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SOME EFFECTS OF SHEARING MOTION ON THE PROPAGATION OF WAVES IN THE PREVAILING WESTERLIES

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INTRODUCTION

In recent years great interest has developed in the study of the role of the semi-permanent centers of action as they affect weather conditions averaged over several days. An examination of the daily synoptic and upper air charts indicates that although these centers of action have closed isobaric systems at the ground (e.g., the Aleutian and Icelandic lows) they often appear at upper levels as sinusoidal disturbances of the average zonal distribution of pressure.

In a recent paper* Rossby found an important relationship between the velocity of movement (c) of these sinusoidal disturbances, the wave lengths (L) and the velocity of the undisturbed geostrophic wind (U) e.g.: $c=U-(\beta L^2/4\pi^2)$ where β is the rate of change of the Coriolis parameter ($f=2\Omega \sin \theta$) along a meridian circle, Ω is the angular velocity of the earth's rotation and θ is the latitude.

The importance of this formula resides in the fact that it indicates when the centers of action will move eastward, westward or remain stationary ($c \gtreqless 0$). Such movements are of fundamental importance in forecasting procedure since these centers are controlling factors in weather situations (e.g.: a trough at the three kilometer level is associated with precipitation, hence the prediction of the movement of this trough enables a prediction as to where precipitation will spread).

The case treated by Rossby was that of an infinite horizontal layer. However infinite layers cannot exist on the earth. As a better approximation to actual conditions Haurwitz† treated the case of a finite horizontal layer. He obtained the expression:

$$c = U - \frac{\beta}{4\pi^2}\left\{\frac{L^2D^2}{L^2+D^2}\right\}$$

where D is the horizontal width of the disturbance (see Fig. 2).

Another important factor which has been neglected heretofore is the effect of shear on the movement of perturbations in the prevailing westerlies. It is the primary purpose of this paper to extend the investigations of the horizontal perturbations by determining the effects of shear upon their movements. In treating the case of shear, the method of circulation integrals will be used. V. Bjerknes first made extensive use of the circulation theorems and some fruitful interpretations of them were made by Hoiland.‡ We will show first how these integrals are to be used by applying them to the cases of the infinite and finite horizontal layers where no shear is present. Then the case of shear will be considered for both infinite and finite layers.

The first chapter of the paper deals with the method of circulation integrals, indicating how it is to be applied to wave phenomena. The actual applications of the circulation integral to the single layer of finite and infinite widths, and the double layer are carried out in Chapters II and III. In these chapters, formulas for the velocity, the length of the stationary wave and critical wave length of each case are developed. Tables are computed and diagrams constructed for the more interesting cases of stationary and critical wave lengths.

* Rossby, C.-G.—The Relation Between Variations in the Intensity of the Zonal Circulation of the Atmosphere and the Displacements of the Semi-Permanent Centers of Action.

† Haurwitz, B.—The Motion of Atmospheric Disturbances.

‡ Hoiland, E.—On the Interpretation and Application of the Circulation Theorems of V. Bjerknes.

In the analysis of the single layer problem for both finite and infinite widths, two different formulas for the velocity of a horizontal wave arise, depending upon whether the stream function is a hyperbolic or a trigonometric function of the north-south coordinate. The basis for the choice of stream function is discussed in Chapter II.

The author takes this opportunity to thank Dr. Sverre Petterssen, in charge of the Meteorology Department of Massachusetts Institute of Technology, for his assistance and encouragement in making this work possible. To Prof. Jorgen Holmboe of the University of California, much gratitude must be expressed for many helpful conversations on the subject matter of this paper and related matters. The author's deepest gratitude goes to Helen L. Garstens for her sympathetic cooperation and able assistance in computing the values for the tables and diagrams and in preparing this work for publication.

CHAPTER I. THE METHOD OF CIRCULATION INTEGRALS

Hoiland* has given a new interpretation of the circulation theorems of V. Bjerknes in so far as they can be applied to the investigations of the stability of perturbations in general. In particular, the underlying dynamics of *wave* phenomena is illuminated by this interpretation of the circulation theorems. For the purpose of this investigation, it is necessary to set up the circulation integrals in a form as to apply to a horizontal wave arising from a disturbance of the prevailing westerlies. The Coriolis force is the restoring force of the wave. We will assume throughout that we are dealing with an incompressible fluid ($\rho = \text{constant}$).

The equations of two-dimensional motion for an ideal fluid are given by the expressions:

$$\text{(1)} \qquad \begin{cases} \rho \dfrac{du}{dt} = + \rho f v - \dfrac{\partial p}{\partial x} \\ \rho \dfrac{dv}{dt} = - \rho f u - \dfrac{\partial p}{\partial y} \cdot \end{cases}$$

This in vectorial form is:

$$\text{(1a)} \qquad \rho \dot{\boldsymbol{v}} = - 2\rho \boldsymbol{\Omega}_z \times \boldsymbol{v} - \nabla_2 p$$

where ρ is the density of the air, u and v are the x and y components of the velocity, $\boldsymbol{\Omega}_z$ the vertical component of the earth's rotation, $\dot{\boldsymbol{v}} = (du/dt)\boldsymbol{i} + (dv/dt)\boldsymbol{j}$ the horizontal acceleration, and $\nabla_2 p$ the horizontal component of the pressure gradient. We neglect the effect of gravity ($-\rho \nabla \phi$, where ϕ is the geopotential) since the motion is restricted to the horizontal plane. If (1a) is integrated around a closed path, we get:

$$\text{(2)} \qquad \oint \dot{\boldsymbol{v}} \cdot \delta \boldsymbol{r} = - 2 \oint \boldsymbol{\Omega}_z \times \boldsymbol{v} \cdot \delta \boldsymbol{r}$$

which may be called the circulation theorem for horizontal perturbations. ρ does not appear since it is constant and may be cancelled out. δr is the element of path. Figure 1 indicates the path along which the integrals in (2) are taken in the case of waves. The integral of the pressure term vanishes since the gradient of pressure is a conservative vector (curl $\nabla p = 0$). In evaluating (2) a stream function may be introduced from which we obtain the components of the acceleration $\dot{\boldsymbol{v}}$, given in (1). This stream function will in general be a function of y (the north-south axis). Some interesting properties of the wave motion we are to consider emerge from an analysis of this stream function. These will be discussed in a later chapter.

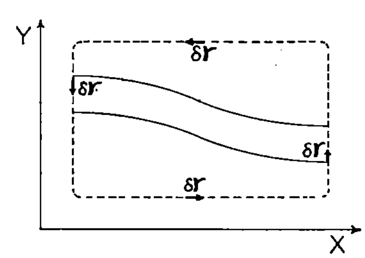

FIG. 1. Showing the path along which the integrals are taken in the case of waves.

* Hoiland, E.—op. cit.

