**WHOI-86-45
1986**

**COURSE LECTURES
ABSTRACTS OF SEMINARS
LECTURES OF THE FELLOWS**

WHOI-86-45

SUMMER STUDY PROGRAM

IN

GEOPHYSICAL FLUID DYNAMICS

WOODS HOLE OCEANOGRAPHIC INSTITUTION

ORDER AND DISORDER IN TURBULENT SHEAR **FLOW**

BY

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WOODS HOLE OCEANOGRAPHIC INSTITUTION

WOODS HOLE, MASSACHUSETTS

DECEMBER, 1986

TECHNICAL REPORT

Prepared for the Office of Naval Research under contract N00014-82-G-0079
and the National Science Foundation under Grant DMS-85-04166.

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A handwritten signature in dark ink, appearing to read "Charles D. Hollister", written over a horizontal line.

Charles D. Hollister
Dean of Graduate Studies

1986 SUMMER STUDY PROGRAM

IN

GEOPHYSICAL FLUID DYNAMICS

AT

THE WOODS HOLE OCEANOGRAPHIC INSTITUTION

ORDER AND DISORDER IN TURBULENT SHEAR FLOW

GEOPHYSICAL FLUID DYNAMICS PROGRAM

Summer 1986

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EDITOR'S PREFACE

Our thanks go to Martin Landahl for his stimulating expository lectures on recent development in "lab scale" turbulent flow. This subject was covered by him and other staff members from the experimental, analytic, and numerical point of view. The seminars on two-dimensional coherent structures provided a nice connecting link for subsequent lectures on large scale ocean eddy dynamics (e.g. warm core rings detaching from the Gulf Stream).

The purpose of these notes is to indicate the range of the topics covered, and the degree of beneficial **communication**.

It is something of a wonder, with all these lectures going on, that our GFD Fellows are still able to come up with a (sometimes) admirable research project! For this our thanks go to those staff members who have given so generously of their time and spirit.

We acknowledge with sincere appreciation the financial support provided by the Office of Naval Research and the National Science Foundation.

Melvin E. Stern

TABLE OF CONTENTS

ORDER AND DISORDER IN TURBULENT SHEAR FLOW

Page

Martin E. Landahl - ORDERED AND DISORDERED STRUCTURES IN SHEAR FLOW

Lecture 1. What is Coherent Structure	2
Lecture 2. Statistical Measures of Turbulence.	12
Lecture 3. Review of Experiments	24
Lecture 4. Coherent Structures in the Laboratory and in Nature.	29
Lecture 5. Equations of Motion.	39
Lecture 6. Shear Flow Instability.	44
Lecture 7. Turbulence and Flow Instability.	51
Lecture 8. Euler Description of Fluctuating Field.	57
Lecture 9. Coherent Structure Modeling.	62
Lecture 10. Modeling. Numerical Simulation. Final Questions.	65

ABSTRACTS OF SEMINARS

	<u>Page</u>
Merger, Binding, Axisymmetrization and Other Basic Processes in 2D Incompressible Flows. (Applications of Pseudospectral and Contour Dynamical DC Algorithms)	
Norman Zabusky	.76
Horizontal Entrainment and Detrainment In Large Scale Eddies	
Melvin E. Stern.	.77
Isolated Eddy Models for Geophysical Flows	
Glen Flierl.	.78
Experiments on Internal Wave Critical Layers	
Donald Altman.	.87
Simple Models of the General Circulation of the Ocean	
Rick Salmon	.89
Chaos and Turbulence	
Alan C. Newell	.90
Instabilities in Channel Flows: Three-dimensionality, Geometry, and Resonance	
Anthony T. Patera	.104
Stability Bounds on Turbulent Poiseuille Flow	
Willem V. R. Malkus (in collaboration with Glenn R. Ierly)	.105
Formation and Destruction of Inviscid Large Eddies	
Raymond T. Pierrehumbert.	.106
Vortex Dynamics - Computer Simulations and Mathematical Models	
Mogens V. Melander.	.107
Normal Forms for Partial Differential Equations Near to the Onset of Instabilities	
E. A. Spiegel	.108
Weir Flows, Pouring Flows, and Sink Flows	
Joseph Keller and Jean-Marc Vanden-Broeck	.114
Numerical Experiments in Forced Stably Stratified Turbulence	
Jackson R. Herring.	.115
Review of Baroclinic Lenses in the Ocean	
Thomas Rossby	.116

	<i>Page</i>
Amplitude Propagation in Trains of Orr-Sommerfield Waves John M. Russell	.118
Movements and Interactions of Isolated Eddies Doran Nof	.120
Instabilities of Salt Fingers George Veronis.	.123
Vortex Methods for Stratified Flows Gretar Tryggvason	.124
Deep-Ocean Circulation and Topography Peter Rhines	.125
On Large Scale Structures in Free Shear Layers Israel Wygnanski	.129
Negative Energy Waves William Young	.131
Coherent Structures in a Baroclinic Atmosphere: Theory for Atmospheric Blocking and Comparison with Data Paola Malanotte Rizzoli	.132
A Model for Fastest-Growing Fingers in a Gradient Eric Kunze.	.132
Shear-Layer Structures in Near-Wall Turbulence, Their Evolution and Connection to Wall-Pressure Peaks Arne Johansson.	.133
Optical Microstructure During C-Salt Eric Kunze	.133
Stability of Streamwise Vortices C. Frederick Pearson	.134
Flow Structures in the California Current Pierre Flament	.135
Chaos off the Shelf: Topographically Generated Mean Flow on the Continental Slope Roger Samelson.	.137
Simple Models of the General Circulation of the Ocean Rick Salmon	.138
Effects of Condensation and Wind-Induced Surface Heat Fluxes on Baroclinic Instability Kerry Emanuel	.139

LECTURES OF FELLOWS

	Page
Interactions Between a Point Vortex and a Shear-Flow Katherine Hedstrom.	.142
Stability of Parallel Shear Flows with Mean Spanwise Variation Dan Henningson.	.153
Convection with Spatial Forcing and Mean Flow Generation Ya Yan Lu	.168
The Lives of an Eddy-Experimental Study of Planetary Eddies Yehuda Agnon.	.180
Self-Propagation of a Barotropic Circular Eddy Gavin L. Bievuyck	.193
Stability of Stationary-Enstrophy Beta-Channel Flow Joseph Barsugli	.202
Non-hysteretic Non-intermittent Transitions Directly Related to Chaos R. Van Buskirk.	.212
Instability Waves in Galatic Slab Zhongshan Qian.	.232
Barotropic Shear Flow Instability Over the Continental Shelf Uwe Send	.241
Hexagons, Harmonics and Helmoltz Nicholas H. Brummell.	.251

**GFD PROGRAM SUMMER 1986 ORDERED AND
DISORDERED STRUCTURES IN SHEAR FLOW**

Introductory lectures

by

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Part I: THE FACTS • OBSERVATIONAL AND EXPERIMENTAL DATA

Lecture 1

WHAT IS A COHERENT STRUCTURE

1.1 Some early findings

We wish to examine the physical nature of so called coherent structures and what they imply regarding turbulence. There is a great deal of information but not much is **known** in the ~~sense~~ well **understood**. This is because a great deal of the 'knowledge' is **not** reliable (due to lack of theoretical and experimental understanding) and because the theoretical and **conceptual** models have to be oversimplified (due to the complexity **involved**).

There is no precise, or agreed upon, definition of a **coherent** structure. Generally it is meant to describe a localized region of enhanced velocity **fluctuations**, usually **containing** a vorticity anomaly. Simple examples of coherent structures (not found in **turbulent** shear flows) are a vortex ring (**i.e.** a limited region of enhanced vorticity **propagating** through the flow) and a Hill's spherical vortex (which has bipolar **axisymmetry**). In the literature the terms eddy and typical eddy are **often** used. The term coherent structure, however, usually implies a typical structure one which recurs at (usually) random **intervals**. This typical shape will tend to dominate the entire structure. A liberal **definition** is: **any** structure deduced by a suitably chosen sampling **technique**. **Townsend** (1976) states: "**Since** the experimental data is always **incomplete**, the identification of eddy type must be **informed** guesswork followed by **measurements** designed to **confirm** the guess, **and then** to fit the **inferred** structure into a **coherent dynamical** account of the matter". In that respect, **turbulence, like pornography, is in the eyes of the beholder...or rather** - there is a personal bias in **interpreting** the observations.

The first account of turbulence is perhaps a drawing by **Leonardo DaVinci**, portraying the outlet of a sewer in **Milan**. The wealth of eddy sizes in that drawing is remarkable. Early ideas on structures were formed by Richardson (**1922**), who observed the cascade of "whorls", paraphrasing **on** Swift's hierarchy of fish. **Prandtl** (1925) used the term "lumps" for structures that carry and propagate some property of the fluid. **Theodorsen** (1952) observed that a three **dimensional** boundary layer near a wall **consists** of "horseshoe" (or hairpin) vortices of different **signs**, that he believed were attached to the surface (we know that they rather are attached to **lines** of **spanwise** vorticity in the mean flow. **Finally, Grant** (1958) quantified large eddies **in** the wake **behind** a **blunt** body. More "**modern**" ideas

where formed by Lumley (1965) who tried to devise an objective procedure, not depending on preconceptions of experimentalists. He looked for eigensolutions of

$$\int_{-\infty}^{\infty} R_{ij}(\mathbf{x}) \phi_j^{(n)}(\mathbf{x} + \mathbf{r}) d\mathbf{r} = \phi_i^{(n)}(\mathbf{x}) \quad (1.1)$$

where

$$R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r}) \rangle \quad (1.2)$$

is the two-point velocity correlation function. In three dimensions it is a difficult numerical task to determine the eigenfunctions and eigenvalues. It is simplified when some homogeneity exists. The idea is that the lowest eigenvalue will dominate the flow. The method was applied to the flow behind a circular cylinder.

The takeoff of the "modern" era of coherent structure research occurred with the visualization experiments of Kline et al (1967) who used hydrogen bubbles as markers in a low speed turbulent channel flow. The typical scales in the region near the wall of a turbulent boundary layer can be expressed in terms of the viscous wall length unit: $l_* = \nu/u_*$ i.e. the length scale corresponding to a Reynolds number unity based on the friction velocity: $u_* = \mu(dU/dy)_{wall}$, which may be 1/30-1/20 of the free stream velocity, U_∞ . For the streamwise length scale typically $l_1^+ = 200 - 500$ for ejections and ≈ 1000 for sweeps where $+$ denotes number of wall units, for the cross stream scale $l^+ \approx 50$ and for the spanwise scale $l^+ \approx 100$. Typical values: for water with $\nu = 1.110^{-6} \text{ m}^2/\text{s}$ we take $u_* = 50 \text{ cm/s}$ (for a ship sailing at $U_0 = 20 \text{ knots} = 10 \text{ m/s}$) and get $l_* = 0.002 \text{ mm}$. The actual length of the streaks in such a flow is about 2 mm and their width is about 0.2 mm. These very small scales place limitations on the our measuring ability.

In the experiments of Kline et al (1967) they saw a low-speed region growing, the flow lifting up and subsequently the onset of a rapid instability (see Figure 1). They measured the Reynolds stress: $(= - \langle u'v' \rangle$, where u' and v' are the velocity fluctuations in the x and y directions, respectively) and found that the bulk of the turbulent momentum transfer occurs during those intermittent bursts of vorticity. Blackwelder and Kaplan (1976) studied boundary layer flow using an array of ten hot wires. The limitations of this technique are due to the finite size of the hotwires and to the likely possibility of misinterpreting the streak lines picture that in fact shows the past history of the flow. Brown and Roshko (1974) have studied the mixing of fluids past a splitter plate. Localized organised structures are formed which are associated with the instability of the flow (the density difference is not important in the dynamics). We see "vortex merging" which has been attributed to subharmonic resonance or to a selection of scales mechanism (see Figure 2).

1.2 Geophysical observations

Coherent structures can also be observed in geophysical flows. Some examples in meteorology include: Cumulus clouds (which are thermals), in which buoyancy differences drive the thermal convection, fronts and cyclones (where the lows in a weather map are better

Figure 1.1: Conceptual burst model.(a)Mechanism of streak formation,(b) Mechanism of streak breakup (from Kline et al., 1976)

Figure 1.2: Shadowgraphs of a mixing layer (from Brown and Roshko, 1974)

Figure 1.3: Weather map for 26 Jan 1952 (from Tritton, 1977)

defined than the highs since they are regions of concentrated vorticity, see Figure 3). In the atmosphere there is vorticity at all scales. Solar granulation is another example with complicated dynamics • shear flow, heating and MHD (Figure 4). Finally, in oceanography, Robinson (1984) describes similar coherent structures, or eddies, in the ocean. The Gulf Stream develops kinks which leave isolated eddies that exist on their own for long times (see Figure 5).

1.3 Intermittency

Turbulent flow may contain many different scales which may interact among them. In an experiment by Filipsson et al., (1979) a jet flow was visualized, and then a little asbestos fiber was added to the fluid inhibiting the small scale vortices, thus emphasizing the larger basic structure (see Figure 6) (drag reducing polymers had a similar effect). The overall characteristic of spreading rate was unaffected (in a boundary layer the whole flow would have been drastically affected). Such pictures support the classical view that the small scales have primarily a passive role of draining energy. In contrast, for wall bounded flows we now believe that they play a crucial role. When the small scales are eliminated by the addition of certain polymers, for example, the drag is greatly reduced.

