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VOLUME I

COURSE LECTURES

SEMINARS

ABSTRACTS

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Summer Study Program
in
GEOPHYSICAL FLUID DYNAMICS
at
The WOODS HOLE OCEANOGRAPHIC INSTITUTION

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Editor's Preface

The central topic of this seventeenth Geophysical Fluid Dynamics program was fluid motion in the earth's mantle and core. Our principal lecturer, Dan McKenzie, first addressed himself to the task of separating solid behavior of the mantle from fluid behavior. When the level of protest diminished Dan advanced to his numerical studies of mantle convection. The relationship of these numerical experiments and geophysical observables was impressive indeed for this first generation of mantle modeling.

Intertwined seminars from P. Molnar, B. Parsons, J. Sclater and T. Atwater exposed us to data gathering and its rationale at the frontiers of geophysics.

The fluid properties of the core may be less suspect than those of the mantle, but how and why the core fluid moves is still a mystery. Our associate principal lecturer, Fritz Busse, discussed the geomagnetic evidence for core motion. Then moving quickly to the more abstract problems of model geodynamos, Fritz described in five lectures his achievement of a first complete dynamic dynamo driven by convection.

The fellows of the summer program are responsible for the notes of the principal lecturers. The abstracts of other seminars recorded in this first volume were prepared by the speakers. The twelve fellowship lectures are recorded in a second volume.

Once again Mary C. Thayer has gathered and typed all the abstracts, lecture reports, and fellowship papers. Fellows and staff alike will remember her skill at graciously maintaining a taut ship.

We thank also the National Science Foundation and the Woods Hole Oceanographic Institution for support and encouragement.

Willem V.R. Malkus

Dr. Dan P. McKenzie, Principal Lecturer (slowly drifting westward).

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by

Dr. Dan P. McKenzie
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by

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COURSE LECTURES

by

Dan P. McKenzie
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Lecture #1.

PLATE TECTONICS: DEFINITIONS

Introduction to Plate Tectonics:

The field of plate tectonics encompasses an enormous mass of observational material. Present research is aimed at extracting some order from some of this data and then constructing appropriate models. This lecture will introduce plate tectonics and lead to our discussion of mantle convection.

The concepts of plate tectonics originally came from the examination of the earth's seismicity (Fig.1).

Fig.1 (Ref.1) Worldwide distribution of all earthquake epicenters for the period 1961 through 1967 as reported by U.S.Coast and Geodetic Survey (after Barazangi and Dorman, 1968). Note continuous narrow major seismic belts that outline seismic blocks; very narrow, sometimes step-like pattern of belts of only moderate activity along zones of spreading; broader very active belts along zones of convergence; diffuse pattern of moderate activity in certain continental zones.

Earthquakes are largely confined to narrow belts, especially in the oceanic trenches (e.g., the Aleutian) and in the oceanic ridges (e.g., the Mid-Atlantic). The spread in locations of epicenters is somewhat greater in the Southern Hemisphere, most probably because the data is less accurate there. In Asia, the earthquakes are spread over a region about 2000 km wide. The reasons for this are only now beginning to be understood, and the deformation of continents is a very active field of research.

The map of earthquake epicenters suggests dividing the earth into plates. One such hypothesized division is shown in Fig.2. A plate is defined to be a rigid cap on the earth's surface and is the key concept of plate tectonics.

Fig.2 (Ref.1) The crust is divided into units that move as rigid spherical caps. The boundaries between the caps are rises, trenches (or young fold mountains). The boundaries drawn in Asia are tentative, and more may be required. (This figure is based on Sykes's, 1969, map of the ridge system with additional features from Heezen and Tharp's, 1965, tectonic map).

As an example, consider plate B moving relative to plate A (which surrounds B).

Boundaries 1 and 2 are transform faults, denoted by single lines, along which there is pure slip. A and B separate along 3, which is a ridge, denoted by a double line. The plates converge at 4, which is a trench, denoted by a single line with short perpendicular lines.

Sea floor is produced on both sides of a ridge, on the average symmetrically. Fixing ourselves with respect to a ridge, the cross section would be

At a trench, material is consumed from one side. The plate being consumed is marked by a line with an arrow.

Suppose that in the above example, plate B is being used up. Then it is underthrusting A.

In the first diagram A was fixed, but the ridge was spreading symmetrically. Hence the ridge must move relative to A, and plate B becomes smaller.

Thus the sizes and shapes of plates are constantly changing simply because of the symmetry of ridges and the asymmetry of trenches.

The following could also occur, in which A is underthrusting B along 1 and B is passing under A along 2:

The top half of plate B becomes wider and the bottom half narrower.

One can see that even this simple description of ridges, trenches, and transform faults has all kinds of evolution built in.

So far we have considered translational motions in a plane. These ideas are easily extended to rotations, since any motion of B with respect to A can be represented by a rotation about a pole P and transform faults between A and B must be arcs of circles about P. A method of finding a pole of relative motion is to draw perpendiculars to several transform faults and see where they intersect.

The extension to a spherical surface is straightforward. The motion of one plate with respect to another can again always be represented by a rotation, and if the plate is to stay on the earth's surface the rotation axis must pass through the earth's centre. Transform faults must be small circles with respect to the pole of rotation, and the instantaneous motion of plate B with respect to A can be represented by an angular velocity ω_{A-B} . By considering circuits crossing three plates it is easily shown that

$$\omega_{A-B} + \omega_{B-C} + \omega_{C-A} = 0$$

and similarly for a greater number of plates. Such relations are useful in obtaining ω_{A-B} when it cannot be measured directly.

The extension of these ideas to include finite, rather than infinitesimal, motions is also straightforward. Any movement can be represented by a rotation through a finite angle about a pole, but since the instantaneous poles are not in general fixed with respect to the two plates involved there is not generally a simple relationship between the instantaneous and finite poles. Furthermore finite rotations must be represented by matrices, and they cannot be combined by the simple rules of vector addition. For these reasons most extensions of these simple ideas have been obtained from infinitesimal rotational vectors.

The actual determinations of relative velocities of plates depends almost entirely on data from the magnetic anomalies near ridges. These are measured by towing a magnetometer along the sea surface to get the variations in the total magnetic field at the ocean bottom, and then subtracting out the earth's mean magnetic field as measured by satellites. An example of the resulting pattern shows in Fig.3.

Such symmetric patterns are produced by a process first suggested by Vine and Matthews (1963). New sea floor is produced along a ridge axis by the upwelling of hot mantle material. As the pressure decreases partial melting occurs. Then the molten basalt meets the cold ocean water and solidifies into a very fine-grained rock. As they cool these rocks are magnetized in the prevailing direction of the earth's magnetic field. This magnetic field reverses itself occasionally (with a

Fig.3 (Ref.1) Summary diagram of the magnetic anomalies observed over Reykjanes Ridge. Straight lines indicate the axis of the ridge and the central positive anomaly (Heirtzler et al., 1966).

time scale of the order of several million years), but the upwelling basalt "remembers" and retains the direction of the sense of the magnetization from the time it was chilled. As material spreads away from the ridge, a pattern of magnetic stripes appears, showing how the earth's magnetic field alternated between being "normal" and "reversed".

The times at which the changeovers occurred has been established by radioactive-decay dating of lava flows near volcanoes, since solidifying lava also obtains its original magnetization. Then using the geomagnetic time scale and the magnetic anomaly pattern at a ridge, one can compute the spreading rate, the relative velocity of the two plates at the ridge. This is possible because the reversal pattern is aperiodic with certain magnetic anomalies having distinctive shapes. For simplicity, a constant spreading rate is commonly assumed.

These ideas have been extensively applied, and magnetic anomalies have now been used to date most parts of the world's oceans, and to investigate their evolution. A major difficulty, however, arises near the magnetic equator. There the magnetization is horizontal, and if the spreading is east-west, no anomalies appear on the sea surface. Dating in these cases can only be done by drilling from the ocean floor.

Another interesting problem is that segments of a few ridges show asymmetric spreading. This may be due to the ridge axis "jumping" to a new position once in a while, or may be due to true asymmetric plate production.

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Notes submitted by William A. Facinelli.

Lecture #2. SEISMOLOGY AND PLATE TECTONICS

Seismic data are used in plate tectonics for locating plate boundaries and for ascertaining the direction of motion of the plates. Only current information is yielded by such a study, instruments having been used in earthquake detection only since 1900. Reliable records date only from about 1960.

To locate earthquakes, one assumes that the pressure wave and shear wave velocities (respectively v_p and v_s) in the earth, considered as an elastic solid, are functions of the radius only. Using this, one constructs travel timetables, computing $\Delta(t)$, the angle between the epicenter and the station location as a function of duration of time between the event and its detection by the station, and inverting the relationship to give $t(\Delta)$. These generally give the location of the epicenter accurate to within 30 km though, for shocks in island arcs, the error may be much greater.

Joint hypocenter determination (JHD) is a method for determining the relative locations of several earthquakes occurring in a small area. One chooses a master event, and then determines the time and the (three-dimensional) location of all the other events relative to the master event.

Fault plane solutions are useful paradigms for describing earthquakes. We illustrate with an example.

This sketch is a cross section of a strike-slip fault, the plane of the paper being perpendicular to the fault plane. An earthquake occurs at X, with slipping along the fault plane, represented here by a solid line. The arrows parallel to the fault plane indicate the relative motion. The auxiliary plane is defined to be the plane perpendicular to the motion, and is represented here by a dotted line. The fault plane and the

auxiliary plane divide the region into four quadrants. The quadrants containing a black dot experience motion away from the earthquake center, i.e., compression, the quadrants containing an open circle experience motion toward the earthquake center, i.e., dilatation. Waves originating from regions experiencing compression are 180° out of phase with those originating from areas experiencing dilatation.

initial compression wave

initial dilative wave

Fault-plane solutions are obtained in the following fashion: draw a hemisphere through the earth with center at the epicenter. If the wave intersecting the hemisphere is compressive, place a black dot at the projection of the point of intersection onto the surface; if the wave is dilative, place an open dot there (the projection normally used is not the one depicted here, but an equal-area projection). For the strike-slip fault above, the corresponding fault-plane diagram is:

It is clear from the diagram that the fault and auxiliary planes are completely equivalent, and that additional information is required to decide which is which. One obtains the identical diagram by switching the fault and auxiliary planes, and changing the slip vectors appropriately.

The other two types of faulting, and their associated fault plane solutions, are given below.

Normal Fault (Observed along oceanic ridges)

Thrust Fault (Observed in trenches)

The darker solid lines represent the boundary of the faulted material. In the fault plane solutions, the boundaries between the compressive and dilative regions always intersect at right angles, and correspond to the fault and auxiliary planes, with the same sort of ambiguity as in the strike-slip case.

Fault plane solutions can be constructed directly from seismic data, using observations of the phase of incoming waves from a large number of stations. The ambiguity mentioned above is a persistent problem in using them for plate tectonics, where one wants to determine the direction of plate motion, i.e., the slip vector, which is normal to the auxiliary plane.

There are a number of ways to resolve this ambiguity. If the fault intersects the surface of the earth, one can look for displacement of geographical features along the fault plane. After an earthquake of considerable size, there is often a sequence of aftershocks in the same vicinity; studying them will often give the fault plane. One example is of a sequence of shocks occurring along a plane, which would then be the fault plane. Another example is given by the following events:

Since normal faults are often associated with ridges, the ambiguity in the strike-slip fault can be resolved as follows:

however, the ambiguity in the normal faults remain unresolved.

We now look at some examples.

Ridges and Transform Faults

One observes symmetric, continuous spreading from oceanic ridges. Oblique spreading (1) is what one might expect in several situations, particularly in the Atlantic. Though oblique spreading sometimes occurs, more commonly the boundary changes to a series of ridges perpendicular to the spreading direction, connected by transform faults parallel to the spreading direction (2).

In the ridge and transform fault system sketched here, the following topography and seismicity are observed.

The x's represent earthquakes: strike-slip along the transform fault, normal faulting along the ridges. One sees a notch at the center of the ridge. In cross section,

The topography of the transform fault, in cross section is

Trenches

Seismic information tells us that oceanic plate is subducted as trenches, maintaining its integrity to depths as great as 700 km in the mantle. The angle between the subducted plate and the horizontal varies from 25° to 90° . Along the surface where the underthrust plate meets the upper plate, one sees thrust faulting.

Deeper down, one observes earthquakes inside the slabs. Looking at the fault planes, one notices that the slip vectors are not in the direction of the dip of the slab, which would indicate slip between the descending slab and the surrounding mantle. The conclusion is that earthquakes occur along faults in the plate itself.

In the sinking slab, one sees both normal and thrust faulting. Where JHD (Joint Hypocenter Determination) has been used, the steeper plane has been the fault plane.

Normal Fault (Tension Earthquake)

Thrust Fault, (Compressional Earthquake)

If one assumes the axes at 45° to the fault and auxiliary planes are the axes of maximum and minimum stress, the T directions are the directions of minimum compressive stress, and the P directions are the directions of maximum compressive stress. So thrust faults correspond to compression along the slab, and normal faults to tension along the slab.

The following sketch summarizes the observations, and suggests an interpretation of earthquakes in the sinking slabs. Open dots represent compressional earthquakes, solid dots represent tensional earthquakes.

In (1), the dense slab sinking through the hotter lighter mantle material results in failure of the slab in tension. In (2) this mechanism is still causing tensional earthquakes in the upper part, but the increased strength of the surrounding mantle resists the intrusion of the slab, causing compressional earthquakes in the lower part. In (3) the slab is supported from below by the high strength part of the mantle, and there are compressive earthquakes throughout. Diagram (4) represents an absence of seismic activity, which is interpreted as a discontinuity in the plate itself. This might be caused by an actual discontinuity in the consumption of plate, or by the sinking rate being faster than the rate of subduction.

Notes submitted by
Phillip Colella.

Lecture #3 THERMAL MODELS: CREEP AND SIMILARITY SOLUTIONS

The observational data pertinent to plate tectonics have been summarized in the previous lectures. Assuming the plates act as a thermal boundary layer, we will first attempt to 'see' through the plates and deduce the motions below from our observations of the boundary. During the second part of today's lecture, we will discuss the mantle as a fluid.

1. A simple model of plate creation

We will attempt to explain the observed ridge topography and heat flow with the simple two-dimensional model diagramed below:

The z-axis, which coincides with the ridge, is an axis of symmetry. The temperature boundary conditions are: $T = T_1$ on $z = 0$ and on $x = 0$; $T = T_0$ on $z = d$. (There are no lateral variations in these temperature boundary conditions.) The spreading velocity v is a constant. In contrast to several previous models of mantle convection, the ridge is not the surface expression of the rising limb of a convection cell extending through the upper mantle.

The equation of heat conduction for steady-state two-dimensional flow is:

$$\int C_p \frac{DT}{Dt} = \int C_p v \frac{\partial T}{\partial x} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \quad (1)$$

Non-dimensionalizing (1) by $x = dx'$, $z = dz'$, and $T = T_0 + (T_1 - T_0)T'$, we have:

$$\frac{\partial^2 T'}{\partial x'^2} - 2R \frac{\partial T'}{\partial x'} + \frac{\partial^2 T'}{\partial z'^2} = 0 \quad (2)$$

The Rayleigh number $R = \frac{\text{advective heat flux}}{\text{diffusive heat flux}} = \frac{\int C_p v d}{2k}$, where k is the thermal conductivity ($\text{W m}^{-1} \text{ } ^\circ\text{C}^{-1}$) and C_p is the specific heat at constant pressure ($\text{J kg}^{-1} \text{ } ^\circ\text{C}^{-1}$). Typical values of R range from 30 for the mid-Atlantic to 110 for the mid-Pacific, depending on the spreading velocity v .

The solution of (2) is:

$$T' = 1 - z' - \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi} \exp \left[\left(R - \sqrt{R^2 + n^2 \pi^2} \right) x' \right] \cdot \sin n\pi z' \quad (3)$$

For $R \gg 1$ this may be written as

$$T' = 1 - z' - \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi} \exp \left[\left(-\frac{n^2 \pi^2}{2R} \right) x' \right] \cdot \sin n\pi z' \quad (3a)$$

The resulting isotherms are shown below.

It is not possible to verify this predicted temperature profile by direct measurement. One must use either the observed heat flow or the observed ridge elevation.

The predicted heat flow is:

$$-\left(\frac{\partial T'}{\partial z'}\right)_{z'=1} = 1 + 2 \sum_{n=1}^{\infty} \exp \left[\left(\frac{1}{2n-1} \right) x' \right] \quad (4)$$

where the quantity in brackets is the same as in (3) or (3a). Note that the integrated heat flux is finite only when the conductive horizontal flux is ignored.

(This is the large R approximation, where $\frac{\partial^2 T'}{\partial x'^2}$ is dropped from Eq. 2.) Comparison with the observed heat flow requires that $d \approx 50$ km and $T_1 \approx 550$ °C. This temperature is too low to detach the plate from material below or to account for the molten basalt with $T_1 \approx 1200$ °C. However, the heat-flow measurements are suspect (Parsons and Sclater, 1975). It appears that jets of hot water carrying heat by convection pass through the porous rocks of the oceanic crust. Thus measurements of the conductive flux, which is the product of temperature gradient and rock conductivity, underestimate the total heat flow. Indeed, heat-flow measurements are consistently higher at carefully selected spots that are capped by sediments to prevent convection of water.

This difficulty may be avoided by comparing the result of (3) or (3a) with the observed ridge topography instead. No gravity anomalies are observed across a ridge, indicating that the density per unit area is uniform. We can thus compute the elevation, knowing the coefficient of thermal expansion:

$$e'(x) \cdot d = \frac{4\alpha \rho T_1 d}{(\rho - \rho_w) \pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \exp \left[\left(\frac{1}{2k-1} \right) x' \right] \quad (5)$$

The dimensionless ridge elevation is given by $e'(x)$; α is the coefficient of thermal expansion (°C⁻¹) for the plate; ρ and ρ_w are the densities (kg m⁻³) of the plate and water, respectively. Here we have assumed the chemical composition to be homogeneous. Comparison with topographical data yields $d = 117$ km $\pm$ 15 and T_1 equals roughly 1200 °C. In addition, the good (carefully selected) heat-flow data are consistent with these values.

The model discussed above requires the temperature at a depth d to be maintained at T_1 . This boundary condition provides a lower limit on the temperature gradient and an upper limit on the ocean depth. An alternative model has been proposed in which the plate thickness increases without limit as heat is conducted to the sea bottom. This model gives exactly the same results as the previous one near the ridge axis but provides no lower bound on the temperature or upper bound on ocean depth, which increases at a rate proportional to $\sqrt{t}$. Parsons and Sclater (1975) have recently demonstrated that the depth observations deviate from a $\sqrt{t}$ dependence on age and closely follow Eq. (5).

The simple plate model discussed today is basically a diagnostic tool. Given a presumed spreading rate, the resulting temperature distribution can be calculated. The results indicate that:

(1) There must be heating from below away from the ridge.

(2) Convection below the plate is required to maintain a constant temperature at the lower plate boundary.

The most important conclusion provided by this model is that there is no reason, on the basis of surface observations alone, to require a hot upwelling plume in the mantle directly beneath the ridge.

2. The mantle as a fluid

The mantle material is basically a crystalline solid. Sometimes, however, it behaves as a Newtonian fluid. We write experimentally

$$\dot{\epsilon} \propto \sigma^n \quad (6)$$

where $\dot{\epsilon}$ is the creep rate and σ is the shearing stress. A commonly used stress-strain relation implied by (6) is

$$\dot{\epsilon}_{ij} \propto (\tau_{kl} \tau_{kl})^{\frac{n-1}{2}} \cdot D(T) \cdot \tau_{ij} \quad (7)$$

In this notation, $\dot{\epsilon}_{ij}$ is the rate of strain tensor, $D(T)$ is a temperature-dependent diffusion coefficient, and τ_{ij} is the deviatoric stress tensor.

A regular crystal can only be deformed elastically. For fluidlike motions, we must consider defects in the crystal lattice. These may be in the form of edge, screw, or point dislocations (Read, 1953). In general, relatively little stress is required to move dislocations. They may, however, lock into each other, resulting in work hardening. Illustrated below is a crystal lattice having a pair of edge dislocations that became locked in with each other.

Further deformation of the above lattice occurs only through dislocation climb.

At low temperatures, vacancies in the crystal lattice can creep diffusively along the boundaries of crystal grains. At somewhat higher temperatures, the creep occurs through the grain body. In either case, the stress-strain relation is like that for a Newtonian fluid ($n = 1$). Under stress, grain sizes do not change very much. The shapes, however, adjust much as soap bubbles do. Metals can deform more easily than other crystals because there are no forbidden slip planes due to the distribution of charges.

Creep controlled by dislocation climb can occur at high stress ($n \geq 1$). Under these conditions, the viscosity is non-Newtonian, not independent of stress.

There is need for experimentation on olivine, which emerges in Kimberlite pipes from depths of about 250 km and thus indicates the stress state of material within the convecting region (Stocker and Ashby, 1973). In particular, there is a lack of data in the borderline region with low stress, where diffusive creep and dislocation climb become about equally important.

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Rev. Geophys. Space Phys., 11: pp. 391-426.

Notes submitted by
 Kirk S. Hansen and
 Howard C. Houben.

Lecture #4

MODELS OF MANTLE CONVECTION

The previous lectures have illustrated the difficulty in extrapolating from surface observations to what goes on below. Today we will construct a dynamical model of the crust and upper mantle from first principles.

It is a hopeless task to solve the equations of fluid dynamics for the mantle in three dimensions. Adequate resolution of the temperature field requires too many grid points. It is even more difficult when non-Newtonian creep is to be considered.

We will attempt a simplified conceptual model of the situation. The elements of this model are:

(1) Two-dimensional flow (because of the computational limitations).

(2) Lower boundary of convection at 700 km. (This is the observed maximum depth at which earthquakes occur; also, the phase change to the very hard mineral stishovite occurs near this level.)

(3) Plane-layer model (due to the small vertical scale compared with the Earth's radius).

The relevant dimensionless parameter is the Rayleigh number, which is defined here in terms of constant flux rather than constant temperature gradient.

$$R = \frac{g \alpha d^4}{\kappa^2 \nu} (f + \epsilon d) \quad (1)$$

The thermometric conductivity $\kappa = K/\rho c_p$, while $f = F/\rho c_p$ and $\epsilon = H/\rho c_p$ are the thermometric heat fluxes. The flux through the bottom boundary is given by F and the internal (radiogenic) heat generation by H . It should be noted that if there is convection within the Earth, all original heat from the accretion of the Earth and from core formation has been lost, since the convection is so efficient.

The Rayleigh number may be evaluated from the following parameters:

g is known and (coincidentally) nearly constant in the region of interest.

α is known to within a factor of two except perhaps in the region of the phase change from olivine to spinel. If α jumps by an order of magnitude in this layer, the Boussinesq approximation may be in jeopardy.

