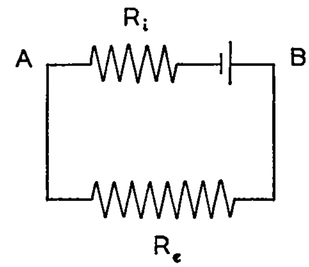


FIGURE 15

there is a considerable potential gradient across the English Channel which can lie between 17 and 71 per cent of the maximum mean potential gradient. Values for $\rho'$, deduced from the above observations, lie between $1{\cdot}3 \times 10^4$ and $2{\cdot}7 \times 10^4$ ohm-cm (see Longuet-Higgins, 1949).

(b) *Currents in the deep ocean.* The conductivity of the different layers in the bed of the deep ocean is also little known. Let us assume that the effective conductivity over the regions concerned is of the same order as for the English Channel, so that the ratio of the conductivities of the water and of the sea bed is of the order of $10^{-3}$. If the depth of water is, say, 3 km, it appears that the conductivity of the sea bed will be of importance only if the width of the current considered is of the order of 3,000 km. For a narrow current such as the Gulf Stream, for example, the effect of the sea bed conductivity is probably negligible (in contrast to the English Channel, where it is critically important). However for very broad currents, if they exist, the conductivity may be important.

In the English Channel the tidal stream velocity is not, of course, uniform with depth (in the Straits of Dover the mean velocity is about 0·83 times the surface velocity; see Longuet-Higgins, 1949). In the deep oceans the difference between mean velocity and surface velocity may be much greater. We shall now consider the effect of variation of water velocity with depth.

## 2.2 The Effect of a Vertical Variation of Velocity

Suppose now that the velocity $V$ of the stream, instead of being uniform, is a function of the vertical co-ordinate $z$. Consider the field due to $H_z$. If the width $D$ of the stream is large compared to the depth $h$, the electrical current-density **i** will be mainly horizontal, in the direction of the induced e.m.f. and hence the equipotential lines in the ocean will be vertical. Hence, in the water, $\nabla \phi$ is nearly horizontal and independent of the depth. The horizontal component $i_x$ of the current-density is found from equation 1.1.4:

$$\frac{\partial \phi}{\partial x} = V(z)\, H_z - \rho i_x, \qquad 2.2.1$$

and hence the total current $I$ is given by

$$\rho I = \int_{-h}^{0} \rho i_x \, dz = \int_{-h}^{0} V(z)\, H_z \, dz - h\frac{\partial \phi}{\partial x}, \qquad 2.2.2$$

that is

$$\rho I = h\left(\overline{V}H_z - \frac{\partial \phi}{\partial x}\right), \qquad 2.2.3$$

where

$$\overline{V} = \frac{1}{h}\int_{-h}^{0} V(z)\, dz. \qquad 2.2.4$$

Thus the total current, and so the potential gradient, are the same as if the water moved with its mean velocity $\overline{V}$.

The circuit analogy of Section 2.1 may be extended to this case also. Imagine the sea to be divided into horizontal layers of infinitesimal thickness $dz$, in each of which the velocity is constant. The vertical component of field will induce in each layer a horizontal e.m.f. $V(z)H_zD$. The system may be compared to a circuit of the same kind as in Figure 15, except that the cell and internal resistance $R_i$ must now be replaced by a number of cells connected in parallel, each of strength $V(z)\, H_zD$ and of resistance $\rho D/dz$ (see Figure 16). If $\phi_{AB}$ denotes the potential difference between $A$ and $B$, the current flowing in each cell is

$$\frac{dz}{\rho D}(\phi_{AB} - VH_zD). \qquad 2.2.5$$

The total current $I$ is given by

$$I = \frac{1}{\rho D}\int_{-h}^{0} (\phi_{AB} - VH_zD)\, dz$$

$$= \frac{h}{\rho D}(\phi_{AB} - \overline{V}H_zD). \qquad 2.2.6$$

This is the same as for a single cell of voltage

$$\overline{E} = \overline{V}H_zD \qquad 2.2.7$$

and internal resistance

$$\overline{R}_i = \rho D/h \qquad 2.2.8$$

Thus $I$ is given by

$$I = \frac{\overline{E}}{\overline{R}_i + R_e} \qquad 2.2.9$$

and the potential difference between $A$ and $B$ by

$$\overline{E} - IR_i = \frac{\overline{E}}{1 + \overline{R}_i/R_e} \qquad 2.2.10$$

Thus the induced potential is the same as if the stream moved with its mean velocity $\overline{V}$.