CHAPTER II. SINGLE LAYERS

1. SINGLE LAYERS OF FINITE WIDTHS

The closed path which we consider in the case of a horizontal wave is that through the trough, crest and along the two fixed boundaries. (For a wave of infinite extent the boundaries are at infinity.) In Fig. 2 the path of integration is along *ABCDA*, *AB* being the trough and *CD* the crest of the wave. The amplitude of the wave is indicated as Δy, assumed to be infinitesimal in magnitude. It is assumed that there is an undisturbed west-east velocity of flow U, and that the streamlines indicated in Fig. 2 arise as a result

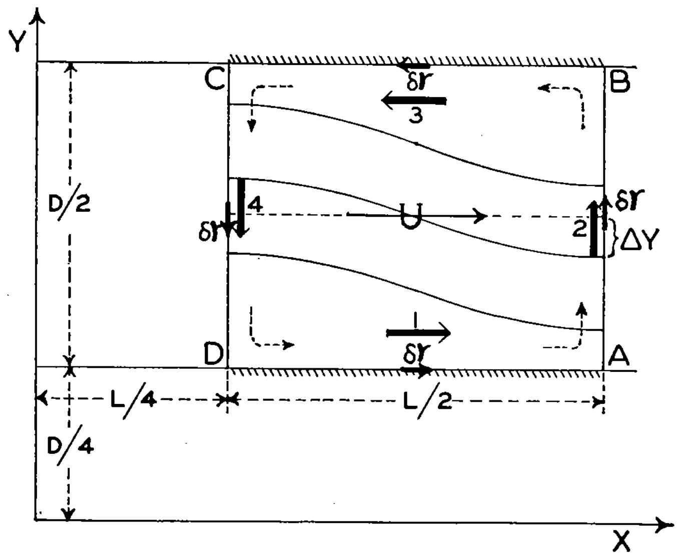

FIG. 2. Showing the amplitude of the wave and the directions of the accelerations within it.

of an infinitesimal perturbation of the undisturbed flow. If *AB* and *CD* move with the velocity of the wave, then they will continue to pass through the trough and crest respectively. Thus within the area *ABCD*, steady state conditions will prevail. The heavy arrows indicate the direction of the acceleration along each of the four sides. Thus arrow 1 points in the direction of converging streamlines. The fluid obviously accelerates in that direction. Similarly for arrow 3. The arrows 2 and 4 indicate the direction of the accelerations along the trough and crest due to the curvature of the streamlines there. Close to the fixed walls *BC* and *DA* there are no components of the acceleration perpendicular to them. Along *AB* and *CD* the accelerations are perpendicular to the fixed walls and have no component parallel to them since the streamlines do not converge or diverge there. It is thus clear that the integral of the acceleration in the direction *ABCDA* will yield the same sign for each of the four components (*AB*, *BC*, *CD*, *DA*) of the path (δr being parallel to arrows 1, 2, 3 and 4 respectively). With these points in mind, it follows immediately that equation (2) in this instance reduces by means of equation (1) to the form:

(3)
$$\oint_{(ABCDA)} \dot{v}\cdot\delta r = \int_A^B \frac{dv}{dt}\delta y + \int_B^C \frac{du}{dt}\delta x + \int_C^D \frac{dv}{dt}\delta y + \int_D^A \frac{du}{dt}\delta x$$
$$= -2\oint_{(ABCDA)} \Omega_z \times v\cdot\delta r.$$

For a *compressible* fluid like the atmosphere, the integral around a closed path of equation (1a) may be written in the form:

$$\oint \dot{v}\cdot\delta r = -2\oint \Omega_z \times v\cdot\delta r - \oint \alpha\nabla_2 p\cdot\delta r$$

where $\alpha = 1/\rho$ is the specific volume. Here

$$-\oint \alpha\nabla_2 p\cdot\delta r$$

is the well known expression for the number of solenoids contained within the closed path of integration (*ABCDA*). This is a small quantity for an average distribution of temperature and pressure from north to south and therefore may usually be neglected. However, close to a coastline where a large concentration of solenoids may occur, this term may no longer be negligible. These solenoids constitute an external force which remains fixed at a point. Such a force will cause a stationary wave to occur. The Coriolis force however is a free force and can, of course, give rise to propagating waves.

Returning to equation (3) it is clear that a stream function ψ_r may be introduced since the effect of compressibility is being neglected. For example, assuming that ψ_r is a trigonometric function of the north-south variable y, we have:

(4)
$$\frac{\psi_r}{(U-c)} = C_0 \cos\frac{2\pi}{D} y \sin \mu x_r - y$$

where the subscript r indicates motion relative to a system of coordinates moving at the velocity c of the wave (e.g.: $x_r = x - ct$). D is the width of the flow and $\mu = 2\pi/L$, where L is the wave length.

The stream function for relative coordinates may be introduced into equation (3) making use of the relations:

$$\dot{v}_r = \frac{d}{dt}(v - c), \qquad u_r = -\frac{\partial\psi_r}{\partial y_r} \quad \text{and} \quad v_r = \frac{\partial\psi_r}{\partial x_r}\cdot$$

The resulting equation simplifies considerably when second order terms are dropped. Thus, since $u_r = (u-c) + u'$ and $v_r = v'$ (primes denoting disturbed values of the velocities) we get:

$$\frac{d(\;)}{dt} = \frac{\partial(\;)_r}{\partial t} + u_r\frac{\partial(\;)_r}{\partial x_r} + v_r\frac{\partial(\;)_r}{\partial y_r} = (U-c)\frac{\partial(\;)_r}{\partial x_r}\cdot$$

Equation (3) therefore becomes:

$$\oint \dot{v}\cdot\delta\boldsymbol{r} = \oint \dot{v}_r\cdot\delta\boldsymbol{r} = \int_A^B \frac{dv_r}{dt}\,\delta y + \int_B^C \frac{du_r}{dt}\,\delta x + \int_C^D \frac{dv_r}{dt}\,\delta y + \int_D^A \frac{du_r}{dt}\,\delta x$$

$$= (U - c)\left\{\int_A^B (c\mu^2 C_0 \cos\lambda y - U\mu^2 C_0 \cos\lambda y)\delta y\right\}$$

(5) $$+ (U - c)\left\{\int_B^C (- c\lambda\mu C_0 \cos\mu x_r + U\mu\lambda \cos\mu x_r\right\}$$

$$= \frac{4\mu^2}{\lambda} C_0(c - U)^2 + 4\lambda C_0(c - U)^2 = 4(c - U)^2 C_0\left(\frac{\mu^2 + \lambda^2}{\lambda}\right)$$

$$= -2\oint \boldsymbol{\Omega}_z \times v\cdot\delta\boldsymbol{r}$$

since

$$\int_A^B \frac{dv_r}{dt}\,\delta y = \int_C^D \frac{dv_r}{dt}\,\delta y, \qquad \int_B^C \frac{du_r}{dt}\,\delta x = \int_D^A \frac{du_r}{dt}\,\delta x$$

$$\sin\mu x_r = 1, \quad \text{and} \quad \sin\lambda y = 1$$

along BC and AB respectively.

To evaluate the right hand part of equation (3) we note that $\boldsymbol{\Omega}_z\times v\cdot\delta\boldsymbol{r}$ vanishes along the walls and hence need only be integrated along the trough and crest of the wave. This

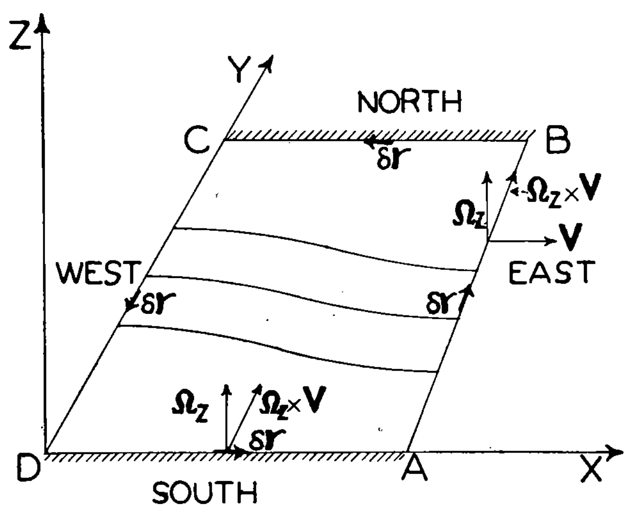

FIG. 3. The direction of the vertical component $\boldsymbol{\Omega}_z$ of the earth's angular velocity and of the cross product $\boldsymbol{\Omega}_z\times v$ within the wave.