$(f + \epsilon d)$ is just the total heat flow, which is measurable. Petrological evidence concerning the abundance of radioactives (K, Th, U) in basalts from the mid-ocean ridges indicates that about half of this is from the upper mantle (less than 700-km depth). This figure may, however, be adjusted by a factor of 2, since it is based on an estimate of the partial melting of the mantle material.

The appropriate value of $K(T)$ is disputed but is probably known to within a factor of two also. We use $K(T) = \text{constant}$, which appears to be a valid approximation. Evaluating the kinematic viscosity $\nu(T)$ presents the greatest difficulty. It may vary by several orders of magnitude. For these calculations, however, $\nu(T)$ is taken as constant.

Some numerical values are: $\alpha = 2 \times 10^{-5} \text{ } ^\circ\text{C}^{-1}$; $K = 1.5 \times 10^{-6} \text{ m}^2 \text{ sec}^{-1}$; and $\nu = 2 \times 10^{17} \text{ m}^2 \text{ sec}^{-1}$. A wide range of Rayleigh numbers is generated by varying the heat flux ($f + \epsilon d$). Three different cases are investigated for each R: all heating within, half within and half below, and all heating from below.

The plates are considered to be part of the system, because they contribute to the energetics, as in the case of a cold descending plate. This aids in the study of stability.

The dimensionless heat and momentum equations have been derived by McKenzie, Roberts, and Weiss (1974).

$$\frac{\partial T'}{\partial t'} = -\nabla \cdot (T' \underline{u}') + \epsilon' + K' \nabla^2 T' \quad (2)$$

$$\nabla^4 \psi' = -\partial T' / \partial x' \quad (3)$$

The non-dimensional energy generation rate is $\epsilon' = \epsilon \nu / g \alpha T_1 d = H \nu / \rho C_p \alpha T_1 d^2$ and non-dimensional conductivity $K' = \frac{\nu}{g \alpha T_1 d^2}$. In deriving (2) and (3), the time variable was scaled using $T = \frac{\nu}{g \alpha T_1 d}$, which is like an overturn time for the temperature T_1 at the lower boundary.

Reflection symmetry is imposed at the vertical boundaries. The temperature is fixed at the top, while (for most cases) the heat flux is fixed at the bottom. A few runs were made for the Rayleigh-Bénard problem (fixed temperature at lower boundary). Also, most calculations were for free boundaries.

Solutions are obtained for 700-km squares (24 X 24 grid points) and 1400 X 700-km rectangles (48 X 24 grid points). A few of the calculations with the square region, primarily for the Rayleigh-Bénard problem, have to be carried out on a 48 X 48 mesh to insure adequate resolution of the boundary layers. The stream function ψ and the temperature T are stored at alternate grid points to speed the calculations, and the ψ -points and T -points are interchanged at each successive time step.

When the heat source is purely internal, the relaxation time for the calculations is on the order of the age of the Earth. It is only 0.1 to 0.25 times as long for the case of heating from below. Typical results for the 700-km square indicate a single convective cell. For internal heating, the streamline pattern is asymmetrically distributed, especially at high Rayleigh numbers, with an intense descending branch. For heating from below, the resulting flow pattern is much more symmetrical. There is very little difference between the fixed-flux problem with heating from below, and the Rayleigh-Bénard problem.

At still higher Rayleigh numbers, and with all heating from within, the convective pattern in the square region forms two or three distinct cells.

When the 1400 X 700-km rectangle is utilized, the circulation always breaks up into two or more cells. We have thus not explained, on the basis of mantle convection, any horizontal length scale sufficiently long to explain the length scale of the major plates. (No time-dependent studies have found stable solutions of this kind.)

Vorticity, streamline, and temperature distributions for the various cases discussed above may be found in McKenzie, et al., (1974). Some additional results of the calculations are:

- (1) Non-Newtonian creep laws do not seriously affect the results (for reasonable values of n).
- (2) The turnover time for mantle convection is about 100×10^6 years.
- (3) The mean temperature profile (averaged over a cell) is always stably stratified. This result has been obtained also by other investigators in studies of Rayleigh-Bénard convection.
- (4) Variation of viscosity as $\eta = \eta_0 e^{-\alpha T}$ leads to a strong concentration of
  - (a) the sinking region when heating is from within
  - (b) the hot, rising jet when heating is from below.

Reference

McKenzie, D.P., J.M. Roberts, and N.O. Weiss, 1974: Convection in the earth's mantle: towards a numerical simulation. J. Fluid Mech., 62: 465-538.

Notes submitted by
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Lecture #5

MODELS AND GEOPHYSICAL OBSERVABLES

The major problem which has beset research into mantle convection is that of deciding what observations are relevant. Most geophysical and geological observations are concerned with the mechanical and thermal properties of the plates, rather than with the movements in the mantle below.

Flow in the mantle is required by the plate motions themselves, and their velocities and sizes are clearly of importance. Long wavelength gravity anomalies measured either on the earth's surface or by satellite motions cannot be supported by elastic forces within the plate. Since there is no evidence that rocks possess finite strength at high temperatures, such anomalies must be maintained by flow in the mantle. The topography of ocean ridges can be simply accounted for by the creation of a hot plate which cools with time. However this model can only account for the depth of sea floor older than 100 My if the plate thickness tends to maximum value (see Parsons' abstract). To prevent old plate from becoming thicker requires heat to be supplied from below, presumably by convection. (Though the plate model can account for the mean variation of depth with age by thermal contraction as the plate moves away from the spreading axis, significant regional depth anomalies remain after the mean variation has been removed. Sclater et al. (1975) has demonstrated that these anomalies correlate with the long wavelength gravity anomalies, and believe that they are of convective origin.

The most widely published evidence of mantle motion is the existence of volcanoes, such as Hawaii, far from plate boundaries. Following Wilson (1965) and Morgan (1971) it has become popular to believe that such structures lie above sites of mantle upwelling, called plumes. It is not, however, yet clear whether the mantle below such features is anomalously hot or whether it merely contains a larger fraction of minerals with low melting points. Furthermore the position of the volcanoes is likely to be controlled by lines of weakness in the plates as well as by mantle processes.

Besides these geophysical observations, there is now some geological evidence relevant to mantle motions. Recently attempts have been made to determine the creep behaviour of mantle rocks by studying the fabric of nodules in kimberlite pipes. It is not easy to separate the deformation associated with the emplacement from earlier less intense deformation which occurred while they were part of mantle flow. Despite this difficulty some progress has been made, and the creep appears to be controlled by dislocation climb and to be non-Newtonian. Geology has also provided the only evidence for any change in the form of the flow as the rate of radioactive heat generation decreased. The style of Early Precambrian tectonics (older than 2,800 My) appears to be different from that of present-day tectonics. The scale of the structures is smaller, and the temperature of the lavas often 200°C hotter.

Considerable efforts have recently been devoted to attempts to understand these observations with simple convective models (see McKenzie, Roberts and Weiss, 1974, for a summary of some of these). The simplest convective models consist of a layer of fluid with constant properties in which two-dimensional flow occurs. The heat is supplied from below or from within, is convected close to the upper boundary where it is lost by conduction. The various parameters involved can be estimated from field and laboratory estimates, and the resulting flow can produce depth anomalies, gravity anomalies and velocities of the right order of magnitude. The main difference between internal heating and heating from below occurs in the form of the flow. The former consists of slow upward motion almost everywhere with narrow rapidly moving jets of cold fluid descending from the surface. When the heat is supplied from below rising sheets of hot fluid and sinking sheets of cold fluid transport the heat. The principal shortcoming of the numerical solutions is that they suppress certain classes of two-dimensional, and all classes of three-dimensional, instability. Laboratory experiments suggest that the form of the flow is three-dimensional at high Rayleigh numbers. The gross features of the heat transport and velocity field do not, however, appear to depend on the details of the motion.

The main shortcoming of the simple models is their inability to produce cells whose width is large compared with the depth of the layer. Even if such cells are artificially introduced they break up into smaller cells whose width is comparable to the depth of the layer. Laboratory experiments show the same behaviour.

Similar experiments were carried out with a strongly temperature-dependent viscosity. When the heat was supplied internally the flow was not strongly affected by the viscosity variation, though the sinking sheets became somewhat thinner. When the heat was supplied from below, however, narrow rising sheets resulted with broad sinking regions. The viscosity variation did not, however, stabilize cells with a width much greater than the depth, and therefore the scale problem remained. These experiments suggest that two scales of convection must exist, one with a width comparable or smaller than the depth and with little surface expression, the other corresponding to the plate motions. The motions are almost certainly three-dimensional, and plates only exist if the viscosity varies strongly with temperature. It is not simple to study such convection either numerically or in the laboratory. At present only the depth of old plates gives us any information about the small-scale flows, and it is very desirable to find some other evidence of its existence and plan form.

The fabric of kimberlite nodules is the result of the integrated deformation as the rock is convected. To extract comparable information from numerical experiments, we used a Lagrangian description of the motion. It is not difficult to convert from the usual Eulerian description used to generate the numerical models to Lagrangian description of the motion. Unfortunately the results of such experiments show that there is no simple relation between the fabric and other features of the flow. It is not even simple to discover the importance of dislocation climb from such observations.

Though some problems still remain it is now clear that there are no obvious objections to maintaining the plate motions by mantle convection, and that models now exist which can account for all observations known to be relevant. The major factor which has permitted this considerable progress is the high viscosity of the mantle. Unlike the atmosphere, the oceans and the core, motions whose scale is less than a few hundred kilometers dissipate more energy than is available from thermal convection. The resulting motions must therefore be relatively simple and furthermore convect negligible moment. Therefore many of the complications of other branches of Geophysical Fluid Mechanics are absent and considerable progress can be expected in this field over the next few years.

Notes submitted by
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HYDRODYNAMICS OF THE EARTH'S CORE

by

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The radial structure of the Earth is believed to consist of a number of layers. The outermost layer is the mantle, (1), which consists of a number of subdivisions. (See previous lectures on plate tectonics,] The next inner layer is the outer core, (2), with an outer radius of 3483 km. (The ellipticity of the Earth causes variations of about 10 km in this value.) Finally, there is an inner core, (3), of radius 1216 km, which value is less accurately known than the radius of the outer core. These radii are obtained from seismic wave data.

It is presumed that the Earth's magnetic field is generated by motions in the Earth's core; there are a number of phenomena associated with the observed magnetic field.

<u>Phenomenon</u>	<u>Time Scales</u>	<u>Origin</u>
Reversals	10^5 - 10^6 yrs	Intrinsic property of the dynamo process; otherwise difficult to explain
Long period variations (dipole field)	10^4 yrs	Redistribution of magnetic energy between magnetic modes, with nonlinear periodicity.
"Westward" drift of non-dipole field.	10^2 - 10^3 yrs (approximately .2 per yr.)	This drift could indicate a differential rotation of the core, or it may be an indication of the propagation of waves. (There is some debate as to whether westward drift really exists, but general consensus is that it does, although there does appear to be an eastward drift in high latitudes.)
Changes of period and changes of direction of rotation (wobble)	approx. 10 yrs.	Coupled electromagnetic and mechanical mechanisms, possibly excited by earthquakes. (Pre-earthquake movements are of large period and amplitude and could vary the moment of inertia tensor.)
Gravitational oscillations at the core surface	< 1 day	These are expected but have yet to be observed. Such perturbations in the core should lead to relative changes in gravitational field strength of 1 part in 10^6 .
Elastic free oscillations	< 1 hr	Excited by earthquakes, and result from the elastic nature of the restoring forces on these time scales.

Core Paradox. (Higgins and Kennedy, 1971, 1973).

The argument of Higgins and Kennedy (1971) commenced with the reasonable assumption that the inner-outer core boundary is a melting point boundary. The resulting temperature profile in the outer core depends on the value of the Gruneisen

parameter $\Gamma = \frac{\alpha C_p}{\rho K_s} = \frac{\alpha C_v}{\rho K_T}$. Assuming a typical value of 1.66 for Γ , Higgins and

Kennedy at first believed that the outer core temperature profile was subadiabatic everywhere. Hence there could be no motions with which to sustain a dynamo: Herein lies the paradox.

However, in their second paper (1973), Higgins and Kennedy realized that this was not true for an 800 km thick region near the inner core boundary, where the adiabatic gradients are quite small. For the entire outer core to be superadiabatic, α must take on a value of 0.25. This argument therefore seems to imply that convection is possible in the outer core, but in a restricted region only. This restriction can be relaxed to some extent by allowing condensation of iron in the outer core, which enables the material to change adiabats while moving isentopically, and so remain on the melting point curve. In this way, the convective region can be extended even to the whole outer core for the right composition of outer core material.

Turbulence driven by either convection or precession has a hard time against a stably stratified core. In the case of precession, we know the energy source, although it has not been answered yet whether this is enough to power the geodynamo. On the other hand, we read some other energy source for driving convection. It is known that precession occurs in the core, but it is not known if convection exists there.

Notes submitted by
Stephen Jones

Lecture #2   HYDRODYNAMICS OF THE EARTH'S CORE - A GRAVITATING AND ROTATING SYSTEM.

The two important nondimensional parameters of fluid motion in a rotating system are:

The Ekman number $E = \frac{\nu}{\Omega r_0^2}$
 and The Rossby number $\mathcal{E} = \frac{U}{\Omega r_0}$ {Note: the Reynolds number $Re = \frac{\mathcal{E}}{E}$ }

In the Earth's core, both parameters are small, so the Coriolis forces are the most important, and balance the pressure gradients to good approximation.

If the fluid in the core is taken to be Boussinesq, then the governing equations are:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} + 2 \underline{\Omega} \times \underline{v} = -\nabla P + \underline{g} \frac{\rho}{\rho_0} + \nu \nabla^2 \underline{v} \quad (1)$$

where $P = p/\rho_0$

$$\nabla \cdot \underline{v} = 0 \quad (2)$$

For a basic state when $\underline{E}$, $\underline{E}$ and the stratifications are small, the resulting balance is geostrophic:

$$2 \underline{\Omega} \times \underline{v} = -\nabla P$$

Taking the curl, yields, for steady motion

$$-2 \underline{\Omega} \cdot \nabla \underline{v} = 0.$$

This is the mathematical statement of the Taylor-Proudman theorem, which states that $\underline{v}$ cannot vary in the direction of $\underline{\Omega}$ for small amplitude, steady motion:

$$\underline{v} = \underline{v}(x, y) \text{ where } \underline{\Omega} = \Omega \underline{\hat{z}}.$$

This theorem imposes a severe restriction on convection in a rotating fluid because convection requires a $\underline{\hat{z}}$ velocity, yet this velocity must be zero at the top and bottom boundaries:

However, it can be seen that convection is easier to maintain if $\underline{g}$ is at an angle to $\underline{\Omega}$, since cylindrical columns can form with axes coaxial to $\underline{\Omega}$ such that there is both a zero normal velocity at the boundaries and motions independent of the $\underline{\Omega}$ -ordinate (i.e., the Taylor-Proudman theorem is satisfied).

Examples:

This suggests the following laboratory problem of convection in the annulus between two rapidly, co-rotating cylinders whose end-walls slope so that the inner length is larger than the outer length. The system is assumed to be rotating fast enough so that the centrifugal force is the only effective "gravity" in the system. The outer cylinder is heated, and a temperature difference $\Delta T = (T_2 - T_1)$ is main-

tained between the cylinders. The heating is expected to drive convection against the centrifugal force, and we are interested in non-axisymmetric disturbances, or

rolls. We assume the narrow gap approximation, $D/r_0 \ll 1$, so that a local Cartesian coordinate system may be used. The equation of state relating the density field to the temperature field is

$$\rho = \rho_0 \{1 - \alpha (T - T_0)\} \quad (3)$$

The scales used are: length $[L]$
time $[\Omega^{-1}]$
temperature $[(T_2 - T_1) L/D]$
velocity $[L\Omega]$

The resulting nondimensional equations are:

$$\frac{\partial \hat{\underline{u}}}{\partial t} + (\hat{\underline{u}} \cdot \nabla) \hat{\underline{u}} + 2 \hat{\underline{k}} \times \hat{\underline{u}} = -\nabla \hat{p} - B \hat{\underline{i}} \hat{\theta} + E \nabla^2 \hat{\underline{u}} \quad (4)$$

$$\nabla \cdot \hat{\underline{u}} = 0 \quad (5)$$

$$\frac{\partial \hat{\theta}}{\partial t} + \hat{\underline{u}} \cdot \nabla \hat{\theta} = P^{-1} E \nabla^2 \hat{\theta} \quad (6)$$

where $B = \frac{\alpha (T_2 - T_1) r_0}{D}$ (Buoyancy parameter) (7)

$$E = \nu / \Omega L^2 \quad (8)$$

$$P = \nu / \kappa \quad (\text{Prandtl Number}) \quad (9)$$

A static solution for the flow in the cylindrical annulus is

$$\hat{V}_0 = 0, \hat{\theta}_0 = \frac{(T_2 + T_1)}{2(T_2 - T_1)} \frac{D}{L} + \chi \quad (10)$$

If we perturb $\hat{V}$ and $\hat{\theta}$: $\hat{V} = \hat{V}_0 + \underline{v}$, $\hat{\theta} = \hat{\theta}_0 + \theta$, the following linearized perturbation equations result

$$\frac{\partial \underline{v}}{\partial t} + 2\mathbf{k} \times \underline{v} = -\nabla p - B \hat{i} \theta + E \nabla^2 \underline{v} \quad (11)$$

$$\nabla \cdot \underline{v} = 0 \quad (12)$$

$$\frac{\partial \theta}{\partial t} = -\underline{v} \cdot \hat{i} + E P^{-1} \nabla^2 \theta \quad (13)$$

The y-t structure of the solution is expressed in the form

$$\underline{v}, \theta \propto \exp\{i\alpha y + i\omega t\}$$

The nondimensional parameters B and E are both assumed small, yielding an approximate geostrophic balance:

$$\underline{v} = -\frac{1}{2} \nabla \times p \mathbf{k} \quad \text{where} \quad p = p(x) e^{i\alpha y + i\omega t} \quad \text{is the pressure field.} \quad (14)$$

The appropriate boundary conditions are zero normal velocity at the side walls, and a combination of geometrical and Ekman boundary layer effects at the top and bottom boundaries. To simplify the geometrical effects, we assume that φ is small. The Ekman effects in the boundary layer come from the requirement that the vorticity parallel to the rotation axis must go to zero on the boundary. This results in a pressure drop in the boundary layer leading to suction or expulsion depending upon the sign of the relative vorticity. (See Greenspan).

Therefore, the boundary conditions are:

$$\underline{v} \cdot \hat{i} = 0 \quad \text{at} \quad x = \pm \frac{1}{2} D/L$$

which implies

$$p' = 0 \quad \text{at} \quad x = \pm \frac{1}{2} D/L \quad (15)$$

and

$$\underline{v} \cdot \mathbf{k} = -\eta_T \underline{v} \cdot \hat{i} - \frac{1}{2} E \mathbf{k} \cdot \nabla \times \underline{v} \quad \text{at} \quad z = \frac{1}{2} \quad (16)$$

$$\underline{v} \cdot \mathbf{k} = \eta_B \underline{v} \cdot \hat{i} + \frac{1}{2} E \mathbf{k} \cdot \nabla \times \underline{v} \quad \text{at} \quad z = -\frac{1}{2} \quad (17)$$

where

$$\eta_T = \tan \varphi_T, \eta_B = \tan \varphi_B \quad \text{are small.}$$

The vorticity equation derived from (11) is

$$i\omega \nabla \times \underline{v} - 2\mathbf{k} \cdot \nabla \underline{v} = B \hat{i} \times \nabla \theta + E \nabla^2 (\nabla \times \underline{v})$$

Indicating integration over $\int_{-1/2}^{1/2} ( ) dz$ by $(\overline{\quad})$, this equation becomes, upon taking the scalar product with $\mathbf{k}$

$$\underline{k} \cdot \nabla \times \{E \nabla^2 \underline{v} - i\omega \underline{v}\} + B \frac{\partial}{\partial z} \theta = [2\eta_T \underline{v} \cdot \underline{i} + E^{1/2} \underline{k} \cdot \nabla \times \underline{v}]_{z=1/2} + [2\eta_B \underline{v} \cdot \underline{i} + E^{1/2} \underline{k} \cdot \nabla \times \underline{v}]_{z=-1/2} \quad (18)$$

Now we know that the flow is approximately geostrophic, so to first order, (18) is the geostrophic evolution equation, with $\underline{v}$, independent of z . Thus

$$\underline{k} \cdot \nabla \times \underline{v} = \frac{1}{2} \left(\frac{d^2}{dx^2} - \alpha^2 \right) p(x) e^{i\alpha y + i\omega t}$$

$$\underline{v} \cdot \underline{i} = -\frac{1}{2} p_y = -\frac{1}{2} i\alpha p(x) e^{i\alpha y + i\omega t}$$

and (18) becomes

$$\frac{1}{2} (E \Delta_2 - 2E^{1/2} - i\omega) \Delta_2 p(x) e^{i\alpha y + i\omega t} + i\alpha (\eta_T + \eta_B) p(x) e^{i\alpha y + i\omega t} + B \theta_y = 0 \quad (19)$$

Similarly, the heat equation, (13) becomes

$$[P^{-1} E \Delta_2 - i\omega] \theta = i \underline{i} \cdot \underline{v} = -\frac{1}{2} i\alpha p(x) e^{i\alpha y + i\omega t} \quad (20)$$

where

$$\Delta_2 \equiv \frac{d^2}{dx^2} - \alpha^2 \quad (21)$$

Eliminating θ between the two equations yields

$$(P^{-1} E \Delta_2 - i\omega) \left[(E \Delta_2 - 2E^{1/2} - i\omega) \Delta_2 + 2i\alpha (\eta_T + \eta_B) \right] p(x) = -\alpha^2 B p(x) \quad (22)$$

In the limit $\varphi \rightarrow 0$ we have $\eta_T + \eta_B = 0$. There are two cases of interest here. First, if $E^{-1/4} \ll L/D$, then $2E^{1/2}$ is small compared to $E \Delta_2$, and Bénard convection results, with rolls, at the critical Buoyancy parameter $B_c = P^{-1} \left(\frac{L}{D} \right)^4 E^2 (1708)$. The rolls are aligned along the rotational axis (Ω), and they possess a wavenumber $\alpha_c = 3.11$ and $\omega = 0$ (marginal stability). In the opposite limit of $E^{1/4} \gg L/D$, the $E^{1/2}$ term dominates over the $E \Delta_2$ term, so that the problem is of second order and $p(x) = \cos \frac{\pi L}{D} x$. The critical value of the Buoyancy parameter is $B_c = 8 \left(\frac{\pi L}{D} \right)^2 E^{3/2} P^{-1}$, and $\alpha_c = \pi L/D$ and $\omega = 0$. In the limit where we must allow for the inclination of the boundary, ($\eta_T + \eta_B \gg E^{1/4}$) we obtain

$$B_c = 3 \left\{ 4 P E^2 \left(\frac{\eta_T + \eta_B}{1+P} \right)^4 \right\}^{1/3}$$

$$\alpha_c = \left\{ \frac{\sqrt{2} (\eta_T + \eta_B) P}{E (1+P)} \right\}^{1/2} \quad \text{and} \quad \omega = \frac{-2 (\eta_T + \eta_B)}{\alpha (1+P)}$$

This implies that there is an eastward drift in phase, so the presence of the "sloping" boundary is generating some type of Rossby-like wave. Essentially, the sloping boundaries have disturbed the geostrophic balance and have made it harder for convection to develop.