Figure 1.4: Solar granulation (from Bahng and Schwarzschild, 1961)

Figure 1.6: Spreading of a jet in water with and without addition of asbestos fibres (from Filipsson et al., 1977)

Figure 1.7: Wall pressures in a turbulent boundary layer measured in four consecutive pressure transducers (from Schewe, 1983)

There is no direct or easy way of detecting turbulent events by measuring the streamwise velocity. However, using very small microphones Schewe (1983) measured wall pressures which showed strong isolated bursts on very small scales which resembled travelling wave packets (see Figure 7). The intermittent nature of fully developed turbulent boundary layer flow is that the production of Reynolds stress is associated with those large pressure fluctuations which are localized events occurring now and then.

1.4 Order-partial order-disorder

There are all sorts of scales in a turbulent flow. The smaller scales are generally disordered while the larger scales are partially, or "half" ordered. Only the mean flow is ordered, and we are looking for the connection between these scales. Our task is to separate the different scales and try to find order in the combined flow. The coherent structure concept is a means to that end.

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Notes submitted by Yehuda Agnon

Lecture 2

STATISTICAL MEASURES OF TURBULENCE

Mark Twain once noted that there are three kinds of lies: little lies, big lies, and statistics. Today's lecture will cover statistics, in particular, statistical measures of chaotic phenomena. First we consider the nature of the coherent structures and chaotic phenomena we are trying to understand **as** a clue to what sort of statistical measures we will need. Coherent structures (lumps, or what have you) have a life; they are born, grow, and decay, and this should be reflected in their statistics. Spatial structure must also be represented. Consider shear flow in an inviscid, homogeneous fluid. This is always unstable, **as** can be shown by looking at a bump on the interface. The fluid velocity increases above the bump and decreases below it, which by irrotationality and Bernoulli's relation implies a pressure gradient which acts to further accelerate the flow upwards.

The shear layer can also be represented by vortices along the interface. The "induction effect" tends to roll these up into concentrated vortex cores. We are likely to see such intermittent thin shear layers in fully developed turbulent shear flow as the result of stretching of vortices in **3-D** flow.

Thus statistics need to incorporate: time evolution on different scales, spatial structure, intermittency.

2.1 STANDARD STATISTICAL METHODS

Let us first look to standard statistical methods. These methods are usually used for phenomena which in some sense exhibit "stationarity", and "homogeneity". For measurements at one point we can use the probability density. For example, consider the probability that the velocity u is less than some other velocity U . We can write this as $Pr[u < U] = F(U)$, which may look something like in Figure 1.

The probability density is defined as $f(U) = dF/dU$. For comparisons we often use the Gaussian distribution, which for a zero mean is of the form:

$$f(U) = (1/2\pi\sigma^2)^{1/2} \exp(-U^2/2\sigma^2) \quad (2.1)$$

Observations confirm that velocities at a point are often nearly Gaussian. It is the deviations from the normal distribution which give us information on the non-linear phenomena

Figure 2.1: Distribution function

present. Two measures of this deviation are the skewness $S = \langle u^3 \rangle / \sigma^3$, and kurtosis, $K = \langle u^4 \rangle / \sigma^4$, where $\langle u^n \rangle = \int_{-\infty}^{\infty} U^n f(U) dU$. For a Gaussian distribution $S = 0$ and $K = 3$. Note that $\langle \rangle$ represents the ensemble average.

To elucidate spatial structure we need to use a joint probability density between two points. **From** this we get the covariance, etc. For example, consider the 2-point **velocity-velocity** correlation (i, j components):

$$R_{ij}(\mathbf{x}, \mathbf{x} + \boldsymbol{\xi}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x} + \boldsymbol{\xi}) \rangle \quad (2.2)$$

which has nine components. Symmetry arguments can reduce this to six independent components. This formalism has been used for homogeneous, isotropic turbulence. Consider Townsend's (1976) model of turbulence. (All the figures in this section are from **Townsend**, 1976) Suppose we have a velocity field with eddies of the following form (constant density):

$$u_1 = -\frac{\partial}{\partial x_2} [e^{-\frac{1}{2}\alpha^2 x^2} \cos(l_1 x_1) \cos(l_2 x_2) \cos(l_3 x_3)] \quad (2.3)$$

$$u_2 = \frac{\partial}{\partial x_1} [e^{-\frac{1}{2}\alpha^2 x^2} \cos(l_1 x_1) \cos(l_2 x_2) \cos(l_3 x_3)] \quad (2.4)$$

$$u_3 = 0 \quad (2.5)$$

where

$$\alpha^2 x^2 = \alpha_1^2 x_1^2 + \alpha_2^2 x_2^2 + \alpha_3^2 x_3^2 \quad (2.6)$$

Figure 2.2: Isolated model eddy (from **Townsend**, 1976)

that is, a "Gaussian hat" eddy with some periodic behavior. If we superpose these eddies randomly in space with the same strength we get the following correlations:

$$R_{11}(\mathbf{r}) = -\frac{\partial^2}{\partial r_2^2} f(\mathbf{r}) \quad (2.7)$$

$$R_{22}(\mathbf{r}) = -\frac{\partial^2}{\partial r_1^2} f(\mathbf{r}) \quad (2.8)$$

$$R_{12}(\mathbf{r}) = R_{21}(\mathbf{r}) = \frac{\partial^2}{\partial r_1 \partial r_2} f(\mathbf{r}) \quad (2.9)$$

where

$$f(\mathbf{r}) = A e^{-\frac{1}{2} \alpha^2 r^2} (\cos l_1 r_1 + e^{-l_1^2 / \alpha_1^2}) \times (\cos l_2 r_2 + e^{-l_2^2 / \alpha_2^2}) (\cos l_3 r_3 + e^{-l_3^2 / \alpha_3^2}) \quad (2.10)$$

where **A** is a **constant** specifying the intensity of the eddy system. Now consider the special case of an isolated eddy with $l_1 = l_2 = l_3 = 0$. The velocity distribution looks like the following:

The scale of the eddies show up in the curvature. We get the wavenumber spectrum by taking the Fourier transform

For a spherical eddy with $l_1 = l_2 = l_3 = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = a$ we find

$$\phi_{11} = k_2^2 g(\mathbf{k}) \quad (2.11)$$

$$\phi_{22} = k_1^2 g(\mathbf{k}) \quad (2.12)$$

Figure 2.3: Spectrum of an isolated eddy. **(a)Velocity** distribution ,**(b)3-D** spectrum (from Townsend, **1976**)

$$\phi_{12} = \phi_{21} = -k_1 k_2 g(\mathbf{k}) \quad (2.13)$$

where

$$g(\mathbf{k}) = A \exp\left(-\frac{k^2}{\alpha^2}\right) \quad (2.14)$$

If we graph the energy as a function of wavenumber we get:

For time behavior we define $R_{ij}(\mathbf{x}, \xi, t, \tau) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \xi, t + \tau) \rangle$. If R_{ij} is stationary then it is not a function of t . If it is homogeneous then it is not a function of $\mathbf{x}$. Some examples:

Figure 2.4: Correlation functions for isotropic turbulence with eddies of two distinct sizes (from Townsend, 1976)

- Isotropic turbulence: must stir continuously to have stationarity
- Shear flow: can have stationarity, but not necessarily homogeneity. Spatial dependence depends on geometry of problem.
- Channel flow: Homogeneous in x (streamwise); for a free shear layer the dominating scales grow downstream.
- Time spectrum expands to low frequency end, high frequency end filled with dissipating eddies.

Consider the following graphs of R : Large curvature at the axis means that there are small (coherent) scales present which can only be seen if the separation is small. If two scales are present, such as in the following graphs this would imply two peaks in the Fourier spectrum. There is no such behavior in shear flow.

Let's look at the time delay correlations: We **can** obtain a "convective velocity" (sometimes also called celerity) of the eddies by following the peak of the graph with time. This velocity varies with scale though. This procedure may or may not pick out the dominant

Figure 2.5: Time delay correlations, as functions of streamwise separation for various time intervals (from Townsend, 1976)

Figure 2.6: Wave-number-frequency spectrum for system with lightly damped waves

scale eddies. To get a more accurate picture we take the Fourier transform in both space and time:

$$S_{ij}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} R_{ij}(\tau) d\tau \quad (2.15)$$

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int \int_{-\infty}^{\infty} \exp[-i(\mathbf{k}\xi)] R_{ij}(\xi, 0) d\xi \quad (2.16)$$

$$\Psi_{ij}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4} \int \int \int_{-\infty}^{\infty} \exp[-i(\mathbf{k}\xi - \omega\tau)] R_{ij} d\xi d\omega \quad (2.17)$$

Assuming homogeneity in the direction we are interested in and stationarity in time we get a set of **curves** looking qualitatively as follows:

The phase and group speeds are derived from the values of c_{phase} and k along the crest of the ridge. $c_{\text{phase}} = \omega/k$, $c_{\text{group}} = d\omega_{\text{peak}}/dk$. Note that damping is necessary to ensure stationarity of the statistics, else the energy at a point would be unbounded. The width of the ridge on the above graph is a measure of the strength of the damping.

2.2 HOW LITTLE A SPECTRAL DENSITY TELLS US

For unit impulses random in time the power spectrum $S(w) = \text{constant}$. If we now delete every other **impulse**, but increase the amplitude of each pulse by 2 the power spectrum is unchanged. Repeat this many (say 10^6) times and we come to the conclusion that a big bang now and then is equivalent to raindrops. Power spectra are not unique!

Much of turbulence theory has been centered on predicting wavenumber spectra. The question remains: If we can predict this, so what?. One can get only limited information on physical processes from spectra, but this may be enough to spur further research. Spectra can indicate if a system is chaotic and can be used to investigate ideas of universality if, for example, the spectrum at one scale is the same as the spectrum at another scale.

What can we do to further understand the physical processes? One answer is to understand the detailed time and space evolution of a single system. This gives no information on the differences between one realization and the next. So, some sort of statistics are needed to find out what is in some sense "typical". One such method is conditional sampling.

2.3 CONDITIONAL SAMPLING TECHNIQUES

Conditional sampling techniques use some sort of trigger to turn on the sampling, eg. $u \geq u_0$ or $\partial u / \partial t > 0$. This establishes a time frame of reference. Some conditional sampling results can be obtained directly from the probability density. For example if we wish to sample when $u > u_{mean}$, we integrate the probability density for values greater than the mean. We will look at two conditional sampling methods. Quadrant-decomposition is used to look at the types of physical processes that contribute to Reynolds stress.

First sample the data with the constraint that $uv > H$, where H is called the "hole size" (for the hole it cuts out around the origin of a graph of u vs. v) and determines the threshold of sampling. (See diagram below) H is usually taken to be of order unity. Then separate the data according to which quadrant of the u - v plane it lies in. Contributions to the Reynolds stress from different quadrants may dominate in different regions of the flow. For example in the boundary layer quadrant IV dominates close to the wall while quadrant II dominates further away. Quadrant II ($u < 0, v > 0$) may be associated with the "ejection" of low speed fluid away from the wall. Quadrant IV ($u > 0, v < 0$) may be associated with "sweep" of high speed fluid towards the wall. Both processes contribute positively to the Reynolds stress ($-\rho < uv >$). Quadrant decomposition can be calculated directly from a joint probability distribution.

Variable Interval Time Averaging-(VITA) (Blackwelder and Kaplan, 1976) consists of sampling the data conditionally when the short-time variance, defined by

$$\text{var}(u, T) = \frac{1}{2T} \int_{-T}^T u^2 dT - \left[\frac{1}{T} \int_{-T}^T u dT \right]^2 \quad (2.18)$$

is greater than some threshold variance k_{rms}^2 . Thus, two parameters must be specified, the threshold k , usually chosen to be $O(1)$, and the time interval T , usually chosen to be the duration of some characteristic event.

Figure 2.7: Quadrant decomposition

Figure 2.8: Orbit in phase space of a one-dimensional system. a) Decaying oscillations, b) limit cycle, c) limit cycle with two periods

The VITA technique can be viewed as a filter. The time window filters out events of duration less than T . It turns out (Johansson and **Alfredsson**, 1982) that this technique picks out rapid accelerations and decelerations, that is, the passage of some structure. If the results are scaled with $k^{1/2}$ then they all collapse onto the same graph.

More complex pattern recognition techniques may be used, but these usually leave many parameters to vary.

2.4 MODERN IDEAS ON CHAOS

Turbulence is chaos, but chaos is not necessarily turbulence. Chaotic systems so far investigated in detail have a small number of degrees of freedom whereas turbulence has a nearly infinite number of degrees of freedom.

The **dynamical** system description is usually applied to systems with finite total energy by describing orbits in phase space. In a 1-dimensional system with x and $\dot{x} = dx/dt$ as coordinates the orbits for a decaying oscillation look something like (a) :

Non-linear systems can equilibrate to limit cycles (b) which may have more than one periodicity (c). The system may bifurcate (undergo period doubling). One well-known

Figure 2.9: Portrait in phase space of Lorene' strange attractor

example of a chaotic system is the Lorene attractor, a system with three degrees of freedom. The geometry of this can be described by fractal dimensions. A picture of the orbit in phase space is shown below:

2.4.1 Fractal dimensions

Suppose we measure them with a measuring stick of length $l(\epsilon)$. Then the fractal dimension is defined as

$$\lim_{\epsilon \rightarrow 0} (dN(\epsilon)/d\epsilon) \quad (2.19)$$

where $N(\epsilon)$ is the length of the curve in units of $l(\epsilon)$. For Koch's curve $dN/d\epsilon = \ln 4 / \ln 3$.