follows from the fact that $\boldsymbol{\Omega}_z\times v$ is at right angles to $\delta\boldsymbol{r}$ along the walls, whereas it is parallel to $\delta\boldsymbol{r}$ along the trough and crest (see Fig. 3). Along AB and CD, $\boldsymbol{\Omega}_z\times v\cdot\delta\boldsymbol{r}$ may be written in the form $\Omega_z u\,\delta y$ so that we merely have to evaluate the integrals

$$2\int_A^B \Omega_z u\delta y + 2\int_C^D \Omega_z u\delta y.$$

Ω_z is assumed to vary linearly in the y direction.* We may therefore make the substitution: $2\Omega_z = 2\Omega_{z_0} + \beta y$ where

$$\beta = \frac{\partial f}{\partial y} = \frac{2\Omega \cos\theta}{a}$$

is constant. Here a is the earth's radius and Ω_{z_0} is the value of the vertical component of the earth's angular velocity at the origin. Thus:

$$2\int \Omega_z u\delta y = 2\Omega_{z_0}\int u\delta y + \beta\int yu\delta y.$$

The first integral on the right taken across AB and CD vanishes because of the continuity assumptions (e.g.: the fluid is homogeneous and incompressible). Equation (3) therefore reduces to the following form:

(6)
$$4(c-U)^2 C_0\left(\frac{\mu^2+\lambda^2}{\lambda}\right) = \beta\left\{\int_A^B yu\delta y + \int_C^D yu\delta y\right\}.$$

Since

$$u = u_r + c = -\frac{\partial\psi_r}{\partial y} + c = \pm\lambda(c-U)C_0 \sin\lambda y + U$$

along AB and CD respectively, we may integrate equation (6) by parts, e.g.:

$$\beta\int_A^B yu\delta y = (c-U)\left\{\beta\lambda C_0\int_A^B y\sin\lambda y\delta y\right\} = -(c-U)\beta C_0\int_A^B y\delta(\cos\lambda y)$$

$$= -(c-U)\beta C_0\left\{y\cos\lambda y\Big|_A^B - \int_A^B \cos\lambda y\delta y\right\}$$

$$= -(c-U)\beta C_0\left\{y\cos\lambda y - \frac{1}{\lambda}\sin\lambda y\right\}_A^B$$

$$= -\frac{2\beta C_0}{\lambda}(c-U)$$

and similarly for the integral from C to D. The integral of $Uy\delta y$ vanishes around a closed path. Equation (6) therefore becomes:

(6a)
$$4(c-U)C_0\left(\frac{\mu^2+\lambda^2}{\lambda}\right) = -\frac{4\beta C_0}{\lambda}$$

or:

(7)
$$c = U - \frac{\beta}{4\pi^2}\left\{\frac{L^2D^2}{L^2+D^2}\right\}$$

which is Haurwitz's formula.

* See Rossby, C.-G.—op. cit., p. 42, for justification of this assumption.

2. Single Layers of Infinite Widths

From this analysis we can see the role played by the different forces entering into the formation of the wave. As we go on to more complicated cases of waves (e.g.: double layers) the same type of analysis will make it possible to see clearly the physical basis of the phenomenon. Thus equation (3) indicates that there is a balance of three forces involved in producing the inertial wave we have been considering. They are:

1. The Curvature Forces (along *AB* and *CD*, see Fig. 4)
2. The Kinematic Restraint (along *BC* and *DA*)
3. The Coriolis Force.

The forces due to curvature along the trough and crest are given by the integral of the

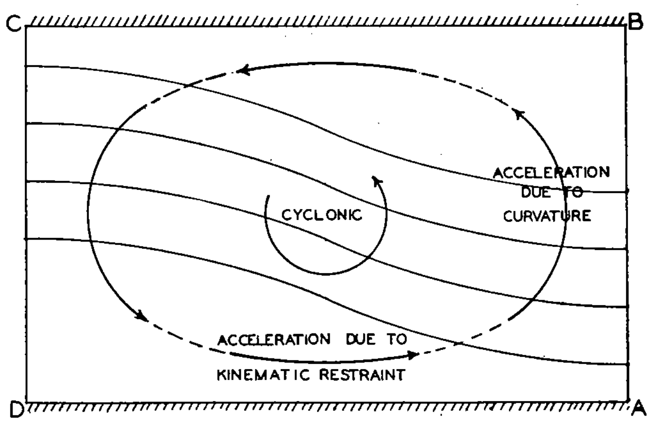

Fig. 4. The accelerations due to the curvature of flow and to the presence of a boundary.

acceleration along *AB* and *CD* and are equal to $(4\mu^2/\lambda)C_0(c-U)^2$. For sufficiently deep flow this is the only one which is important in balancing out the Coriolis force. This is quite clear when we consider the term in equation (3) due to the kinematic restraint $4\lambda C_0(c-U)^2$. This term arises from the integration of the acceleration along *BC* and *DA*. For wide layers $\lambda \to 0$. For such layers, the following equation holds:

$$\frac{4\mu^2}{\lambda} C_0(c - U)^2 = -2\oint \mathbf{\Omega}_z \times \mathbf{v}\cdot\delta\mathbf{r} = -\frac{4\beta C_0}{\lambda}(c - U)$$

or

$$c = U - \frac{\beta L^2}{4\pi^2} \tag{8}$$

which is Rossby's formula. In other words, Rossby's formula states the existence of a balance between the Coriolis force:

$$2\oint \mathbf{\Omega}_z \times \mathbf{v}\cdot\delta\mathbf{r}$$

and the forces due to streamline curvatuve along the trough and crest. From Fig. 4

we note that the accelerations along the trough, crest and walls are cyclonic. The Coriolis force is the third force which alone balances the other two.

With the introduction of the walls, a force which is due to the kinematic restraint arises. This one is also cyclonic in direction and therefore shares with the curvature force the task of balancing the Coriolis force. It is therefore to be expected that the frequency ν of the wave or its velocity for a given wave length will be affected by the kinematic restraint introduced by the boundaries.

Before we proceed to the discussion of double layers, it is important to point out some very interesting consequences of the assumption made in equation (4) that the stream function is a trigonometric function of the north-south variable y.

3. The Stream Function ψ

Assuming a perturbation which is independent of the north-south direction and considering only the effect of the latitudinal variation of the Coriolis force, the velocity c of the perturbation is given by equation (8). The assumption that underlies the above result is that the perturbation velocity u', in the x' (or east-west) direction is independent of the latitude (e.g.: u' is constant along the y axis).

From the law of the conservation of absolute vorticity, it is not difficult to obtain the expression for the propagation of horizontal waves when u' is not constant in the y direction. Thus from

$$\frac{d(\zeta + f)}{dt} = 0 \quad \text{where} \quad \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{9}$$

is the relative vorticity and f the Coriolis parameter, we obtain by neglecting terms of order higher than the perturbation velocities u' or v':

$$\frac{\partial^2 v'}{\partial t \partial x} - \frac{\partial^2 u'}{\partial t \partial y} + U\left\{\frac{\partial^2 v'}{\partial x^2} - \frac{\partial^2 u'}{\partial x \partial y}\right\} + \beta v' = 0. \tag{10}$$

Assuming horizontal incompressible motion, we may introduce a stream function ψ such that:

$$(U + u') = -\frac{\partial \psi}{\partial y}, \qquad v' = \frac{\partial \psi}{\partial x}.$$