The dependence of B_c on E is now $E^{2/3}$, which makes it less dependent on the value of the viscosity which the convection must overcome, and furthermore, it does

not depend on D any more. (This gives us confidence in applying the theory to the interior of the spherical Earth, which possesses no natural D . B_c also increases with $(\eta_T + \eta_g)$, implying that the onset of convection becomes more difficult as the inclination of the boundary increases.

The motivation behind the above problem is that we can break a sphere up into coaxial cylinders with sloping upper and lower boundaries.

For the sphere, the temperature and gravity fields are described by

$$\nabla T = -\beta_0 \hat{r}$$

$$\underline{g} = -g \hat{r}$$

Therefore

$$B = \gamma \beta_0 \sin^2 \theta r_0 = B_0 \sin^2 \theta$$

$$E = \frac{\gamma}{\Omega (2 r_0 \cos \theta)^2} = E_0 \sec^2 \theta$$

In regions where the slope of the upper and lower boundaries is important, it follows that

$$B_{oc} = 3 \left\{ \frac{P E_0^2}{4} \left(\frac{2 \tan \theta}{(1+P) \cos \theta} \right)^4 \right\}^{1/3} / \sin^2 \theta$$

and

$\sin \theta_c = \frac{1}{\sqrt{5}}$ or $\theta_c = 26.6^\circ$ where $\frac{\tan^4 \theta}{\sin^6 \theta \cos^4 \theta} = \text{minimum}$. (This treatment is not exact; see Busse, 1970, for a more exact analysis).

We conclude that instabilities are only likely to develop near $\theta = 30^\circ$ because near the equator the inclination of the boundaries is too large, while near the poles, the centrifugal force is too small.

Experiments show that a ring of rolls develops. As the buoyancy parameter increases, the stabilizing effects of the sloping boundaries is overcome and the columns extend further outwards in the sphere. In addition, the column drifts eastwards with an angular velocity (relative to the rotating apparatus) which depends on latitude.

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Notes submitted by
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BASIC DYNAMO THEORY

Friedrich H. Busse

Introduction

The fundamental equations governing electrodynamics are Maxwell's equations which (in cgs units) are:

$$\nabla \cdot \underline{B} = 0 \quad (1)$$

$$\nabla \times \underline{B}/\mu = \frac{4\pi}{c} \underline{j} + \frac{1}{c} \frac{\partial \epsilon \underline{E}}{\partial t} \quad (2)$$

$$\nabla \cdot \epsilon \underline{E} = \rho_{\text{free}} \quad (3)$$

$$\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} \quad (4)$$

where all symbols have their usual physical connotations. In addition we may relate the current density to the fields by a modified form of Ohm's law:

$$\underline{j} = \sigma \left(\underline{E} + \frac{\underline{v}}{c} \times \underline{B} \right), \quad (5)$$

σ being the electrical conductivity of the medium. For nonrelativistic motion we may simplify Eq. (2) as follows:

The left-hand side of (2) is of order B/L while the displacement current $\sim \frac{v}{c} \frac{E}{L}$, L being a characteristic length of the system. From (4), however, we see that $\frac{E}{L} \sim \frac{v}{c} \frac{B}{L}$ hence, $\frac{v}{c} \frac{E}{L} \sim \left(\frac{v}{c}\right)^2 \frac{B}{L}$. Thus the displacement current is negligible compared with j/c , as j/c must be of order B/L so as to balance the left-hand side of (2). Neglecting the displacement current in (2) is known as the 'Magnetohydrodynamic Approximation'.

Eliminating $\underline{j}$ between (2) and (5), taking the curl of this expression and using (4) to eliminate curl $\underline{E}$ we obtain

$$\lambda \text{curl} (\text{curl} \underline{B}) = -\frac{\partial \underline{B}}{\partial t} + \text{curl} (\underline{v} \times \underline{B}), \quad (6)$$

where $\lambda = \frac{c^2}{4\pi\mu\sigma}$ is the magnetic diffusivity.

Expanding $\nabla \times (\nabla \times \underline{B}) = \nabla (\nabla \cdot \underline{B}) - \nabla^2 \underline{B} = -\nabla^2 \underline{B}$, Eq. (6) becomes

$$\frac{\partial \underline{B}}{\partial t} - \lambda \nabla^2 \underline{B} = \nabla \times (\underline{v} \times \underline{B}). \quad (6')$$

For an incompressible fluid $\text{div} \underline{v} = 0$ and so we may write the right-hand side of (6') as $(\underline{B} \cdot \nabla) \underline{v} - (\underline{v} \cdot \nabla) \underline{B}$. Equation (6') is known as the 'Induction Equation' or the 'Dynamo Equation'.

If we take the dot product of (6') with $\underline{B}$ and integrate over all space using the fact that $\underline{B} \rightarrow 0$ on the bounding surface at infinity we can obtain the following energy equation for an incompressible fluid:

$$\frac{\partial}{\partial t} \langle |\underline{B}|^2/2 \rangle = -\lambda \langle |\nabla \underline{B}|^2 \rangle + \langle \underline{B} \cdot (\underline{B} \cdot \nabla) \underline{v} \rangle. \quad (7)$$

The terms on the right-hand side of (7) represent the diffusion (or Ohmic dissipation) and the stretching of magnetic field lines. If we consider the ratio of the

magnitudes of the stretching to diffusion, we may define the magnetic Reynolds number of the system as $R_m = \frac{UL}{\lambda}$ where U is a characteristic velocity. It is clear from (7) that R_m must be greater than one to obtain magnetic field generation.

Decomposition of the Magnetic Field

Any solenoidal vector field is specified by two free components; the third being obtained from the vanishing of its divergence. We may therefore, decompose this field $\underline{B}$ into parts.

$$\underline{B} = \nabla \times g \underline{r} + \nabla (\nabla \times h \underline{r}) \quad (8)$$

The two degrees of freedom in $\underline{B}$ are contained in the scalars g and h . $\underline{r}$ is a unit vector in the radial direction. The decomposition in (8) is a natural one as it divides the $\underline{B}$ field into its toroidal and poloidal parts. Consider the first term in (8). As $\text{curl}(g \underline{r}) = (\text{curl } \underline{r})g + \nabla g \times \underline{r} = \nabla g \times \underline{r}$ we see that this component of the field lies in a surface everywhere normal to the radial direction. Hence, it is denoted as the toroidal component of $\underline{B}$. Any radial part of $\underline{B}$ is associated with the poloidal component as seen from dotting (8) with $\underline{r} = \underline{B} \cdot \underline{r} = -L^2 h$ where the surface Laplacian $L^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$ in spherical polars. We see that $\underline{B} \cdot \underline{r}$ is a function of h only. Taking the curl of (8) we see that the toroidal part of the field generates a poloidal current and vice versa, i.e.

$$\nabla \times \underline{B} = \frac{4\pi\mu}{c} \underline{j} = \nabla \times (\nabla \times g \underline{r}) + \nabla \times (-\underline{r} \nabla^2 h).$$

Theorems on Kinematic Dynamos

In the kinematic dynamo problem we are not concerned with the consistency of the velocity field which drives the dynamo and the effects of the Lorentz forces on the velocity field itself. We concern ourselves with arbitrary velocity fields; insofar as their origin is undefined, however, we demand that they be solenoidal in nature. In addition we exclude sources of magnetic field at infinity and hence $|\underline{B}| \rightarrow 0$ as $r \rightarrow \infty$ at least as fast as $1/r^3$ due to the non-existence of magnetic monopoles. We might mention here that a possible solution to the dynamo equation is $\underline{B} = \text{constant}$ for $\underline{v} = 0$, however the boundary condition on $\underline{B}$ at infinity then implies $\underline{B} = 0$.

Cowling's Theorem: An arbitrary, steady velocity field cannot maintain either an axisymmetric nor a two-dimensional magnetic field.

Toroidal Theorem: Steady toroidal velocity fields cannot maintain magnetic fields in the case of constant conductivity.

Here we will proceed to prove the toroidal theorem on an infinite domain. It can be shown that it holds in a sphere and other simple finite domains as well.

Consider a velocity field which everywhere lies in a plane normal to the $\underline{k}$ direction where $(\underline{i}, \underline{j}, \underline{k})$ are unit vectors corresponding to a right-handed coordinate system (x, y, z) .

$$\underline{v} = \nabla \times \psi \underline{k} \quad (9)$$

Decompose $\underline{B}$ into toroidal and poloidal components

$$\underline{B} = \nabla \times g \underline{k} + \nabla (\nabla \times h \underline{k}). \quad (10)$$

(Here, toroidal means contained in a plane perpendicular to the $\underline{k}$ direction while poloidal means having a component in the $\underline{k}$ direction as well.) We can rewrite the

dynamo equation as two equations governing the components of $\underline{B}$. This separation is achieved by taking the curl of (6') and dotting it with $\underline{k}$ to obtain the toroidal equation and for the poloidal equation simply dot $\underline{k}$ with (6').

The poloidal equation is

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla\right) \Delta_2 h = \lambda \nabla^2 (\Delta_2 h) \quad (11)$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. This is an ordinary diffusion without sources and has exponentially decaying solutions with time for $\Delta_2 h$ and hence for $h(t)$ as well. The toroidal equation is

$$\left(\frac{\partial}{\partial t} - \lambda \nabla^2\right) \Delta_2 g = -\underline{k} \cdot \nabla \times (\text{curl } \underline{v} \times \underline{B}) + \text{function of } h. \quad (12)$$

As we've shown that h decays to zero we neglect the term dependent on h in (12) as well as in $\underline{v} \times \underline{B} = -\underline{v} \times (\underline{k} \times \nabla g) = -\underline{k} (\underline{v} \cdot \nabla g)$. Substituting this into (12) we obtain

$$\Delta_2 \left(\frac{\partial}{\partial t} - \lambda \nabla^2 + \underline{v} \cdot \nabla\right) g = 0 \quad (13)$$

We may integrate (13) over x and y to remove the Δ_2 operator. As the resulting integration constants on the right-hand side must be bounded for all x, y and t it can be a function of z only. We will denote it as $f(z)$. If we multiply through by g and integrate over all space we obtain

$$\frac{\partial}{\partial t} \langle g^2/2 \rangle + \lambda \langle |\nabla g|^2 \rangle = \langle f g \rangle = 0 \quad (14)$$

The reason that $\langle f g \rangle = 0$ is that doing the integration over the x - y plane first, as $f = f(z)$ only, we need only to consider $\bar{g}$ (overbars indicate x - y averaging). However in the decomposition of $\underline{B}$ in (10) we are interested only in curl $g \underline{k}$ and as this will involve derivatives with respect to x and y only we may subtract from any function independent of x and y . We will choose this function to be $\bar{g}(z)$. Then $\langle f g \rangle$ becomes $\langle f (g - \bar{g}) \rangle = 0$. Hence $j \rightarrow 0$ as $t \rightarrow \infty$ as well.

The implication of the toroidal theorem is that for a finite time the poloidal field can enhance the toroidal field, however after a diffusion time $t \rightarrow 0$ and hence, so will g .

Cowling's theorem holds even if the diffusivity is a scalar function of the spatial coordinates, not simply a constant. However, the Toroidal Theorem does not hold for non-axisymmetric diffusivities. As an example of the application of these theorems we cite the case of the homopolar generator.

A homopolar generator consists of a conducting disk mounted on a conducting shaft and rotated normal to an external magnetic field $\underline{B}_{\text{ext}}$. The Lorentz forces drive a nearly radial current which is picked up at the edge by a contact brush. This current is then led down a wire which is wound around the shaft. One finds that the $\underline{B}$ field produced by the winding is equivalent at sufficiently large Ω , in magnitude to the external field and hence one may remove the $\underline{B}_{\text{ext}}$ and the generator will maintain the field which will, in turn, maintain the current. The way in which this simple example is governed by the above theorems is as follows: In the case of Cowling's theorem we note that the magnetic field produced is not axisymmetric due to

the vertical piece of wire. In the case of the Toroidal Theorem, although the rotating disk represents a toroidal velocity field, the wire represents a non-axisymmetric conductivity.

This example of a dynamo requires a continuous energy source as does any dynamo. Independent of any viscous dissipation we see that there is an additional Lorentz force acting on the radial currents in the disk. This provides a torque on the disk which tends to slow it down.

A. Possible Solution to the Dynamo Equation

We now show that we can obtain a solution to the dynamo equation given that the velocity field has a non-toroidal component.

$$\text{Let } \underline{v} = \nabla \times (\underline{h} \psi(x, y)) + \underline{h} \omega(x, y) \quad (1)$$

We decompose the magnetic field into its mean and fluctuating components in the following manner.

$$\underline{B} = \underline{B}_H \left\{ \underline{i} F(\underline{z}) - \underline{j} G(\underline{z}) + \nabla \times [\nabla \times (\underline{h} \underline{h})] + \nabla \times (\underline{h} \underline{g}) \right\} \quad (2)$$

(No $\underline{h}$ -component of the mean field as $\overline{\nabla \cdot \underline{B}} = 0 \Rightarrow \frac{d \overline{B_z}}{dz} = 0$ and if $B \rightarrow 0$ as $r \rightarrow \infty$, $\underline{B}_H = 0$. Overbar $\equiv$ horizontal average).

Here $F(\underline{z})$ and $G(\underline{z})$ are the $\underline{i}$ and $\underline{j}$ components of the field having averaged over the (x,y) plane. The fluctuations are decomposed into poloidal-like and toroidal-like parts. By definition $\underline{h} = \underline{g} = 0$. We assume that the fluctuations are small compared to the mean field. In addition, the length scales of the problem are assumed to be such that

$$|\partial/\partial z| \ll |\partial/\partial x| \ll |\partial/\partial y| \quad (3)$$

We begin by writing the dynamo equation

$$\left(\frac{\partial}{\partial t} - \lambda \nabla^2 \right) \underline{B} = \nabla \times (\underline{v} \times \underline{B}) \quad (4)$$

Substituting (1) and (2) into this and averaging over the x-y plane, we obtain the "mean field" equations:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2} \right) F &= - \frac{\partial}{\partial z} (\underline{v} \times \underline{B})_y \\ &= - \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial y} \Delta_z \underline{h} + \omega \left[\frac{\partial \underline{g}}{\partial y} + \frac{\partial^2 \underline{h}}{\partial x \partial z} \right] \right) \end{aligned} \quad (5)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2} \right) G &= - \frac{\partial}{\partial z} (\underline{v} \times \underline{B})_x \\ &= - \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial x} \Delta_z \underline{h} - \omega \left[- \frac{\partial \underline{g}}{\partial x} + \frac{\partial^2 \underline{h}}{\partial y \partial z} \right] \right) \end{aligned} \quad (6)$$

$\frac{\partial}{\partial y} (\underline{v} \times \underline{B})_z$ is 0 as y is the azimuthal variable and integration is over a closed path.

$\frac{\partial}{\partial x} (\underline{v} \times \underline{B})_z$ requires the evaluation of $(\underline{v} \times \underline{B})_z$ at $x = 0, \infty$. However, the fields must vanish at $x = \infty$ and B_y and v_y must vanish, by symmetry, at the origin. We have also taken $\omega = 0$.

The z-component of (4) and the z-component of the curl of (4) yield the "Fluctuating field" equations:

$$\left(\frac{\partial}{\partial t} - \lambda \nabla^2\right) \Delta_2 h = - \left(F \frac{\partial}{\partial x} - G \frac{\partial}{\partial y}\right) w + \dots \quad (7)$$

$$\left(\frac{\partial}{\partial t} - \lambda \nabla^2\right) \Delta_2 g = \Delta_2 \left[\left(F \frac{\partial}{\partial x} - G \frac{\partial}{\partial y}\right) \psi \right] - \frac{\partial w}{\partial y} \frac{\partial F}{\partial z} - \frac{\partial w}{\partial x} \frac{\partial G}{\partial z} + \dots \quad (8)$$

where, on the RHS's, we have neglected terms in g and h as being small compared with F and G .

We now assume that w and ψ can be broken into harmonics by satisfying

$$\begin{aligned} \frac{\partial^2}{\partial y^2} \begin{pmatrix} w \\ \psi \end{pmatrix} &= -\alpha^2 \begin{pmatrix} w \\ \psi \end{pmatrix} \\ \frac{\partial^2}{\partial x^2} \begin{pmatrix} w \\ \psi \end{pmatrix} &= -j^2 \begin{pmatrix} w \\ \psi \end{pmatrix} \quad \text{with } \alpha^2 + j^2 = a^2 \end{aligned} \quad (9)$$

Equations (7) and (8) are linear relations between h and g with w and ψ and hence h and g must also satisfy (9).

We can now write down the steady solutions to (7) and (8) for $\Delta_2 h$ and g :

$$\begin{aligned} \Delta_2 h &= - \left(F \frac{\partial}{\partial x} - G \frac{\partial}{\partial y}\right) \frac{w}{\lambda a^2} \\ g &= \left(F \frac{\partial}{\partial x} - G \frac{\partial}{\partial y}\right) \frac{\psi}{\lambda a^2} + \left(\frac{dF}{dz} \frac{\partial}{\partial y} + \frac{dG}{dz} \frac{\partial}{\partial x}\right) \frac{w}{\lambda a^2} \end{aligned} \quad (10)$$

but now neglect the second term in g with respect to the first by the orderings in Eq. (3).

We can now insert these solutions into the "mean field" equations (5) and (6). The equations for steady fields become

$$\begin{aligned} \lambda \frac{d^2 F}{dz^2} &= \frac{d}{dz} \left[- \frac{\partial \psi}{\partial y} \left(F \frac{\partial}{\partial x} - G \frac{\partial}{\partial y}\right) \frac{w}{\lambda a^2} + w \left(F \frac{\partial^2}{\partial x \partial y} - G \frac{\partial^2}{\partial y^2}\right) \frac{\psi}{\lambda a^2} \right] \\ \lambda \frac{d^2 G}{dz^2} &= \frac{d}{dz} \left[- \frac{\partial \psi}{\partial x} \left(F \frac{\partial}{\partial x} - G \frac{\partial}{\partial y}\right) \frac{w}{\lambda a^2} + w \left(F \frac{\partial^2}{\partial x^2} - G \frac{\partial^2}{\partial x \partial y}\right) \frac{\psi}{\lambda a^2} \right] \end{aligned} \quad \dots \quad (11)$$

using (9) and the fact that $\overline{w} = 0 = \overline{\psi}$, unless $\overline{w\psi} \neq 0$, these equations can be written in the following form:

$$\left. \begin{aligned} \lambda \frac{d^2 F}{dz^2} &= \frac{d}{dz} \left\{ G \cdot \frac{2\alpha^2 \overline{w\psi}}{\lambda a^2} \right\} \\ \lambda \frac{d^2 G}{dz^2} &= \frac{d}{dz} \left\{ F \cdot \frac{-2j^2 \overline{w\psi}}{\lambda a^2} \right\} \end{aligned} \right\} \quad (12)$$

These equations can be solved by introducing the complex mean field function, H , where

$$H = \sqrt{\frac{\alpha}{j}} G + i \sqrt{\frac{j}{\alpha}} F$$

whence equations (11,12) become

$$\lambda \frac{d^2 H}{dz^2} = i \frac{2\alpha j \overline{w\psi}}{\lambda a^2} \frac{dH}{dz} \quad (13)$$

Solutions of this equation have the form

$$H \sim e^{im\theta}$$

where

$$m = \frac{2\alpha_j \overline{w\psi}}{\lambda^2 a^2} \quad (14)$$

Equation (14) shows that H possesses opposite helicity to the velocity field. For example, if $\overline{w\psi} > 0$, the velocity field possesses right-handed helicity and the magnetic field left-handed. This dynamo process is basically equivalent to one of the cases considered by Roberts (1972).

PHILOSOPHIES OF APPROACHING THE HYDROMAGNETIC PROBLEM

As the dynamo equations are linear and homogeneous in $\underline{B}$, nothing can be said about the amplitude of the generated magnetic field. We may obtain exponentially growing fields, however this cannot go on indefinitely as Lorentz forces will eventually come into play and restrain fluid motions across field lines. Ideally we would like to solve both the hydrodynamic and hydromagnetic problems simultaneously but this is not generally feasible. There are basically two philosophies of approach to solving the full problem.

The 'Cautious' Approach:

In this approach one considers the convection problem first without a magnetic field and hence no Lorentz forces in the equations of motion. One can then insert the convective velocity field into the Dynamo equations to see if the convection can support dynamo action. If so, one may then compute the Lorentz forces and treat them as a perturbation on the convection problem. This is clearly only good for a small induced $\underline{B}$ field although one hopes that this perturbation on the velocity field will prevent further growth of the magnetic field and help equilibrate the growth and decay rates for both the velocity and magnetic fields.

The 'Good Luck' Approach:

An alternative approach is to specify the form of the magnetic field and include it in the convection problem from the beginning. One then takes the velocity field found from the hydromagnetic convection problem and inserts it into the dynamo equations. We hope that it can generate the same magnetic field as was originally assumed to influence the convection. As we increase the number of free parameters which specify the form of the $\underline{B}$ field we also increase our chances of having 'Good Luck'.

A MODEL FOR THE GEODYNAMO

We have already shown that a purely toroidal velocity field cannot support dynamo action over long time scales (as compared to the diffusion time). However, we have shown that a velocity field which possesses helicity can maintain a magnetic field. Moreover, we have seen that purely hydrodynamic convection in a rotating annulus with inclined boundaries is described by a helical velocity field. We need only to compute the form of the field generated by this velocity field and then we will be able to compute the Lorentz forces. Hence, we will follow the 'Cautious' approach first.