What is the connection to turbulence? High Reynolds number flows possibly have **self**-similar structures but viscosity prevents reaching the smallest scales. As the Reynolds number goes to infinity the range of scales also goes to infinity. We can establish the fractal nature of turbulence by using volume elements instead of measuring sticks and we ought to get a dimension between 2.5 and 3 (Sreenivasan and Meneveau, 1986). This would **mean** that a shear layer, for example, would evolve into a structure which does not quite **fill** space. But the question remains: What is the (explicit) connection of turbulence to **dynamical** systems?

Figure 2.10: **Koch** curves

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Notes submitted by Joseph H. Barsugli

Lecture 3

REVIEW OF EXPERIMENTS

In this lecture I will present a lot of facts as the experimentalists see them. In particular I will be concerned with the wall region of a turbulent boundary layer. Since theorists tend to believe too much when they see experimental results it is useful to discuss the limitations of the experimental techniques. I will review four established techniques: visual observation, hot wires, hot films, and laser anemometry.

The first thing to notice about visual observation of flow patterns is that it is a kind of conditional sampling. This is because the human eye is very sensitive to visual patterns and so we sometimes see patterns where there are none. Secondly one usually fills the region one wants to observe with smoke or dye. But fluid markers in an unsteady flow will follow streaklines rather than streamlines, so that one sees an integrated history of the flow which can be misleading.

But despite these difficulties pictures make for good art. In Figure 1. we see interesting holes in a side view of a turbulent boundary layer. Oil drops were used for visualization and the Reynolds number is 4000. In Figure 2. we see a "typical eddy", a mushroom looking vortex ring. In both cross-sections we must use our imagination (**i.e.** conditional sampling) to see the structure. Figure 3. shows us turbulence full of non-turbulent holes or **pockets**- a "fractal cheese" if you like. This would seem to indicate a rapid mixing independent of viscosity effects. We will now leave these nice artistic pictures with this warning: you may not be seeing what you think you see.

Hot wires are the most commonly used tool for studying flow in the Eulerian frame of reference. They are typically of length $L = 1\text{mm}$ and diameter $d = 0.0025\text{mm}$, thinner than a human hair. One **heats** the wire and then **measures** how much current is needed to keep the temperature constant, thus finding the velocity. Taking a free stream velocity of 5m/s , $u_\tau = 0.15\text{m/s}$ and $\nu = 1.5 \times 10^{-6}\text{m}^2/\text{s}$ one finds $l_* = 0.1\text{mm}$. So the diameter is fine but the length is about 10 viscous wall units, and since streaks are about 20 wall units wide there is a resolution problem. It is also important to note that the viscous time scale $t_* = 0.00075\text{sec}$. For $U^+ = 10$ one needs a resolution of about one t^* , so 1000 samples per second are necessary.

Hot wires work well in air but in water one must use a coating to insulate the wire. A wire coated with a quartz layer of insulation is a hot film. Typical parameters are

Figure 3.1: Side view of a turbulent boundary layer visualized by smoke (from Falco, 1977).

Figure 3.2: "Typical eddy" in form of a vortex-ring like structure in lower left of figure (from Falco, 1980).

Figure 3.3: Smoke visualization of a turbulent boundary layer developing on a flat plate (from Sreenivasan and eneveau, 1986).

$L = 1.25\text{mm}$ and $d = 0.07\text{mm}$. In water with $u_* = 1\text{cm/sec}$ if one calculates the Reynolds number one finds that shedding occurs for $d = 0.07\text{mm}$. This means that velocity must be kept small and $t_* = 0.01\text{sec}$, so there is an advantage from the time point of view to using water rather than air. A more recent technique is that of laser anemometry, which consists of two laser beams which produce an interference pattern. Particles then pass through this region and show up as light blips, by counting the number of blips per second velocities may be obtained. Typical dimensions are $L = 1\text{mm}$ and $d = 0.2\text{mm}$ which give $L^+ = 10 - 100$ and $d^+ = 5 - 50$. So the resolution is not too much better than with hot wires.

There are problems with these techniques which must be kept in mind. For example a hot wire near a wall may transfer heat to the wall and the resulting signal will be wrong. With water and a hot film there is less problem of this kind as long as the water is kept free of contaminants. It is also necessary to be able to control water temperatures to within about one one hundredth of a degree Celsius. Long time averages in water require several hours monitoring time, so one ends up having to put about 1kW into the water to keep the temperature constant. With laser anemometry the problem is to have the seed particles as small as possible and yet big enough to be visible in laser light. With air at low speeds convection currents can be a problem. In general one must be careful that one is not measuring the wrong things. In Figure 4 we see a plot of u_{rms} versus distance from the wall for different Reynolds numbers. Figure 5 shows the maximum turbulent intensity as a function of probe length, and Figure 6 the pressure versus transducer diameter. The moral is that one should be very careful in choosing experimental equipment parameters and correct for finite probe size, if possible.

Figure 3.4: Root-mean-square streamwise component of fluctuating velocity in a turbulent boundary layer (from Johansson and Alfredsson, 1983).

Figure 3.5: The maximum turbulence intensity as function of the probe length in viscous wall units (from Johansson and Alfredsson, 1983).

Figure 3.6: Measured intensity of wall pressure fluctuations as function of normalized transducer size (from Schewe, 1983).

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Notes submitted by Gavin B. Biebuyck

Lecture 4

COHERENT STRUCTURES IN THE LABORATORY AND IN NATURE

Recall from the previous lecture on statistics that the VITA (variable interval time averaging) technique looks for values of the variance larger than a threshold value,

$$\text{var}(u, T) = \frac{1}{2T} \int_{-T}^T u^2 dT - \left[\frac{1}{T} \int_{-T}^T u dT \right]^2 > k < u^2 > \quad (4.1)$$

This technique will pick out inclined shear layers since events with large u accelerations (or decelerations) will have large values of the **variance**. **VITA** might be a good method for looking at structures in a boundary layer since one would expect inclined shear layers to form there. In a high Reynolds number flow the range of sizes of flow structures would be large, which could be determined by varying the averaging time T . Counting the total number of events as a function of T (which acts as a filter) and the threshold k leads to the graphs shown in Figures 1 and 2. A change in T changes the time scale of the events picked out by the filter, **from** which one can estimate their spatial scale. The maximum number of events is seen to occur for T between 10 and 100 t_* . Since there are more positive events than negative events, let us concentrate on seeing what sort of structures the positive ones correspond to. The velocity profiles for the mean and for a positive event look like (Figure 3): The angle θ is a function of the age of a structure and decreases with time as the structure is rotated by the mean shear. The average angle is found to be about twenty degrees. Such a shear layer is unstable according to the criterion for an **inviscid** parallel flow, but conditions for the instability may not manifest themselves in the flow at hand. For instance, the size of the shear layer may be smaller than the wavelength of the **fastest-growing** unstable mode, or the instability wave may propagate faster or more slowly than the shear layer. The spectrum of the Reynolds stresses compared to the energy intensity of the fluctuations shows that the Reynolds stresses are at lower frequencies corresponding to larger scales. This would imply that most of the momentum exchange is taking place farther from the wall than where the highest fluctuation velocities are. Flow visualization can provide a different type of information which can be useful in understanding what is going on. Pictures taken with a camera moving at the speed of a structure can follow the evolution of the structure. This was the method used by Praturi and Brodkey (1978) to study the large

Figure 4.1: Number of events with positive $\partial u / \partial t$ ($\circ$) and negative $\partial u / \partial t$ ($\times$) detected per unit time as function of the integration time; $Re = 13800$, $y^+ = 12.9$ (from Johansson and Alfredsson, 1982).

Figure 4.2: Total number of events ($\circ$) and events with positive $\partial u / \partial t$ ($\square$) per unit time as function of the threshold level; $Re = 13800$, $T^+ = 10.7$, $y^+ = 12.9$ (from Johansson and Alfredsson, 1982).

Figure 4.3: Boundary layer velocity profiles a) Mean b) Instantaneous

scale processes (Figure 4). They found velocity fronts within a turbulent boundary layer associated with growing distortions of the boundary layer turbulent-nonturbulent interface. The wave troughs were found to carry with them the irrotational fluid in closer to the wall and the crests to move fluid away from the wall with **spanwise** vortices embedded. This motion contributes substantially to the Reynolds stress. Another technique is to use hydrogen bubbles for flow visualization. C.R. Smith (1983) used a hydrogen bubble wire and a fiber optic lens, both moving with the flow. With simultaneous top and side views he was able to watch as streaks became unstable and to develop hairpin vortices. He also studied a synthesized model by producing a hairpin vortex behind a hemispherical roughness element from which he constructed the conceptual model illustrated in Figure 5. Streaks are long streamwise regions of alternating **high**- and low-speed flows with a **spanwise** spacing of about $l_{\lambda}^+ = 100$. An instability of a low-speed streak starts as a series of small **bumps** and indentations whose growth can be explained by Bernoulli's equation. Above a bump there is an increased velocity by continuity and therefore a lowered pressure. The depressions in between, on the other hand, feel an increased pressure. The induced pressures will hence tend to enhance the distortion of the interface between the **low**- and high-speed fluid above it. As streamline distortions grow they are pushed over by the shear and they evolve to resemble hairpins. The small angle of the structure near the wall in the synthesized model was found to be close to the 20° found by use of the VITA technique, confirming the dominating effect of the mean shear on the evolution of the structure.

4.1.1 Free shear layers

Brown and Roshko (1974) used shadowgraphs to visualize the structures in a free-shear layer. They observed two-dimensional vortices which paired up to increase the **boundary**-layer thickness while remaining essentially two-dimensional. Chandrsuda et al. (1978) did similar experiments in air flow that was not as undisturbed and found that near the splitter plate the vortices appear to be two-dimensional while farther from the plate they become three-dimensional due to the residual turbulence. They argued that in any practical

Figure 4.4: Progression of flow illustrating (conceptual) formation of spanwise vortices (from Praturi and Brodkey, 1978).

Figure 4.5: Formation and breakup of hairpin vortices during streak "bursting" according to Smith (1983).

Figure 4.6: Shadowgraph of a mixing layer taken at random times (from Brown and Roshko, 1974).

Figure 4.7: Vortex pattern in a free shear layer in a turbulent background **as** visualized by smoke (from Chadrusuda et al., 1978)

situation the level of turbulence would lead to a three-dimensional flow. Gaster et al. (1985) also did a similar experiment where the tip of the splitter plate was oscillated to select one wavelength of the instability and to force the two-dimensionality. They found that the instability initially grew very fast, then the growth stopped and finally the instability grew again at the unforced rate. Weisbrot (1984) explained this in his thesis where the theory agrees quite well with the experiments except for the actual value of the growth rate. A fractal interface has structure at all scales so that the perimeter is always longer when measured with a shorter measuring stick. The number of measuring sticks N is proportional to the stick length raised to some power which is minus the fractal dimension:

$$N(l) = l^{-fd} \quad (4.2)$$

The length of the coastline of Britain is an often cited example. Sreenivasan and Meneveau (1986) considered the interface between a turbulent boundary layer and the irrotational fluid outside could be a fractal. By putting smoke in the boundary layer and illuminating it in a **sheet** they obtained photos such **as** shown in Figure 8. They measured the length of the interface for a wide range of scales and found that between the viscous Kolmogorov scale and a large scale set by the experiment, the interface is indeed a fractal of dimension 1.38. From this one can estimate that the entire three-dimensional surface has fractal dimension of 2.38.

Figure 4.8: Smoke photograph of a turbulent boundary layer developing on a flat plate. The momentum thickness Reynolds number is around 2000. The thickness of the intersecting light sheet is between one and two Kolmogorov thicknesses (from Sreenivasan and Meneveau, 1986).

4..2 Coherent structures in Nature

There are isolated coherent structures which can survive on their own in a number of different environments. One well known example is the warm and cold Gulf Stream rings (Robinson, 1982). The Gulf Stream meanders in such a way that it occasionally pinches off. The rings thus formed may interact with each other and with the Gulf Stream and have either cold or warm cores (see Figure 9). The cold rings spin cyclonically while the warm rings spin anti-cyclonically. Both types have been observed to last for over a year. Their formation and evolution can be observed from satellite pictures of the sea-surface, temperature. "Meddies" are similar features which are sub-surface rotating lenses of water of possibly Mediterranean origin. Their exact formation mechanism **is** not understood. A **SOFAR** float was accidentally dropped into **such** a lens and its path was observed to be a spiral around the core of the structure. A neighboring float was outside the structure and did not spin around it, but otherwise moved in a nearly parallel path (Figure 10). Other coherent structures observed in nature include atmospheric blocking as studied by Malguzzi and **Rizzoli (1984)**, who proposed an explanation involving **Rossby** waves. There is also the Great Red Spot on Jupiter. Maxworthy and Redekopp (1976) favor a soliton model. Spiral galaxies can also be thought of as coherent structures. "Pick your **own** scale."

Figure 4.9: Gulf Stream rings (from Robinson, 1982).