If the stream pattern ψ is propagated at the constant velocity c in the x direction, then we may express it in the form:

$$\psi = P \sin \mu(x - ct) - U_y$$

where P is now a function of y. Substituting this in (10) we get:

$$\frac{d^2 P}{dy^2} - P\left(\mu^2 - \frac{\beta}{U - c}\right) = 0. \tag{11}$$

If P is constant in the y direction then

$$\frac{d^2 P}{dy^2} = 0 \quad \text{and} \quad \left(\mu^2 - \frac{\beta}{U - c} = 0\right) \quad \text{or} \quad c = U - \frac{\beta}{\mu^2}$$

which is Rossby's result. The most general solution of (11) is:

$$P = Ae^{Ky} + Be^{-Ky},$$

where

$$K = \left\{\frac{\beta}{U-c} - \mu^2\right\}^{1/2}$$

is imaginary or real depending upon whether

$$\frac{\beta}{U-c} < \quad \text{or} \quad > \mu^2.$$

It is of some interest to consider the meaning of these inequalities. If we take the first one:

$$\frac{\beta}{U-c} < \mu^2$$

then K is imaginary and P can be expressed as a trigonometric function, e.g.:

$$P = C \cos Ky.$$

If we define the lateral extent of the flow as D we can set $K=\lambda$ or

$$\lambda^2 = \frac{\beta}{U-c} - \mu^2$$

from which we get as the velocity c of the wave, the expression:

$$c = U - \frac{\beta}{4\pi^2}\left\{\frac{L^2D^2}{L^2+D^2}\right\}.$$

When

$$\frac{\beta}{U-c} > \mu^2$$

K is real and P can be expressed in terms of exponential or hyperbolic functions. If two finite lateral boundaries are desired (e.g.: $\psi=0$ along two different lines, see Fig. 5) it is clear that a single hyperbolic function is not sufficient since the latter can vanish only along a single line. This difficulty may be overcome by using two hyperbolic functions instead of one. Thus we may let:

(12a) $$\psi_1 = C_1 \sinh K(y+h) \cos \mu(x-ct)$$

for negative y (below the x axis), and

(12b) $$\psi_2 = C_2 \sinh K(y-h) \cos \mu(x-ct)$$

for positive y (above the x axis).

Here $\psi_1=0$ at $y=-h$ and $\psi_2=0$ at $y=+h$ which are the conditions required.

This distinction between the use of hyperbolic and trigonometric functions does not arise in the case of purely gravitational waves. In the latter case β does not occur at all

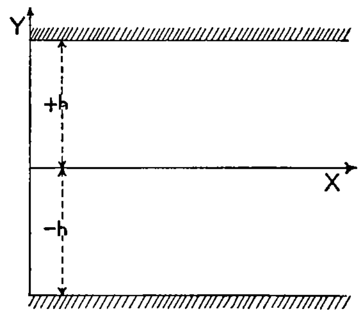

FIG. 5

($\beta=0$). This gives rise to an equation similar to (11) except that the term $[\beta/(U-c)]$ is not present. The resulting equation:

$$\frac{d^2P}{dy^2} - P\mu^2 = 0$$

can only have hyperbolic solutions. It is for this reason that hyperbolic functions occur in the velocity formulas for gravitational waves. Thus the velocity of a wave in relatively deep water is given by the expression:

$$C^2 = \frac{g}{\mu} \tanh h\mu$$

where h is the depth of the water and g the acceleration of gravity.

In the case of waves whose restoring force is the earth's rotation, we have two types of streamlines to consider, each of which will give rise to a different expression for the velocity.

For gravitational waves $K=\mu$. But this is not the case as we have seen, when the Coriolis force is considered. Thus when

$$\frac{\beta}{U-c} < \mu^2,$$

$K=\lambda$ (see Page 12). When

$$\frac{\beta}{U-c} > \mu^2$$

K is a measure of the damping effect (e.g.: as in $P(y) \sim e^{-Ky}$). Besides K in (12) we have h and h' which now measure the separation of the fixed walls instead of D (see Fig. 5).

It is worthwhile to obtain the amplitude of the streamlines and trajectories from the stream function we have considered (equation (4)).

4. Amplitude of the Trajectories and Streamlines

In a recent paper* Rossby showed that for the case of an infinite layer the ratio of the amplitude of the trajectory to that of the streamline is given by $U/(U-c)$. It follows from this that the particles move farther north than the streamlines indicate for eastward moving waves, and not as far north as the streamlines indicate for westward moving waves. It is evident that for finite layers, the amplitudes for both the trajectories and the streamlines are influenced by the presence of the walls. We will determine just what that influence is.

It should be borne in mind that the perturbations we are considering are infinitesimal, so that all amplitudes are likewise infinitesimal and of the same order as C_0. The amplitude of the trajectory of a given particle within a fluid whose streamlines are given by equation (4) is $C_0 \cos \lambda y$. This can be shown in the following manner: A given particle of fluid at X, Y will in general describe some closed path, as the disturbance ψ, defined by equation (4), is propagated from west to east (see Fig. 6). If X, Y are the coordinates of

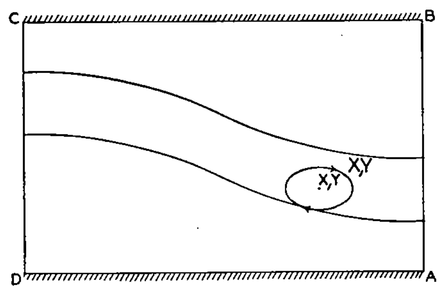

FIG. 6. Showing the trajectory X, Y of a particle around its mean position x, y.

this particle and x, y the geometric center of the closed path it describes, then it is possible to express X, Y in terms of $\psi(x, y, c)$ by the relationship:

$$u = \frac{dX}{dt} = -\frac{\partial \psi}{\partial y} + \left\{\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)\delta x\right\}$$

$$v = \frac{dY}{dt} = +\frac{\partial \psi}{\partial x} + \left\{\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)\delta y\right\}.$$

However, the expressions within the parenthesis are second order terms and may therefore be dropped. Hence:

$$-\frac{\partial \psi}{\partial y} = \frac{dX}{dt} \quad \text{and} \quad \frac{\partial \psi}{\partial x} = \frac{dY}{dt}$$

which with (4) yields:

$$\text{(13)} \quad \begin{aligned} &\text{(a)} \quad X = \int \frac{dX}{dt}\,dt = -\int \frac{\partial \psi}{\partial y}\,dt = (U-c)\int C_0 \lambda \sin \lambda y \sin \mu(x-ct)\,dt + Ut \\ &\text{(b)} \quad Y = \int \frac{dY}{dt}\,dt = \int \frac{\partial \psi}{\partial x}\,dt = (U-c)\int C_0 \mu \cos \lambda y \cos \mu(x-ct)\,dt. \end{aligned}$$

* Rossby, C. G.—Planetary Flow Patterns in the Atmosphere.

Now the point x, y around which the particle described the trajectory indicated in Fig. 6 is swept along with the mean velocity U, so that $x = Ut$ in (13b) and Y is therefore given by the expression:

$$Y = C_0 \cos \lambda y \sin \mu(U - c)t.$$

Neglecting the first term to the right in (13a) this may be written as:

$$Y = C_0 \cos \lambda y \sin \frac{(U - c)}{U} X. \tag{13c}$$

This satisfies the required kinematic conditions. Thus the amplitude ($C_0 \cos \lambda y$) of the trajectory varies from zero at the walls ($y = D/4$, $3D/4$) to a maximum value of C_0 at the center of the flow ($y = D/2$). (See Fig. 2.)