We consider a rotating annulus with a flat bottom, a slightly sloping top, with a ratio of height to gap width $L/D \ll 1$, and in the narrow gap approximation. This allows us to replace our cylindrical coordinate system with a locally Cartesian coordinate system. The origin being centered at height $L/2$, distance $D/2$ from the outer boundary and (x, y, z) represent the radial, azimuthal, and vertical coordinates.

$$\frac{L}{D} \ll 1, \beta \ll 1, D \ll (R + D/2).$$

Our velocity field is composed basically of a geostrophic part and a higher order component due to the combined effects of the inclined upper boundary and the Ekman suction in the boundary layers.

$$\underline{v} = \nabla \times \psi \underline{k} + \nabla \times (\nabla \times \phi \underline{k}) \quad (1)$$

$\underline{v} \cdot \underline{k} = -\Delta_z \phi$ and we note that though $\overline{\underline{v} \cdot \underline{k}}$ is small, it is not zero. We maintain the ordering used before, i.e.

$$\frac{\partial}{\partial z} \ll \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}. \quad (2)$$

In a previous lecture on hydrodynamic convection we found a steady solution corresponding to a basic geostrophic balance. The inclined boundaries and Ekman suction destroy the steadiness of the velocity field in (1). The solution to the convective role problem is then

$$\psi = -\frac{1}{2} P(x) e^{(i\alpha y + i\omega_0 t)} \quad (3)$$

where $P(x) = -2A \sin Y (X + D/2L)$ and A is the arbitrary perturbation amplitude. In addition $\underline{v} \cdot \underline{k}$ had to satisfy the following boundary conditions at $z = \pm \frac{1}{2}$ (z having been non-dimensionalized by length D).

$$\underline{v} \cdot \underline{k} = -\Delta_z \phi = \mp \beta_T \underline{v} \cdot \underline{i} \mp E \underline{k} \cdot (\nabla \times \underline{v}) \quad (4)$$

where in this case $\beta_T = \beta$ and $\beta_0 = 0$. The first term on the right-hand side of (4) being due to the inclined boundaries, the second due to the Ekman suction. The expression for ω_0 in (3) is found from the dispersion relation of the hydrodynamic problem. Note that the ordering in (2) implies that $Y \ll \infty$ in (3). We proceed to find the magnetic field induced by this velocity field.

We non-dimensionalize the magnetic field through the expression for the Alfvén velocity. We scale lengths and times by L and Ω^{-1} respectively and define the following quantities:

$$\underline{H} = \frac{B}{\sqrt{4\pi e \mu \rho_0}} \cdot \frac{1}{\Omega L}, \quad \tau = \frac{\lambda}{\Omega L^2}.$$

The dynamo equation becomes

$$\left(\frac{\partial}{\partial \tau} - \tau \nabla^2 \right) \underline{H} = \nabla \times (\underline{v} \times \underline{H}). \quad (5)$$

We decompose the $\underline{H}$ field into its mean and fluctuating parts,

$$\underline{H} = H_A \{ \underline{i} F(z, t) - \underline{j} G(z, t) + \nabla \times (\nabla \times h \underline{k}) + \nabla \times g \underline{k} \} \quad (6)$$

where the fluctuations, h and g , are small compared to the mean fields F and G , and $h = \underline{g} = 0$. The boundary conditions on the magnetic fields arise from the assumption

that the annulus is surrounded by a perfect insulator ($\tau = \infty$) and hence $G(z) = 0$ at $z = \pm \frac{1}{2}$ while $F(\frac{1}{2}) = -F(-\frac{1}{2})$. The condition on G stems from the fact that a non-vanishing G at the boundaries would imply either currents flowing in the insulator or sources of magnetic field at infinity. The asymmetry of F at the boundaries ensures closure of the radial field component. This radial component is supposed to represent the poloidal field of a conducting sphere. The sphere has been distorted into a cylinder to simplify the problem.

Had we not thrown out the time dependence in the old expressions for g and h we would have obtained

$$g = \frac{(F \frac{\partial \psi}{\partial x} - G \frac{\partial \psi}{\partial y})}{i\omega_0 + \tau a^2}, \quad h = \frac{(F \frac{\partial \phi}{\partial x} - G \frac{\partial \phi}{\partial y})}{i\omega_0 + \tau a^2}. \quad (7)$$

For the case of steady mean fields we obtain the dynamo equation

$$\tau \frac{d^2 F}{dz^2} = \frac{d}{dz} \left\{ \frac{1}{4} G \left(-\Delta_2 \phi^* \frac{\alpha^2 \psi}{i\omega_0 + \tau a^2} + \frac{\partial \psi^*}{\partial y} \frac{(-\frac{\partial}{\partial y} \Delta_2 \phi)}{i\omega_0 + \tau a^2} \right) + C.C. \right\} \quad (8)$$

where $C.C.$ denotes the complex conjugate of the written expression before it. A similar equation is obtained for $\frac{d^2 G}{dz^2}$. Integrating these expressions over $-\frac{1}{2} \leq z \leq \frac{1}{2}$ and using the above boundary conditions on F and G , we obtain equations of the form,

$$\tau \frac{dG}{dz} = \{ --- \} + \text{constant} \quad (9A)$$

$$\tau \frac{dF}{dz} = \{ --- \}. \quad (9B)$$

These equations represent the separated Ohm's laws for the radial and azimuthal magnetic fields (whose curls are related to the currents via Maxwell's equations). The constant in Eq. (9A) is non-vanishing corresponding to $\underline{E} \cdot \underline{I} \neq 0$. However (9B) has no such constant as $\underline{E} \cdot \underline{I} = 0$.

If we define $\hat{H} = \sqrt{\frac{\alpha}{\gamma}} G + i\sqrt{\frac{\gamma}{\alpha}} F$ and add Eqs. (9), we obtain

$$\frac{d\hat{H}}{dz} = -i f(z) \hat{H} + \sqrt{\frac{\alpha}{\gamma}} C \quad (10)$$

where C is the constant in (9) and

$$f(z) = \frac{\alpha \gamma A^2 E^{1/2} \alpha^2 a^2}{\omega_0^2 + \tau^2 a^4} z \equiv \mu z. \quad (11)$$

We can integrate (10) to find

$$\hat{H}(z) = e^{-i\Phi(z)} \left\{ b + \int_z^{\frac{1}{2}} C \sqrt{\frac{\alpha}{\gamma}} e^{i\Phi(\xi)} d\xi \right\} \quad (12)$$

where $\Phi(z) = \int_z^{\frac{1}{2}} f(\xi) d\xi$. From (11) we see that $f(z)$ is an odd function of z , hence $\Phi(z)$ is even. As $F(z)$ satisfies antisymmetric boundary conditions at $z = \pm \frac{1}{2}$ Eq. (12) must possess this antisymmetry. Thus $b = 0$ in (12).

The real part of (12) yields

$$G(z) = C \sqrt{\frac{\pi}{\mu}} \left\{ \cos\left(\frac{\mu}{2} z^2\right) C\left(\sqrt{\frac{\mu}{\pi}} z\right) + \sin\left(\frac{\mu}{2} z^2\right) S\left(\sqrt{\frac{\mu}{\pi}} z\right) \right\} \quad (13)$$

where $C(x) = \int_0^x \cos \frac{\pi t^2}{2} dt$ and $S(x) = \int_0^x \sin \frac{\pi t^2}{2} dt$ are the Fresnel integrals. From (13) it is clear that $G(z) = 0$. We are interested in values of μ for which $G(\pm \frac{1}{2})$ vanishes. As μ is a function of the convection amplitude A (see Eq. (11)) we are interested in the smallest value of μ such that $G(\pm \frac{1}{2}) = 0$. This value of μ is given by

$$\mu_1 = 18.42.$$

We conclude that for $\mu \geq \mu_1$ field generation occurs, for $\mu < \mu_1$, any fields may decay.

One can visualize the qualitative nature of $F(z)$ and $G(z)$ as in the figure below.

We may now evaluate the Lorentz forces due to the mean fields to see what effect they might have on the convective velocity field. Averaging over the x-y plane we find that

$$(\nabla \times \mathbf{H}) \times \mathbf{H} = \frac{1}{2} \frac{d}{dz} (F^2 + G^2) \equiv -\nabla \Pi^{(1)} \quad (14)$$

and hence the mean field Lorentz forces may be balanced by a pressure perturbation $\Pi^{(1)}$. Thus the mean field should not affect the convection.

If we proceed to find corrections to the eigenvalue B due to finite amplitude nonlinear convection and then treat the Lorentz force as an additional perturbation we will obtain the following expression for the boundary parameter.

$$B = B_0 \left(1 + \frac{1}{8} \frac{\mathcal{L}^2 a^2 A^2}{\omega_0^2 + a^2 E_0^2} \right) + H_A^2 B^{(1)}$$

We would expect this expression for B to be valid only near the critical value of $B = B_0$ for the onset of convection. This expression implies a partitioning of the energy into convective energy and magnetic energy. In a physically realised system the value of B is to be taken as a given parameter of the system. If the field becomes large enough that A decreases to the point where μ (of Eq. (11)) drops to $\mu_1 = 18.42$, then no further generation of the magnetic field will occur.

THE 'GOOD LUCK' GEODYNAMO

We will now try to include the Lorentz force in the equations of motion from the onset. We will postulate a magnetic field of the same form as in Eq. (6). The amplitude of the field being a free parameter for now. As there exists a linear relationship between g and ψ , as well as h and ϕ (see Eq. (7)), we need only to consider the effects of the g component as ϕ is of higher order than ψ in the small quantity η .

With appeal to the ordering in (2) we see that the only important currents are generated by the fluctuating fields. Thus the largest contribution to the Lorentz force is due to the interaction of these currents with the mean fields. We express this Lorentz force as

$$\underline{L}_1 \cong H_A^2 [\nabla \times (\nabla \times g \underline{k}_z)] \times [\underline{i} F - \underline{j} G] . \quad (15)$$

We now consider the z-component of the vorticity equation including this Lorentz force. Using (7) for g in (15) we find

$$\begin{aligned} (E \nabla^2 - i\omega) \underline{k}_z \cdot (\nabla \times \underline{v}) + 2 \underline{k}_z \cdot \nabla (\underline{k}_z \cdot \underline{v}) - B \underline{j} \cdot \nabla \theta &= \underline{k}_z \cdot (\nabla \times \underline{L}_1) \\ &= -H_A^2 (F^2 \frac{\partial^2}{\partial x^2} + 2FG \frac{\partial^2}{\partial x \partial y} + G^2 \frac{\partial^2}{\partial y^2}) \frac{\Delta_z \psi}{i\omega + \tau a^2} . \end{aligned} \quad (16)$$

We once more make use of the ordering in (2) to simplify the right-hand side of (16). As $\frac{a}{\partial x} \ll \frac{a}{\partial y}$ we may neglect the term $2FG \frac{\partial^2}{\partial x \partial y}$ as compared to $G^2 \frac{\partial^2}{\partial y^2}$. We may, on the same grounds, neglect the F^2 term. In addition to this simplified form of (16) we need the linearized heat equation as well.

$$\frac{\partial \theta}{\partial t} = EP^{-1} \nabla^2 \theta + \underline{v} \cdot \underline{i} \quad (17)$$

With $\underline{v}$ given in terms of ψ and ϕ as in (1) we may solve (17) for θ in terms of ψ where the form of ψ is given by (3) with ω replacing ω_0 . With the usual boundary conditions of $\psi = \theta = 0$ at $x = 4D$ we see that ψ in (3) must be integer multiples of $\pi L/D$. We find for the ratio

$$\psi/\theta = \frac{i\omega + a^2 EP^{-1}}{i\alpha} \quad (18)$$

where $\alpha^2 = \alpha^2 + \gamma^2$. Substitution of (3) and (18) into (16) followed by an integration over z with the boundary conditions in (4) on $\Delta_z \psi$ leads to the following relation.

$$\begin{aligned} (i\omega + a^2 EP^{-1})(i\omega + a^2 E) a^2 - B \alpha^2 - (2\eta i\alpha + 2E^{1/2} a^2) \chi (i\omega + a^2 EP^{-1}) &= \\ &= -\alpha^2 a^2 K \frac{(-i\omega + \tau a^2)(i\omega + a^2 EP^{-1})}{\omega^2 + \tau^2 a^4} \end{aligned} \quad (19)$$

where $K = H_A^2 \int_{-1/2}^{1/2} G^2 dz$. In the limit $|\omega| \gg \tau a$ the imaginary part of (19) reduces to a quadratic in ω with the solution

$$\omega = \frac{-\eta \alpha}{(1+P)a^2} \pm \sqrt{\left[\frac{\eta \alpha}{(1+P)a^2} \right]^2 - \frac{(\lambda/K - 1)\alpha^2}{(1+P)} K} . \quad (20)$$

In obtaining (20) we have neglected the number $\frac{2P}{(1+P)} \frac{E - 1/2}{a^2}$ as compared to unity. In the square root of Eq. (20) the first term is the effect of the non-geostrophic part of the flow on the steadiness of the motion while the second term is the effect of the Lorentz forces.

As has been assumed real we must have

$$K(\lambda/K - 1) \leq \frac{\eta^2}{(1+P)a^4} \quad (21)$$

If a growth rate σ had been assumed in addition to the frequency ω we would have found that growing solutions require a lower upper bound than given by (21) while solutions exceeding the bound (21) must decay. This indicates that the equality sign in (21) corresponds to a stable equilibrium. This equilibrium still depends,

of course, on the parameter 'a'. Numerical dynamo solutions (Gubbins, 1973) indicate that there exists a lower bound for the order 'a' of the order 10. The real part of (19) yields the following expression for the buoyancy parameter.

$$B = E^2 P^{-1} \frac{a^4}{\alpha^2} - \frac{\omega^2 a^2}{\alpha^2} + \frac{2\eta\omega}{\alpha} - 2a^2 \frac{E^{3/2}}{\alpha^2} P^{-1} + H_A^2 B_1 \quad (22)$$

where

$$B_1 = \frac{a^4(\omega^2 + \tau E P^{-1} a^4)}{\omega^2 + \tau^2 a^4} \int_{-1/2}^{1/2} G^2 dz.$$

Now we would like to see if the velocity field which satisfied (16) and (17) is capable of driving a dynamo. As $\mathbf{k} \cdot \mathbf{v} = -\Delta \phi = a^2 \phi$, we can substitute the solutions for ψ and θ into the modified form of (16) and applying the boundary conditions on ϕ we find,

$$\phi = \psi \left\{ \frac{-i\alpha\eta}{a^2} \left(z + \frac{1}{2} \right) + \frac{E^{1/2}}{a^2} \left(z + \frac{1}{2} \right) + \frac{1/2 H_A^2}{i\omega + \tau a^2} \left[\int_{-1/2}^z \alpha^2 G^2 dz - \left(z + \frac{1}{2} \right) \int_{-1/2}^{1/2} \alpha^2 G^2 dz \right] \right\}. \quad (23)$$

It turns out that the magnetic term contributes a real positive part to ϕ and therefore reinforces the Ekman suction term. Hence the inclusion of the Lorentz forces from the start does not suppress the helicity of the velocity field. As we have already shown, a velocity field which possesses helicity can support dynamo action.

The magnetic term in (23) will modify $f(z)$ in (11). The new criterion for dynamo action is that

$$\frac{\alpha^2 A^2 \alpha^2 a^2}{\omega^2 + \tau^2 a^4} \left(E^{1/2} + \frac{b H_A^2 \tau a^4}{\omega^2 + \tau^2 a^4} \right) \geq \mu_1 = 18.42 \quad (24)$$

where b is of order .10, and A is again, the amplitude of the convection.

It is surprising that the magnetic field increases the helicity and hence enhances the dynamo action. This is an additional reason that we must assume that the strength of the magnetic field is limited by dynamical constraints.

GENERATION OF PLANETARY MAGNETISM

The goals of any theory of geomagnetism may be broken into sets of primary and secondary goals, as follows:

Primary Goals: The theory should be able to explain

1. the dipole character of the observed fields,
2. the alignment of the magnetic poles with the rotation axis,
3. the stationary nature of the field,
4. the amplitude of the field (in terms of physical parameters),
5. the fields of other planets.

Secondary Goals:

1. the existence of higher multipoles,
2. the tilt of the dipole field,

3. the secular variations,
4. the long period variations, i.e. magnetic reversals.

In these lectures we have addressed ourselves to the first four of the primary goals. We will now try to show that the theory presented above is consistent with the observed poloidal fields of the planets.

Below we list observed physical characteristics of the Earth, Jupiter and Mercury. From them we form the nondimensional quantity H_A^2 as well as $\lambda/K H_A^2$.

	<u>Earth</u>	<u>Jupiter</u>	<u>Mercury</u>
$\Omega (s^{-1})$	7.3×10^{-5}	1.75×10^{-4}	1.22×10^{-6}
$R_{core} (10^8 cm)$	3.47	49.5	1.8
$\lambda (cm^2 s^{-1})$	1.6×10^4	1.2×10^4	10^4
$L (cm)$	R_{core}	10^9	R_{core}
$K (cm^2 s^{-1})$	0.07	0.4	0.04
$\rho (g \cdot cm^{-3})$	10	1.5	9
$B_{poloidal} (gauss)$	3.7	23	0.019
H_A^2	1.7×10^{-10}	9×10^{-10}	6.6×10^{-11}
$\lambda/K H_A^2$	3.9×10^{-5}	2.8×10^{-5}	1.7×10^{-5}

If we assume that the fields of the planets are at the maximum allowed by any dynapical constraints, we may use the equality in Eq.(21) to estimate the wavenumber 'a' at which the convection is stable. We find, using these observed quantities that $a \approx 10$. This is in close agreement with the lower limit to 'a' established by numerical kinematic dynamo solutions.

We should point out an important difference between the theory presented here and other dynamo theories. Though we have taken $\gamma < \alpha$ here, the number $\sqrt{8/\alpha}$ is of order unity. This implies that the toroidal and poloidal components of the field are of the same order of magnitude. Hence, if this theory is a description of what is occurring in nature, observations of the poloidal fields of the planets allow us to infer information about the internal toroidal fields. Other dynamo theories take the rate of poloidal to toroidal field strengths as a parameter and hence can predict nothing about the toroidal field.

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SEMINARS and ABSTRACTS

DETAILS OF SPREADING RIDGES:

CONSTRAINTS ON DYNAMIC MODELS FOR THE EMPLACEMENT OF OCEANIC LITHOSPHERE

Tanya M. Atwater

The purpose of this presentation was, hopefully, to entice a fluid dynamicist into exploring the problem of how oceanic lithosphere is emplaced under the centers of mid-ocean ridges. The problem is of considerable interest to the marine geological community since all oceanic lithosphere is created at ridge crests and most of its structures and characteristics are inherited from this origin. Also, it is a fun problem for its own sake, since it includes several quite unusual features and since the problem has quite a few constraints placed upon it by direct observations and their straightforward inferences.

The basic problem is to construct a mathematical description of the manner in which asthenospheric material moves upward to fill the gap and to add new material to the lithospheric plates as they move apart. Following are the constraints and some of their uses and implications.

Mature oceanic lithosphere is made of a crust, about 6 km thick, and the solid upper mantle, 50-100 km thick. The crust is probably entirely of basaltic composition and is emplaced in a molten state. The mantle is probably peridotite with eclogite (high pressure basalt) inclusions and is emplaced as a crystal mush with some interstitial basaltic liquid (see Forsyth and Press, 1971, for the constraints on these compositions). In the upwelling slot the crustal basalt is thought to escape from the mantle rocks at about 30 km depth and to move to the surface quite quickly. The mantle crystal mush may move upward much more slowly. The model should deal with these two fluids and their interrelationship (Lachenbruch and Thompson, 1972).

Surface topography is very highly fractured (Klitgord and Mudie, 1974; MacDonald and Luyendyk, in prep., 1975), so that the crust near the center can be thought of as a layer of no strength. Thus, the gross profile of the ridge crest probably accurately reflects the dynamic stresses at depth.

The gross topographic profile of mid-ocean ridges varies from a deep rift valley on slow spreading ridges to a small distinct uplifted ridge on fast ones (Anderson and Noltimier, 1973). Presumably the rift valley reflects the difficulty which material is encountering flowing upward in the slot. The width of the rift valley, about 30 km, must reflect the width of the slot in some way. Apparently, fast spreading ridges have a much wider zone of new, hot, soft material (a much wider slot?) so that there is no such difficulty. Perhaps the width of the horst, about 20 km, shows the width of the region containing the most melt (less dense so the column stands higher), or perhaps some other explanation is needed.

Ninety percent of the surface material is emplaced within a few kilometers of the center; in the rifted centers it is emplaced within the central floor of the rift valley (Atwater and Mudie, 1973, and MacDonald, in prep., 1975). This requires that at rifted centers, the surface rocks (and perhaps the entire crust) is uplifted as long fault blocks while it moves outward, until it has attained rift mountain heights. This is an uplift of 1 to 1.5 km. Independent evidence for such an uplift is given by the existence of deep basin turbidite sediments presently found perched upon some of the highest rift mountain blocks on the Gorda Ridge (Atwater and Mudie, 1973). The uplift apparently reflects the continued upward movement of materials at depth as far out as the rift mountain tops. Beyond the rift mountain tops, the ocean floor gradually deepens in a way which can be simply explained by cooling and shrinking of the rigid lithosphere as it ages, e.g., Sclater and Francheteau, 1970.

Ridge crests are aligned approximately perpendicular to their transform faults (and to the spreading direction). This configuration is the one which minimizes the length of the spreading sections at the expense of lengthening the transform faults to whatever length is necessary to maintain the inherited shape of the spreading system. This observation has been assumed to imply that perpendicular spreading is a least energy configuration and that spreading centers are much more energy consuming than transform faults (Lachenbruch and Thompson, 1972, and Lachenbruch, 1973b). On the other hand, perpendicular ridge-transform systems appear in some wax models in which the "ridge" had a molten gap so that the energy involved there seemed very low. Oldenburg and Brune, 1975, believe that this invalidates the energy argument and that the shear strength of the wax (and the crust?) determines how it breaks. Recent detailed observations show that in some places on slow and medium rate spreading centers the ridge crests are actually aligned 15°-20° different from the perpendicular direction and the sense of this difference is such that the transform faults are shorter than they would have been (Atwater and MacDonald, 1975). It may be that this obliquity is a more widespread effect than was previously realized and that it is an important constraint.

The shape of the "slot" in which material is moving upward is a great unknown in this problem. Various thermal calculations (e.g., McKenzie, 1967, Parker and Oldenburg, 1973) show that the region which is at or very near the basalt melting point widens very quickly with depth under a ridge crest and is not at all slot-like. On the other hand, it may be that the percentage of magma present might be the slot-defining parameter, not temperature. In this case, upwelling might be confined to a very narrow central slot where magma is released by the decrease of pressure right under the concentrated center of opening. As soon as the magma is lost, the refractory remains would become part of the rigid plate nearby. Models using both vertically walled slots and flared ones should be examined.

For previous attempts at the solution of this problem, see Lachenbruch, 1973b, and Sleep, in prep., 1975.