Figure 4.10: A pair of neighbor SOFAR float tracks, one of which is trapped in a small eddy discovered in the POLYMODE local dynamics experiment: —, float 52, - - - float 53 (from Robinson, 1982).

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Notes submitted by Katherine S. Hedstrom

Part II: EFFORTS AT THEORETICAL MODELING

Lecture 6

EQUATIONS OF MOTION

The starting point for our theoretical modeling of turbulence phenomena is the Navier-Stokes equations

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} \quad (5.1)$$

By taking the curl of the above equations an equation for the vorticity can be found,

$$\frac{\partial \Omega_i}{\partial t} + U_j \frac{\partial \Omega_i}{\partial x_j} = \Omega_j \frac{\partial U_i}{\partial x_j} + \nu \frac{\partial^2 \Omega_i}{\partial x_j \partial x_j} \quad (5.2)$$

where $\Omega_i = -\epsilon_{ijk} \frac{\partial U_k}{\partial x_j}$ is the vorticity. The first term on the right hand side represents vortex stretching by the mean shear and is only non-zero in the three-dimensional case. (In the two-dimensional case only Ω_3 is non-zero but then all derivatives in x_3 -direction are zero.) Using the Navier-Stokes equations some conservation relations will now be derived. First the ensemble averaged kinetic energy will be considered. If the equations are multiplied with U_i and the ensemble average taken we find

$$\frac{1}{2} \frac{\partial}{\partial t} \langle U_i U_i \rangle + \frac{1}{2} \frac{\partial}{\partial x_j} \langle U_j U_i U_i \rangle = -\frac{1}{\rho} \frac{\partial}{\partial x_i} \langle U_i P \rangle + \quad (5.3)$$

$$\frac{1}{2} \nu \frac{\partial^2}{\partial x_j \partial x_j} \langle U_i U_i \rangle - \nu \langle \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} \rangle \quad (5.4)$$

Where $\frac{1}{2} \langle U_i U_i \rangle$ is the ensemble averaged kinetic energy per unit mass. Note that the divergence terms will disappear if integration over a material volume is **considered**. Note also that the last viscous term may be rewritten as

$$-\nu \langle \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} \rangle = -\nu \langle \Omega_i \Omega_i \rangle - \nu \langle \frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} \rangle \quad (5.5)$$

Thus in a material volume only rotational fluid is dissipated within the boundaries while irrotational fluid is dissipated by the motion of the material interface. For an **inviscid** flow the energy equation simply means that the total kinetic energy of the fluid in a material volume is conserved. The second quantity we will consider is the helicity, i.e. $H = \langle \Omega_i U_i \rangle$.

If the Navier-Stokes equations are multiplied with Ω_i and the vorticity equation is multiplied with U_i , the results are added, and the **ensemble** average is taken we find

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x_j} \langle U_j \omega_i U_i \rangle = \frac{\partial}{\partial x_j} \langle \Omega_j U_i U_i \rangle - \quad (5.6)$$

$$\frac{1}{\rho} \frac{\partial}{\partial x_i} \langle \Omega_i P \rangle + \nu \langle \Omega_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} \rangle - \nu \langle U_i \frac{\partial^2 \Omega_i}{\partial x_j \partial x_j} \rangle \quad (5.7)$$

High helicity means that Ω_i and U_i - vectors are aligned, this can be shown to imply low Reynolds stress **gradients**, ($H = \epsilon_{ijk} U_i \partial U_j / \partial x_j / \partial x_k$). For an **inviscid** flow the helicity equation simply means that the total helicity of the fluid in a material volume is conserved. The third quantity we will consider is the enstrophy, $E = \frac{1}{2} \langle \Omega_i \Omega_i \rangle$. If the vorticity equation is multiplied with Ω_i and the ensemble average is taken we find

$$\frac{\partial E}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x_j} \langle U_j \Omega_i \Omega_i \rangle = \langle \Omega_i \Omega_j \frac{\partial U_i}{\partial x_j} \rangle + \nu \frac{\partial^2 E}{\partial x_j \partial x_j} - \nu \langle \frac{\partial \omega_i}{\partial x_j} \frac{\partial \Omega_i}{\partial x_j} \rangle \quad (5.8)$$

For an **inviscid** fluid the enstrophy equation simply means that the total enstrophy of the fluid in a two-dimensional material volume is conserved. To gain some insight into the mechanics of turbulence it is useful to divide the flow quantities into a mean and a fluctuating component, **i.e.**,

$$U_i = \bar{U}_i + u_i \quad (5.9)$$

$$P = \bar{P} + p \quad (5.10)$$

where

$$\bar{U}_i = \langle U_i \rangle, \langle u_i \rangle = 0 \quad (5.11)$$

$$\bar{P} = \langle P \rangle, \langle p \rangle = 0 \quad (5.12)$$

If this is introduced into the Navier-Stokes equations and the ensemble average is taken we find the so called Reynolds averaged equations,

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial \bar{U}_i}{\partial x_j} + \frac{1}{\rho} r_{ij} \right) \quad (5.13)$$

where

$$r_{ij} = -\rho \langle u_i u_j \rangle \quad (5.14)$$

is the so called Reynolds stress tensor. If this equation is subtracted from the full **Navier-Stokes**, with the mean and the fluctuating components introduced, we find

$$\frac{\partial u_i}{\partial t} + \bar{U}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \bar{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial u_i}{\partial x_j} - u_i u_j - \frac{1}{\rho} r_{ij} \right) \quad (5.15)$$

This is the equation for the fluctuating components. If it is multiplied by u_i and the ensemble average is taken an equation for the kinetic energy of the fluctuating components per unit mass, **i.e.**, $q = \frac{1}{2} \langle u_i u_i \rangle$ may be found'

$$\rho \frac{\partial q}{\partial t} = P - \epsilon + \frac{\partial F_i}{\partial x_i} \quad (5.16)$$

Figure 5.1: Energy balance in the outer region of a turbulent boundary layer (from Townsend, 1976)

where

$$F_i + - \langle p u_i \rangle + \mu \left\langle \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\rangle - \rho q \overline{U_i} - \frac{1}{2} \rho \langle u_j u_i u_i \rangle \quad (5.17)$$

is the transfer due to pressures, viscous stresses, and advection, respectively,

$$\epsilon = \frac{\mu}{2} \left\langle \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle \quad (5.18)$$

is the dissipation, and

$$P = \tau_{ij} \frac{\partial \overline{U_i}}{\partial x_j} \quad (5.19)$$

is the production of kinetic energy by the Reynolds stresses working against the mean shear. The magnitude of these different terms have been measured in a boundary layer. The results (Figure 1) show that the production term and the dissipation term balance through out most of the boundary layer, while the other terms are small. In the next picture which shows how these terms balance closer to the wall we have a different picture (Figure 2). Here the pressure term and the convection terms are also large. The energy exchange following **Townsend (1976)** is illustrated schematically in Figure 3. The production term in the energy equation can be better understood if the following simplified case is considered. **Assume** that there is no acceleration of an individual fluid element, **i.e.** the viscosity and the pressure gradients are small. Then we have

$$\frac{Du_i}{Dt} = 0 \quad (5.20)$$

Figure 5.2: Energy balance in the near-wall, constant-stress region of a turbulent boundary layer according to measurements by Laufer (1955). (from Townsend, 1976)

Figure 5.3: Conversion and flow of energy in a boundary layer (from Townsend, 1976).

giving

$$u_i(x_j) = \bar{U}_i(x_j) - \bar{U}_i(x_j + \ell_j) \quad (5.21)$$

where

$$\ell_j = \int_0^t u_i D t \quad (5.22)$$

i.e., the distance the fluid element has traveled during the time interval zero to t. Using this we find

$$\rho \frac{1}{2} \frac{D}{Dt} (u_i u_i) = \rho u_i \frac{D u_i}{Dt} = -\rho u_i u_j \frac{\partial \bar{U}_i}{\partial x_j} = \tau_{ij} \frac{\partial \bar{U}_i}{\partial x_j} \quad (5.23)$$

Thus, the production term in this simple model can be seen to arise **from** the variation of the disturbance velocities, associated with the mixing of fluid elements, resulting from the assumption of constant momentum of the element. Note that this accounts only for two of the disturbance velocities (usually the two horizontal ones) since continuity also has to be satisfied. The vertical velocity gives the **liftup** ℓ_2 , which in turn gives the horizontal velocities. Note also that this is purely a three-dimensional effect since in the two-dimensional case the only horizontal velocity is u_1 which would be given **from** continuity if $\mathbf{v}$ were known. Finally we will calculate the maximum of the production term in a turbulent boundary layer. If we assume a zero pressure gradient boundary layer and neglect non-parallel effects, this gives the following streamwise disturbance momentum equation:

$$0 = \frac{\partial}{\partial y} \left(\mu \frac{\partial \bar{U}}{\partial y} + \tau_{12} \right) \quad (5.24)$$

which in terms of wall variables gives

$$\frac{dU^+}{dy^+} + \tau_{12}^+ = 1 \quad (5.25)$$

where the value of the constant is found through the use of the boundary condition at the wall. This gives the following production term:

$$P^+ = \tau_{12}^+ \frac{dU^+}{dy^+} = \left(1 - \frac{dU^+}{dy^+} \right) \frac{dU^+}{dy^+} \quad (5.26)$$

maximizing this we find

$$P_{\max}^+ = \frac{1}{4} \quad (5.27)$$

which occurs for the value $y^+ = y_m^+$ for which $\frac{U^+}{dy^+} = \frac{1}{2}$, which is about $y^+ = 13$.

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Lecture 6

SHEAR FLOW INSTABILITY

We consider the instability theory in this lecture for the reason that much of the **dynamics** of turbulent shear flow and coherent **structures** is associated or could be associated with instability. The linear instability problem is also the simplest flow model incorporating the interaction between unsteady fluctuations and a background shear or density distribution.

6.1 Linear instability of shear flow

Let us write down the equation of motion for fluctuation terms as in the previous lectures,

$$\rho \left(\frac{\partial u_i}{\partial t} + \bar{U}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \bar{U}_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \bar{U}_i}{\partial x_j} - \rho u_i u_j - \tau_{ij} \right) \quad (6.1)$$

where $\tau_{ij} = -\rho \langle u_i u_j \rangle$, with the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (6.2)$$

where $\bar{U}_j$ is the ensemble average of velocity and all the low case variables are fluctuations from the mean quantities. The difficulty of solving these equations is the nonlinearity and the Reynolds stresses which lead to the closure problem.

We study the linear problems for flows in channel and boundary layer. The mean flow is assumed to be two dimensional, **i.e.** (see Figure 1)

$$\bar{U}_i = U(y) \delta_{1i}$$

Small velocity fluctuations in x, y and a direction are denoted by u, v and w , respectively. Here we linearize by dropping all the quadratic terms in the basic equations. However, the terms neglected may have dominant effects, depending on the scale of perturbation and the time scale we are considering. If we follow a small disturbance for a long time, for example, we may have to consider the nonlinear terms. The linearized equations are

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (6.3)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \quad (6.4)$$

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w \quad (6.5)$$

and the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (6.6)$$

Taking the divergence of the momentum equations we obtain

$$\nabla^2 p = -2\rho \frac{\partial U}{\partial x} U' \quad (6.7)$$

where $U' = dU/dy$. Combining the $\partial/\partial y$ of the equation above with the Laplacian of equation (4), we get a single equation for v ,

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \nabla^2 v - U'' \frac{\partial v}{\partial x} - \nu \nabla^4 v = 0 \quad (6.8)$$

We could **nondimensionalize** this equation with length scale d and velocity scale U_0 for channel flow, length scale δ and velocity scale U_∞ for boundary layer flow, where d is the width of the channel, U_0 is the maximum velocity of the mean velocity, δ is the boundary layer thickness, U_∞ is the free stream velocity. The following boundary conditions are needed:

$$v = 0, \quad \frac{\partial v}{\partial y} = 0 \quad (6.9)$$

at $y = 0$ and $y = d$ for channel flow or $y = \infty$ for boundary layer flow. The condition $\frac{\partial u}{\partial y} = 0$ can be seen easily from the continuity equation. The initial condition is

$$v(x, y, z, t = 0) = v_0(x, y, z)$$

for some specific function v_0 which must be consistent with the continuity equation. In principal, this problem can be handled by some standard mathematical methods like Fourier or **Laplace** transform. Since the coefficients are independent of x, y, z, t , Fourier transform with respect to x, z, t leads to an equation depending on y only, but **Laplace** transform with respect to t can also be used. The problems so formulated are not so difficult from a mathematician's point of view, but complicated to in their details.

Instead of trying to solve the initial-boundary value problem, we study the problem for eigensolutions of the form

$$v = \hat{v}(y)e^{i(\alpha x + \beta y - \omega t)}$$

where $\omega = \alpha c$ (c is the phase velocity) is the frequency. Assume α, β real, and let $\omega = \alpha(c_r + i c_i)$. The sign of c_i tells whether the wave like disturbance will grow or decay. The growth rate is αc_i . Introducing this eigensolution into the equation we get the following equation:

$$(U - c)(\hat{v}'' - k^2 \hat{v}) - U'' \hat{v} - \frac{1}{i\alpha R} \left(\frac{d^2}{dy^2} - k^2 \right)^2 \hat{v} = 0 \quad (6.10)$$

where $k^2 = \alpha^2 + \beta^2$, with boundary conditions

$$\hat{v}(0) = \hat{v}'(0) = 0 = \hat{v}(\infty) = \hat{v}'(\infty)$$

for boundary layer flow (similarly for channel flow).