It is interesting to note that the ratio of the amplitude of the trajectory to that of the streamline is not affected by the introduction of the boundary conditions thus far discussed. The amplitude of the streamline is easily obtainable from (4). Consider any

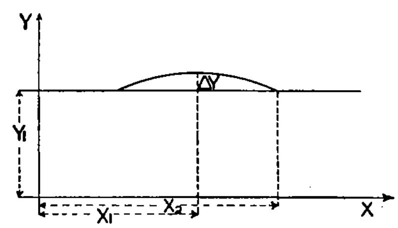

FIG. 7

streamline ψ_1, and let y_1 represent the mean ordinate of this line. If Δy represents the amplitude of the streamline we may, by neglecting higher order terms, write:

$$\begin{aligned}\psi_1 &= (U - c)C_0 \cos \lambda y \sin \mu(x_1 - ct) - Uy = - Uy \\ &= (U - c)C_0 \cos \lambda(y + \Delta y) \sin \mu(x_2 - ct) - U(y + \Delta y) \\ &= (U - c)C_0 \cos \lambda y - U(y + \Delta y)\end{aligned}$$

since

$$\sin \mu(x_1 - ct) = 0,$$

and

$$\sin \mu(x_2 - ct) = 1.$$

Subtracting these two equations from each other we get, for the amplitude of the streamline:

$$\Delta y = C_0 \frac{(U - c)}{U} \cos \lambda y.$$

This varies in the same way as the amplitude of the trajectory between the walls $y = D/4$ and $\frac{3}{4}D$. The ratio of the two is $U/(U - c)$. *The lateral extent of the disturbance has therefore no effect on the relative magnitude of the two amplitudes.*

CHAPTER III. DOUBLE LAYERS

1. Double Layers with Uniform Undisturbed Velocity in Each

Up to this point we have neglected the effect of shear completely. As an approximation to the most general case of an arbitrary velocity profile, that of a double layer with a uniform velocity in each will be considered. The important effect of shear is that of being a destabilizing force. If sufficiently strong, shear will give rise to an unstable wave. In the absence of this force, as can be seen for the case of the inertial waves just treated, only stable waves can occur. Thus the velocity c of the disturbance just considered (equation (8)) cannot be imaginary so that instability is clearly impossible.

2. Instability in a Double Layer with No Restoring Force

To see how instability arises, we will consider the case of pure shear in a double layer, *with no restoring force.* The disturbed state of the originally straight line flow is represented by Fig. 8. We may, for the purpose of this discussion, neglect the kinematic restraints and consider only those forces due to the curvature of the streamlines along the trough and crest. The geostrophic wind velocity U' of the current to the north is assumed to differ from that to the south, which is U. We assume that there is no variation in density from one current to the other and that both are homogeneous and incompressible.

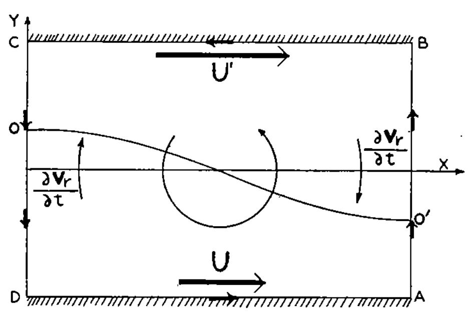

Fig. 8. The action of the local accelerations $\partial v_r/\partial t$ in causing the amplitude of the wave to increase (instability).

It is clear from the directions of the convective accelerations along AB and CD (heavy arrows in Fig. 8) that the present shape of the wave cannot persist since all of the accelerations are acting in the same direction (counterclockwise) and there is nothing to counterbalance this force except the local accelerations which will be in the opposite direction (clockwise, $\partial v_r/\partial t$ in Fig. 8). To illustrate this point more fully, let $\dot{v} = \dot{u}i + \dot{v}j$ represent the acceleration of any particle with respect to a fixed system of coordinates and $\dot{v}_r$ that with respect to a system moving with the velocity c of the wave. Then:

$$\dot{v}_r = \frac{d}{dt}(v - c) = \dot{v}.$$

For a stable wave there are no local accelerations with respect to the moving coordinate system, so that

$$\dot{v}_r = \frac{\partial v_r}{\partial t} + v_r \cdot \nabla v_r = v_r \cdot \nabla v_r.$$

This will only occur if the convective acceleration $v_r \cdot \nabla v_r$ is balanced by some other force such as the Coriolis force (e.g.: $v_r \cdot \nabla v_r = -2\Omega_z \times v$) or that of gravity. If no other balancing forces exist, then local accelerations $(\partial v_r/\partial t)$ set in and the direction of these accelerations are such as to cause the wave to grow in the same direction as the perturbation. A crest grows taller and a trough deeper. For if we integrate $v_r \cdot \nabla v_r$ along the path *ABCDA*, taken as the positive direction, we get a positive value as Fig. 9 shows. Since

$$\frac{\partial v_r}{\partial t} + v_r \cdot \nabla v_r = 0$$

in this case, the integral of $\partial v_r/\partial t$ in the same direction must be negative, so that along *AB* the local acceleration will be downward (deepening trough) and along *CD* upward (growing crest).

If the convective term $v_r \cdot \nabla v_r$ is integrated around the closed path in a case when instability is possible, then the velocity c and frequency ν turn out to be imaginary. Thus it is easy to show that the convective acceleration along the trough and crest is equal to $(U-c)^2/r$ in the lower layer and $(U'-c)^2/r'$ in the upper, where r and r' are the radii of curvature for the streamlines with respect to the moving system of both respectively. To demonstrate this, we first note that:

$$v_r \cdot \nabla v_r = (U - c)\frac{\partial v_r}{\partial x_r}$$

in the lower layer, and

$$v_r \cdot \nabla v_r = (U' - c)\frac{\partial v_r'}{\partial x_r}$$

in the upper, neglecting second order terms (e.g.: $u_r(\partial v_r/\partial x_r)$). $v_r = v$ is the perturbation velocity in the y direction and likewise for v_r'. Since $v = +\partial\psi/\partial x$, it can be shown by use of equation (4) that

$$\frac{\partial v_r}{\partial x_r} = \frac{\partial v}{\partial x} = -\mu^2 C_0 \cos \lambda y$$

along the trough and crest. It is also a simple matter to prove that

$$\frac{1}{r} = \frac{\mu^2 C_0}{(U - c)} \cos \lambda y.$$

It follows therefore that

$$(U - c)\frac{\partial v_r}{\partial x_r} = \frac{(U - c)^2}{r}.$$

For sufficiently deep layers we can assume that $(U-c)^2/r$ and $(U'-c)^2/r'$ remain constant throughout, so that the integral of the acceleration around the closed path *ABCDA*, with the accelerations along the walls neglected since they are small for deep layers, yields:

$$\text{(14)} \qquad \oint \dot{v} \cdot \delta r = \left\{\frac{(U - c)^2}{r} + \frac{(U' - c)^2}{r'}\right\} = 0.$$

Solving this for the velocity c we get:

$$\text{(14a)} \qquad c = (Ur' + Ur') \pm \frac{1}{(r + r')}\sqrt{- rr'(U - U')^2}.$$

The last term is always imaginary which means that the wave is unstable.

It is not difficult to see that the addition of the accelerations along the wall will not remove the instability since, as was previously pointed out, they act in the same counterclockwise direction as the centrifugal acceleration.

3. The Velocity of a Wave in a Double Layer

The introduction of the Coriolis term provides a balancing force against the shearing force. The critical point which divides the stable from the unstable wave depends on the wave length which we may call L_c. We will determine this critical point now. In order to do this we first obtain the velocity c of the wave. The stream functions for the upper and lower layers with respect to coordinates moving at the speed of the wave can be represented by:

(14b)
$$\begin{cases} \dfrac{\psi_r}{c - U} = C_0 \cos \lambda y \sin \mu x_r + y \\ \dfrac{\psi_r'}{c - U'} = C_0 \cos \lambda y \sin \mu x_r + y \end{cases}$$

respectively. The lateral breadths of the upper and lower layers are assumed to be equal and the equation of separation of the two is given by:

$$y = C_0 \sin \mu x_r.$$

In integrating the equation of motion $\dot{v}_r = -2\Omega_z \times v$ around the path $ABCDA$ (see Fig. 8) we first compute the integral along $O'BCO$ using ψ_r', then along $ODAO'$ using ψ_r. Points O and O' may be considered to be at a distance exactly midway between AB and CD respectively. The error involved is of the second order and is therefore negligible. Thus the integral of the terms in the equation of motion from the walls to O or O' is of the order of C_0, which by assumption is infinitesimal. The integral of the same terms from O or O' to the central point is of the order $C_0 \Delta y$ where Δy is the infinitesimal distance from O or O' to the center.