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CONTINENTAL COLLISIONS IN ASIA

Peter Molnar

Between about 70 and 40 million years ago India moved north with respect to (the rest of) Eurasia at about 100 to 150 mm/year. At approximately 40 million years ago, the two continents collided with one another, and since this time the rate of motion has been only about 50 mm/year. The low density of continental crust makes it very buoyant. Therefore, presumably, India could not descend to great depths into the asthenosphere, and its relative motion with Eurasia was slowed. As only a small portion of India is likely to have been subducted beneath Eurasia, approximately 2000 km of shortening between them must have occurred since 40 million years ago, solely by deformation within continental crust. Some deformation appears to occur by thrust faulting in the Himalayas and other mountain belts further north causing a thickening of the crust there. Perhaps as much as 1000 km of shortening could have occurred this way, but this is probably an extreme upper limit. Major strike slip faults appear, carry material eastward and, thus, out of the way of the impinging continental blocks. The deformation field bears a striking resemblance to the slip lines in the plasticity problem of plane indentation by a rigid die. We infer that stress is transmitted over a broad area, causing deformation in weaker areas. This stress is presumably transmitted both across regions of high strength and across regions of low strength but with high altitude. The high altitude induces a pressure gradient capable of transmitting a force across the region.

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THE VARIATION OF OCEAN FLOOR BATHYMETRY WITH AGE

Barry E. Parsons

Many surface features on the earth can be explained in terms of the properties of the plates, the quasi-rigid boundary layer of the large scale circulation. No assumptions need be made about the nature of the flow in the mantle except that some driving mechanism exists to provide the relative motions. Hence any broad constraints on the convective motions beneath the plates are welcome. The variations in ocean floor bathymetry and heat flow with the age, t , of the plate are thought to be explicable in terms of the thermal structure of the plates. Several different models have been proposed, all of which include a rigid plate moving with a uniform velocity away from the ridge crest, with an upper temperature boundary condition,

$T = 0^\circ\text{C}$. The differences lie in the assumptions made about the nature of the bottom boundary of the plate. With one exception the models predict that the depth of the ocean should increase linearly as $\sqrt{t}$, as one moves away from the ridge crest. The exception, generally known as the plate model, includes an isothermal boundary at a fixed depth as the bottom boundary condition, with the same temperature applied as the side boundary condition at the ridge crest. The plate model predicts that the depth should increase linearly as $\sqrt{t}$ initially, but eventually this relation should break down. In the older regions the depth should decay exponentially to a constant asymptotic depth. The age at which the linear relation between depth and $\sqrt{t}$ breaks down is determined by the fixed thickness of the plate assumed in this model.

Depth values are among the most abundant and accurate geophysical data. The age of the ocean floor is provided by identifying magnetic anomalies or by using an age determined by using the microfossils obtained from drill cores in the location of the depth measurement. In this way compilations of mean depth versus age have been built up for the age range 0 to 160 m.y.b.p. Of the various models the plate model provides the best description of the data. If we measure an elevation of the ocean floor relative to the constant asymptotic value to which the depths tend, then the data can be described by two simple empirical relationships. In the age range

$0 < t \leq 60$ m.y.b.p. we find that

$$e(t) = 3700 - 360\sqrt{t} \text{ metres}$$

where t is measured in m.y.b.p. (million years before present), and for the overlapping age range > 25 m.y.b.p. the asymptotic relation is

$$e(t) = 2400 \exp\left(-\frac{t}{54.3}\right) \text{ metres}$$

The forms of these relationships are as predicted by the plate model. The point of departure from the $\sqrt{t}$ relation and the values of the numerical constants in the above expressions fix the thickness of the plate at 115 km, (with about a 15% error).

The above empirical expressions can be used as reference levels in studying more local variations in depth. Also if one regards the good agreement found between the observed relationships and those predicted by the plate model as a justification of that model, then a constraint on mantle processes can be obtained. In older regions, where the plate is close to thermal equilibrium, one needs a relatively uniform supply of heat at the bottom boundary. It does not seem possible to provide this heat by conduction, and hence this is one of the reasons that it has recently been proposed that convection exists in the upper mantle on a much smaller horizontal scale than that of the major plates. Its function is to provide an efficient means of vertical heat transport, and it is this mode of convection that brings heat up to the base of the plate.

THE RELIABILITY OF OCEANIC HEAT FLOW DATA

John G. Sclater

One of the major problems in ocean heat flow is the variability of closely-spaced values. The photographs of widespread cracks and fissures on the floor of the Mid-Atlantic Ridge are evidence that the oceanic crust is effectively permeable to fluid flow. From the observation of a hydrothermal vent at the crest of the Galapagos spreading center and the cracks and fissures on the Mid-Atlantic Ridge we argue that hydrothermal circulation in the permeable crust and the loss of heat by mass transport across the crust/sea water interface is the cause of much of the variability of the oceanic heat flow. As a consequence, only where this crust is sealed from the seawater by an extensive and thick layer of impermeable sediment can reliable estimates be made of the flux at depth. We analyze heat flow data from such areas and show that the scatter in the data is low and that in most areas the mean heat flow through a province of known age is close to that predicted by plate models which account for the increase in depth with increasing age of the ocean floor. We argue from our analysis that these heat flow means are a reliable estimate of the flux at depth. In order to be able to use heat flow measurements to investigate plate thickness, upper mantle convection and local anomalies, it is necessary to substantiate this analysis and to find an adequate explanation of the low mean heat flow observed at the center of the sediment bulge in the equatorial Pacific and in the Guatemala Basin.

ON HYDROMAGNETICALLY DRIVABLE DYNAMOS WITH HELICAL SYMMETRY

Edward R. Benton

(extended abstract)

Abstract

The class of steady kinematic dynamos with helical symmetry (i.e. Lortz dynamos) is given a rational, motivated derivation. Four potential members of this class are examined to determine whether or not they could be driven in a self-consistent hydromagnetic fashion by forces corresponding to buoyancy or precession.

Derivation of Helical Dynamos

In 1968, D. Lortz supplied dynamo theory with a new page. He presented a class of dynamos with helical symmetry for which exact, closed-form solutions of the induction equation exist in an unbounded conductor: all of the electric current flows within an infinite circular cylinder. Unfortunately, Lortz provided no rational motivated derivation of these dynamos, nor did he explicitly display any of the solutions. Some unpublished work of G. O. Roberts, now indicates the possibility that Lortz's infinite cylinder can be deformed into a torus so as to make a finite (i.e. relevant) dynamo (see Paul H. Roberts in Zmuda below, as well as Lortz's own attempt in 1972). Here we first attempt to understand where Lortz dynamos come from.

Key steps in the derivation include the following:

(A) Proposition: a desirable property of any motion advanced as a candidate for admission to the family of self-excited dynamos is that it possess non-zero helicity density (see Moffatt, 1970): $|\vec{V} \cdot (\nabla \times \vec{V})| \neq 0$.

(B) Construction: inbed this desirable helicity into the coordinate system onto which the velocity field (and magnetic field) are resolved.

(C) Fact: the simplest geometrical shape for a coordinate axis possessing non-trivial helicity is a simple helix.

(D) Decomposition: separate the motion and magnetic fields into their parts ("components") parallel to and orthogonal to a system of helices.

Mathematically, the above (with suitable amplification of details) leads to the following forms for $\vec{v}$ (=non-dimensional fluid motion) and $\vec{a}$ (=non-dimensional Alfvén vector = $\vec{B}/(\rho_0 \mu)^{1/2}$):

$$\vec{v} = \alpha \vec{h} + \nabla \beta \times \vec{h}, \quad (1)$$

$$\vec{a} = \gamma \vec{h} + \nabla \delta \times \vec{h}. \quad (2)$$

Here $\vec{h}$, the basic helical coordinate vector, is found to be fixed by the following properties: $\vec{h}$ is

- (i) bounded and continuous
- (ii) solenoidal ($\nabla \cdot \vec{h} = 0$)
- (iii) helical (i.e. field-lines of $\vec{h}$ are helices)
- (iv) Beltrami ($\vec{h} \times (\nabla \times \vec{h}) = 0$, this property is not essential; adopted for convenience.

$$(3) \vec{h} = \frac{ms\hat{\phi}}{e^{\alpha t} m^2 s^2} \quad \text{where } \ell, m = \text{integers}$$

s = dimensionless cylindrical radius
 $\hat{\phi}, \hat{z}$ = unit vectors in directions of increasing azimuth and axis

In (1), (2) $\alpha, \beta (\gamma, \delta)$, the "defining scalars", are the parallel and orthogonal parts of the motion (magnetic field).

The field lines of $\vec{h}$ are $s = \text{constant}$, $\psi = \ell\phi - mz = \text{constant}$ where s, ϕ, z are dimensionless cylindrical coordinates,

Comment: In item D above, the word "separate" means that $\alpha \vec{h}$ and $\nabla \beta \times \vec{h}$ in (1) must separately be solenoidal (no kinematic coupling between α and β). Hence,

$$\nabla \cdot (\alpha \vec{h}) \equiv \alpha (\nabla \cdot \vec{h}) + (\nabla \alpha) \cdot \vec{h} = 0 \quad (3)$$

so α must be constant on the field lines of $\vec{h}$. Thus $\alpha = \alpha(s, \psi)$, i.e. α is a function of only two variables. Similarly for β, γ, δ . Thus, Lortz' "similarity variable" ψ reduces everything by one dimension - a major step forward.

In the above construction, with $\alpha, \beta, \gamma, \delta$ functions only of s, ψ there is no choice but that the electrostatic potential is also a function only of s, ψ . In consequence, the two equations that determine γ, δ (given α, β) are the $\vec{h}$ component of Ohm's law and the $\vec{h}$ component of the induction equation. With much algebra, these two equations are found to be

$$R_m^{-1} \left[\frac{1}{s} \frac{\partial}{\partial s} (h^2 s \frac{\partial \delta}{\partial s}) + \frac{1}{s^2} \frac{\partial^2 \delta}{\partial \psi^2} \right] + \frac{h^2}{s} \frac{\partial (\beta, \delta)}{\partial (s, \psi)} = 2 R_m^{-1} \ell m h^4 \gamma \quad (4)$$

$$R_m^{-1} \left[\frac{1}{s} \frac{\partial}{\partial s} (h^2 s \frac{\partial \gamma}{\partial s}) + \frac{1}{s^2} \frac{\partial^2 \gamma}{\partial \psi^2} \right] + \frac{1}{s} \frac{\partial (\beta, h^2 \gamma)}{\partial (s, \psi)} = \frac{1}{s} \frac{\partial (h^2 \alpha, \delta)}{\partial (s, \psi)}. \quad (5)$$

Here $R_m = \mu \sigma V L$ is the magnetic Reynolds number and V, L are motion and length scales. The (linear) terms multiplied by R_m arise from Ohmic diffusion whereas the (non-linear) Jacobians describe induction. Exact closed form solutions of these equations exist because of the presence of the important coupling term on the right of (4) - it arises from the helicity of the coordinate system. Note that (4), (5) are invariant to the transformations

$$\begin{aligned}\alpha &\rightarrow \alpha + h^{-2} \eta(\delta), & \beta &\rightarrow \beta; \\ \alpha &\rightarrow \alpha - \gamma \xi'(\delta), & \beta &\rightarrow \beta + \xi(\delta)\end{aligned}$$

where η, ξ are functionals (the induction equation only fixes the motion across field lines).

Four Possible Dynamos

I. **Lortz's Case** (simplest magnetic field structure, no radial field, mean field plus a harmonic of a spatially fluctuating field, no mean motion).

$$\begin{aligned}\alpha &= \alpha_1(s) \sin n\psi + \alpha_2(s) \sin 2n\psi, & n &= \text{integer} \\ \beta &= \beta_1(s) \sin n\psi, \\ \gamma &= \gamma_0(s) + \gamma_1(s) \cos n\psi, \\ \delta &= \delta_0(s).\end{aligned}$$

The six functions $\alpha_1, \alpha_2, \beta_1, \gamma_0, \gamma_1, \delta_0$ have to satisfy five equations. All functions but δ_0 can be found explicitly in terms of γ_0 which can be chosen arbitrarily (see Benton 1975 for a specific example).

This is certainly a dynamo, and Taylor's consistency criterion is satisfied for any choice of γ_0 . This dynamo is therefore drivable by a force system lying in s, ϕ planes.

II. A Possible Dynamo with Simplest Motion

(No radial motion, parallel motion only, no mean motion, magnetic field possesses radial component).

$$\begin{aligned}\alpha &= \alpha_1(s) \sin n\psi, \\ \beta &= 0, \\ \gamma &= \gamma_0(s) + \gamma_1(s) \cos n\psi, \\ \delta &= \delta_0(s) + \delta_1(s) \cos n\psi.\end{aligned}$$

The five functions must satisfy five equations. It is not yet known whether non-trivial solutions exist, but if so, they automatically give a drivable dynamo in the sense that Taylor's criterion is satisfied.

III. Simplest Second Order "Dynamo" (a counter-example).

$$\begin{aligned}\alpha &= \alpha_0 + \alpha_{11} \sin n\psi + \alpha_{21} \sin 2n\psi, \\ \beta &= \beta_0 + \beta_{11} \sin n\psi,\end{aligned}$$

$$\gamma = \gamma_0 + \gamma_{11} \sin n\psi + \gamma_{12} \cos n\psi + \gamma_{21} \sin 2n\psi,$$

$$\delta = \delta_0 + \delta_{11} \sin n\psi.$$

The eleven functions here must obey ten dynamo equations. However, Taylor's criterion supplies two more equations which are

$$\delta'_{11} \gamma'_{12} = 0, \quad (h^2 \gamma'_{12}) \delta_{11} = 0.$$

This produces a complete collapse. Thus, we have an example of a kinematically possible dynamo, but one that is not dynamically drivable.

IV. The General Second Order "Dynamo"

$$\alpha = \alpha_0 + \alpha_{11} \sin n\psi + \alpha_{12} \cos n\psi + \alpha_{21} \sin 2n\psi + \alpha_{22} \cos 2n\psi,$$

$$\beta = \beta_0 + \beta_{11} \sin n\psi + \beta_{12} \cos n\psi,$$

$$\gamma = \gamma_0 + \gamma_{11} \sin n\psi + \gamma_{12} \cos n\psi + \gamma_{21} \sin 2n\psi + \gamma_{22} \cos 2n\psi,$$

$$\delta = \delta_0 + \delta_{11} \sin n\psi + \delta_{12} \cos n\psi.$$

There result twelve dynamo equations for the sixteen functions. Satisfaction of Taylor's criterion produces the following two ~~more~~ equations which may allow for a drivable dynamo.

$$\ell h^2 \gamma'_{12} \delta'_{11} - \delta_{12} (\ell h^2 \gamma'_{11} + m s h^2 \delta'_{11})' = 0,$$

$$\ell h^2 \gamma'_{11} \delta'_{12} - \delta_{11} (\ell h^2 \gamma'_{12} + m s h^2 \delta'_{12})' = 0.$$

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NON-PERIODIC REVERSAL IN A MODEL DYNAMO

Kay Ann Robbins

There is a one-to-one correspondence between certain low order moment equations of the alpha dynamo and a reversing disk dynamo proposed by Malkus (1971). This relationship suggests that many features of magnetic reversal can be understood by analysis of representative moments of the large scale field equations. The observed dominance of the dipole component of the poloidal field is central in the selection of these moments. Motivated by geostrophic prominence in rotating flows and some basic symmetries of the alpha dynamo, we choose the azimuthal angular momentum to be representative of the large scale velocity field. Consistency then requires that the quadrupole moment of the toroidal field be included to complete the set.

The disk dynamo under consideration is similar to a model of Bullard (1955) with brush resistance included. This addition permits interaction between the currents in the disk and coil. The non-dimensional model equations which result are:

$$\frac{dw}{dt} = R - zy - w,$$

$$\frac{dz}{dt} = wy - z,$$

$$\frac{dy}{dt} = \sigma(z - y)$$

w is the angular velocity of the disk which can be identified with the volume averaged azimuthal angular momentum of the alpha dynamo. The current in the coil, y , gives rise to a dipole-like field. An antisymmetric quadrupole field is produced by z , the current in the disk. R is a measure of the external forcing and σ is a dimensionless combination of circuit resistances and inductances. The disk dynamo equations described above were studied by Lorenz (1963) as a model of atmospheric convection.

As the external torque is increased from zero, the model exhibits a series of stability transitions which ultimately result in wildly oscillating non-periodic motions. For $R < 1$ all solutions approach a zero-field state: $(w, z, y) = (R, 0, 0)$. For $R > 1$ this solution is unstable and two steady solutions, $(1, \pm\sqrt{R-1}, \pm\sqrt{R-1})$ bifurcate from it. The new steady solutions have a subcritical bifurcation at

$R = R_c = \frac{\sigma(\sigma+4)}{(\sigma-2)}$. This type of bifurcation is marked by hysteresis effects and limited domains of attraction. A novel aspect of this study is the one-dimensional characterization of the subcritical instability and the transition to non-periodicity. There is a critical value of R , R_{sc} , above which the system has solutions which do not approach any of the critical points.

The set of such solutions remains small until R is increased beyond a certain point, R_{cc} . For $R > R_{cc}$ the set of initial conditions corresponding to solutions not approaching any of the critical points has non-zero measure. For $R > R_c$ all critical points are linearly unstable and non-periodic motion ensues.

A striking non-uniformity in the number of oscillations between reversals is associated with the transition R_{cc} . Observations of similar non-uniformities during geomagnetic reversal indicate that subcritical transitions are important in the geodynamo as well. The transitions described above have not been previously observed and can be interpreted geometrically in terms of invariant surfaces of the system. Similar transitions may be present in other subcritical configurations such as shear flow.

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INTEGRABLE SYSTEMS OF NONLINEAR EVOLUTION EQUATIONS

Alan C. Newell

The inverse scattering transform (IST) has proven to be a powerful tool for analyzing and solving the initial value problem for a broad class of nonlinear evolution equations. The method was first developed by Gardner, Greene, Kruskal, and Miura (1,2) and applied by them to the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0 \quad (1)$$

The basic idea is to map the unknown function $u(x,t)$ of (1) into the scattering functions of the Schrodinger equation

$$\psi_{xx} + (\zeta^2 + \frac{1}{2} u(x,t))\psi = 0, \quad (2)$$

in which $u(x,t)$ plays the role of the potential. This mapping is a canonical one (3,4) in which certain combinations of the scattering functions are action and angle variables and hence their time evolution is simple. In particular, the discrete eigenvalues of (2) are constants of the motion. The method of solution, then, is (a) map the initial data $u(x,0)$ into the scattering data $S(0)$, (b) follow the evolution of $S(0)$ to $S(t)$ and (c) use the inverse mapping to reconstruct $u(x,t)$! From this point of view, the method is nothing but a generalization of the ideas of a Fourier transform.

An elegant perspective on the relation between (1) and (2) was provided by Lax (5) at an early stage in the development of the theory. Lax observed that the requirement that all operators of a family $\{L(t)\}$ are unitarily equivalent, and thereby share a common spectrum, can be written as an operator equation

$$L_t = (B, L). \quad (3)$$

For the Korteweg-de Vries equation, let

$$L = -\frac{\partial^2}{\partial x^2} - \frac{1}{6} u(x,t), \quad B = -4 \frac{\partial^3}{\partial x^3} - u \frac{\partial}{\partial x} - \frac{1}{2} u_x, \quad (4)$$

and (3) becomes (1). Lax also provided an algorithm for generating a sequence of operators B each of which when used in conjunction with the L of (4) leads to an integrable partial differential equation in the Korteweg-de Vries family.

The difficulty in general, of course, is in finding an appropriate L and a B (if they exist at all) for a given evolution equation of interest. Such ingenuity was supplied by Zakharov and Shabat (6) who solved the nonlinear Schrodinger equation

$$q_t - iq_{xx} + 2iq^2q^* = 0 \quad (5)$$

using the matrix operator

$$L = \begin{pmatrix} i \frac{\partial}{\partial x} - iq \\ -iq^* - i \frac{\partial}{\partial x} \end{pmatrix} \quad (6)$$

In a series of papers shortly thereafter, Ablowitz, Kaup, Newell and Segur (7,8,9) provided a systematic procedure for finding and classifying (by means of the dispersion relation of the associated linear problem) all the evolution equations which could be solved by using a generalization of the Zakharov and Shabat operator. The Hamiltonian and operator theoretic structure of these equations is discussed by Flaschka and Newell (10).

In the last year there has been an explosion in the amount of literature on this subject. Many new problems, including difference equations (11-15), equations with periodic boundary conditions (16,17,18) evolution equations corresponding to matrix operators of higher order (19,20) have been solved and even certain partial differential equations of more than one space dimension have been put in the Lax formulation (20,21,22).

What is so exciting about the theory is not so much its elegant mathematical structure but rather its direct applicability to so many areas of classical and modern physics. In this regard we mention the work on nonlinear optics (23,24,25), water waves (26,27) elementary particle theory (28,29,30) and plasma waves (31,32, 33) and recently there has been much attention paid to possible applications to internal gravity and atmospheric waves (34).

The most striking feature which is common to the solutions of all these problems is the soliton, a truly singular and nonlinear entity whose remarkable properties were first noted by Zabusky and Kruskal (35). These special solutions are each identified with one of the discrete eigenvalues (which are usually but not necessarily time invariant (36)) of an appropriate operator L . If an isolated soliton collides with other solution components, it will behave in a very nonlinear way during the interaction. However, after the interaction, it reemerges with its former identity (speed, shape, height) intact with only a phase shift to show for its collision experience. The close resemblance with particles caused Zabusky and Kruskal to coin the name soliton. The possibility that solutions of similar structure of certain universal field equations may exhibit elementary particle behavior is intriguing.

In any event, there is much to be said for studying these "perfect" nonlinear equations. At the very least, they provide an alternative vantage point from which to study "real" and "nonperfect" problems many of which, heretofore, were studied by perturbation around the known solutions of linear problems, a limit process which often removes the most interesting features from the model.

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STRUCTURAL GEOLOGY AND FLOW INSTABILITIES IN THE EARTH'S CRUST

Ronald B. Smith

This study in theoretical structural geology attempts to explain the formation of folds, boudinage and mullions by treating the rocks as a Newtonian fluid and using the techniques of linear stability analysis. The model problem is a single layer of fluid with viscosity μ_2 surrounded above and below with a different fluid of viscosity μ_1 . In this system pure shear

$$u = U'x, \quad v = -U'y \quad (1)$$

is an exact solution to the governing equations but is unstable to either a fold-like or to a varicose disturbance (boudinage or mullions) unless $\mu_1 = \mu_2$. This shearing instability has several interesting properties:

a) The disturbance growth is caused both by the direct kinematic action of the basic flow and by a secondary flow driven by the discontinuity in normal stress τ_{11} associated with the basic flow. These two effects can either compete or cooperate.

b) The disturbance growth requires cooperation between the two interfaces. The secondary flow driven by stresses at one interface is effective in deforming the other interface. The instability cannot occur on a single interface between two different homogeneous materials.

c) Time is not important in this instability. The growth of the disturbance amplitude "a" depends only on the total strain that the rock package has undergone. We have in fact

$$\ln a/a_0 = \gamma \ln L/L_0 \quad (2)$$

where L and L_0 are the final and initial lengths of a reference line parallel to the layer and γ is the growth rate. The process could occur at a variable rate with no change in the final state. The growth rate γ can be written as

$$\gamma = \left[\frac{1}{R(\lambda/H, m)} \right] (m-1) \quad (3)$$

where m is the viscosity ratio μ_2/μ_1 and λ/H is ratio of the disturbance wavelength to the layer thickness. The function R is positive for varicose (i.e. pinch-and-swell) disturbance and negative for folding. From (2) and (3) it is seen that growth will occur whenever the product

$$\left[\frac{1}{R} \right] (m-1) [\ln L/L_0]$$

is positive. Three of these four cases correspond qualitatively to folding, boudinage, and mullions as they have been described in the field. The fourth case of positive growth has such small rates of growth that it is probably of little interest.