This is the Orr-Sommerfeld equation which only admits solutions for certain values of c . Squire demonstrated that only two dimensional waves need to be considered, since the solution of the Orr-Sommerfeld equation for waves with $\beta \neq 0$ may be obtained from the solution of $\beta = 0$ by solving the problem for $\alpha = k$ and a reduced Reynolds number $R_1 = R\alpha/k$.

Asymptotic solutions can be found, for example, in the expansion in terms of $1/R$. Modern computing techniques give accurate solutions for a large number of cases. Diagrams in Figure 3 are for the Falkner-Skan self-similar boundary layer profiles with a free stream velocity satisfying

$$U_\infty \sim x^\beta$$

where β is a pressure gradient parameter that is negative for flows with positive (adverse) pressure gradient, positive for flows with negative (favorable) pressure gradient. The length scale used in the diagrams of Figure 3 is the displacement thickness S_* , i.e.

$$\delta_* = \int_0^\infty \left(1 - \frac{U}{U_\infty}\right) dy$$

and α_* , R , in these diagrams are defined as $\alpha\delta_*$ and $U_\infty\delta_*/\nu$, respectively. We can see from the diagrams that flows with $\beta < 0$ are considerably more unstable than those with $\beta > 0$, by comparing the growth rates. When $\beta < 0$, the values of c_r and c_i tend to become independent of R_* for large values of R_* , which is related to the fact that there is an inflection point of the mean velocity profile when $\beta < 0$. These results have been verified in experiments on flat-plate boundary layers and in channel flows. The agreement between theory and experiments has been found to be quite good for small wave amplitudes. Note that the Orr-Sommerfeld theory only valid approximately for boundary layer flows since the flow is not strictly parallel. But since the streamwise variation is very small, one may neglect its influence on stability.

Figure 6.1: Boundary layer velocity profiles with a) positive and b) negative pressure gradient

6.2 Destablizing effects of viscosity

By considering the **inviscid** problem, we will demonstrate that feature that the flow may become more unstable with increasing viscosity. We consider the boundary layer problem. The Orr-Sommerfeld equation is reduced to

$$(U - c)(\hat{v}''' - k^2 \hat{v}) - U'' \hat{v} = 0 \quad (6.11)$$

for **inviscid** flow. We consider a long wave deformation of wall which generates a disturbance (see Figure 4). This is a well defined linear problem (the deformation is small). We will determine how much energy we need put into the wall deformation in order to force the wave in the fluid. The fluctuation velocities u , w and fluctuation pressure p are assumed to have the same form as v , i.e.,

$$u = \hat{u}(y)e^{i(\alpha x + \beta y - \omega t)}$$

$$w = \hat{w}(y)e^{i(\alpha x + \beta y - \omega t)}$$

$$p = \hat{p}(y)e^{i(\alpha x + \beta y - \omega t)}$$

The perturbation equations now are

$$i\alpha(U - c)\hat{u} - \hat{v}U' = -i\alpha \frac{\hat{p}}{\rho} \quad (6.12)$$

$$i\alpha(U - c)\hat{v} = -\frac{\hat{p}}{\rho} \quad (6.13)$$

$$i\alpha(U - c)\hat{w} = -i\beta \frac{\hat{p}}{\rho} \quad (6.14)$$

$$i\alpha \hat{u} + i\beta \hat{w} = -\hat{v}' \quad (6.15)$$

where the last equation comes from continuity. From (11-14) we can derive the following single equation for $\hat{p}$:

$$k^2 \hat{p} = -i\alpha \rho (U - c)^2 \frac{d}{dy} \left(\frac{\delta}{U - c} \right) \quad (6.16)$$

Figure 6.2: Curves of c_i and c_r for Falkner-Skan velocity profiles: a) $\beta = 0.05$; b) $\beta = 0$; c) $\beta = -0.05$; d) $\beta = -0.10$ (from Obremski et al. 1969)

Figure 6.3: Wave-like deformation of a wall

In the long-wave approximation we may set $\hat{p} \simeq \hat{p}_\delta$, where δ is the thickness of the boundary layer, and $\hat{p}$ stands for $\hat{p}(\delta)$. Assuming that U is constant outside the boundary layer, we find the following relationship between $\hat{p}$ and $\hat{v}$ at the edge of the boundary layer:

$$i\alpha(U_\infty - c)\hat{v}_\delta = -\frac{\hat{p}_\delta}{\rho} \quad (6.17)$$

If $\hat{p}$ is taken as constant, we have

$$\frac{d}{dy}\left(\frac{\delta}{U - c}\right) = \frac{ik^2\hat{p}_\delta}{\alpha\rho(U - c)'} \quad (6.18)$$

This gives

$$\hat{v}_\delta = -(U_\infty - c)\left(\frac{\hat{v}_0}{c} + k(U_\infty - c)\hat{v}_\delta \int_0^\delta \frac{dy}{(U - c)^2}\right) \quad (6.19)$$

where $\hat{v}_0$ is the value of δ at the wall. We solve $\hat{v}_\delta$ from above equation and find

$$\hat{v}_\delta = -\frac{(U_\infty - c)\hat{v}_0/c}{1 + k(U_\infty - c) \int_0^\delta \frac{dy}{(U - c)^2}} \quad (6.20)$$

Therefore,

$$\hat{p} \simeq \hat{p}_\delta = \frac{-i\alpha\rho(U_\infty - c)^2\hat{v}_0/(ck)}{1 + k(U_\infty - c) \int_0^\delta \frac{dy}{(U - c)^2}}. \quad (6.21)$$

Then the power input is

$$P = \langle pv \rangle_0 = \frac{1}{2} R.P. \langle \hat{p} \hat{v}^* \rangle_0 \quad (6.22)$$

where $*$ denotes complex conjugate, R.P. stands for the real part, and subscript 0 indicates values at the wall. The final result is

$$P = -\frac{1}{2} R.P. \left(\frac{i\alpha\rho(U_\infty - c)^2 |\hat{v}_0|^2 / (ck)}{1 + k(U_\infty - c)^2 \int_0^\delta \frac{dy}{(U - c)^2}} \right). \quad (6.23)$$

In particular, we are interested in the power input needed to make the wave barely grow. Therefore, we let $c_i \rightarrow 0^+$ and take the limit of the expression. This gives

$$\begin{aligned} \int_0^\delta \frac{dy}{(U-c)^2} &= \lim_{U_i \rightarrow 0} \frac{dU}{U'(U-c)^2} \\ &= \lim_{U_i \rightarrow 0} \frac{dU}{(U_i^2 + \frac{U-c}{U_i^2} U_i'')(U-c)^2} \\ &\simeq -\frac{U_c''}{U_c'^2} \int_0^\delta \frac{dU}{U-c} + I_r \\ &= I_r - \pi i \frac{U_c''}{U_c'^2} \end{aligned}$$

Therefore, in that limit

$$P = \frac{1}{2} \alpha \rho \frac{|\hat{u}_0|^2}{c_r U_c'^2} \frac{(U_\delta - c)^2 U_c''}{(I_r^2 + (\frac{\pi U_c'' k}{U_c'^2})^2)} + O(c_i) \quad (6.24)$$

Since U_c'' is everywhere negative for a velocity profile with no inflection, P is negative, i.e. the wavy deformation of wall must actually take energy out of the fluid to make the wave grow. Therefore, in the absence of the wall deformation, but for a noninflected profile in a viscous flow, the viscosity may play the role of the deformed wall in it taking energy out of the flow to make it unstable. We conclude from this analysis that the viscosity may have a destabilizing effect.

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Notes submitted by Ya Yan Lu

Lecture 7

TURBULENCE AND FLOW INSTABILITY

7.1 Nonlinear instability

We consider first infinite wave trains of small but finite amplitude. Let us look at the nonlinear interaction between different waves.

Assume a wave of wave number $\mathbf{k}_1$ interacting with a wave of wave number $\mathbf{k}_2$ and producing two other waves of wave numbers

$$\mathbf{k}_3 = \mathbf{k}_1 \pm \mathbf{k}_2 \quad (7.1)$$

These arise from the quadratic interaction terms. Similarly,

$$\omega_3 = \omega_1 \pm \omega_2 \quad (7.2)$$

and

$$a_3 = a_1 a_2 \quad (7.3)$$

where ω is the frequency and a the amplitude. In the same manner, looking at 3-wave interactions, we get

$$\mathbf{k}_4 = \mathbf{k}_3 \pm (\mathbf{k}_1 \pm \mathbf{k}_2) \quad (7.4)$$

$$a_4 = a_1^2 a_2, \text{ or } a_4 = a_1 a_2^2 \quad (7.5)$$

Here we discuss ~~some~~ examples. 1) $\mathbf{k}_1 = \mathbf{k}_2$ In this case $\mathbf{k}_3 = 2\mathbf{k}_1$ or 0 and $\mathbf{k}_4 = \mathbf{k}_1$ or $3\mathbf{k}_1$ $\omega_4 = \omega_1$ or $3\omega_1$ This describes self-interaction of the wave. Landau's equation gives the evolution of the amplitude,

$$\frac{\partial a}{\partial t} = 2\alpha c_1 a^2 + B a^4 \quad (7.6)$$

αc_1 represents the linear growth rate $B > 0$, which in fact is characteristic of flows in channels, would result in subcritical instability. $B < 0$ causes supercritical stability. $B a^4$ represents the effect of nonlinearity. 2) $\mathbf{k}_2 = 2\mathbf{k}_1$ Then $\mathbf{k}_3 = 3\mathbf{k}_1$ or $\mathbf{k}_1$ which may lead to subharmonic instability. 3) $\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = \mathbf{k}_1$; $\omega_4 = \omega_1 \pm \omega_2 \pm \omega_3 = \omega_1$ (3 waves interacting to produce one of the original waves.) We now look, in some suitable frame of reference

(moving with velocity c_j , for example), at the perturbation around some steady excited state. When the basic state is independent of **time**, we have

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad (7.7)$$

which with $U_i = U_i(x_j)$ is homogeneous in time. We use this plus the continuity equation to look for eigensolutions of the form

$$u_i = \bar{u}_i e^{\sigma t} \quad (7.8)$$

with homogeneous boundary conditions. This may be used to study the instability of an excited state. Generally, one has to employ numerical methods, but one could arrive at a simplified problem by expansion into harmonic components and the like. What use is this for fully developed turbulent flow? One may regard the turbulent flow as one being "kicked" by instabilities occurring locally and at random. For a measuringful model one may need to assume scale separation. **i.e.** the basic flow state has much larger spatial and temporal scale than the instabilities.

7.2 Kinematic wave theory

Suppose the basic flow varies slowly with space and **time** (per wave length of the wave studied). If we measure the nonuniformity of flow by some small number $\epsilon \ll 1$ giving the ratio of instability wave length to the scale of nonuniformity. Then define the dispersion relation as a function of wave number as weak function of spatial coordinates and time, which may include correction for finite ϵ .

$$\omega = \Omega(\alpha, \rho, x, z, t) \quad (7.9)$$

Generally, Ω is complex,

$$\Omega = \Omega_r + i\Omega_i, (\Omega_i \ll \Omega_r) \quad (7.10)$$

Furthermore, we can define group velocity as a vector.

$$c_g = (\Omega_\alpha, \Omega_\rho, x, t) \quad (7.11)$$

Here, the imaginary part is ignored. How k and ω varies along the wave train may be derived as follows: Introduce a phase function $\Theta(x, z, t)$ under the assumption that the oscillation disturbance varies like a $x \exp(i\Theta)$. Then,

$$k = \frac{\partial \Theta}{\partial x}, \omega = -\frac{\partial \Theta}{\partial t} \quad (7.12)$$

Consider

$$\frac{Dk}{Dt} \equiv \frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} \quad (7.13)$$

We obtain

$$\frac{Dk}{Dt} = -\Omega_z, \frac{D\omega}{Dt} = \Omega_t \quad (7.14)$$

and the equation for the ray,

$$\frac{dx}{dt} = c_g \quad (7.15)$$

With these we can follow how the wave number and frequency for a group changes along the ray. Conservation of wave action density A gives

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(c_g A) = 0 \quad (7.16)$$

if it is a conservative system or

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(c_g A) = 2\Omega_t A \quad (7.17)$$

if it is (a weakly) non-conservative system. The definition of the wave action density A is on the basis of the **Lagrangian** function, averaged over a cycle

$$L = \int_0^{2\pi} L d\theta \quad (7.18)$$

L is a quadratic function of amplitude. The wave action density A is defined by

$$A = \frac{\partial L}{\partial \omega} \quad (7.19)$$

The nonconservative equation has a general solution obtained by integration along the ray,

$$\frac{A}{A_{(0)}} = \frac{1}{J} e^{2 \int \Omega_t D t} \quad (7.20)$$

where J is the Jacobian of ray position with respect to the **initial** position $\mathbf{x}(0) = \mathbf{x}_0$. That is, J measures how the volume of a wave group changes along its trajectory. As the ray come together J decreases. We can determine J by ray tracing. Rays which converge together into zero volume will produce infinite amplitudes. This may **happen** in a moving frame of **reference**(**depending** on background flow) if

$$c_g = c_s \quad (7.21)$$

i.e., group velocity = phase velocity. So, waves travelling on a non-uniform background can get stuck on a moving **nonuniformity** with a very large amplitude as a result. This could lead to bursting and creation of coherent structures.