Thus we may write the integral of the equation of motion:

$$\int_{(ABCDA)} \dot{v}_r \cdot \delta r = -2 \int_{(ABCDA)} \Omega_z \times v \cdot \delta r$$

in the form:

(15)
$$\int_{O'BCO} \dot{v}_r \cdot \delta r + \int_{ODAO'} \dot{v}_r \cdot \delta r = -2 \int_{O'BCO} \Omega_z \times v' \cdot \delta r - 2 \int_{ODAO'} \Omega_z \times v \cdot \delta r.$$

In the same manner as in the case of a single bounded layer we can easily show that each of the terms in (15) can be expressed in the following manner:

(15a)
$$\int_{O'BCO} \dot{v}_r \cdot \delta r = \left\{ \int_{O'}^{B} \frac{dv_r'}{dt} \delta y + \int_{B}^{C} \frac{du_r'}{dt} \delta x + \int_{C}^{O} \frac{dv_r'}{dt} \delta y \right\}$$

(15b)
$$\int_{ODAO'} \dot{v}_r \cdot \delta r = \left\{ \int_{O}^{D} \frac{dv_r'}{dt} \delta y + \int_{D}^{A} \frac{du_r}{dt} \delta x + \int_{A}^{O'} \frac{dv_r}{dt} \delta y \right\}$$

(15c)
$$2 \int_{O'BCO} \Omega_z \times v' \cdot \delta r = 2 \int_{B}^{C} \Omega_z u' \delta x$$

(15d) $$2\int_{ODAO'} \mathbf{\Omega}_z \times \mathbf{v}\cdot\delta\mathbf{r} = 2\int_D^A \Omega_z u\delta x$$

where the subscript r indicates motion relative to the moving system of coordinates. Within such a system stationary conditions prevail so that all partial derivatives with respect to time vanish. Thus:

$$\frac{dv_r'}{dt} = (U - c)\frac{\partial v_r'}{\partial x_r}$$

neglecting higher order terms (see Page 17). Similar expressions hold for the other terms. It is important to note that in (15c) and (15d) both u and u' are not expressed in coordinates relative to the moving system. The reason for this is obvious since the Coriolis force is independent of the moving coordinate system. In the case of the convective accelerations (15a) and (15b) however, the introduction of moving coordinates does not affect their value since translation merely adds a constant c which drops out when the time derivative is taken.

Substituting the stream functions (14b) in (15a, b, c, d) the following values are obtained:

(16a) $$\int_{O'}^B \frac{dv_r'}{dt}\delta y + \int_B^C \frac{du_r'}{dt}\delta x + \int_C^O \frac{dv_r'}{dt}\delta y = 2(c - U)^2 C_0\left(\frac{\mu^2 + \lambda^2}{\lambda}\right)$$

(16b) $$\int_O^D \frac{dv_r'}{dt}\delta y + \int_D^A \frac{du_r}{dt}\delta x + \int_A^{O'} \frac{dv_r}{dt}\delta y = 2(c - U')^2 C_0\left(\frac{\mu^2 + \lambda^2}{\lambda}\right)$$

(16c) $$2\int_B^C \Omega_z u'\delta x = \frac{2(c - U)\beta C_0}{\lambda}$$

(16d) $$2\int_D^A \Omega_z u\delta x = \frac{2(c - U')\beta C_0}{\lambda}.$$

Equation (15) therefore becomes:

(17) $$(c - U)^2 + (c - U')^2 = -\frac{\beta}{4\pi^2}\{(c - U) + (c - U')\}\frac{L^2D^2}{(L^2 + D^2)}$$

or

(17a) $$c = \frac{U + U'}{2} - \frac{1}{2}\left\{\frac{\beta L^2D^2}{4\pi^2(L^2 + D^2)}\right\} - \frac{1}{2}\sqrt{\left(\frac{\beta L^2D^2}{4\pi^2(L^2 + D^2)}\right)^2 - (U - U')^2}.$$

Equation (17a) gives the velocity c of the wave where the effect of shear $(U-U')$ and the finite lateral extent of the wave is taken into account. If no shear is present $(U=U')$, then (17a) reduces to the case previously derived (see equation (7)). And for this case, as the width D of the two layers becomes large compared to the wave length L, we obtain the expression for an infinitely wide layer:

$$c = U - \frac{\beta L^2}{4\pi^2}$$

(see equation (8)). For the case of wide layers where the shear does not vanish, we obtain:

$$c = \frac{U + U'}{2} - \frac{1}{2}\left(\frac{\beta L^2}{4\pi^2}\right) - \frac{1}{2}\sqrt{\left(\frac{\beta L^2}{4\pi^2}\right)^2 - (U - U')^2}. \tag{18}$$

4. Length of Stationary Waves in Double Layers

When the disturbance is stationary ($c=0$) equation (17) reduces to:

$$(U^2 + U'^2) = \beta(U + U')\frac{L_s^2 D^2}{4\pi^2(L_s^2 + D^2)}$$

from which the stationary wave length L_s is obtainable:

$$L_s = \frac{1}{\sqrt{\frac{\beta}{4\pi^2}\left(\frac{U + U'}{U^2 + U'^2}\right) - \frac{1}{D^2}}}. \tag{19}$$

For wide layers the stationary wave length is:

$$L_{s\infty} = 2\pi\sqrt{\frac{U^2 + U'^2}{\beta(U + U')}}. \tag{19a}$$

The full significance of equation (19) is indicated in Fig. 9. Given the velocities U and U' and the width D,we can immediately determine whether or not a stationary wave length is possible. Thus, for a given latitude we construct a circle of

$$\text{radius } \frac{D^2\beta}{4\sqrt{2}\,\pi^2} \quad \text{and center } \left(\frac{D^2\beta}{8\pi^2}, \frac{D^2\beta}{8\pi^2}\right).$$

If U and U' fall within this circle, a stationary wave length is possible. As the point approaches the circumference, its stationary wave length becomes larger. At the circumference, it is infinite in size, and outside the circle, it is imaginary (no stationary wave length is possible there). From equation (19) we can obtain the equivalent of the above remarks.

$$\begin{cases} \text{When } \frac{\beta}{4\pi^2}\left(\frac{U + U'}{U^2 + U'^2}\right) > \frac{1}{D^2}, & U \text{ and } U' \text{ are within the region of possible stationary wave lengths.} \\ \text{When } \frac{\beta}{4\pi^2}\left(\frac{U + U'}{U^2 + U'^2}\right) = \frac{1}{D^2}, & L_s \text{ is infinite.} \\ \text{When } \frac{\beta}{4\pi^2}\left(\frac{U + U'}{U^2 + U'^2}\right) < \frac{1}{D^2}, & L_s \text{ is imaginary. No stationary wave length is possible.} \end{cases} \tag{19b}$$

5. Length of Stationary Waves in Infinite Double Layers

In Table I we have tabulated the values of L_s for various cases of shear in deep layers and in Fig. 10 we see the limiting case of Fig. 9 as D becomes infinitely large. The curves of constant L_s are plotted with U and U' as coordinates. These curves are all circles passing through the origin. The limiting circle of Fig. 9 is now a straight line.

All points below this line yield imaginary wave lengths. In Fig. 9 the radius r_{L_s} of the circle of constant L_s is equal to:

$$\left(\frac{L_s^2}{1 + L_s^2}\right)r,$$

where r is the radius of the limiting circle.