THE INFLUENCE OF NON-NEWTONIAN BEHAVIOR ON SHEARING AND GRAVITATIONAL INSTABILITIES

Ronald B. Smith

The Newtonian theory of shearing instability in the crust (see previous lecture) is difficult to check quantitatively as the viscosity ratio " m " is not known. Nevertheless, there are certain aspects of those results which obviously conflict with field observations. For example

1. When the kinematic decay caused by the basic flow is included, the Newtonian theory predicts that boudins will not grow for any " m ". Yet, boudinage is found in the field.

2. The theory predicts growth rates for mullions which are too small to explain the observed cases.

3. Small scale folds are usually seen to have wavelengths from four to six times the layer thickness. The Newtonian theory fails to explain this preferred range for λ/H and predicts that folds will never form with $\lambda/H \leq 6$ or so.

These problems can be eliminated if we allow the fluid to be non-Newtonian.

As the basis for this extension of the theory we use the Reiner-Rivlin constitutive relation

$$\tau_{ij} = -P \delta_{ij} + \mu(\theta_1, \theta_2, \theta_3) \epsilon_{ij} + \nu(\theta_1, \theta_2, \theta_3) \epsilon_{ik} \epsilon_{kj} \quad (1)$$

assuming only that the fluid is

- 1) local and anelastic
- 2) homogeneous and isotropic
- 3) incompressible ($\theta_3 = 0$)

In (1) the three θ 's are the invariants of the rate of strain tensor ϵ_{ij} . The assumption of two-dimensional flow and the linearization about a basic state of pure shear E_{11} greatly simplify (1) and only the effective power law " n " is needed to completely characterize the non-Newtonian behavior. The perturbation flow obeys an anisotropic constitutive relation

$$\begin{aligned} \tau_{11} &= -P + (\mu/n) \epsilon_{11} \\ \tau_{12} &= \mu \epsilon_{12} \\ \tau_{22} &= -P + (\mu/n) \epsilon_{22} \end{aligned}$$

and with this, the shear instability problem can be redone. The goal of such a calculation is the growth rate

$$\gamma(\lambda/H, m, n_1, n_2)$$

which now depends on n_1 and n_2 , the effective power laws for the surroundings and the layer.

This extension of the theory of shearing instability resolves all three conflicts with the field observation mentioned above. If the layer is shearthinning (viscosity decreasing with strain rate, $n_2 > 1$) the growth rates of both boudinage the mullions are increased to reasonable values and the dominant wavelength for folding is decreased. In fact, the range $4 < \lambda/H \leq 6$ is shown to be preferred, for if $n_1 > 5$ or so almost any choice of m, n_1, n_2 will give λ/H in this range.

This method of treating the non-Newtonian effects is also useful in a variety of other problems such as the onset of gravitational instability. The interaction of both simple and pure shear with the classical Rayleigh and Rayleigh-Taylor convection has been carried out and the results have some interesting implications for geophysics.

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A DISCUSSION OF THE PRECESSIONALLY-DRIVEN DYNAMO

David E. Loper

The feasibility of a precessionally-driven dynamo is investigated by consideration of the torque balance and energy budget of the precessing mantle and core. It is shown that at least 10^9 watts must be supplied to sustain an internal field of 100 gauss against ohmic losses. We wish to determine whether this amount of power can be transmitted from the rotational kinetic energy to internal core motions by torques acting on the mantle and core. Both gravitational and pressure torques act perpendicular to the rotation vectors of the core and mantle to dominant order and are therefore incapable of transferring energy to the core motions. We must consider the dissipative torques resulting from stresses at the core-mantle interface generated by viscous and magnetic boundary layers. This requires a detailed analysis of these boundary layers.

Assuming *laminar* flow, three boundary layers occur near the core-mantle interface as a result of precession: a time-dependent Ekman-Hartmann layer of thickness $\nu^{1/2} \omega^{-1/2}$, a magnetic diffusion layer in the core of thickness $(\sigma \mu \omega)^{-1/2}$ and a magnetic diffusion layer in the mantle of thickness $(\sigma_m \mu \omega)^{-1/2}$ where ν , ω , μ , σ and σ_m are respectively viscosity, rotation rate, magnetic permeability, core electrical conductivity and mantle electrical conductivity. The primary forcing for these layers is provided by the jump in the tangential velocity of order $(\omega L \kappa)$ caused by the precession where L is the core radius and κ is the angle between the angular velocity vectors of the core and mantle. Within the Ekman-Hartmann layer, a tangential velocity of order $(\omega L \kappa)$ and a normal velocity of order $(\nu^{1/2} \omega^{1/2} \kappa)$ are induced by the forcing. This in turn induces a magnetic field of order $(\nu^{1/2} \omega^{1/2} \sigma \mu L A \kappa)$ and an EMF of order $(\omega L \mu A \kappa)$ in the Ekman-Hartmann layer where A is the strength of the radial component of the magnetic field at the core-mantle interface. Assuming $\sigma_m \ll \sigma$, this EMF induces a magnetic field in the mantle of order $(\sigma_m \mu \omega)^{1/2} L A \kappa$. Through continuity of magnetic field this induces a magnetic field of the same order within the magnetic diffusion layer in the core. The Lorentz force associated with this magnetic field generates a tangential velocity of order $(\sigma \sigma_m)^{1/2} A^2 L \kappa / \rho$ and, through continuity, a normal velocity of order $(\sigma_m / \mu \omega)^{1/2} A^2 \kappa / \rho$ within the magnetic diffusion layer

in the core. The ratio of this normal velocity to that of the Ekman-Hartmann layer is $(\sigma_m / \mu \nu)^{1/2} A^2 \rho \omega$ which may be large within the Earth's core. The detailed structure of these layers is presented in Loper (1975) where it is seen that the breakdown of the Ekman layer at the critical latitudes ($\theta = \pi/6, 2\pi/3$ for spin-over) is suppressed by the presence of a radial component of magnetic field but that the magnetic diffusion layer exhibits a similar breakdown at the same critical latitudes. This may lead to a stronger and more unstable rectified flow than predicted by Busse (1968).

It is shown that laminar dissipative torques are capable of reducing the rotational kinetic energy at a rate of 3×10^7 watts, much less than needed to power the dynamo. It is further shown that, to dominant order, all this energy is dissipated in the laminar Ekman-Hartmann layer and in the magnetic diffusion layer of the mantle, clearly demonstrating why a laminar precessionally-driven dynamo is not possible. The possibility remains that power is fed to the dynamo by torques resulting from turbulent motions. However, turbulent motions are much more dissipative than laminar motions, placing a severe drain on the power supply.

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EVIDENCE FOR NON-NEWTONIAN RHEOLOGY IN THE ASTHENOSPHERE

H. Jay Melosh

Summary

This talk falls naturally into two parts. The first part is theoretical, and extends Elsasser's (1967) theory of stress propagation in the upper mantle to an asthenosphere with nonlinear rheology. The second part uses these theoretical results as a basis for a measurement of the rheology of the asthenosphere. The aftershock migration patterns of the February 4, 1965 Rat Island earthquake are analyzed, and strong evidence is found for a non-Newtonian stress-strain relation in the asthenosphere. The stress propagation coefficients calculated from laboratory data are in excellent agreement with those derived from the aftershock migration patterns.

I. Theory of Stress Propagation in a Nonlinear Asthenosphere

A. Derivation of the Fundamental Equation.

Suppose that stress and strain within the asthenosphere are related by:

$$\dot{\epsilon}_{ij} = A \sigma^{n-1} \sigma_{ij} \quad (1)$$

where

$\dot{\epsilon}_{ij}$ = strain rate tensor

j = deviatoric stress tensor

A = Dorn Parameter, a strong function of temperature

$$\sigma = \left[\frac{1}{2} \text{Tr} (\sigma_{ij} \sigma_{jk}) \right]^{1/2}, \quad \text{Second Invariant of } \sigma_{ij}$$

We shall follow the approach of Elsasser (1967), but we shall assume $n \neq 1$. Assume that we have an elastic lithosphere of thickness h_1 , Young's modulus E

overlying a fluid asthenosphere of thickness h_2 . A stress $\sigma_x(x)$ is applied in the $\hat{x}$ direction of the lithospheric plate (Fig.1). The net force per unit area of the plate is $F = h_1 \left(\frac{\partial \sigma_x}{\partial x} \right)$. This force is balanced by drag forces in

Fig.1

the asthenosphere, σ_{xz} per unit area, where σ_{xz} is the shear-stress across the asthenosphere.

In an elastic material $\sigma_x = E \frac{\partial u_x}{\partial x}$ where u_x is the displacement of the lithosphere in the $\hat{x}$ direction. Thus,

$$\sigma_{xz} = h_1 \frac{\partial \sigma_x}{\partial x} = h_1 E \frac{\partial^2 u_x}{\partial x^2} \quad (2)$$

Since σ_{xz} is constant through the thickness of the asthenosphere, Eq.(1) requires

$$\frac{\partial u_x}{\partial t} = 2A h_2 |\sigma_{xz}|^{n-1} \sigma_{xz} \quad (3)$$

hence

$$\frac{\partial u_x}{\partial t} = \chi \left| \frac{\partial^2 u_x}{\partial x^2} \right|^{n-1} \frac{\partial^2 u_x}{\partial x^2} \quad (4)$$

where χ is the stress propagation coefficient

$$\chi = 2A h_2 (h_1 E)^n \quad (5)$$

These equations (4) and (5) reduce to Elsasser's equations if we set $n=1$ and $A = \frac{1}{2\eta}$.

Equations (4) and (5) are the fundamental equations for stress propagation in the upper mantle. They are rigorously valid when the following conditions hold.

a) Displacement u_x is constant along a line parallel to $\hat{y}$ which is much longer than relevant distances perpendicular to that line (one-dimensional approximation).

b) Displacements do not change significantly over distances comparable to the thickness of the asthenosphere (so that σ_{xz} is constant across the asthenosphere).

Both of these conditions are sufficiently well satisfied by the 1965 Rat Island Earthquake to allow the application of Eqs.(4) and (5).

B. Solution for a step function initial displacement.

This solution models the stress and displacement propagation following a sudden displacement of magnitude u_0 at the edge of the lithospheric plate. The solution thus describes the effect of a large decoupling earthquake, during which the lithospheric plate may be displaced by tens of meters along a fault up to 1000 km long.

The boundary conditions are:

$$\begin{aligned} u_x(x, 0) &= 0 \\ u_x(0, t) &= u_0 \end{aligned}$$

Since the boundary conditions contain no quantities with the dimensions of time, $u_x(x,t)$ can only be a function of

$$\eta \equiv \frac{x}{(\alpha t u_0^{n-1})^{1/n}} \quad (6)$$

and

$$u_x(x,t) = u_0 I_n(\eta) \quad (7)$$

where $I_n(0) = 1$ and $I_n(\eta) = 0$ for $\eta \geq \eta_{max}$. The dimensionless η_{max} has a value between 2 and 3, depending (weakly) upon n . In general, $I_n(\eta)$ is a complex integral which can only be evaluated numerically. However, it is a good approximation to write

$$I_n(\eta) \simeq \left(1 - \frac{\eta}{\eta_{max}}\right)^2 \quad (8)$$

which is accurate to about 10%.

The most interesting behavior is given by the $\frac{1}{t^{1/n}}$ factor in η . If the displacement front takes time t_1 to move distance x_1 , it takes $2^{2n} t_1$ to move a distance $2x_1$ (e.g., for $n=3$ it takes $2^6=64$ times as long to move 200 km as it does to move 100 km). This is a characteristic feature of nonlinear stress propagation. Shortly after the initial displacement the front moves very rapidly away from $x=0$, but later slows down into comparative immobility.

In observing aftershocks, which we shall assume roughly delineate the stress field, it is important to be able to trace a given stress contour's motion. The typical behavior of a stress contour is to move rapidly out to a maximum excursion x_m from the initial fault, and then to slowly retreat to the origin, arriving after a time t_0 from the initial event. Using x_m, t_0 as parameters, the $\sigma_x(x,t) = \text{const.}$ contours are given by

$$\frac{x}{x_m} = c_d \left(\frac{t}{t_0}\right)^{1/n} \left[1 - \left(\frac{t}{t_0}\right)^{\frac{n-1}{2n^2}}\right]^{\frac{n}{n+1}} \quad (9)$$

where $c_d \approx 4$ for all n . The parameters x_m, t_0 can also be used to determine

$$\alpha = c_k \frac{x_m^{2n}}{u_0^{n-1} t_0} \quad (10)$$

$$\frac{\sigma_x}{E} = c_s \left(\frac{u_0}{x_m}\right) \quad (11)$$

where $c_s \approx \frac{1}{2}$ for all n , and c_k depends more strongly upon n ($c_k = 0.026$ for $n=3$). u_0 is the initial displacement, and E is Young's modulus for the lithosphere.

C. Solution for a Steady Displacement of the Plate Boundary.

Another solution of geophysical interest is that of a steadily moving plate boundary of velocity v . This solution models the average effect of a large number of decoupling earthquakes. The boundary conditions for this case are

$$u_x(x,0) = 0$$

$$u_x(0,t) = vt$$

It can be shown that the solution is of the form

$$u_x(x,t) = K t g(\phi) \quad (12)$$

$$\phi = \frac{1}{(x^{n-1} \tau)^{1/2n}} \frac{x}{\sqrt{t}} \quad (13)$$

and to about 10% accuracy,

$$g(\phi) \simeq \left(1 - \frac{\phi}{\phi_{\max}}\right)^2 \quad (14)$$

$g(0)=1$ and $g(\phi)=0$ for $\phi \geq \phi_{\max}$. $\phi_{\max}$ is a function of n , but its value is between 1.5 and 2 for $3 \leq n \leq 7$.

In contrast to the solution for a single impulse, we see from the $\phi \sim x/\sqrt{t}$ dependence that a steady motion is rapidly propagated into the plate (as rapidly as in a Newtonian $n=1$ asthenosphere) for any n value.

Thus, for large n the plate motion becomes progressively less sensitive to the effects of short-lived motions of its boundary, and more accurately reflects the long-term average stresses acting upon it.

II. Determination of the Rheology of the Asthenosphere

We have seen that the stress and displacement propagation patterns following a sudden displacement of a plate boundary are diagnostic of the rheology of the asthenosphere. Such a sudden displacement is a fairly accurate representation of the effect of a large decoupling earthquake in an oceanic trench. Thus, by measuring $u_x(x,t)$ following such an earthquake, we might hope to obtain information on the rheology of the asthenosphere.

Since sufficiently accurate geodetic data on the ocean floor is not available, $u_x(x,t)$ cannot be measured directly. We are thus forced into the tricky art of using small earthquakes (after shocks) to delineate zones of regional stress.

As a first approximation, suppose that aftershocks occur only when stresses exceed some critical stress σ_x^c . By following the migration of the aftershocks away from the zone of initial rupture we thus trace out the motion of the σ_x^c contour.

This procedure was followed for the February 4, 1965, Rat Island aftershocks, and a very clear pattern emerged. The aftershocks from the event filled a rectangular region 650 km long and 300 km wide at their maximum extent. The fault dips at only 18° to the horizontal, and 2.5 m of slip occurred during the event. The aftershocks of the first day fell in a 50 km wide zone 500 km in length. This zone served to define the line of initial rupture, and subsequent aftershock's distances from the line were taken as a measure of their distance from the initial rupture.

Aftershock data for nine years, from 1965-1973 were obtained from USCGS tapes and the aftershock distance from the initial rupture was plotted in 50 km bins as a function of time. The edge of the distribution, as defined by extrapolating the number vs. distance plots to zero for seven logarithmically increasing time intervals, was plotted vs. time on a log-log plot and compared to the curves predicted by theory. The result is shown in Fig.2.

It is clear from Fig.2 that a linear asthenosphere, $n=1$, cannot be made to fit the data. On the other hand, the data is not sufficiently accurate to differentiate between the various $n=2-7$ curves, and so we quote the result simply as $n=4 \pm 2$.

The result also yields values for the σ_x^c contour of

$$\begin{aligned} X_m &= 175 \pm 25 \text{ km} \\ t_0 &= 2600 \pm 500 \text{ days} \end{aligned}$$

Fig.2

which, for $n = 4$, implies

$$\chi = 4 \times 10^{44 \pm .7} \text{ (c.g.s.)}$$

$$\sigma_x^c = 11 \pm 2 \text{ bars}$$

Note that for $h_1 = 70 \text{ km}$, $h_2 = 50 \text{ km}$, $E = 1.6 \times 10^6 \text{ bar}$ together with the laboratory creep data of Kohlstedt and Goetze at a temperature of 1300°C we calculate

$$\chi_{calc} = 1.6 \times 10^{42} \text{ (cgs)}$$

which is in very satisfactory agreement with the observed stress propagation coefficient.

These results correspond to an effective viscosity of ca. 4×10^{19} poise for the asthenosphere at a stress appropriate to the average 6 cm/yr rate of underthrusting in this region of the Aleutians.

III. Conclusions

We have seen that the aftershock migration data from the 1965 Rat Island Earthquake rules out the possibility of a Newtonian asthenosphere. Although the exact value of n is yet uncertain, the agreement between calculated and observed values of the stress propagation coefficient χ (for each of the possible n) is strong evidence in favor of the interpretation of aftershock migration patterns we have presented. The effective viscosity of the asthenosphere at stresses necessary to drive a 6 cm/yr motion is 4×10^{19} poise.

The stresses due to an individual earthquake such as the Rat Island event die out as we proceed into the oceanic plate, falling to the level of the average stresses at ca. 300 km from the plate edge. Beyond this distance plate motion is chiefly governed by the average subduction rate rather than by the effects of individual earthquakes.

MACRODYNAMICS OF α -EFFECT DYNAMOS
(an extended abstract)

Michael R. E. Proctor

This study takes as its starting point the ' α -effect' model of magnetic field regeneration, in which motions and fields in a domain of electrically conducting fluid such as the Earth's core exist on two widely-differing length scales. Provided the two scales are sufficiently disparate, the transfer of energy from the small scale to the large scale may be expressed in the form of an additional e.m.f. (electromotive force)

$$\mathcal{E}_i = \alpha_{ij} B_j$$

where $\underline{B}$ is the large scale magnetic field and α_{ij} is a symmetric tensor which depends only on averaged properties of the small scale flow $\underline{u}'$, notably the mean helicity $\underline{u}' \cdot \nabla \wedge \underline{u}'$. Provided the motions are sufficiently vigorous this e.m.f. can maintain the large scale field against ohmic dissipation. (Steenbeck, Krause and Radler (1966); see also Roberts and Stix (1971) and Moffatt (1970a, b; 1972)). The linear kinematic problem that results when α_{ij} is a given function is now well understood. Moffatt (1972) has initiated a study of the fully dynamic problem that results when the small scale motions are driven by a given force-field. The growing large scale field then changes the nature of the flow, and hence the form of α_{ij} , eventually $\|\alpha_{ij}\|$ decreases to a value at which the energy fed in is just sufficient to balance ohmic dissipation, and field growth ceases. Moffatt was able to ignore large scale flows since in his unbounded domain no Lorentz forces or Reynolds stresses were present. Unfortunately, it can be shown that in a bounded domain such as the core, non-irrotational Lorentz forces must arise, and the large scale flows which are driven by these forces may act to stabilize the dynamo. Here we isolate this mechanism by suppressing the dependence of α_{ij} on $|\underline{B}|$, and ignore the effect of Reynolds stresses, although these could easily be included in a more sophisticated model. The resulting Navier-Stokes equation for the large scale flow is then scaled to take account of the 'magnetostrophic' balance between Coriolis and Lorentz forces (and pressure gradients) that probably prevails in the interior of the core. The resulting equation contains small terms representing inertial and viscous forces. Neglect of the former assumes that energy transport by Alfvén waves is effectively instantaneous. If the latter is also neglected, the magnetic Lorentz forces are subject to a constraint due to Taylor (1963), and in principle this constraint can be satisfied by suitable choosing of a free geostrophic azimuthal flow. At very low amplitudes, however, viscous forces become important, and Taylor's constraint is no longer relevant. The existence of these two regimes implies two very different prescriptions for the eigenvalue problem that underlies the nonlinear theory. In the viscous case, all velocities are suppressed by viscosity, but when viscosity can be ignored a geostrophic flow of scale amplitude must be found as part of the eigenvalue problem. Although no analytic solutions to the nonlinear problem exist at present (the one presented in Malkus and Proctor (1975) is in need of some revision), the eigenvalue problem has been studied in both limits and in the transition region at intermediate parameter ranges, and a numerical study of the full equations indicates that the proposed scaling is correct. The theory leads to amplitudes for the evolved field proportional to $\Omega^{1/2}$, where Ω is the angular velocity of the core, and its predicted magnitude appears to be in fair agreement with observations. Another question concerns the attainability of a state in which Taylor's constraint holds from an arbitrary initial state. This involves the resaturation of the inertial terms; it can be shown that torsional Alfvén waves result; these decay on the ohmic decay time scale (at least in the parameter range relevant to the Earth) and lead to a magnetostrophic state at large times. The numerical

analysis appears to bear out these predictions, although it is possible that the system may exhibit persistent small oscillations under some circumstances about the geostrophic state.

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TURBULENT CONVECTION

Dan E. Fitzjarrald

The results of an experiment done at the University of California at Los Angeles were discussed and compared with relevant theories, previous experiments, and atmospheric measurements. The experiments were performed in a large convection box 3.5m x 3.5m with height variable from 6 to 180 cm. Air is the working fluid. Rayleigh numbers from $R_a = 4 \times 10^4$ up to 7×10^9 were attained.