Figure 7.1: Rays focussing with $c_g = c_0$

7.2.1 Wave structure of boundary layer turbulence

Write the equation for the vertical fluctuating velocity component with all non-linear terms included, assuming parallel flow,

$$L(v) \equiv \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 v - U'' \frac{\partial v}{\partial x} - \nu \nabla^4 v = q(u_i, u_j) \quad (7.22)$$

Her, q contains all the nonlinear **terms** but no mean component. We can find the discrete set of eigenfunctions of the operator L . It turn out that they form an incomplete set in the y -direction for the boundary layer since there is also a continuous set. We treat unknown right hand side q **as** a random force. Consider now a simple system having resonance at some frequency

$$\omega = \Omega_r^{(o)} + \Omega_i^{(o)} = \Omega^{(o)} \quad (7.23)$$

Then, if that system has some random forcing, the power spectrum is

$$\equiv S_{uu} = \frac{S_{qq}(\omega)}{|W|^2} \quad (7.24)$$

W is a transfer or response function to harmonic excitation of unit amplitude. In a ordinary mechanical system, the response function W behaves near resonance **as**

$$W \simeq im(\omega - \Omega^{(o)}) \quad (7.25)$$

Here, m is a generalised **mass**. **This** gives for the power spectral density

$$S_{uu} = \frac{S_{qq}(\omega)}{m^2[(\omega - \Omega_r^{(o)})^2 + (\Omega_i^{(o)})^2]} \quad (7.26)$$

Figure 7.2: Measured $S_{uu}(\omega^+, k_x^+)$ spectrum in a pipe flow (from Morrison et al., 1971).

From the peak position and the width of the peak one can determine the phase velocity and the decay rate for each wave component (the system must not have self-excited waves, only damped ones). Bark (1975) determined the phase speed and decay rates for waves in the wall region of a turbulent pipe flow in this manner using the experimental results of Morrison and Kronauer (1971). The values formed from the experiments agree quite well with the calculated eigenvalues found by solving the Orr-Sommerfeld problem for the mean velocity profile in the wall region. (see Figures 2 - 4)

7.3 REFERENCES

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Notes submitted by Z.S. Qian

Figure 7.3: $\circ$, measured values of $S_{uu}(\omega^+, k_x^+)$ for $k_x^+ = 0.01$ (from Morrison et al. 1971). $\cdot$ three-parameter fit according to the equation above (from Bark, 1975)

Figure 7.4: Real and imaginary parts of the phase velocity. $\square$, $\circ$, determined from measurements, $\cdot$ computed from the Orr-Sommerfeld equation (from Bark, 1975)

Lecture 8

EULER DESCRIPTION OF FLUCTUATING FIELD

8.1 Intermittency from perturbation waves in a nonuniform background

In boundary layers, the transition to fully developed turbulence is usually observed to be very sudden and not like a successive series of instabilities/bifurcations generating successively smaller scales of activity (as known in other systems, e.g., convective ones). These sudden changes in the nature of the flow give rise to the characteristic **intermittency** of boundary layer flow. One potential mechanism for the behavior may be rationalized starting from kinematic wave theory (short waves on a slowly varying background).

It was shown above that the conservation of wave action density A in that framework is

$$\frac{DA}{Dt} = -A \frac{\partial c_g}{\partial x} + 2\Omega_i A \quad (8.1)$$

The dispersion relation can be extended to include weak non-linearities by introducing a term quadratic in the wave amplitude a (or proportional to A) like $B(\mathbf{k}; \mathbf{x}, t)a^2$, to yield

$$\omega = \Omega_r(\mathbf{k}; \mathbf{x}, t) + a^2 B(\mathbf{k}; \mathbf{x}, t). \quad (8.2)$$

At a caustic, i.e., a location in space where $c_g = 0$, the wave action conservation then turns out to lead to the Landau-Ginzburg equation, and also the regime changes **from** convective instability to absolute instability across the caustic. If $a^2 B$ has the appropriate sign, it is known that the Landau-Ginzburg equation has certain solutions that give rise to chaotic behavior (Moon et al, 1983). Thus focussing of disturbances by appropriate kinds of background flows may lead to chaos and thus the generation of all scales of motion, which potentially is an important factor in sudden transitions to turbulence.

8.2 Evolution of localized/zero-wavenumber perturbations in parallel shear flow

Consider a perturbation of the form shown in Figure 1. This is localized (not wave-like)

Figure 8.1: Initial localized disturbance

and has non-zero mean (average over x), in a parallel background flow $U = U(y)$. This type of disturbance may be more appropriate for consideration as a "coherent structure" than a wave-like disturbance. The linearized equations are, as before, for the **inviscid** case

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v U' = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (8.3)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (8.4)$$

These equations formed the basis of **Rayleigh's** inflection point criterion for small wave-like perturbations in v , which provides some information about the possible v -behavior. Now we will consider the case for zero wavenumber in x (streamwise direction), i.e., the **x -averaged** quantities, and will learn about the behavior of other variables as well, in particular u .

$$\bar{u}(y, z, t) = \int_{-\infty}^{\infty} u dx, \bar{v}(y, z, t) = \int_{-\infty}^{\infty} v dx, \bar{p}(y, z, t) = \int_{-\infty}^{\infty} p dx \quad (8.5)$$

Integrating the equations from $x = -\infty$ to $x = +\infty$ and assuming rapid enough decay of the perturbation to $\pm\infty$ we obtain

$$\frac{\partial \bar{u}}{\partial t} + v U' = 0 \quad (8.6)$$

$$\frac{\partial \bar{v}}{\partial t} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} \quad (8.7)$$

However, $\bar{p} = 0$, since taking the divergence of the above momentum equations yields $\nabla^2 \bar{p} = -2\rho U'(\partial v / \partial x) = 0$ and $\bar{p}_y = 0$ at the boundary, since $v = 0$ there. Thus, $\partial \bar{v} / \partial t = 0$ and $\bar{v}(y, z, t) \simeq v_o(y, z)$, and we have

$$\bar{u} = \bar{u}_o - \bar{v}_o U' t \quad (8.8)$$

This result has the following implications: A localized perturbation must contain a collection of nonzero-wavenumber contributions. The non-zero perturbation wave number components will generally decay, as shown by initial-value calculations for the boundary shear flow profile without inflection point. Then the system will tend to spread the zero- k content out over

Figure 8.2: Behavior of u (conceptual) as time increases

2, conceptually as shown in Figure 2. Equation (8.10) shows that the mean part of the v - perturbation forces a steadily increasing mean u at the corresponding locations in y, z and can do so only in the overall **x -range** of the disturbance. Thus, with the correct type of disturbance, regions with anomalous **streamwise** velocity decreasing in amplitude but growing in their streamwise extent may be generated. One might speculate that these are closely related to the 'streaks' observed in wall-bounded flows.

8.3 Torque on a localized region of activity in a shear flow

Start from the Navier-Stokes equations for incompressible flow, and divide the flow and the pressure into their ensemble averages U and P , where by assumption $U = \mathbf{U}(y)\delta_{11}$, and the fluctuating components (u, v, w) and p have zero mean. From the Navier-Stokes equations, we subtract their averages, using the familiar trick of adding $\mathbf{u}(\nabla \cdot \mathbf{u}) = 0$ to the u -equation and $\mathbf{v}(\nabla \cdot \mathbf{u}) = 0$ to the v -equation. Then the equations for u, v become

$$\begin{aligned} \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{\partial U}{\partial y} = & -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u - \frac{\partial}{\partial x} [u^2 - \langle u^2 \rangle] - \\ & \frac{\partial}{\partial y} [uv - \langle uv \rangle] - \frac{\partial}{\partial z} [uw - \langle uw \rangle] \end{aligned} \quad (8.9)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = & -\frac{1}{\rho} \frac{\partial p}{\partial y} + \\ \nu \nabla^2 v - \frac{\partial}{\partial x} [uv - \langle uv \rangle] - \frac{\partial}{\partial y} [v^2 - \langle v^2 \rangle] - \frac{\partial}{\partial z} [vw - \langle vw \rangle] \end{aligned} \quad (8.10)$$

Now consider the torque around an axis in the **spanwise** (z -) direction (Figure 3): On the assumption that the disturbances vanish sufficiently rapidly away from the localized region the only torque exerted on the fluid element is that produced by the wall pressures induced by the localized activity. Thus

$$\dot{M}_3 = - \int \int x p_{\text{wall}} dx dz = \int \int \int_0^\infty xy \nabla^2 p dx dz dy \quad (8.11)$$

Figure 8.3: Torque on fluid around z-axis due to localized activity

By taking the divergence of the momentum equations for the fluctuating velocities we find the **3-D** Laplacian of p to be

$$\nabla^2 p = -\rho U' \frac{\partial v}{\partial x} - \rho \frac{\nabla^2 [u_i u_j - \langle u_i u_j \rangle]}{\partial x_i \partial x_j} \quad (8.12)$$

Substitution into the equation for $\dot{M}_3$ yields

$$\begin{aligned} \dot{M}_3 &= - \int \int U' \int x \frac{\partial v}{\partial x} dx dy dz - 2 \int \int \int x y \nabla^2 [u v \partial x - \langle u v \rangle] dx dy dz = \\ &= 0 + 2 \int \int \int x \frac{\partial [u v - \langle u v \rangle]}{\partial x} dx dy dz = -2 \int \int \int [u v - \langle u v \rangle] dV \end{aligned} \quad (8.13)$$

This is the desired result. Since the disturbance was assumed to be isolated in space (only one disturbance at a time) and the disturbances occur only intermittently, the instantaneous $\int \int \int u v dV$ will be much larger than $\int \int \int \langle u v \rangle dV$ at a time of activity. Thus $\dot{M}_3 \approx -2 \int \int \int u v dV$, and since in a localized region of instability the average of uv will be negative we find that $\dot{M}_3 > 0$, i.e., the fluid in this region experiences a torque tending to lift up the downstream end of the disturbed region and press down the upstream end (see Figure 3), which is in accordance with observations in the boundary layer ("sweep" generally following an "ejection".)

A similar derivation for the x-component of the torque yields

$$\dot{M}_1 = -2 \int \int \int [v w - \langle v w \rangle] dV \quad (8.14)$$

where now the ensemble average vanishes identically due to statistical homogeneity in the z -direction (spanwise). This component has no preferred sign. It can generate longitudinal vorticity and may cause locally enhanced streamwise momentum.

8.4 Generation of shear layers by a passive advected disturbance

In a boundary shear flow U that only varies in y , it is possible to have a disturbance only in u and w , which thus does not interact directly with the shear, but is only advected by

Figure 8.4: Distortion of a localized region of activity by the mean shear (from Landahl, 1977)

$U(y)$. Such a passive disturbance $[u(x, y, z, t), 0, w(x, y, z, t)]$ may be described by functions f, g such that

$$u = f(\xi, y, z), w = g(\xi, y, z), \xi = x - U(y)t \quad (8.15)$$

where f and g are related by continuity so that

$$\frac{\partial f}{\partial \xi} + \frac{\partial g}{\partial z} = 0 \quad (8.16)$$

The evolution of the x - and z -components of vorticity is then given by

$$\omega_x = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{\partial g}{\partial y} + tU' \frac{\partial g}{\partial \xi} \quad (8.17)$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial f}{\partial y} + tU' \frac{\partial f}{\partial \xi} \quad (8.18)$$

Since f, g are **fixed**, these vorticity components will grow linearly **downstream**, and may generate shear layers. This is easily understood by following how a region of local activity is distorted by the mean shear (Figure 4). In the **inviscid** approximation the shear will grow linearly with time, indefinitely. However, viscosity will act to limit the shear for long times.

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Lecture 9

COHERENT STRUCTURE MODELING

In the following, we are going to develop an approximate theory for the dynamics of a coherent structure in a turbulent flow.

To do this, we divide the flow field into three components,

$$U_i = \bar{U}_i + \tilde{u}_i + u'_i \quad (9.1)$$

where $\bar{U}_i$ is the mean field, $\tilde{u}_i$ are the correlated fluctuations (found by conditional sampling), and u'_i are the uncorrelated fluctuations (those fluctuations not picked out by the conditional sampling criterion). Also we have $\langle \tilde{u}_i \rangle = \langle u'_i \rangle = 0$. In a dynamical or momentum transfer sense, viscosity plays a different role for these three components: for the mean flow, viscosity is important, whereas for the fluctuations, viscosity plays a small or unimportant role, at least in their initial phase of development.