It is interesting to note the following points which both Table I and Fig. 10 indicate. The stationary wave length does not depend on shear alone as does the critical wave

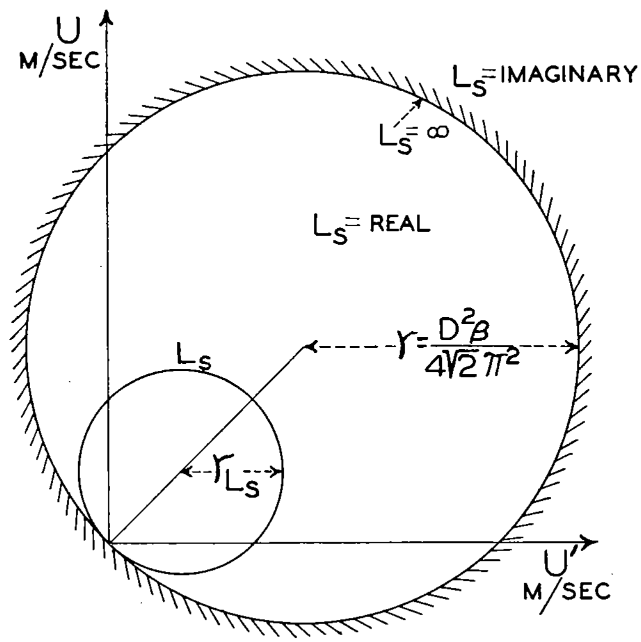

FIG. 9. Lengths of stationary waves for double layers of varying lateral event.

length L_c (the wave length at which instability sets in. See page 26). It obviously depends on the values of the velocities themselves. Thus for $(U - U') = 4$ m/sec, we may have the following values of L_s for latitude 30°:
when

$$U = +\ 8, \quad U' = +\ 4, \quad L_s = 3640 \text{ km}$$
$$U = +\ 12, \quad U' = +\ 8, \quad L_s = 4550 \text{ km}$$
$$U = +\ 16, \quad U' = +\ 12, \quad L_s = 5350 \text{ km} \quad \text{etc.}$$

This increase in analogous to the ordinary increase in stationary wave length with velocity when there is but a single unbounded layer.

For the infinite double layer with both U and U' positive, L_s lies in between the values of L_s for single layers of velocity U and U'. Furthermore it is impossible to obtain a stationary wave length in a single layer if the geostrophic wind is negative, since L_s becomes imaginary in that case (equation (8)). This is not so for the double layer. As long as $(U + U')$ is not negative a stationary wave length is possible. *Thus it is possible* *for easterly winds to exist either to the north or south of westerlies and still get stationary waves*, provided the absolute value of the westerlies is greater than that of the easterlies. When one of the winds is easterly, there is a marked increase in L_s for the double

TABLE I

LENGTH OF THE STATIONARY WAVE AS A FUNCTION OF THE VELOCITIES U AND U' FOR AN INFINITE DOUBLE LAYER AND DIFFERENT LATITUDES.

LATITUDE	U \ U'	0 m/sec	+4 m/sec	+8 m/sec	+12 m/sec	+16 m/sec	−4 m/sec	−8 m/sec	−12 m/sec	−16 m/sec
30°	4	2822	2822	3650	4450	5220	∞	i	i	i
	8	3990	3650	4020	4550	5300	6310	∞	i	i
	12	4888	4450	4550	4900	5350	6310	10200	∞	i
	16	5644	5220	5300	5350	5650	6640	8950	14100	∞
	20	6310	5870	5760	5830	6030	7200	8760	11640	18100
45°	4	3120	3120	4050	4930	5770	∞	i	i	i
	8	4412	4050	4412	5040	5840	6970	∞	i	i
	12	5405	4930	5040	5405	5910	6970	11250	∞	i
	16	6241	5770	5840	5910	6241	7430	9860	15600	∞
	20	6978	6490	6360	6430	6660	7960	9690	12850	20700
60°	4	3713	3713	4830	5900	6910	∞	i	i	i
	8	5252	4830	5252	5990	6960	8350	∞	i	i
	12	6432	5900	5990	6432	7050	8350	13390	∞	i
	16	7428	6910	6960	7050	7428	8890	11760	18560	∞
	20	8304	7760	7580	7660	7930	9530	11520	15300	23700

i = imaginary

layer. Thus when $U=+20$ m/sec and $U'=-4$ m/sec at latitude 45°, $L_s=7960$ km as compared to 6978 km for the single layer. When $U'=-8$ m/sec, $L_s=9690$ km, an increase of more than 40% over the single layer value.

6. LENGTH OF STATIONARY WAVES IN FINITE DOUBLE LAYERS

Table II and Fig. 11 show the stationary wave lengths for several cases of shear where the lateral extent of the disturbance stretches from the pole to the equator. The same remarks which were made with regard to Table I and Fig. 10 apply to Table II and Fig. 11. Thus L_s of the double layer is in between those for single layers with velocities U and U'. Again L_s is imaginary for easterly geostrophic winds for the single layer whereas easterly winds with real values for L_s are again possible for double layers providing $(U+U')$ is positive.

A comparison of Figs. 10 and 11 or equations (19) and (19a) shows that the effect of the introduction of walls in the case of double layers is to increase the size of the stationary wave length for any given values of U and U'. Thus at latitude 45° for: $U = +12$ m/sec and $U' = +8$ m/sec

L_s = 5040 km, for the infinite double layer,

L_s = 5800 km, for the finite double layer.

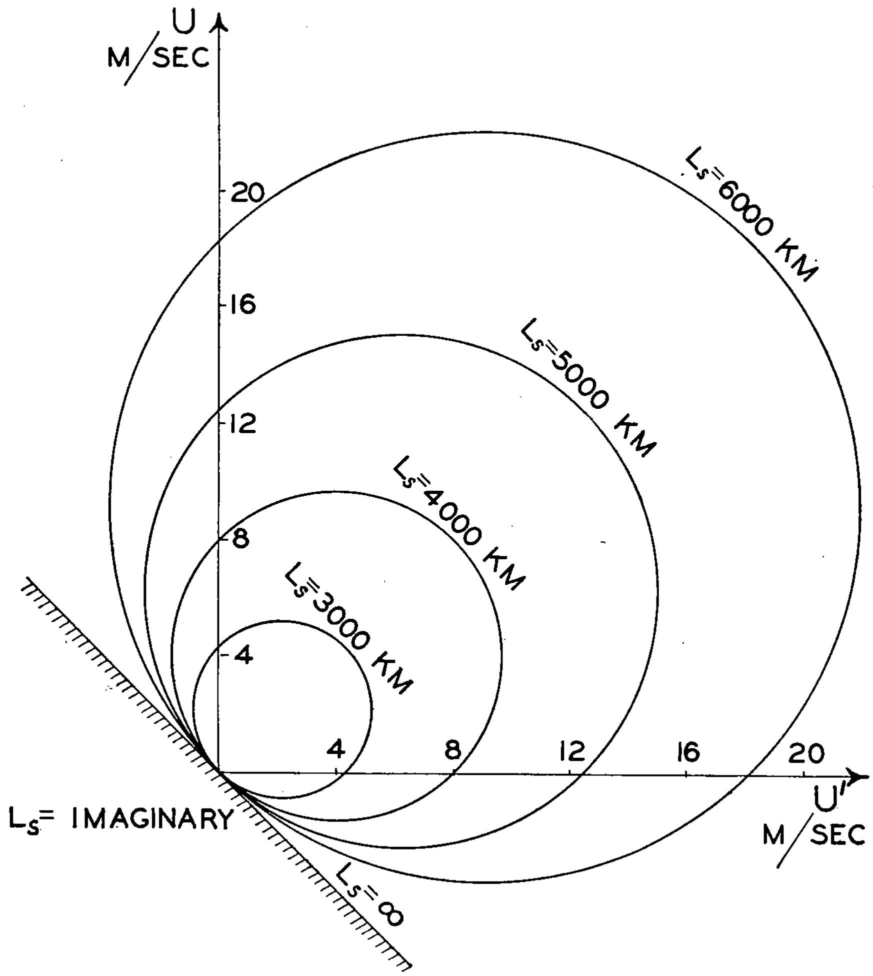

FIG. 10. Lengths of stationary waves for an infinite double layer.

For: $U = +8$ m/sec and $U' = -4$ m/sec

L_s = 6970 km, for the infinite double layer,

L_s = 9750 km, for the finite double layer.

In the latter case the introduction of walls at a distance of 10,000 km from each other causes an increase in wave length of about 30%.