If the conductivity of the sea bed is relatively high ($\rho D/\rho' h >> 1$) the potential gradient will be small; the current density **i** at the free surface

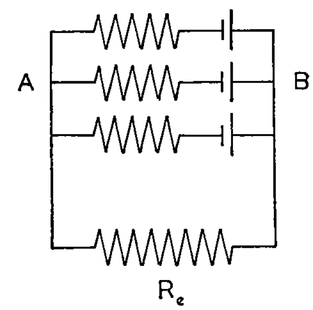


FIGURE 16

(which is what is measured by towed electrodes) will depend only on the surface stream velocity.

Suppose now that the conductivity of the sea bed is low, so that $I$, the return electric current, is negligible. Then we have from 2.2.6.

$$\phi_{AB} = \overline{V}H_zD. \qquad 2.2.11$$

The potential gradient is given by

$$\frac{\partial \phi}{\partial x} = \overline{V}H_z \qquad 2.2.12$$

and the current-density $i_x$ by

$$\rho i_x = VH_z - \frac{\partial \phi}{\partial x} = (V - \overline{V})\, H_z. \qquad 2.2.13$$

If the stream extends uniformly to the bottom, $\overline{V} = V$ and $\rho i_x = 0$; the towed electrodes would then record zero signal (from the vertical field, at least). On the other hand the largest velocities in ocean currents are usually confined to the surface, so that $\overline{V} << V_0$ where $V_0$ is the surface velocity. In this case

$$\frac{\partial \phi}{\partial x} = \overline{V}H_z << V_0 H_z, \qquad 2.2.14$$

i.e. most of the induced e.m.f. is short-circuited, not through the sea bed, but through the lower, relatively slow-moving, parts of the ocean itself. We then have

$$\rho i_x = V_o H_z \qquad 2.2.15$$

very nearly, so that the towed electrodes give the "expected" signal. We may note that the conductivity of the sea bed, so far as it has any effect, tends to diminish the potential gradient and to increase the current-density, making it more nearly equal to the "expected" value.

Suppose that the streams extend uniformly to a depth $h'$ below the upper surface, with strength $V_o$, and are zero below this depth (see Figure 19). Then we have

$$\overline{V} = V_o h'/h \qquad 2.2.16$$

and

$$\rho i_x = V_o H_z (1 - h'/h) \qquad 2.2.17$$

Thus the "expected" signal is reduced by a relative amount $h'/h$, which is small when the surface streams are very shallow.

Even if the velocity-distribution is not of exactly the above form, but the velocities are concentrated mainly near the upper surface, it is sometimes convenient to define the *equivalent depth* $h_{\text{equ}}$ of the streams by the equation

$$\overline{V} = V_o h_{\text{equ}}/h \qquad 2.2.18$$

so that $\rho i_x$ is given in terms of $V_o$ and $h_{\text{equ}}$ by

$$\rho i_x = V_o H_z (1 - h_{\text{equ}}/h). \qquad 2.2.19$$

## 2.3 The Field Due to $H_x$.

The horizontal component of magnetic field $H_x$ will induce a vertical component of e.m.f. equal to $VH_x$ (acting downwards). In the case considered in Section 2.2., where the streams extend

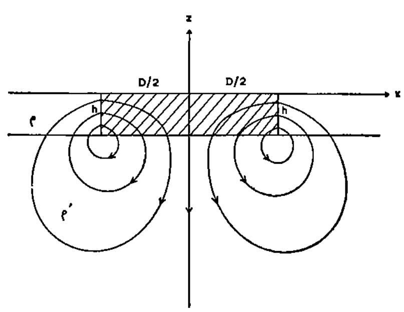


FIGURE 17. The electrical current-density due to $H_x$ (sketch only).

for a width $D$, and may vary vertically. this will produce a symmetrical circulation of current as shown in Figure 17. Each half of the system may be compared to a simple circuit similar to that of Figure 16, where now

$$E \sim \overline{V} H_x h, \quad R_i \sim \rho D/h, \quad R_e \sim \rho'. \qquad 2.3.1$$

Since the resistances are of the same order as before, but the e.m.f. $E$ is reduced by a factor $(h/D)$ $(H_x/H_z)$, the electrical current $I$ will also be

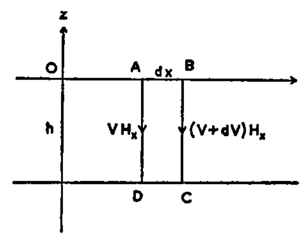


FIGURE 18

reduced by this factor, in general. When $H_x/H_z$ is of order unity, the factor is small, since $h/D << 1$ by hypothesis.