Cospectra of vertical velocity and temperature were calculated from hot-wire measurements in the convection chamber. The results for a large number of traverses through the chamber showed that the dominant horizontal wavelength L was larger than the depth of the layer d , at $R_a > 10^6$ giving $L/d \sim 6$. In addition the dominant wavelength was seen to change from $L/d \sim 4$ at $R_a \sim 10^5$ to $L/d \sim 6$ at $R_a \sim 10^6$. This transition to larger horizontal scale is similar to that observed in other fluids (silicon oil $P_r = 50$, and water $P_r = 7$), and is called the "collective" or "spoke pattern" instability by Friedrich H. Busse.

Recent analysis of scales of motion in atmospheric free convection also indicates large horizontal scales relative to the convecting layer depth. When compared with the laboratory experiments such data suggest that such scales of motion are ubiquitous in turbulent convection.

In addition to the spectral data, heat transport was measured and a systematic investigation of smaller-scale motions made. Neither of these investigations produced any surprises: the heat transport curve was similar to previous measurements and no evidence of multi- α type motions was found.

SPECTRAL ANALYSIS OF POLAR MOTION

Masanari Saito

Abstract The ILS/IPMS coordinates for the earth's instantaneous pole of rotation have been spectral analyzed by sophisticated digital data processing techniques. The spectrum obtained for the full data set (1900.0 ~ 1969.0) shows a single, well-defined Chandler peak and an annual peak. The apparent Q of the Chandler peak is estimated to be about 100, three times as large as previous estimates. The possibility of the atmospheric excitation of the polar motion is demonstrated.

Data and Method of Analysis

The most homogeneous set of ILS/IPMS data compiled by Vincente and Yumi (1969, 1970) is the starting point of our analysis. The data provide the monthly values of x (along the Greenwich meridian) and y (90°W) coordinates of the pole since 1900. Unfortunately, they are non-uniformly sampled in time and unsuitable for analysis. Thus an equi-spaced time series was generated from the original data set in the following way:

(i) x and y series were combined to form a single complex time series, $z = x - iy$, then interpolated by a least squares spline interpolation technique, and sampled at every 1/48 year.

(ii) Secular drift was removed by fitting a quadratic in time.

(iii) High frequency components were attenuated by a zero phase-shift filter, and the results were resampled at every 1/12 year.

Power spectrum was estimated by the maximum entropy method (MEM). In MEM the length of the prediction error filter is of vital importance to the quality of estimates. The optimum filter length was determined according to Akaike's criterion (see, for example, Fryer et al., 1975).

Result and Discussion

Power spectrum of z has been determined in the frequency range, $-1.5 < f < 1.5$ cpy. According to the definition of $z = x - iy$, negative frequency represents east-to-west (retrograde) motion of the pole, and positive frequency prograde motion. As expected in the dynamics of the problem, no appreciable power was found for retrograde motion, except for a very low peak near $f \approx -1$ cpy.

The power spectrum of the polar motion can be written in the following form:

$$P(f) = |H(f)|^2 I(f) \quad (1)$$

$$|H(f)|^2 = \frac{f_{ch}^2}{(f - f_{ch})^2 + \alpha_{ch}^2} \quad \alpha_{ch} = \frac{f_{ch}}{2Q_{ch}}$$

where $H(f)$ denotes the transfer function of the earth, $I(f)$ the excitation spectrum, $f_{ch} = 1/1.24$ the Chandler frequency, and Q_{ch} the Chandler dissipation factor. The observed $P(f)$ consists of two well-defined peaks, Chandler and annual. Thus we can fit the observed $P(f)$ with

$$I(f) = \frac{\alpha_A}{\pi} \frac{S}{(f - f_A)^2 + \alpha_A^2} \quad \alpha_A = \frac{f_A}{2Q_A} \quad f > 0 \quad (2)$$

where $f_A = 1$ cpy and $I(-f) \approx I(f)$. Parameters appearing in (1) and (2) were consistently determined to be

$$Q_{ch} \doteq 100 \quad Q_A \doteq 60 \quad S \doteq (20 \text{ msa})^2$$

(msa = millisecond of arc).

Q_{ch} obtained here is considerably larger than previously-determined values. For example, Currie (1974) obtained $Q_{ch} \approx 30$. Although he used the same data set, his method of analysis differs slightly; (i) he used Burg's MEM algorithm which could result in misleading estimates, and (ii) he used fairly long prediction error filter (about 10 years) without any objective criterion. According to Akaike's criterion, one can not use prediction error filter longer than five years. Thus it is probable that Currie's error bounds were too optimistic.

The large Q_{ch} and small Q_A favor the long-abandoned hypothesis that the polar motion is excited by the seasonal movement of air mass (Munk and Hassan, 1961). Integrating $I(f)$ we find that the parameter S represents the total power of excitation, $\sqrt{S}$ the root-mean-square amplitude of excitation. According to Munk and Hassan the r.m.s. amplitude of the atmospheric excitation function is 7 msa, which compares quite well with the required excitation, $\sqrt{S} = 20$ msa, considering the large amount of uncertainty involved in their estimate.

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THE VISCOUS-GRAVITATIONAL RELAXATION OF "REAL" NEWTONIAN EARTH MODELS (An Extended Abstract)

W. R. Peltier

The principle of correspondence is invoked to determine Laplace transform domain solutions to the surface mass loading problem for radially stratified visco-elastic (Maxwell) spheroids. These Laplace transform domain solutions are expressed in terms of visco-elastic analogues of the conventional surface load Love numbers of elasticity (Peltier, 1974) and have the form

$$\begin{bmatrix} h_2(r,s) \\ l_2(r,s) \\ k_2(r,s) \end{bmatrix} = \frac{1}{\Phi_{2,l}(r)} \begin{bmatrix} g U_2(r,s) \\ g V_2(r,s) \\ \Phi_{1,l}(r,s) \end{bmatrix} \quad (1)$$

where U_2 , V_2 , and $\Phi_{1,l}$ are respectively the coefficients of the radial displacement, tangential displacement, and gravitational potential perturbation due to density variations within the spheroid. $\Phi_{2,l}$ are the coefficients of potential perturbation due to the surface mass load and s is the Laplace transform variable.

These visco-elastic Love numbers may be approximately transformed to the time domain using an extremal technique. Application of this technique shows that the Love number time histories may be well approximated by the linear superposition of a discrete set of purely exponential relaxations plus a constant term. These Love number time histories may be combined to produce Green functions for the various signatures of the response of the earth to the point mass load. For example the perturbation in radius has the form

$$\Delta R(\gamma, t) = \frac{a}{m_e} \sum_l h_l(t) P_l(\cos \gamma) \quad (2)$$

where γ is the angular distance from the applied load. These Green functions have been derived in Peltier (1974) for several illustrative visco-elastic earth models. They have been employed in Peltier and Andrews (1975) to simulate the process of glacial isostatic adjustment. Such solutions of the direct problem strongly suggest that the mantle of the earth should have a viscosity which is on the order of 10^{21} P (c.g.s.) throughout its volume if it is to be in accord with the Newtonian model of glacial isostatic adjustment.

As an alternative to the approximate solutions discussed above the discrete spectrum of relaxation times involved in the synthesis of each Love number time history may be found exactly by solving the associated homogeneous problem. Such solutions determine the discrete set of eigen-decay times which are associated with the discrete set of normal modes of viscous gravitational relaxation appropriate to the decay of each harmonic or order l in the spherical harmonic decomposition of an initial distortion of the planet from a gravitational equilibrium configuration. The solution of the inhomogeneous problem may be expressed in the form of a normal mode expansion.

This normal mode expansion may be employed as a basis for the construction of a rigorous first order perturbation theory for the inference of the viscosity of the deep interior of the planet (Peltier, 1975). A variational principle is derived which determines to first order the shift in position of a free decay pole in the relaxation spectrum which is forced by the addition of a radially distributed perturbation of viscosity to the starting model. This determines the differential kernels required for the solution of the inverse problem. The uniqueness of the state of isostatic equilibrium for the viscously incompressible Maxwell models employed in this analysis is demonstrated and this uniqueness allows us to calculate the shift in the residue which is associated with the shift in position of a free decay pole for the inhomogeneous problem. The inverse theory for the inhomogeneous problem is thus closed to first order. A formula is derived for the differential kernels appropriate to the inhomogeneous problem. The time domain form of these kernels may be calculated analytically. The main characteristics of the Quaternary data set of relative sea level curves are discussed and these characteristics demand a special form for the inverse theory which we proceed to obtain. The structure of the full theory is sufficiently simple that it may be employed to rigorously test the compatibility of the simple Maxwell model with the observed relaxation of the earth's shape which accompanied deglaciation at the end of the last ice age.

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FREE BOUNDARY PROBLEMS

Joseph B. Keller

1. Introduction

A free boundary problem is a problem in which the boundary of a region, or an interface inside a region, must be found together with the solution of a boundary value problem within the region. Such problems are common in continuum mechanics. In this note some general aspects of free boundary problems are described, and then a variety of particular free boundary problems are considered.

Mathematically it is clear that the formulation of a free boundary problem must involve additional conditions, besides the usual boundary or interface conditions specified in a boundary value problem. These additional conditions are necessary to fix the location of the free boundary or interface. In continuum mechanics these additional conditions follow the laws of mass, momentum and energy conservation applied to an interface or boundary.

A boundary or interface in continuum mechanics is called a contact discontinuity if no matter crosses it, i.e. if the normal velocity of the matter relative to the interface or boundary is zero. It is called a shock or shock type discontinuity, if matter does cross it and if the pressure is discontinuous across it. When matter crosses the interface or boundary but the pressure is continuous across it, the interface or boundary is called a phase front.

Examples of contact discontinuities are the air-water interface at the ocean surface, the water-solid interface at the ocean bottom and the air-solid interface between the ground and the atmosphere. Shock type discontinuities include ordinary shocks, detonation fronts, deflagration fronts, acoustic discontinuities and fronts of elastic pulses. Examples of phase fronts are the moving interfaces between a freezing liquid and its solid, between a melting solid and its liquid, between an evaporating liquid and its vapor, between a condensing vapor and its liquid, between a sublimating solid and its vapor, between a condensing vapor and its solid, between the growing superconducting and receding normally conducting phases of a metal, etc.

We shall now consider some particular examples.

2. Ice on a lake

Let us determine the steady state thickness h of ice on a lake when the upper surface is kept at a temperature T_1 which is below the freezing temperature $T_f = 0$ and the bottom surface is kept at a temperature T_2 which is above the freezing temperature. Let y denote the depth with $y = 0$ at the top and $y = H$ at the bottom. Then the temperature $T(y)$ satisfies the steady state heat equation in the ice, occupying the region $0 \leq y \leq h$, and in the water, which occupies the region $h < y \leq H$. Thus

$$(k_i T_y)_y = 0 \quad , \quad 0 \leq y < h \quad , \quad (2.1)$$

$$(k_w T_y)_y = 0 \quad , \quad h < y \leq H. \quad (2.2)$$

Here k_i and k_w are the thermal conductivities of ice and water respectively. The boundary conditions on T are

$$T(0) = T_1, \quad T(h) = 0, \quad T(H) = T_2, \quad T_1 < 0 < T_2. \quad (2.3)$$

In addition the heat flux must be continuous at the ice-water interface, which yields

$$k_i T_y(h-) = k_w T_y(h+) \quad (2.4)$$

The notations $T_y(h-)$ and $T_y(h+)$ denote the limits of $T_y(y)$ as y tends to h through smaller and larger values of y , respectively. This distinction is necessary because T_y is discontinuous at $y = h$.

From (2.1) it follows that $T_y = a + by$ where a and b are constants. Then the first two conditions in (2.3) yield

$$T(y) = T_1(h-y)/h \quad 0 \leq y \leq h. \quad (2.5)$$

Similarly (2.2) and the last two conditions in (2.3) yield

$$T(y) = T_2(y-h)/(H-h), \quad h \leq y \leq H. \quad (2.6)$$

Now the additional condition (2.4) yields the following equation for h :

$$-k_i T_1/h = k_w T_2/(H-h) \quad (2.7)$$

From (2.7) we find that the thickness of ice is

$$h/H = -k_i T_1 / (k_w T_2 + k_i T_1)^{-1}. \quad (2.8)$$

3. Freezing of a lake

Now we shall consider how the thickness $h(t)$ of the ice on a lake increases with time after the upper surface temperature is suddenly reduced to the value $T_1 < 0$. We suppose that initially there is no ice and that the water is at the constant temperature $T_0 > 0$. Furthermore, to simplify the analysis, we assume that the lake is infinitely deep. This problem was formulated and solved by Stefan about 100 years ago, so similar problems are often referred to as Stefan problems.

The temperature $T(y, t)$ must satisfy the heat equation in the ice and in the water, so

$$T_t = (K_i T_y)_y, \quad 0 \leq y < h(t) \quad (3.1)$$

$$T_t = (K_w T_y)_y, \quad h(t) < y. \quad (3.2)$$

Here K_i and K_w are the thermal diffusivities of ice and water, respectively. They are related to the conductivities k_i and k_w by $K_i = k_i/c_i$, $K_w = k_w/c_w$ where c_i and c_w are the specific heats per unit volume of ice and water.

The initial and boundary conditions on T are

$$T(y, 0) = T_0 \quad y > 0 \quad (3.3)$$

$$T(0, t) = T_1, \quad t > 0 \quad (3.4)$$

$$T[h(t), t] = 0 \quad (3.5)$$

In addition the heat fluxes into and out of the interface must differ by the latent heat of fusion which is released in the freezing process. Thus

$$k_i T_y[h(t)-, t] - k_w T_y[h(t)+, t] = \lambda \frac{dh}{dt}. \quad (3.6)$$

Here λ is the latent heat per unit volume. Finally the ice thickness is initially zero so

$$h(0) = 0 \quad (3.7)$$

We shall now find the ice thickness $h(t)$ by dimensional analysis, except for a dimensionless constant factor. To do so we note that the dimensions of K_i and K_w are $(\text{length})^2/\text{time}$ and the dimensions of T_o and T_i are degrees. The only other data of the problem are the ratios k_i/λ and k_w/λ in (3.6), which have the dimensions $(\text{length})^2/(\text{time})(\text{degrees})$. Since $h(t)$ has the dimensions of length, and $(K_i t)^{1/2}$ also has the dimensions of length, the ratio $h(t)/(K_i t)^{1/2}$ is dimensionless. This dimensionless ratio is a function of t , but no dimensionless function of t can be formed from the data of the problem, so the ratio must be independent of t . If we denote it by the dimensionless constant z_o we have

$$h(t) = (z_o K_i t)^{1/2} \quad (3.8)$$

Thus as (3.8) shows, the ice thickness grows like $t^{1/2}$.

Next we note that $T(y, t)/T_i$ is a dimensionless function of y and t . The only independent dimensionless combination of y and t which can be formed with the data of the problem is $z = y^2/K_i t$. Therefore $T(y, t)/T_i$ must be a function of z so we write

$$T(y, t) = T_i f(z), \quad z = y^2/K_i t. \quad (3.9)$$

We now substitute (3.9) into (3.1) - (3.6) to find the equations which must be satisfied by $f(z)$. In doing so we note that $z = 0$ when $y = 0$, that $z = z_o$ when $y = h(t)$ and that $z = \infty$ when $t = 0$. Then (3.1) - (3.6) become

$$4z f'' + (2+z) f' = 0, \quad 0 \leq z < z_o. \quad (3.1')$$

$$4z f'' + (2 + \frac{K_i}{K_w} z) f' = 0, \quad z_o < z, \quad (3.2')$$

$$f(\infty) = T_o/T_i, \quad (3.3')$$

$$f(0) = 1 \quad (3.4')$$

$$f(z_o) = 0 \quad (3.5')$$

$$f'(z_o-) - \frac{k_w}{k_i} f'(z_o+) = \frac{\lambda K_i}{4 T_i k_i} \quad (3.6')$$

The solution of (3.1') satisfying (3.4') and (3.5'), obtained by integrating (3.1') twice, is

$$f(z) = 1 - \frac{\int_0^z z^{-1/2} e^{-z/4} dz}{\int_0^{z_o} z^{-1/2} e^{-z/4} dz}, \quad 0 \leq z \leq z_o. \quad (3.10)$$

Similarly, the solution of (3.2') satisfying (3.3') and (3.5') is

$$f(z) = \frac{T_o}{T_i} \left[1 - \frac{\int_{z_o}^{\infty} z^{-1/2} e^{-z/4} dz}{\int_{z_o}^{\infty} z^{-1/2} e^{-z/4} dz} \right], \quad z_o < z \quad (3.11)$$

Finally by using (3.10) and (3.11) in (3.6') we get the following equation for z_o :

$$-\frac{z_o^{-1/2} e^{-z_o/4}}{\int_0^{z_o} z^{-1/2} e^{-z/4} dz} - \frac{k_w T_o}{k_i T_i} \left(\frac{K_i z_o}{K_w} \right)^{-1/2} \frac{e^{-K_i z_o/4 K_w}}{\int_{z_o}^{\infty} z^{-1/2} e^{-z/4} dz} = -\frac{\lambda K_i}{4 T_i k_i} \quad (3.12)$$

This equation can be solved numerically for z_o . When z_o is either large or small, approximate analytic solutions can be obtained.

If the lake is of finite depth H , the solution is approximately that given here for a short time. Ultimately it tends to that given in section 2. To find the

solution at intermediate times, other methods, such as numerical ones, must be used.

4. An ice machine

In Fig.1 is shown a machine for the continuous conversion of water into ice. It consists of two plates at $y=0$ and at $y=H$ maintained at the temperatures T_1 and T_2 respectively, with $T_1 \leq 0 < T_2$, where 0 is the freezing temperature. Water is fed in continuously from $x = \pm \infty$ and ice is withdrawn continuously at $x = \pm \infty$. The water is adjacent to the warmer plate at $y=H$ and the ice is adjacent to the colder plate at $y=0$. The machine is assumed to be symmetric about $x=0$ and to operate in a steady state.

Fig.1

At $x = \pm \infty$ the ice thickness $h(x)$ and the temperature $T(x, y)$ tend to the values $h(\infty)$ and $T(\infty, y)$ determined in section 2. They are independent of the velocity V with which the ice is withdrawn. Then the velocity $u(\infty)$ of the incoming water assumed to be independent of y , is determined by mass conservation, which yields

$$u(\infty) = \frac{-h(\infty)}{H - h(\infty)} V. \quad (4.1)$$

To find the ice thickness $h(x)$, the temperature $T(x, y)$ and the fluid velocity $u(x, y)$, $v(x, y)$ we shall make the simplifying assumption that these quantities all vary slowly with x . As a consequence we shall set $v=0$ and assume that $u = u(x)$ is independent of y . Then mass conservation yields

$$u(x) = \frac{-h(x)V}{H - h(x)}, \quad x \geq 0 \quad (4.2)$$

Furthermore at each value of x , $T(x, y)$ is given by the piecewise linear solution (2.5) and (2.6) of section 2 with h being $h(x)$. Thus it only remains to find $h(x)$, and for that purpose we shall consider the energy balance in that part of the machine from $x=0$ to x .

The heat balance for this region is

$$\int_0^x [k_i T_y(x', 0) - k_w T_y(x', H)] dx' - u(x) \int_{h(x)}^H c_w T(x, y) dy - V \int_0^{h(x)} c_i T(x, y) dy + \lambda V h(x) = 0. \quad (4.3)$$

The first term denotes the difference between the heat flux in across $y=0$ and that out across $y=H$, the second term denotes the heat convected in by the water, the third term represents the heat convected out by the moving ice and the last term is the latent heat released within the region. Integration of (2.5) and of (2.6) yields

$$\int_0^{h(x)} T(x, y) dy = h(x) T_1 / 2, \quad \int_{h(x)}^H T(x, y) dy = [H - h(x)] T_2 / 2 \quad (4.4)$$

Furthermore (2.5) and (2.6) yield $T_y(x, 0) = -T_1 / h(x)$ and $T_y(x, H) = [H - h(x)]$. Substitution of these values and of (4.2) and (4.4) into (4.3) yields

$$\int_0^x \left[\frac{-k_i T_1}{h(x')} - \frac{k_w T_2}{H-h(x')} \right] dx' + \left[\frac{c_w T_2}{2} - \frac{c_i T_1}{2} + \lambda \right] U h(x) = 0 \quad (4.5)$$

This is an equation which determines $h(x)$.

To solve (4.5) we first convert it into a differential equation by differentiating it with respect to x , to obtain

$$-\frac{k_i T_1}{h} - \frac{k_w T_2}{H-h} + (c_w T_2 - c_i T_1 + 2\lambda) \frac{U}{2} \frac{dh}{dx} = 0 \quad (4.6)$$

By setting $x=0$ in (4.5) we find that $h(0)=0$. The solution of (4.6) satisfying this condition is given by

$$x = (c_w T_2 - c_i T_1 + 2\lambda) \frac{U}{2} \int_0^h \left[\frac{k_i T_1}{h'} + \frac{k_w T_2}{H-h'} \right]^{-1} dh' \quad (4.7)$$

Evaluating the integral in (4.7) yields

$$x = (c_w T_2 - c_i T_1 + 2\lambda) \frac{U}{2} \left[\frac{h^2}{2(k_i T_1 - k_w T_2)} + \frac{k_i H(2k_i T_1 - k_w T_2)}{(k_i T_1 - k_w T_2)^2} + \frac{H^2 k_i T_1 (2k_i T_1 - k_w T_2)}{(k_i T_1 - k_w T_2)^3} \log \left\{ 1 + \frac{(k_w T_2 - k_i T_1)}{k_i T_1 H} \right\} \right] \quad (4.8)$$

This result gives the ice-water interface explicitly in the form $x = x(h)$.

For small values of h/H , (4.8) yields

$$h \sim \left[\frac{-4k_i T_1}{U} (c_w T_2 - c_i T_1 + 2\lambda)^{-1} \right]^{1/2} x^{1/2}, \quad (4.9)$$

As x tends to ∞ , (4.8) yields

$$\frac{h}{H} \sim \frac{k_i T_1}{k_w T_2 - k_i T_1} \left\{ 1 - \exp \left[\frac{2(k_i T_1 - k_w T_2)^2 x}{UH^2 k_i T_1 (2k_i T_1 - k_w T_2) (c_w T_2 - c_i T_1 + 2\lambda)} \right] \right\}. \quad (4.10)$$

Thus $h(x)$ grows initially like $x^{1/2}$ and then tends to $h(\infty)$, with the difference decreasing exponentially.

The solution given in this section may be applicable to the determination of the thickness of the plates forming the earth's crust, with x denoting the distance from a ridge.

EFFECT OF GRAVITATIONAL TIDES ON A MODEL EARTH'S CORE (an extended abstract)

Steven T. Suess

The prototype model of core dynamics in the presence of precession is a rigid, fluid-filled, spheroidal container rotating about its symmetry axis and precessing about some other axis. The known parameters of this model are ω , Ω , a , φ , and i , the angular velocity of the fluid, the precession rate, equatorial radius of the container, and angle between Ω and ω , (the inclination) respectively. The fluid is treated as inviscid. This problem was solved by Poincaré (poorly quoted by Lamb ("Hydrodynamics", Dover, 1936: 789), and misinterpreted subsequently to imply there is no relative motion between the fluid and container.