7. The Critical Wave Length L_c

It is obvious from (18) that for sufficiently strong shear, instability will set in, since $(U-U')$ will become larger than the Coriolis term $(\beta L^2/4\pi^2)^2$. If we hold $(U-U')$ fixed, then instability occurs if the wave length L_s is made sufficiently small. Conversely, the

TABLE II

Lengths of Stationary Waves as a Function of the Velocities U and U', for a Finite Double Layer and Different Latitudes, with D=10,000 km.

Latitude	U \ U'	0 m/sec	+4 m/sec	+8 m/sec	+12 m/sec	+16 m/sec	+20 m/sec	−4 m/sec	−8 m/sec
30°	4	2,940 km	2,940	3,910	4,950	5,950	7,230	*i*	*i*
	8	4,350	3,910	4,350	5,130	6,010	7,010	7,930	*i*
	12	5,600	4,950	5,130	5,600	6,290	7,140	7,930	*i*
	16	6,850	5,950	6,010	6,290	6,850	7,480	8,180	17,900
	20	8,120	7,230	7,010	7,140	7,480	8,120	11,300	18,350
45°	4	3,280	3,280	4,400	5,710	7,000	8,500	*i*	*i*
	8	4,820	4,400	4,820	5,800	6,970	7,800	9,750	*i*
	12	6,440	5,710	5,800	6,440	7,320	8,420	9,750	*i*
	16	8,000	7,000	6,970	7,320	8,000	8,990	11,050	70,750
	20	9,750	8,500	7,800	8,420	8,990	9,750	41,990	42,750
60°	4	4,000	4,000	5,460	7,270	8,430	12,430	*i*	*i*
	8	6,160	5,460	6,160	7,500	8,620	11,550	15,000	*i*
	12	8,410	7,270	7,500	8,410	9,900	11,950	15,000	*i*
	16	11,100	8,430	8,620	9,900	11,100	13,100	19,100	*i*
	20	14,900	12,430	11,550	11,950	13,100	14,900	88,300	*i*

i=imaginary

larger the wave length and the smaller the shear, the less the instability which occurs. The stabilizing effect of the Coriolis force is clearly indicated in (18). If we neglect this force, we get:

$$c = \frac{U+U'}{2} - \tfrac{1}{2}\sqrt{-(U-U')^2}$$

which is imaginary for all values of U and U' except when $U=U'$. This is the case of pure shear.

In the more general solution (17a) we can see the effect of the finite boundaries on the stability of the wave. The first term underneath the radical can be written:

$$\left\{\frac{\beta L^2}{4\pi^2\left(\frac{L^2}{D^2}+1\right)}\right\}^2$$

If we compare this with the corresponding term in (18) it is clear that the introduction of the walls leads to less stability.

8. Critical Wave Lengths for Infinite Double Layers

In the case of disturbances of infinite lateral extent, equation (18) enables us to

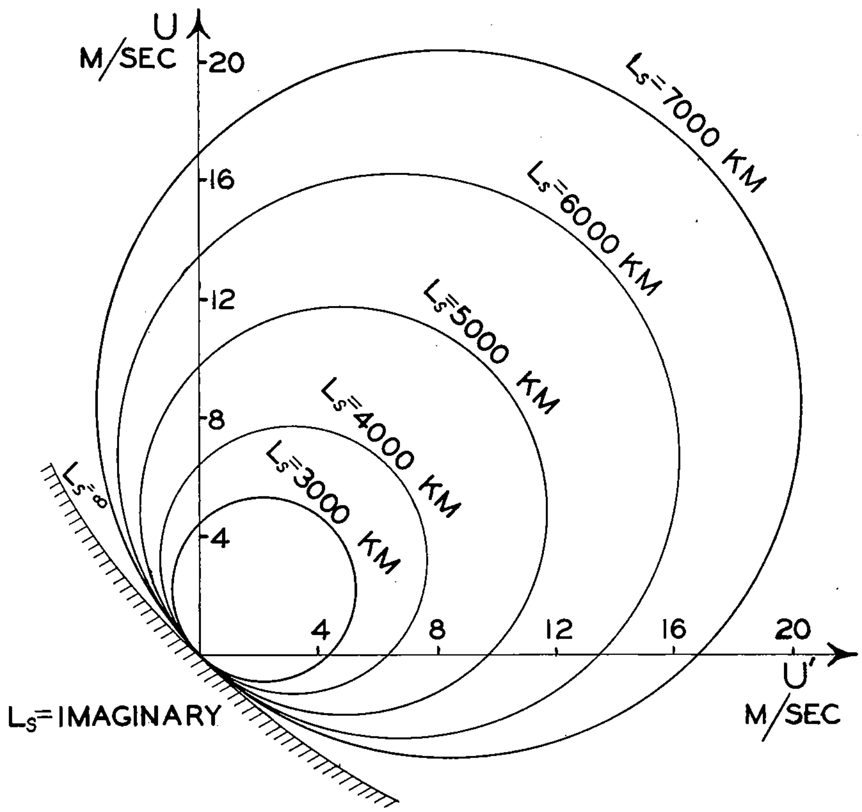

Fig. 11. Lengths of stationary waves for a double layer where D= 10,000 km.

obtain the critical wave lengths L_c at which instability sets in. Thus, if the radical in (18) is not to be imaginary it is necessary that

$$\left(\frac{\beta L}{4\pi^2}\right)^2 \geqq (U - U')^2.$$

Hence

$$L_c = \sqrt{\frac{4\pi^2(U - U')}{\beta}}. \tag{20}$$

Table III indicates the variation of this critical value for different latitudes and shear. It is evident from (20) that L_c depends only on the difference in wind velocities and not on the actual wind velocities in each layer.

TABLE III

CRITICAL WAVE LENGTHS AS A FUNCTION OF SHEAR $(U-U')$, FOR AN INFINITE DOUBLE LAYER AND DIFFERENT LATITUDES.

Latitude \ $U-U'$	4 m/sec	8 m/sec	12 m/sec	16 m/sec	20 m/sec
30°	2,820 km	4,020	4,880	5,650	6,310
45°	3,120	4,450	5,390	6,250	7,000
60°	3,712	5,270	6,420	7,430	8,330

9. Critical Wave Lengths for Finite Double Layers

From equation (17a) we can obtain the critical wave lengths where the disturbance is of finite lateral extent. Setting the expression under the radical equal to zero we get.

$$L_c = \frac{1}{\sqrt{\frac{\beta}{4\pi^2}\frac{1}{(U-U')} - \frac{1}{D^2}}}\,.$$

Fig. 12 shows the variation of L_c for different values of shear, latitude and lateral extent. Table IV contains some tabulated values for L_c when $D=10{,}000$ km. The effect of

TABLE IV

CRITICAL WAVE LENGTHS AS A FUNCTION OF SHEAR $(U-U')$, FOR A FINITE DOUBLE LAYER AND DIFFERENT LATITUDES.

Latitude \ $U-U'$	4 m/sec	8 m/sec	12 m/sec	16 m/sec	20 m/sec
30°	2,940 km	4,350	5,600	6,830	8,130
45°	3,290	4,925	6,440	8,025	9,780
60°	4,010	6,180	8,425	11,630	15,000

boundaries on critical wave lengths is similar to that on stationary ones. In both cases the wave length increases as the lateral extent of the wave diminishes. For small values of shear, the increase in wave length is slight, whereas for large values the increase is quite appreciable. Thus for $(U-U')=4$ m/sec,

$L_c = 2820$ km for the deep layer,

$L_c = 2940$ km for the shallow one,

while for $(U-U')=20$ m/sec,

$L_c = 6310$ km for the deep layer,

$L_c = 8130$ km for the shallow one,

where both cases are in latitude 30°. The effect of boundaries is much more pronounced for strong shear than for weak. The dotted lines in Fig. 12 shows this effect very clearly.