Thus $H_x$ will contribute little, in general, to the horizontal electrical current-density or to the horizontal potential gradient. It will however produce a vertical potential gradient equal, very nearly, to $VH_x$. This is approximately the gradient that would be measured by a pair of electrodes suspended, one below the other, in the water.

Suppose, however, that we are in a region where the water velocity varies with $x$. In any vertical line $x$ = constant, there is a total e.m.f. equal to $\overline{V}hH_x$ acting vertically downwards (see Figure 18). At a distance $dx$ to the right the e.m.f. equals $(\overline{V} + d\overline{V})\, hH_x$. The total circulation of $\mathbf{v} \wedge \mathbf{H}$ round the circuit $ABCD$ in Figure 18 equals $d\overline{V} \cdot hH_x$. This must equal the circulation of $\rho\mathbf{i}$ round the same circuit. The contribution to this circulation from the horizontal component $i_x$ is

$$\rho \left( i_x \right)_{z=-h}^{z=o} dx \qquad 2.3.2$$

and from the vertical component $i_z$ is

$$-\int_{-h}^{o} \left( \frac{\partial i_z}{\partial x}\, dx \right) dz. \qquad 2.3.3$$

Assuming the second term is not large compared with the first, we see that

$$\rho \left( i_x \right)_{z=-h}^{z=o} dx \sim d\overline{V}\, h\, H_x. \qquad 2.3.4$$

Thus the value of $\rho\mathbf{i}$ at the upper surface will differ from that at the lower surface by an amount of the order of $h\, H_x\, d\overline{V}/dx$. If the velocity gradient $d\overline{V}/dx$ is large, a strong horizontal component of current-density will be recorded. At a discontinuity in $\overline{V}$, such as at the edge of the current in Figure 19, $i_x$ is theoretically infinite.

## 2.4 Conclusions, and Some Integral Theorems

To sum up, we shall consider the form of the signal recorded by towed electrodes when the ship crosses a stream of the form shown in Figure 19, that is a stream of width $D$ with sharply defined edges, and of an equivalent depth $h'$ which is small compared with the total depth $h$.

In Figure 20 the dotted line shows the actual surface velocity as a function of the horizontal

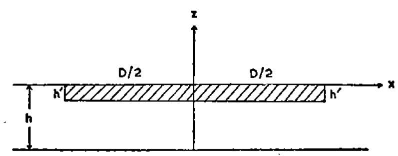


Figure 19. A cross-section of the rectangular stream.

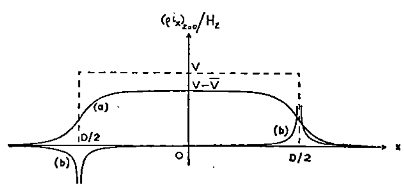


Figure 20. Form of the "apparent velocity" as measured by towed electrodes at the surface, when crossing the stream shown in Figure 19: (a) the signal due to $H_z$, (b) the signal due to $H_x$.

coordinate $x$. Thus the velocity equals $V_o$ when $|x| < D/2$, and equals zero when $|x| > D/2$. The signal due to the vertical component of magnetic field is shown by the full line (a). It is a symmetrical curve which at the mid-point $x = O$ lies just below the line $V = V_o$; the departure from the theoretical value at this point is roughly equal to $\overline{V}$ (assuming that the sea-bed is effectively non-conducting). Outside the area of the stream the curve falls away to zero. The transition across the edge of the stream, however, is not as abrupt as the discontinuity in the actual velocity, since the horizontal component of current-density, which produces the signal, must always be continuous. At the discontinuity, the signal is about half its value at the centre, since if an exactly similar velocity-distribution were placed next to the first, the signal at that point would be doubled and become nearly $(V_o - \overline{V})$, as at the centre of the first stream.