This problem is reexamined and reappraised with embellishments (see Suess, J. Geophys. Res., 1970 75: 6650-6661, and J. Fluid Mech., 1971 45: 189-201). First, an inviscid, **nondissipative** model of the earth-moon system is constructed, having the simple structure of a homogeneous elastic mantle and a homogeneous fluid core for the earth, and a point moon. The system is specified by giving the inclination

of the model earth to the earth-moon line, the masses of the earth and moon, densities and volumes of the core and mantle, angular velocity of the mantle, and the distance to the moon. Also, although the figure shapes of the core and mantle are assumed determined by hydrostatic equilibrium, elasticity is introduced empirically (via a "Love number") to control the amount of latitude distortion produced by the tides. Solution of this problem indicates that the amount of relative motion predicted by the prototype problem is reduced to about 2/3 of its previous value by finite elasticity effects in the mantle of the earth.

Further appraisal of this model, by allowing a small amount of viscosity in the core, indicates that the semidiurnal forcing of the fluid, due to the ellipticity of the streamlines, in a tidally distorted core produces a vortical shear zone along the rotation axis of the fluid due to nonlinear rectification of the oscillatory motion of the fluid. This is in addition to the cylindrical shear zones occurring in the presence of the diurnal forcing of precession (Busse, 1968 J. Fluid Mech. 33: 789).

Experimental evaluation of this result was possible by building a clear rubber ball, filling it with water, and rotating while deforming it mechanically from the outside to simulate gravitational tides. This method was also used to observe the stability of the flows, and it was found that the axial shear zone could easily be driven unstable by sufficiently large deformation of the container.

The reappraisal of precession and tides caused by the moon (or sun), and their consequences on core dynamics indicates that

- (i) Finite elasticity of the mantle reduces precessional forcing by about 33%.
- (ii) Nonlinear effects of the semidiurnal (tidal) and diurnal (precessional) forcing lead to large internal shear layers.
- (iii) The internal shear layers are experimentally unstable in laboratory simulations, if driven with sufficient strength.
- (iv) Present mathematical models of flow in the interior of the core, in this model, describe the shear layers poorly since they are treated only in the inviscid limit, and as a perturbation away from solid rotation.

ON THE AMPLITUDE OF BAROCLINIC WAVE-TRIADS
AND MESO-SCALE MOTION IN THE OCEAN
(extended abstract)

Joseph Pedlosky

A review of the finite amplitude theory of baroclinic instability reveals that the capacity of a single unstable wave to equilibrate weakens as the cross-stream scale of the disturbance increases. In the limit of very broad currents the capacity for self-equilibration essentially vanishes. Yet linear theory indicates that the most unstable wave on a horizontally uniform baroclinic current has little or no cross-stream variation and hence no finite amplitude equilibration mechanism.

This lecture reported a study of an equilibration mechanism for such modes. These modes which have the Rossby deformation radius as their spatial downstream scale with much larger cross-stream scales are shown to be linearly unstable. The analysis is done in the context of the conventional two-layer, β -plane, quasi-geostrophic dynamical model suitable for the oceanic meso-scale. Within that model it is shown that the marginally unstable baroclinic wave is unstable to "daughter"

waves whose cross-stream scale is also of the order of the Rossby deformation radius. The daughter waves and the parent wave then form a resonant triad. Only the parent wave extracts energy from the mean baroclinic shear flow. The triad as a whole now has both downstream and cross-stream scales of the order of the Rossby radius of deformation.

The finite amplitude dynamics of the resonant triad in a slightly unstable current is examined. The triad consists of the wave which is unstable on the mean (and which has little cross-stream variation) and those two daughter waves which are most unstable on the flow consisting of the mean shear and the parent wave. The dynamics of the triad is studied by the method of multiple time scales. It was shown that the triad cooperatively equilibrates. However a class of initial conditions corresponding to very small initial daughter wave amplitudes yields solutions possessing large velocity amplitudes for all three members of the triad ("large" means large compared to the shear supercriticality of the mean shear flow).

It was suggested that this mechanism of triad instability may be a mechanism for the production of intense meso-scale oceanic eddies with the Rossby deformation radius as the relevant spatial scale in both horizontal directions.

A MODEL OF INTERNAL WAVE INSTABILITIES

Stephen H. Davis

Consider a semi-infinite body of a density-stratified fluid with constant buoyancy frequency, N , bounded by a vertical plane boundary. When this rigid wall undergoes vertical, harmonic oscillations with angular frequency ω , $\omega \neq N$, a periodic boundary layer motion in the fluid results. This problem can be slightly generalized by inserting a second, stationary, plane boundary, parallel to the oscillating one. In either case exact solutions, which can be called buoyancy-Stokes layers, for the velocity and density fields of the Boussinesq equations are obtainable.

Buoyancy-Stokes layers appear (in modified form) as a result of internal wave interactions with plane boundaries that are either vertical or oblique (Wunsch, 1969; Cacchione and Wunsch, 1974). Thorpe (1968, McEwan (1971) and Orlanski (1971) study finite amplitude forced, standing internal waves in closed containers. The growth and decay of such internal waves depend to some extent on the nature of the boundary layers at the container walls. The boundary layers at the vertical walls are of the buoyancy-Stokes type if the scale height of the density stratification is much larger than the boundary layer thickness. It is important to know if such boundary layers are laminar or turbulent in order to estimate the energy dissipation there. It is also important to know if an unstable disturbance in such a flow could penetrate outside of the boundary layer.

Hart (1971) has observed the buoyancy-Stokes layer on an inclined oscillating plane. For large enough oscillation amplitude, this basic flow has density inversions and so instabilities presumably associated with buoyancy effects are seen. The experiment of Robinson and McEwan (1975) deals directly with the buoyancy-Stokes layer on a vertical wall. At no instant is there a density inversion in this basic flow yet when $\omega \approx N$, the flow is highly unstable. The secondary flow induced takes the form of regular wave motions superposed on the basic boundary layer motion. These secondary waves can be two-dimensional, three-dimensional or a combination of these, depending on the value of $B = N/\omega$. For both $B \gg 1$ and $B \ll 1$, the basic state is highly stable.

The stability of (unstratified) Stokes layers is pertinent here. It has been investigated by von Kerczek and Davis (1974) who find that Stokes layers are highly stable to infinitesimal disturbances. Thus, it would seem that the statically stable stratification provides the catalyst for instability of the buoyancy-Stokes layer.

We consider here the linear stability theory of the buoyancy-Stokes layer against two-dimensional disturbances and at high Schmidt number, $N_{sc} = 100$. Critical values of a boundary layer Reynolds number are obtained. In the range of B where Robinson and McEwan (1975) observe two-dimensional instability waves, excellent agreement is obtained between theory and experiment. Furthermore, the eigenfunctions that correspond to these points on the neutral curve are obtained and the relevant energetics of the instability are examined. The theory does, indeed, show that the predicted instability is a result of the oscillating plate producing instantaneous horizontal density gradients that give rise to a buoyancy instability. The Reynolds stresses are small so that the concomitant shear is relatively unimportant as an instability source. The highly unstable nature of buoyancy-Stokes layers for B near unity is due to the low rate of density diffusion, i.e. $N_{sc} \gg 1$.

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DISPLACEMENT AND RECTIFICATION OF PLANETARY FLUIDS

Melvin E. Stern

The lateral displacement of fluid on a rotating sphere gives rise to motions (e.g. Rossby waves) whose Reynolds stress produces a rectification in the surrounding fluid. Such an effect may be produced by the meanders in the Gulf Stream, thereby providing a mechanism for its recirculation and the downstream increase in mass transport. The purpose of this investigation, done in collaboration with Colin Shen of the University of Rhode Island, is to examine the planetary rectification under such laboratory conditions as can be analyzed theoretically. Reference is made to previous related work by Whitehead, Schmitz, Brink, and Rory Thompson.

We first give a simple illustration of the effect by considering the β -plane vorticity (ζ) equation

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \zeta + \beta v + \zeta/T = 0, \quad 0 = \nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad (1)$$

where T is the spin-up time due to bottom friction. Suppose motions at $y > 0$ are generated by an oscillating flexible wall at $y = 0$, with $\Delta R \sin \omega t \sin x/a$ denoting the displacement of the boundary at latitude $y = 0$. The mean field equation obtained by averaging (1) over all $x-t$ is

$$\frac{\partial}{\partial y} \overline{v \zeta} - \frac{\partial \bar{u}}{\partial y} = 0 \quad \text{or} \quad \bar{u}(y) = \overline{v \zeta} T \quad (2)$$

since all motion vanishes at $y \rightarrow \infty$. A "quasi-linear" approximation is now used to compute $\overline{v \zeta}$. Thus if $v = R_0 e^{i\omega t} \hat{v}(x, y)$, $\zeta = R_0 e^{i\omega t} \hat{\zeta}$ then the linearized form of (1) is $i\omega \hat{\zeta} + \beta \hat{v} + \hat{\zeta}/T = 0$, and therefore

$$\hat{\zeta} = -\frac{\beta \hat{v}}{\omega^2 T (1 + \frac{1}{\omega^2 T^2})} + \frac{\beta \hat{v}}{\omega (1 + \frac{1}{\omega^2 T^2})} \quad (3)$$

$$\overline{v \zeta} = -\frac{\beta}{\omega^2 T} \frac{\bar{v}^2}{1 + \frac{1}{\omega^2 T^2}} \quad (4)$$

Equation (2) then gives $\bar{u}(y) = -(\beta/\omega^2) \bar{v}^2 (1 + 1/\omega^2 T^2)^{-1}$ as the far field rectified flow. The value of $\bar{v}^2(0)$ is given, and $\bar{v}^2(y)$ can be computed from the partial differential equation (3). In particular we note that when $T \gg 1/\omega$ the maximum $(-\bar{u})$ is proportional to $\beta (\Delta R)^2$.

In the remaining part of the talk we mention our current efforts to realize such rectification effects. Following Brink's report last year in this journal, we translate and oscillate a cylinder of water along a straight line drawn on the rotating turntable. The velocity $\mathbf{V}_x$ of the fluid relative to the cylinder is then given by

$$\begin{aligned} d\mathbf{V}_x/dt + \mathbf{f} \times \mathbf{V}_x &= -\nabla(p/\rho - g h_{eq}) + \gamma \nabla^2 \mathbf{V}_x - \mathbf{B} \times \mathbf{V}_x - \mathbf{f} \times \mathbf{R}_T \\ g \nabla h_{eq} &= f^2/4 (\mathbf{R} + \mathbf{r}) \end{aligned}$$

where $\mathbf{R}$ is the position of the center of the cylinder relative to the rotating table and $\mathbf{r}$ is position of a fluid parcel relative to the center of the cylinder.

A "spin-up" experiment was done first, wherein $\mathbf{R}$ was changed by a small amount and the free surface changed from one h_{eq} to another. The displacement of

a fluid parcel at $r=0$ was observed and compared with theory. The displacement spin-up depends on $N = f^2 a^3 / 4g (\nu / 2f)^{1/2}$ where a is a radius of the cylinder, and good agreement with a simple theory is obtained at large N . We continue with this work and plan to investigate the rectification when $\underline{R}$ oscillates sinusoidally about $\underline{R} = 0$.

THERMALLY-DRIVEN LARGE SCALE OCEAN CIRCULATION

George Veronis

A steady model of large scale ocean circulation driven by thermal forces (no wind stress) is developed by parameterizing the thermal driving by an assumed uniform upwelling of water across the interface of a two-layer system. The location of the sinking regions is part of the problem. An assumed western boundary layer provides the transport required to balance the mass fluxes across a latitude circle in an ocean basin which is closed to the north. The assumption that the downstream velocity in the western boundary current is geostrophic yields a relation for the upper layer thickness at the western boundary in terms of the value on the offshore side of the boundary layer. An added assumption that the interior flow is geostrophic then leads to a relatively simple mathematical problem for determining the velocities and layer thicknesses.

Since the real ocean is driven by wind stress as well as by thermal forces and since the stronger velocities are primarily due to the winds, there is no observational data which is pertinent to an exclusively thermally-driven circulation. Therefore, a series of calculations has been made to outline possible types of behavior under a variety of conditions. A key question to be answered is: under what conditions does the thermocline (i.e., the interface between the two layers) rise to the surface and perhaps separate from the coast? (This separation had been studied for a model in which the velocities were driven by wind stress alone and led to realistic results for the Gulf Stream, and Agulhas Currents.) In the present model, surfacing of the thermocline can occur if the upwelling velocity is sufficiently intense and if the given depth of the upper layer along the eastern boundary is sufficiently shallow. Separation of the western boundary current from the coast seems to occur only in the form of a jet extending eastward from the western to the eastern boundary. Hence, the total amount of upper layer water in the ocean is restricted by the separation mechanism.

Results for the southern hemisphere ocean can differ substantially from those for the north because the transport balance involves substantially larger amounts of water. For example, in the case where no sinking occurs in the northern basin, there must be sinking in the south even for relatively weak upwelling. This feature is important for the case where wind stresses and thermal forces drive the circulation because the latitudes of separation of the western boundary currents in the southern hemisphere are strongly influenced by the thermal mechanism.

SOME PROBLEMS IN MANTLE CONVECTION

Donald L. Turcotte

The basic equations for thermal convection* were discussed. The roles of the three relevant nondimensional parameters, the Rayleigh number, the Prandtl, and the dissipation number, were emphasized. The dissipation number determines not only the role of viscous dissipation but also the roles of the adiabatic temperature and density gradients. The dissipation number is of order one for whole mantle convection but its primary effect is to retard convection. The role of non-Newtonian convection was studied by carrying out numerical calculations of two-dimensional convection of a fluid layer heated from below. For a third power relation between rate of strain and stress applicable for dislocation creep in the mantle, it was found that non-Newtonian convection was virtually identical to Newtonian convection. Numerical calculation for a strongly temperature and pressure dependent viscosity were also presented. Up to five orders of magnitude variation of viscosity were included. Major changes in the form of the convection were observed.

In order to Better simulate mantle convection, calculations were carried out in which flow was removed from the upper boundary to simulate the thickening of the elastic, rigid lithosphere. It was found that large aspect ratio convection cells could be obtained in this case.

There are several important questions remaining with regard to mantle convection. What is the depth of convection? Are the primary return flows in the asthenosphere? Is the 700 km seismic discontinuity a barrier to convection? Does whole mantle convection occur? Another range of problems concerns the composition of the mantle. Are there significant compositional variations over the mantle? What happens to the descending lithosphere? Another unanswered question concerns the scale of mantle convection. Certainly the surface plates define a scale of convection. But are there other scales of mantle convection without direct surface evidence?

DOUBLE-DIFFUSIVE TURBULENCE

Herbert E. Huppert

Some recent results on the nonlinear, double-diffusive Bénard problem were outlined. The linear problem indicates that instability first occurs to an oscillatory mode. In the nonlinear range, as the thermal Rayleigh number, R_T , increases, for fixed solutal Rayleigh number, the motion can pass through the following states: i) $0 \leq R_T < R_1$, conduction; ii) $R_1 < R_T < R_2$, "singly-periodic" oscillatory convection of increasing period; iii) $R_2 < R_T < R_3$, "doubly-periodic" oscillatory convection; iv) $R_3 < R_T < R_4$, non-periodic motion which appears to be unstable to linear perturbations in the initial conditions; and v) $R_5 \leq R_T$, monotonic convection. As R_T decreases below R_5 the motion remains monotonic until $R_T = R_6$ below which the solution changes to one of i) - iv) described for increasing R_T . For a part of parameter space v) may follow ii) or iii) and the intervening state(s) does (do) not exist, but a hysteresis loop is always present. The value of R_6 , the minimum possible for monotonic convection, can be less than R_1 , the one obtained from a linear calculation.

CONVECTION EXPERIMENTS - LABORATORY VERSUS MANILE

John A. Whitehead, Jr.

Observations are reported of Rayleigh-Bénard convection which is subjected to both controlled and uncontrolled initial conditions. In the controlled case, stationary bimodal flow is observed at all Rayleigh numbers, the two wavenumbers of stable flow coming together at a Rayleigh number of 200,000. Breakdown of the convection with both wavenumbers the same below 200,000 is observed. In the uncontrolled convection, oscillatory flow is observed to commence occasionally at spots where spoke convection exists above approximately 150,000. Above a Rayleigh number of 500,000, more than half the fluid is in a state of oscillatory flow.

It was then shown that the convection patterns shown so far bear little resemblance to the new global tectonics and it was advocated that new physical processes need to be introduced. Feedback between a rigid moving upper surface and convection below was described as being one possible set of unexplored processes. Studies involving feedback between heat sources and/or insulators near the top of the fluid were reviewed (Howard, Malkus, and Whitehead, G.F.D.I. 123-142, 1970; Whitehead, Phys. Earth Planet. Int., 5: 199-212, 1972), and the instability of a floating heater was described theoretically and experimental movies and data were shown. Lastly, an analysis of the stability properties of fluid with a vertical gradient of heat sources revealed the presence of nonequilibrium "differentiation waves", whose possible presence in the mantle was briefly discussed.

CONVECTION IN THE MOON

S. Keith Runcorn

The great advance of the last decade or so in geophysics has been the replacement of an essentially static model of the Earth by a dynamical one. The theory of the main geomagnetic field in terms of a dynamo driven by motions in the fluid core and the emerging theory of continental drift, sea floor spreading and plate tectonics in terms of thermal convection in the mantle, in which solid state creep is occurring, are the most important aspects of this development. The advance in space experiments now make possible tests of these theories on other planets - an essential step in the scientific method. The Moon provides a most important challenge, not only because so much is known about this body, but because of the complete absence of the two properties on which our knowledge of the dynamic character of the Earth's interior is based, that is, a present global magnetic field and evidence of great horizontal displacements of parts of the lithospheric shell.

The non-hydrostatic figure of the Moon is now known to consist of two kinds of gravitational field anomaly: the anomalies of about 2000 mgal, the contours of which group near the peripheries of the circular maria (termed mascons) and the difference between the moments of inertia (C about the polar axis, A about the mean axis towards the earth, and B about the third axis in the plane of the sky) - about 20 times greater than that on the hydrostatic model. It is clear that the mascons arise from excess lava which filled the circular basins, originally formed by impact, to a level about 1-2 km greater than that corresponding to isostatic equilibrium. Since this occurred 3.2 - 3.8 by ago, the mascons have been retained by the finite strength of the lunar lithosphere.

The non-hydrostatic part of the difference between the moments of inertia - known since Laplace inferred its existence from Cassini's laws - was formerly sup-

posed to represent a distortion of an originally spherical Moon into an ellipsoidal one, on Laplace's theory by inhomogeneous cooling from a liquid state, on Jeffrey's theory by the tidal attraction of the Earth when the Moon was closer. It has also recently been suggested that there is a second harmonic variation in the thickness of the anorthositic gabbroic highland crust (see Runcorn, 1975). Now that it is evident that the anorthositic gabbroic highland crust is thicker on the far side than on the near side, this alternative is superficially attractive. Essential to each of these theories is the supposed property that the Moon has long term finite strength through most of its interior, but Runcorn (1962, 1967) drew attention to the important role played by solid state creep in the Earth's mantle. He argued similarly that solid state creep would be expected to be the dominant mechanical property over 10^8 - 10^9 yr in the Moon: consequently any initial long wavelength gravitational anomaly would have disappeared. Thus the second, nonhydrostatic harmonic gravitational anomaly was ascribed to thermal convection. It was proved that the second harmonic only in the convection pattern would give rise to a difference in the moments of inertia. It was speculated that, according to the marginal theory of convection, the second harmonic convection pattern would have only established itself in the presence of a lunar core of radius between 0.06 and 0.3 of the outer radius. However convection in the Moon is not demanded by lunar tectonics, although the lunar grid system (NW - SE and NE - SW lineaments: crater chains and faults) indicates the existence of a stress pattern in the lithosphere possibly due to convection.

The strongest evidence for the convection theory is that the mean ellipsoidal shape of the Moon is in general agreement with the predictions of the theory and is in disagreement with the theory that the Moon is uniform but distorted.

Since Luna II, it has been known that the Moon does not possess at present a detectable dipole magnetic field and Russell, Coleman and Schubert (1974) have now placed an upper limit of 0.05γ at the equator. The widespread belief that the Moon was "dead" inside and that it had no core - the above argument being the only one advanced to the contrary - fitted perfectly with the absence of a magnetic field. The discovery of stable natural remanent magnetization in the Apollo 11 lavas (3.6 by) (Runcorn *et al.* 1970) was a surprise but the argument that the magnetization resulted when the lavas cooled in a lunar magnetic field of internal origin produced by a dynamo process within the core postulated in the convection theory, was put forward by Runcorn *et al.* (1970) after considering other possible origins. In particular the idea that the Earth's magnetic field was responsible was greatly weakened when natural remanent magnetization was also discovered in the Apollo 12 basalts (Runcorn *et al.* 1971) as they were much younger 3.2 by. The (tidal interaction) dynamics of the Earth-Moon system preclude the bodies being close together for more than a minute fraction of this interval. They argued that the small lunar core, less than about one-tenth of the Earth's, might have acted as a dynamo in the early history of the Moon but with a magnetic Reynolds number closer to the critical value so that in the last 3200 my the motion in the core could have fallen below the velocity corresponding to the critical number. Alternatively they argued that the originally molten core could have solidified due to the cooling of the Moon as a result of solid state convection. Consequently the idea that the dynamo is driven by thermal convection in the iron core seems adequate to explain the facts. However Runcorn and Urey (1973) have pointed out that the lunar magnetizing field might have been due to a primeval magnetization of its interior supposed to have been cold until after 3.2 by when its temperature rose above the Curie point of iron and the global field disappeared.

Relevant to these alternatives, is the study of the palaeointensity of the lunar magnetic field responsible for the natural remanent magnetization of the

lunar rocks. Stephenson, Runcorn and Collinson(1975) has reviewed the palaeointensities as a function of the age of the lunar rocks based on Thellier-Thellier and anhysteretic magnetization experiments. Their observations show that the palaeointensity for rocks of 4.0 by old was 1.2 - 1.4 Oe - probably too large to be explained by a primeval magnetization of the lunar interior. Thus the core dynamo theory is strengthened. The field seems to have decreased logarithmically to 1 - 5,000 Y at 3.2 by ago, This is to be expected if the radioactivity of the interior is the heat source for the convection in the core for the radioactive sources decrease by a factor over 3 - 5 over this interval.

The internal origin of the lunar magnetizing field has been greatly strengthened by the demonstration by Runcorn (1975) that the magnetized lunar shell would produce the observed zero dipole field only if the shell were magnetized by a field of internal origin.

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