Now we can write the equations for the evolution of the coherent structures. For the mean flow equation, it is the same as in previous lectures (where there is only one fluctuation velocity field) except what was the fluctuation velocity is now $\tilde{u}_i$. Thus

$$\frac{\partial \tilde{u}_i}{\partial t} + (\bar{U}_j + \tilde{u}_j) \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial \bar{U}_i}{\partial x_j} = \quad (9.2)$$

$$\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} [\tilde{u}_i u'_j - \langle u_i u_j \rangle] \quad (9.3)$$

and for the fluctuating part,

$$\frac{\tilde{D} \tilde{u}_i}{Dt} + \tilde{u}_j \frac{\partial \bar{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j} \tilde{r}_{ij} \quad (9.4)$$

where,

$$\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + (\bar{U}_j + \tilde{u}_j) \frac{\partial}{\partial x_j} \text{ and } \tilde{r}_{ij} = -(\tilde{u}_i u'_j - \langle u_i u_j \rangle) \quad (9.5)$$

What can you get out of this? Well, if we assume that the flow is parallel, $\bar{U}_i = U(y) \delta_{i1}$, then the equations for the coherent fluctuation are: (using $\tilde{u}, \tilde{v}$, and $\tilde{\omega}$)

$$\frac{\tilde{D} \tilde{u}}{Dt} + \tilde{v} \frac{dU}{dy} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \nu \nabla^2 \tilde{u} + \frac{\partial}{\partial x_j} \tilde{r}_{ij} \quad (9.6)$$

Figure 9.1: Eddy dimensions

$$\frac{\tilde{D}\tilde{v}}{Dt} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial y} + \nu \nabla^2 \tilde{v} + \frac{\partial}{\partial x_j} \tilde{\tau}_{2j} \quad (9.7)$$

$$\frac{\tilde{D}\tilde{\omega}}{Dt} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial z} + \nu \nabla^2 \tilde{\omega} + \frac{\partial}{\partial x_j} \tilde{\tau}_{3j} \quad (9.8)$$

and remember that

$$\frac{\tilde{D}u'}{Dt} + u'_j \frac{\partial}{\partial x_j} (\bar{U}_i + \tilde{u}_i) = -\frac{1}{\rho} \frac{\partial p''}{\partial x_i} + \text{nonlinear terms} \quad (9.9)$$

Then armed with these equations, we do modeling based on some suitable assumptions. Guided by the experimental results we may assume that the eddy or coherent structure has a "surfboard" shape of dimensions $\ell_1 \times \ell_2 \times \ell_3$ as shown in Figure 1, where ℓ_1 is in the stream direction, and ℓ_2 is in the y-direction. We have a surfboard-like shape if $\ell_3 \ll \ell_1$. We also assume that the structure is flat in the sense that

$$\epsilon = \ell_2(\ell_1^{-2} + \ell_3^{-2}) \ll 1 \quad (9.10)$$

Then to within times of the order the convection time scale, $t_c = \ell_1 u_1$ we have $\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x}$ and $\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial z}$ of $O(\epsilon)$. The above equations have been used to predict conditional sampling experiments where it was assumed that

$$1/(\partial \bar{U} / \partial y) = O(\ell_1 / \epsilon U_c) \quad (9.11)$$

The main free parameter in this model is the shape of the Reynolds stress in the y-direction which was assumed to be 15 wall units wide. Then after the mean velocity profile etc. is

Figure 9.2: VITA-sampled u -velocity at $y^+ = 12.9$. Experiments, channel flow (Johansson and Alfredsson, 1982); — Theory (Landahl, 1984)

used to calculate $\langle \tilde{u} \rangle$ vs time for a particular averaging time and y length scale, we get the result shown in Figure 2. What this shows, is that much of the evolution of a coherent structure in a turbulent flow may be governed by linear dynamics.

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Notes submitted by Robert D. Van Buskirk

Lecture 10

MODELLING, NUMERICAL SIMULATION, FINAL QUESTIONS

We shall first recapitulate the "flat eddy" model. The basic concepts of this model are shown in Figure 1. The mean velocity profile gives rise to small scale disturbances which form a "patch" of turbulence which extends in the streamwise and **spanwise** directions, becoming flattened as the flow develops. This patch may give rise to a local conditionally averaged Reynolds stress $-\rho \bar{u}'\bar{v}'$ of the usual kind. The momentum equation implies that downstream the flow is pushed upwards, and the upstream flow is pushed downwards as described in Lecture 8. This gives rise to a large-scale motion having a scale of the order of the size of the patch. The cycle is completed by the formation of a new region of instability through vortex stretching, **e.g.**, the generation of a new region of inflection in the flow.

The $-\rho \bar{u}'\bar{v}'$ Reynolds stresses are not the only ones which may be present in the flow since, if the disturbance is locally three dimensional, it is possible to have $-\rho \bar{v}'\bar{w}'$ nonzero. These stresses can 'kick' the flow to provide rotation about the x-axis (streamwise), but unfortunately have hardly been studied at all.

The main effect of the instability **as** regards the Reynolds stresses is to remove their cause, as the following example illustrates:

Take an entirely **2-D** flow. Using the equations for linear perturbations in a parallel flow one finds (Taylor, 1915).

$$\frac{D\bar{u}}{Dt} = \sigma'' \frac{d}{dt} \left[\frac{1}{2} (\bar{\ell}')^2 \right] \quad (10.1)$$

where $\bar{\ell}' = y - \eta$ and η is the **Lagrangian** co-ordinate in the **y**-direction, with "meaning the conditional average over the coherent structure. Then the following is a sketch of the subsequent behaviour: The instability mechanism thus tends to remove its own cause but produces another mechanism. Momentum is drawn from the mean field through Reynolds stresses at both small and large scales. The flow is primarily controlled by continuity at the later stages as a new instability is created. The basic ingredients in the model are summarized in the section below.

Figure 10.1: Flat-eddy model (conceptual).

Figure 10.2: Acceleration of the mean flow due to Reynolds stresses produced by two-dimensional disturbances

10.1 Coherent structure regeneration model

The model is intended for flows which are stable in the mean, **e.g.**, a boundary layer; it needs to be modified for flows that are unstable in the mean. The different phases with different dominating mechanisms may be listed as follows:

Phase	Description	Mechanism
1	Local instability	Linear and non-linear Stress terms, especially pressure, cannot be omitted
2	Momentum transfer due to instability tends to restore local stability, and initiates a new coherent structure (i.e. new large scale motion on scale of original patch).	Non-linear but primarily inviscid. Time scale given by inverse of local shear $(dU/dy)^{-1}$
3	Interaction with mean flow gives rise to new instability.	Linear and non-linear with typical time scale given by time it takes large scale to move its own length at some typical convection velocity, L_1/U_c
4	Maintenance of mean flow	Through Reynolds stresses in balance with viscous stresses

This is the overall qualitative **picture**. The questions to be asked are of the importance of the various phases. For example, there may be the effect of viscosity on the local instability. Then the evolution of that instability is slow. Now, observation tells us that this is not so, so we would look for the **inviscid** type of instability. Obviously, there must be a cut off somewhere, **i.e.**, the smallest scale that can go unstable on a viscous scale. An estimate of this will be presented in the next section.

10.2 Effects of viscosity on coherent structures

In general the effects of viscosity become important at large times after the initiation of the structure. The stable **inviscid** flow produces $v -$ components which decay, but leave

long-lived 'scars' by the u - and w - components in the flow, which only decay on a viscous time scale. A disturbance with $v = 0$ is a passive one which does not interact with the mean flow except through stretching of the mean shear. The equation governing the **u -component** for a passive disturbance is

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = \nu \nabla^2 u \quad (10.2)$$

Note that the pressure is zero for $v = 0$ (we ignore the effects of the nonlinear terms). If the viscous term is negligible we have

$$u = f(\xi; y; z) \quad (10.3)$$

where $\xi = x - Ut$ is the Lagrangian z -coordinate (in the small-disturbance approximation). The coordinates y and z enter only as parameters. In the absence of viscosity the shear rate for this disturbance thus varies as

$$\frac{\partial u}{\partial y} = f_y - U t f_\xi \quad (10.4)$$

For long times the second term dominates. Now consider the full equation, again with $v = 0$. We introduce the new co-ordinates $\xi = x - \bar{U}t, \tau = t$. Then

$$\frac{\partial u}{\partial t} = \nu [-(u')^2 u_{\xi\xi} - 2\bar{U}' \tau u_{\xi y} + u_{yy} + u_{zz}] \quad (10.5)$$

. The first term will dominate for long times, thus

$$\frac{\partial u}{\partial \tau} \approx \nu (\bar{U}')^2 \frac{\partial^2 u}{\partial \xi^2} \quad (10.6)$$

which may be cast in form of a diffusion equation by introducing the new time

$$T = \frac{1}{3} \nu (\bar{U}')^2 \tau^3 \quad (10.7)$$

So, rescaling with this scale gives

$$\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial \xi^2} \quad (10.8)$$

Thus, the long time scale evolution is stopped on the time scale T . It is possible to work out the smallest value of the shear layer thickness possible due to this viscous cut-off. For example for $L^+ \approx 100 - 150, \delta^+ \approx 5$ The competition of the scales takes place through the mean shear and allows separation of scales also. Note that these are deductions from a simplified model and that it may explain qualitatively the behaviour of the fluctuating flow although no mean flow predictions could be made from it.

Figure 10.3: Competition between viscous dissipation and vortex stretching in a shear layer

10.3 Results from numerical simulations.

Numerical simulations must incorporate the effects of small scales or else the momentum transfer from the mean flow to large scales is lost. This is difficult because the vertical scale is about **5** in wall units, whereas the streamwise is about **100** and the **spanwise** may be **10**.

The best simulations to date are those of the NASA AMES group who did a full **Navier-Stokes** solution on a CRAY XMP using a **128*128*128** spectral method. These are nice simulations and provide interesting results, but are obviously expensive **as** they take many hours of Cray time. Their latest numerical simulation is that of a curved channel flow in three dimensions. Note that there is not much difference between the convex and the concave wall results.

The streaky nature diminishes away from the wall **as** the shear drops off quickly. It is possible to obtain the characteristics further away from the wall by looking at the vorticity. The shear stretches and rotates the fluid elements, and vorticity is obviously strong where the stretching is strong. NASA have looked at this with their Large Eddy Simulations (LES), which models the smallest scale structures in the flow by an eddy viscosity model. Moin and Kim (**1985**) examined the inclination angles of the vortex lines and found them to be near zero close to the wall but that there is a shift to a maximum of **45** degrees and a minimum of **-45** degrees away from the wall, where stretching by the mean shear is operative

Figure 10.4: Curved channel flow mean velocity distributions in local wall coordinates.- concave wall, $\diamond$ plane channel data of Eckelmann(1974), ..., $U^+ = 2.65 \log y^+ + 5$. (From Moser and Moin, 1984)

Figure 10.5: Turbulence intensities in local wall coordinates, --- concave side, — convex side, $\circ$ plane channel data from Kreplin and Eckelmann (1979). (a) $\bar{u}^2^{1/2}$; (b) $\bar{v}^2^{1/2}$; (c) $\bar{w}^2^{1/2}$ (from Moser and Moin, 1984)

Figure 10.6: Contours of streamwise velocity u' in the (θ, z) -plane, (a) near the concave wall, $y^+ = 6.14$, (b) near the convex wall, $y^+ = 5.20$ (from Moser and Moin, 1984)

Figure 10.7: Distribution of the inclination angle of the projection of the vorticity vector in (x, y) -planes; data weighted with the magnitude of the projected vorticity. (s) $y/\delta = 0.006(y^+ = 3.85)$; (b) 0.049; (c) 0.142; (d) 0.193; (e) 0.498; (f) 0.769; (g) 0.922; (h) 1.00 (from Molin and Kim, 1985)

10.4 Still unanswered questions

1. Intermittency. Why is it so strong?

Why is it generated in one case and not another? (**e.g.** Pierrehumbert - all flows having flow in two directions are unstable to small scales, whereas this is not necessarily true in **3-D**)

2. Restoration of stability by small scales.

Fully or only partially restored?

3. Role of pressure.

The pressure is ignored in the flat eddy model. How is it to be included, especially in the **spanwise** gradient?

4. The instability of a truly 3-D flow.

What does this mean and how do we cope?

5. Dynamics of 'sausage-like' structures.

Cross flow development or **streamwise** also?

6. Relation between momentum and energy.

This needs to be better understood. Energy dissipation by small scales needs to be examined, along with the idea that momentum transfer at small scales is inviscid. Is there some coupling?

Of course there are many other questions to be answered, many other courses to be followed and many other blind alleys to be eliminated. Such is the fascination of the subject.

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Notes submitted by **Nic Brummell**

ABSTRACTS OF SEMINARS

MERGER, BINDING, **AXISYMMETRIZATION** AND OTHER BASIC PROCESSES IN 2D INCOMPRESSIBLE FLOWS, (APPLICATIONS OF **PSEUDOSPECTRAL** AND CONTOUR **DYNAMICAL DC** ALGORITHMS)

Norman J. Zabusky

In this talk we emphasize the role of judiciously selected numerical simulations in providing insight into essential processes arising in two dimensional vortex dynamics. We compare results from various models and numerical algorithms and indicate that: There is no algorithm for all seasons. In fact insights that lead to analytical understanding arise in parameter regions where different algorithms "overlap" in their ability to model diverse processes.

The essential problems in two-dimensional hydrodynamics were not posed until computer simulations showed: filamentation of noncircular distributions of vorticity; merger of like-signed vorticity; binding of opposite-signed vorticity; and entrainment in a host vortex of small regions of irrotational fluid or opposite-signed vorticity. Three key questions in evolving two dimensional flows are:

- (1) What are the mechanisms by which smooth vorticity distributions "condense" into near-circular regions of vorticity (that is the mechanism of **axisymmetrization**) and vorticity gradients intensify?
- (2) What are the mechanisms by which two like-signed regions of vorticity merge or the mechanisms by which two opposite-signed regions of vorticity bind or entrain?
- (3) How do the filaments (small scales) which arise in **axisymmetrization**, merger and binding affect the long time evolution of the large-scale structures?