The signal due to $H_x$ is shown in Figure 20 by the curve (b). It is anti-symmetrical about the mid-point $x = O$, since the corresponding electrical current-density is in opposite directions on opposite sides of the stream. There is theoretically an infinity at the edges of the stream, but in fact a slight smoothing of the actual velocity-curve will reduce it to a finite magnitude. In general the order of magnitude of the signal is $V_o\,(h'/h)\,(H_x/H_z)$.

The total signal is the sum of the two curves (a) and (b); this will lie slightly below curve (a) on the left of the stream, and slightly above curve (b) on the right.

One property of the curves in Figure 20 may be mentioned here. From equation 1.1.4 we have

$$\frac{\partial\phi}{\partial x} = VH_z - \rho i_x. \qquad 2.4.1$$

Let this equation be integrated with respect to $x$ right across the current, i.e. from $x = -X$ to $x = X$ where $X - D/2 >> h$. Since $\partial\phi/\partial z = -\rho i_z$, which is small there, we may write $\phi\,(x, z) = \phi_x$ and so

$$\phi_x - \phi_{-x} = \int_{-X}^{X} VH_z\,dx - \int_{-X}^{X} \rho i_x\,dx. \qquad 2.4.2$$

Thus the area beneath the curve representing the surface velocity $V_s$ differs from the area beneath the curve representing the "apparent velocity" $\rho i_x/H_z$ by an amount $(\phi_x - \phi_{-x})/H_z$.

If the sea bed were effectively conducting, e.i. if

$$\rho/\rho' << X/h, \qquad 2.4.3$$

then $\phi_x - \phi_{-x}$ would tend to $O$ as $X$ tended to infinity, and we should have

$$\int_{-\infty}^{\infty} VH_z\,dx = \int_{-\infty}^{\infty} \rho i_x\,dx, \qquad 2.4.4$$

i.e. the total areas beneath the two curves would be equal. If, on the other hand, the sea bed is effectively non-conducting, for which we must have

$$\rho'/\rho >> X/h, \qquad 2.4.5$$

then we may assume

$$\int_{-h}^{o} \rho i_x\,dz = O \qquad 2.4.6$$

Therefore on integrating both sides of 2.4.2 with respect to $z$ from $-h$ to $O$ we have

$$h\,(\phi_x - \phi_{-x}) = \int_{-X}^{X} \int_{-h}^{o} VH_z \, dx \, dz = TH_z \qquad 2.4.7$$

or

$$\frac{\phi_x - \phi_{-x}}{H_z} = \frac{T}{h}, \qquad 2.4.8$$

where $T$ denotes the total transport of the stream.

Hence the difference in area between the two curves in Figure 20 equals $T/h$. Of course, if $X$ is increased to such an extent that the inequality 2.4.5 were reversed, then the total area between the two curves would vanish.

The theorem 2.4.8 was proved by Malkus and Stern (1952). We have seen that its validity depends on the conductivity of the sea bed being negligible.

# 3. ANALYTICAL METHODS

## 3.1 Field Equations and Boundary Conditions

The basic field equations are

$$\nabla \phi = \mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i} \qquad 3.1.1$$

(see equation 1.1.4) and

$$\nabla \cdot \mathbf{i} = O. \qquad 3.1.2$$

Equation 3.1.2 expresses the condition that no electrical charge is created or destroyed in the fluid. At a boundary or a surface of discontinuity we have

$$\Big[\mathbf{i} \cdot \mathbf{n}\Big]_1^2 = O, \qquad 3.1.3$$

where $\mathbf{n}$ is the unit normal to the surface and the suffixes 1 and 2 denote the two sides of the surface.

We have also the following equations:

(a) in the absence of magnetic material

$$\nabla \cdot \mathbf{H} = O; \qquad 3.1.4$$

(b) if the magnetic field associated with the induced current-density $\mathbf{i}$ is small

$$\nabla \wedge \mathbf{H} = O; \qquad 3.1.5$$

(c) if the fluid is incompressible

$$\nabla \cdot \mathbf{v} = O; \qquad 3.1.6$$

(d) if the electrical conductivity is locally uniform

$$\nabla \rho = O \qquad 3.1.7$$

(We may, however, allow surfaces of discontinuity in $\mathbf{v}$ or $\rho$.)

On taking the divergence of both sides of 3.1.1 we have

$$\begin{aligned} \nabla^2 \phi &= \nabla \cdot (\mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i}) \\ &= (\mathbf{H} \cdot \nabla \wedge \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{H}) \\ &\quad - (\rho \nabla \cdot \mathbf{i} - \mathbf{i} \cdot \nabla \rho). \end{aligned} \qquad 3.1.8$$

Using equations 3.1.2, 3.1.5, and 3.1.7 we have

$$\nabla^2 \phi = \mathbf{H} \cdot \nabla \wedge \mathbf{v}. \qquad 3.1.9$$

If the water velocity is irrotational, i.e. if

$$\nabla \wedge \mathbf{v} = O, \qquad 3.1.10$$

then $\phi$ satisfies Laplace's equation

$$\nabla^2 \phi = O. \qquad 3.1.11$$

At a boundary we have from 3.1.2 and 3.1.3

$$\Big[k\mathbf{n} \cdot (\nabla \phi - \mathbf{v} \wedge \mathbf{H})\Big]_1^2 = O, \qquad 3.1.12$$

where $k = \rho^{-1}$, is the electrical conductivity. At the surface of an insulator the above expression is to vanish.

One consequence of the field equations may be noted. On taking the curl of both sides of equation 3.1.1 we have

$$\mathbf{O} = \nabla \wedge (\mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i}) \qquad 3.1.13$$

and therefore

$$\begin{aligned} \nabla \wedge (\rho \mathbf{i}) &= \nabla \wedge (\mathbf{v} \wedge \mathbf{H}) \\ &= (\mathbf{H} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{H}) + \\ &\quad (\mathbf{v} \nabla \cdot \mathbf{H} - \mathbf{H} \nabla \mathbf{v}) \end{aligned} \qquad 3.1.14$$

(see Weatherburn 1943, p. 9). Using 3.1.4 and 3.1.6 we have

$$\nabla \wedge (\rho \mathbf{i}) = \mathbf{H} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{H}. \qquad 3.1.15$$

Thus, even when $\nabla \mathbf{H}$ vanishes ($\mathbf{H}$ is uniform) $\nabla \wedge (\rho \mathbf{i})$ is not in general zero, so that $\rho \mathbf{i}$ and $\mathbf{v} \wedge \mathbf{H}$ are not in general the gradients of potential functions, although their difference is, by equation 3.1.1.

## 3.2 Two-dimensional motion

The solution is particularly simple when the motion is two-dimensional. Let $(x, y, z)$ be rectangular coordinates, and suppose that the velocity is independent of $y$, and has no component in the $y$-direction. Thus

$$\mathbf{v} = (v_x, O, v_z) = \left(\frac{\partial\psi}{\partial z}, O, -\frac{\partial\psi}{\partial x}\right), \qquad 3.2.1$$

where $\psi$ is the stream-function. It is convenient to divide the magnetic field into two parts, one parallel and one perpendicular to the $y$-axis:

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 \qquad 3.2.2$$

where

$$\mathbf{H}_1 = (O, H_y, O), \quad \mathbf{H}_2 = (H_x, O, H_z). \qquad 3.2.3$$

The parts of $\phi$ and $\mathbf{i}$ corresponding to $\mathbf{H}_1$ and $\mathbf{H}_2$ will be denoted by suffixes 1 and 2.

Consider first the field due to $\mathbf{H}_1$. The induced e.m.f. may be written

$$\mathbf{v} \wedge \mathbf{H} = \left(\frac{\partial\psi}{\partial x}H_y, O, \frac{\partial\psi}{\partial z}H_y\right) = \nabla\,(H_y\,\psi). \qquad 3.2.4$$

This vector lies in the $(x, z)$-plane and is everywhere at right-angles to the stream-lines $\psi$ = constant. Further

$$\rho\mathbf{i}_1 = \mathbf{v} \wedge \mathbf{H}_1 - \nabla\,\phi_1 = \nabla\,(H_y\,\psi - \phi_1). \qquad 3.2.5$$

Now the field equation 3.1.9 and the boundary condition 3.1.12, for $\phi_1$, follow from the conditions that $\nabla \cdot \mathbf{i}_1$ shall vanish (3.1.2) and that the normal component of $\mathbf{i}_1$ shall be continuous (3.1.3). These conditions, and 3.2.5, are identically satisfied by

$$\mathbf{i}_1 = O, \qquad \phi_1 = H_y\,\psi, \qquad 3.2.6$$

which is therefore the solution. In other words the e.m.f. is such that no electrical current flows, and the stream-lines are all equipotential surfaces. In particular, on each fixed boundary, which must be a stream-line, the potential is constant, whether or not it is the boundary of a perfect conductor.

Consider now the field due to $\mathbf{H}_2$. We have

$$\mathbf{v} \wedge \mathbf{H}_2 = \left(O, -H_x\frac{\partial\psi}{\partial x} - H_z\frac{\partial\psi}{\partial z}, O\right). \qquad 3.2.7$$

This vector is always parallel to the $y$-axis, and tends to make current flow in that direction. Further, it is independent of $y$. Therefore, if there are no boundaries hindering the path of the current, we have

$$\rho\mathbf{i}_2 = \left(O, -H_x\frac{\partial\psi}{\partial x} - H_z\frac{\partial\psi}{\partial z}, O\right), \quad \phi_2 = O. \qquad 3.2.8$$

In other words, the current flows freely, and the potential gradient is reduced to zero. However, not only must all the boundaries be two-dimensional, but the current must be free to flow to infinity in the $y$-direction. Whether this is true will depend on the conditions of the problem. It will be true if the total current demanded from infinity is zero, i.e. if

$$\iint_S \mathbf{v} \wedge \mathbf{H}_2 \cdot dx\,dz = O \qquad 3.2.9$$

where $S$ is the total area of the $(x, z)$ plane in which there is motion, or a typical part of the plane, supposing the motion is periodic in space (as for waves, see Section 7.1). If 3.2.9 is not satisfied there may still be a free flow of current if there is an external path for the current of sufficiently low resistance. For example, if the streams extend a finite distance $Y$ in the $y$-direction in water of depth $h$, a sufficient condition is

$$Y/h >> \rho'/\rho \qquad 3.2.10$$

where $\rho$ and $\rho'$ are mean resistivities of the sea and sea bed (compare Section 2.1). But in general, when neither 3.2.9 nor 3.2.10 is satisfied, an additional current-density $\mathbf{i}_2'$ must be superposed on that given by 3.2.8.

### 3.3 A Generalization of the Previous Solution

The simplicity of the solution for two-dimensional motion suggests a generalization. By the equation of continuity 3.1.6 we may write in general

$$\mathbf{v} = -\nabla \wedge \Psi \qquad 3.3.1$$

where $\Psi$ is a vector such that

$$\nabla \cdot \Psi = O \qquad 3.3.2$$

(see for example Weatherburn, 1943, p. 44). $\Psi$ is the vector stream function of the motion. In the case of two-dimensional flow we have simply

$$\Psi = (O, \psi, O), \qquad 3.3.3$$

where $\psi\,(x, z)$ is the two-dimensional stream function. $\Psi$ can be regarded as determined by the velocity field $\mathbf{v}$. Since

$$\begin{aligned}\nabla \wedge \mathbf{v} &= -\nabla \wedge \nabla \wedge \Psi \\ &= -\nabla\,(\nabla \cdot \Psi) + \nabla^2\,\Psi \\ &= \nabla^2\,\Psi,\end{aligned} \qquad 3.3.4$$

a general expression for $\Psi$, assuming $\mathbf{v}$ to vanish at infinity, is

$$\Psi\left(x, y, z\right) = -\frac{1}{4\pi} \times \iiint \frac{\nabla \wedge \mathbf{v}\,(x', y', z')\,dx'\,dy'\,dz'}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}, \qquad 3.3.5$$

the integral being taken over all space (see Weatherburn, 1943, p. 45).

From 3.3.4 we see that the differential equation for $\phi$ is

$$\nabla^2\phi = \mathbf{H} \cdot \nabla^2\Psi. \qquad 3.3.6$$

If **H** is uniform, this may be written

$$\nabla^2\chi = 0 \qquad 3.3.7$$

where

$$\chi = \phi - \mathbf{H} \cdot \Psi. \qquad 3.3.8$$

The boundary-condition at an insulator is

$$\mathbf{n} \cdot \nabla\phi = \mathbf{n} \cdot \mathbf{v} \wedge \mathbf{H} = \mathbf{n} \cdot \mathbf{H} \wedge (\nabla \wedge \Psi). \qquad 3.3.9$$

But if **H** is a constant vector we have identically

$$\nabla (\mathbf{H} \cdot \Psi) = \mathbf{H} \cdot \nabla \Psi + \mathbf{H} \wedge (\nabla \wedge \Psi). \qquad 3.3.10$$

Equation 3.3.9 may therefore be written

$$\mathbf{n} \cdot \nabla\phi = \mathbf{n} \cdot [\nabla (\mathbf{H} \cdot \Psi) - \mathbf{H} \cdot \nabla \Psi] \qquad 3.3.11$$

or

$$\mathbf{n} \cdot \nabla \chi = -\mathbf{n} \cdot \nabla (\mathbf{H} \cdot \nabla \Psi). \qquad 3.3.12$$

The problem is thus reduced to that of finding a solution $\chi$ of Laplace's equation 3.3.7 to satisfy the boundary condition 3.3.12. The right-hand side of 3.3.12 does not in general vanish, but when it does so, as in the two-dimensional case we have simply

$$\chi = 0 \qquad 3.3.13$$

and so

$$\phi = \mathbf{H} \cdot \Psi = H_y\psi. \qquad 3.3.14$$

# 4. A STREAM WITH SINUSOIDAL PROFILE

## 4.1 Introduction

In this and the two following Chapters we shall give a precise determination of the electrical field in three cases when the stream-velocity is given by simple analytical expressions. These examples will not only serve to illustrate the general principles stated in Chapter 2, but will also provide quantitative estimates of the different anomalies that are to be expected.

As in Chapter 2 it is supposed that the stream-velocity $\mathbf{v}(x, z)$ is everywhere parallel to the $y$-axis, $(x, y, z)$ being rectangular coordinates, and $z$ being measured vertically upwards from the "sea surface." $\rho$ and $\rho'$ denote the resistivities of the "sea" and "sea bed" respectively, which for simplicity are assumed to be uniform. From 3.1.9 the field equation for the potential is then

$$\nabla^2\phi = H_z \frac{\partial v_y}{\partial x} - H_x \frac{\partial v_y}{\partial z} \qquad 4.1.1$$

in the "sea" and

$$\nabla^2\phi = 0 \qquad 4.1.2$$

in the "sea bed."

In the present Chapter we shall suppose that the velocity $\mathbf{v}$ is given by

$$v_y = \begin{cases} V\cos\alpha x, & 0 > z > -h', \\ 0, & -h' > z > -h, \end{cases} \qquad 4.1.3$$

where $h$ is the total depth, that is, the velocity is confined to a layer of depth $h'$ in the upper part of the sea, being zero below this depth, and the velocity varies sinusoidally across the stream. The field due to $H_z$ has been previously evaluated and partly discussed by Stommel (1948). We shall also evaluate the field due to $H_x$ and give a more complete discussion.

## 4.2 The Field Due to $H_z$

If $\phi_1$, $\phi_2$ and $\phi_3$ denote the values of $\phi$ in the three regions $0 > z > -h'$, $-h' > z > -h$ and $-h > z > -\infty$ we have as field equations

$$\nabla^2\phi_1 = -VH_z\alpha\sin\alpha x, \quad \nabla^2\phi_2 = 0, \quad \nabla^2\phi_3 = 0. \qquad 4.2.1$$

From 3.1.12 and the condition that $\phi$ shall be continuous, we have the boundary conditions

$$\left.\begin{array}{ll} \text{when } z = 0, & \dfrac{\partial\phi_1}{\partial z} = 0 \\ \text{when } z = -h', & \begin{cases} \phi_1 - \phi_2 = 0 \\ \dfrac{\partial\phi_1}{\partial z} - \dfrac{\partial\phi_2}{\partial z} = 0 \end{cases} \\ \text{when } z = -h, & \begin{cases} \phi_2 - \phi_3 = 0 \\ \dfrac{1}{\rho}\dfrac{\partial\phi_2}{\partial z} - \dfrac{1}{\rho'}\dfrac{\partial\phi_3}{\partial z} = 0 \end{cases} \\ \text{when } z \to -\infty, & \phi_3 \to 0 \end{array}\right\} \qquad 4.2.2$$

Equations 4.2.1 and 4.2.2 show that

$$(\phi_1 - VH_z\alpha^{-1}\sin\alpha x),$$

$\phi_2$ and $\phi_3$ must be the sums of functions of the form $e^{\pm\alpha z}\sin\alpha x$. From the last of equations 4.2.2, $\phi_3$ is a multiple of $e^{+\alpha z}\sin\alpha x$ only. It can be verified by direct substitution that the full solution is

$$\phi_1 = \frac{VH_z}{\alpha}\left[1 - \frac{\sinh\alpha(h-h') + (\rho/\rho')\cosh\alpha(h-h')}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h} \times \cosh\alpha z\right]\sin\alpha x$$

$$\phi_2 = \frac{VH_z}{\alpha}\sinh\alpha h' \times \frac{\cosh\alpha(z+h) + (\rho/\rho')\sinh\alpha(z+h)}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h}\sin\alpha x$$

$$\phi_3 = \frac{VH_z}{\alpha}\sinh\alpha h' \times \frac{e^{\alpha z}}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h}\sin\alpha x. \qquad 4.2.3$$

Thus the horizontal component of the potential gradient at the free surface is

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = VH_z(1-\gamma)\cos\alpha x \qquad 4.2.4$$

and the horizontal component of the electrical current-density is

$$(\rho i_x)_{z=0} = VH_z\,\gamma\cos\alpha x \qquad 4.2.5$$

where

$$\gamma = \frac{\sinh\alpha(h-h') + (\rho/\rho')\cosh\alpha(h-h')}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h}. \qquad 4.2.6$$

The "apparent" velocity is everywhere proportional to the actual velocity but reduced by the constant factor $\gamma$.

(a) Suppose the streams are broad compared with the total depth of water. Then $\alpha h$, $\alpha(h-h')$ and $\alpha z$ are small compared with unity and $\sinh\alpha h$, etc., may be replaced by $\alpha h$, etc. We have then

$$\gamma = \frac{\alpha(h-h') + (\rho/\rho')}{\alpha h + (\rho/\rho')} \qquad 4.2.7$$

and

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = \frac{VH_z\,\alpha h'}{\alpha h + (\rho/\rho')}\cos\alpha x. \qquad 4.2.8$$

This shows that the effect of the conductivity of the bottom on the potential gradient is small provided

$$\rho/\rho' << \alpha h \qquad 4.2.9$$

or

$$\rho a/\rho' h << 1, \qquad 4.2.10$$

where $2a$, $= 2\pi/\alpha$, is the "width" of the stream. Equation 4.2.10 is similar to the condition $\rho D/\rho' h << 1$ found in Section 2.1. Further we may write

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = \frac{VH_z(h'/h)}{1 + \pi^{-1}(\rho a/\rho' h)}\cos\alpha x, \qquad 4.2.11$$

which illustrates the theorem of Section 2.2, namely that the potential gradient depends only on the mean velocity in each vertical line (in this case $V(h'/h)\cos\alpha x$). When the conductivity of the bottom is negligible we have simply

$$\gamma = 1 - h'/h \qquad 4.2.12$$

and

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = VH_z(h'/h)\cos\alpha x. \qquad 4.2.13$$

In particular (1) when the stream extends to the bottom, ($h' = h$), we have $\gamma = 0$ and $(\partial\phi_1/\partial x)_{z=0} = VH_z\cos\alpha x$. This is because there is no stationary water to form a return path for the electric current circulation. The "apparent" velocity, proportional to $\rho i_x$, therefore vanishes. (2) when the stream is very shallow (i.e. $h' << h$ and $\alpha h << 1$) we have $\gamma = 1$ and $(\partial\phi_1/\partial x)_{z=0} = 0$. The potential gradient is now completely short-circuited by the water below and the full "apparent" velocity is recorded.

(b) Suppose the streams are very narrow compared with the total depth of the "ocean." We have then $\alpha h >> 1$, so that $\sinh\alpha h$ and $\cosh\alpha h$ can both be replaced by $\frac{1}{2}e^{\alpha h}$. We assume also that $\alpha(h-h') >> 1$. Thus from 4.2.4 – 4.2.6

$$\gamma = e^{-\alpha h'} \qquad 4.2.14$$

and

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = VH_z(1 - e^{-\alpha h'})\cos\alpha x. \qquad 4.2.15$$

The effect of the bottom is then negligible in any case (the ocean, in effect, is infinitely deep) and $\gamma$ depends only on $\alpha h'$, or on the ratio of the depth of the surface streams to their width. When the streams are shallow compared with their width ($\alpha h' << 1$), $\gamma$ is nearly unity, and the maximum "apparent" velocity is recorded. When the streams are very deep and narrow ($\alpha' h >> 1$), $\gamma$ tends to zero. This case, however, is unlikely to occur in practice.