Axisymmetrization, merger and binding may be considered fundamental interactions in two-dimensional turbulent flow. In spectral jargon, merger corresponds to an "upward" energy cascade and filamentation accounts for the "downward" enstrophy cascade. The filamentation of vorticity during **axisymmetrization**, merger and binding is best understood by considering the local corotating or cotranslating streamfunction.

The talk covers the following subjects: A tentative definition of "coherent vortex structures"--a state of vorticity "weakly" perturbed from a stationary state--as illustrated with figures from soap film experiments by Couder et al, pseudospectral and contour **dynamical (CD)** simulations. The pseudospectral simulations include initial value problems with power law energy spectra with random phases (J. **McWilliams**, 1984); single isolated elliptical region and "axisymmetrization" (Melander, **McWilliams** and Zabusky 1985); and symmetric and asymmetric

merger of pairs of like signed vortices (Melander, Zabusky and **McWilliams**). Next, CD is reviewed, including: concepts and diverse applications; rotating and translating stationary states; merger; binding and scattering; and curvature controlled placement of nodes on contours and accuracy. Recent developments included the: automatic cutting-rejoining and elimination of contours or "contour surgery" by D. Dritschel; axisymmetric or coaxial **(r,z)** CD (Shariff **et al**; **Pozrikidis**) and acoustic signature of axisymmetric CD in a colliding ring configuration (Shariff, Leonard and Zabusky). Future directions include: characterization of nature of the dissipation of various contour surgery algorithms; application to multilevel equivalent barotropic flows where potential vorticity is conserved; further development of the modulated vortex method (Zabusky and **McWilliams**)---an invariant core vortex method for the **β -plane**.

For a similar review with references see Melander, **Overman** and Zabusky, "Computational Vortex Dynamics in Two and Three Dimensions". To be published in Applied Numerical Mathematics in 1986.

HORIZONTAL ENTRAINMENT AND DETRAINMENT IN **LARGE** SCALE EDDIES

Melvin E. Stern

The dynamics of a compact barotropic eddy is studied assuming piecewise uniform vorticity in a central core region, in an annular region, and in the irrotational exterior water mass. This circular vortex is known to be unstable when the width of the annular region is sufficiently small, and the evolution of the (most unstable) $n = 2$ azimuthal mode has been obtained using a smoothed version of the vorticity field in a pseudo-spectral numerical calculation. Small perturbations cause the circularly symmetric vortex to split into two oppositely propagating dipoles, and our nonlinear calculations using the piecewise uniform model agree with this. We then investigate the finite amplitude instability of the **vortex** in the linearly stable regime. Although the temporal evolution has an approximate monopolar structure for the stream-function, the vorticity pattern reveals the entrainment of thin filaments of the (irrotational) exterior water mass into the core of the eddy.

An initial disturbance in the $n = 1$ in the stable regime causes "self propagation" of the vortex. For finite amplitudes "wavebreaking" and intrusions occur at the center boundary. The $n = 1$ mode can be generated on a circularly symmetric vortex by the action of an external velocity field, and the finite amplitude evolution for this case also displays **entrainment/detrainment** features on outer boundary of the eddy. The relevance of these features to the decay of **warm/cold** core eddies, and to the mixing with the surrounding water mass, is suggested.

ISOLATED EDDY MODELS FOR GEOPHYSICAL FLOWS

Glenn Flierl

Geophysical fluid flows often appear to be dominated by a strong, but localized **vortical** structure which lasts for many circulation times even when relatively turbulent flows are impinging upon it. A few general statements concerning the existence and nature of nonlinear isolated eddy structures can be made, although it is not yet clear how well the simple solutions known can model such apparently isolated features such as the Red Spot of Jupiter, Gulf Stream rings, or "blocking" events in the atmosphere. Local models for strong geophysical flows fall into three basic classes: the weak amplitude, large aspect ratio **KdV** solitons (Malanotte-Rizzoli and Hendershott 1980); the large amplitude symmetric analytic solutions; and the "modons", which have a region of anomalous potential vorticity.

Far-field structure

We can find general conditions for the existence of isolated **quasi-geostrophic** eddies by considering motions imbedded in a zonal flow

$$\psi = - \int^y \bar{u}(y', z) dy' + \left(\frac{N}{\rho f_0} \right)^{1/2} \phi(x-ct, y, \bar{z}, t) \quad \bar{z} = \int^z \frac{N}{f_0} dz$$

The **stretched** vertical coordinate and the factor multiplying ϕ transform

the $\nabla^2 + \frac{1}{f} \frac{\partial}{\partial z} \left(\frac{f^2}{N^2} \frac{\partial}{\partial z} \right)$ operator into a three-dimensional Laplacian

$\nabla^2 = \nabla^2 + \frac{\partial^2}{\partial t^2}$. A translating reference frame, moving with speed c has also been introduced. With these modifications the QG equations become

$$\frac{\partial}{\partial t} \nabla^2 \phi + J(\bar{\psi} + cy, (\nabla^2 - v^2) \phi) + \left(\frac{N}{\bar{\rho}_0}\right)^{1/2} J(\phi, \nabla^2 \phi) = 0 \quad (1)$$

The term

$$v^2 = \left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \bar{\rho} \frac{\partial^2 \bar{u}}{\partial z^2} \right) / (c - \bar{u}) - \left(\frac{\bar{\rho} \bar{f}_0}{N} \right)^{1/2} \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \bar{\rho} \frac{\partial^2}{\partial z^2} \left(\frac{N}{\bar{\rho}_0} \right)^{1/2} \quad (2)$$

is a generalization of the "potential well" of **Malguzzi** and

Malanotte-Rizzoli (1984); its relationship to wave propagation was first pointed out by Charney and Drazin (1961). Boundary condition equations can be transformed similarly.

In the far field, the motion associated with the feature is weak and (2) can be linearized. If we presume that the disturbance propagates without change in shape $\frac{\partial}{\partial t} = 0$, we have

$$(\nabla^2 - v^2) \phi = 0 \quad (3)$$

The search for QG isolated solutions, then, involves examining (3) and the boundary conditions to find ϕ 's which decay at large distances; nonlinear terms will be introduced later to make the solution well-behaved near the origin.

For solutions which are decaying in x, y , and z it is necessary that f^2 be positive, or

$$\left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \bar{\rho} \frac{\partial^2 \bar{u}}{\partial z^2} \right) / (c - \bar{u}) > \left(\frac{\bar{\rho} \bar{f}_0}{N} \right)^{1/2} \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \bar{\rho} \frac{\partial^2}{\partial z^2} \left(\frac{N}{\bar{\rho}_0} \right)^{1/2}$$

In the case of an isothermal atmosphere ($N = \text{const}$, $\bar{\rho} = \rho_0 \exp(-z/H)$),

the RHS simplifies to $-f_0^2 / 4 N^2 H^2$; for an exponentially stratified ocean ($\bar{\rho} = \text{const}$, $N = N_0 \exp(z/h)$) it becomes $-3 f_0^2 / 4 N^2 h^2$

The equation above serves to restrict the possible phase speeds of isolated disturbances. As an example, consider an isothermal atmosphere with no zonal flows; the isolated, steady-translating disturbances must have $\beta/c > -.25/R_d^2$ so that $c > 0$ or $c < -4BR_d^2$, where the deformation radius is $R_d = NH/f_0$. Like KdV solitons, the isolated disturbances have phase speeds disjoint from those of wave-like solutions; this follows from the imaginary x wavenumber of the linearized part of the disturbance. Note that "propagation" in the vertical or horizontal could be accepted in the presence of boundaries which reflect the energy back to form a normal mode; in this case, trapping in the other direction is more likely.

Point Vortex Models

We can find a large set of isolated solutions which are the singular analogues of modons by examining point vortex solutions to (2) in the case when $\nabla^2 \psi$ is constant. When the potential vorticity of the background state is uniform, we can use standard point vortex theory. The conserved quantity, potential vorticity, is represented as a constant plus a set of delta functions with coefficients for the strength of each vortex. The **inviscid** advection equation for potential vorticity then simplifies because the constant term does not contribute and the singularities translate with the flow. Given the positions of all the sources, the flow at each position can be calculated from the Green's function and the evolution of the flow is found by solving a set of **ODE's**.

For geophysical flows, demanding that $\nabla^2 \psi$ be **zero** is highly restrictive, since it eliminates Rossby wave processes, both in the context originally thought of by Rossby and in the more general sense where waves in shear flows are considered as Rossby waves supported by

the gradient of mean potential vorticity. But we can examine steadily translating states corresponding to a moving set of vortices. These satisfy

$$(\nabla^2 - \nu^2) \phi = \sum_i s_i \delta(x - x_i) \delta(y - y_i) \delta(z - z_i) \quad (4)$$

In addition, however, we must impose a highly restrictive condition that

$$\dot{x}_i = 0 \Rightarrow c = \bar{u} - \left(\frac{N}{f_0}\right) \phi_y ; \dot{y}_i = 0 \Rightarrow \left(\frac{N}{f_0}\right) \phi_x = 0 \quad (5)$$

at each point where a vortex resides.

Monopolar solutions exist only when the mean PV gradients are zero or the vortex resides at a critical layer where $\bar{q}_y = 0$. As an example, consider a model for Jupiter's Red Spot like that of Ingersoll and Cuong (1981). The mean flow is taken to be $\bar{u}(y) = \bar{u}_0 \sin ky$. To make ν^2 constant (here we assume constant N and f), select $c = B/k^2$ so that $\nu^2 = -k^2$. We shall take a normal mode structure in z [so that the vorticity singularity is actually of the

form $\delta(x) \delta(y - y_1) F_1(z)$] to find

$$\phi = -\frac{S}{2\pi} K_0 \left(\left[(\lambda_1^2 - k^2) (x^2 + (y - y_1)^2) \right]^{1/2} \right) F_1(z)$$

with the vortex residing at the point $\sin ky_1 = -B/\bar{u}_0 k^2$.

Dipole solutions, with vortices of strength s and $-s$ at $y = a/2$ and $-a/2$, respectively, exist for barotropic flow when

$$c = \frac{S}{2\pi} \sqrt{\frac{B}{c}} K_1 \left(a \sqrt{\frac{B}{c}} \right)$$

The speed is disjoint from the range of linear wave speeds, depends on the amplitude and the vortices must be sufficiently strong ($s > 1.7 \sqrt{B} a^2$) for a steady solution to exist at all.

In the stratified case, we find dipolar structures related to the "hetons" of Hogg and Stommel (1984). In Fig. 1 are shown various fields for a uniformly stratified, infinitely deep fluid with vortices of strength s at $(0, \pm \frac{a}{2}, \pm \frac{b}{2})$.

Integral Relationships

The fact that a single point vortex satisfies the equations of motion on the f -plane but has no equivalent on the beta-plane is not accidental. Rather, if we integrate the Boussinesq primitive equations, with a zonal flow $u(y,z)$ explicitly factored out from the eddy flow $\tilde{u}(x,y,z,t)$, $\tilde{w}(x,y,z,t)$, vertically between two material surfaces $z = s_0(x,y,t)$ and $z = s_1(x,y,t)$, we find

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int u + \frac{\partial}{\partial y} \int v &= 0 \\ \frac{\partial}{\partial t} \iint u - h \iint v - \beta \iint y \int v &= \iint (p_1 s_{1x} - p_0 s_{0x}) \\ \frac{\partial}{\partial t} \iint v + s_0 \iint u + \beta \iint y \int u &= \iint (p_1 s_{1y} - p_0 s_{0y}) \quad (6) \end{aligned}$$

Here, h is the anomaly in thickness and we also assume that the flow is isolated at all depths $s_1 \geq z \geq s_0$ so that boundary contributions to the integrals vanish (Killworth, 1986). If the boundaries are flat and the flow is isolated at all depths, we derive the "no net angular momentum theorem"

$$\int u = -\Psi_y \quad \int v = \Psi_x \quad \text{and} \quad \iint \Psi = 0$$

For partially isolated, steadily translating solutions, we find

$$\begin{aligned} \int u &= ch - \Psi_y \quad \int v = \Psi_x \\ 0 &= \iint p_1 s_{1x} - p_0 s_{0x} \quad c \iint fh + \beta \iint \Psi = \iint p_1 s_{1y} - p_0 s_{0y} \end{aligned}$$

The results of Nof (1981) are obtained by postulating that the integrals involving $p_0 s_{0y}$ and $p_1 s_{1y}$ vanish; an expression for the translational speed can thereby be derived:

$$c = -\beta \iint \Psi / \iint fh$$

In applying this theory to the oceanic data, we find a number of difficulties: the asymptotic structure is not well defined in either the

density or velocity field and there is no clear argument for a particular isopycnal to be the lower boundary. Indeed, the deep fluid cannot be motionless if it is finite in depth.

Exact Solutions

The **modon** solutions, following the work of Larichev and **Reznik** (1976), build upon an isolated exterior field connected smoothly to an interior region with a different relationship between the vorticity and the streamfunction. The region of anomalous potential vorticity is **assumed** to coincide with the trapped region; for analytic simplicity, the boundary is taken to be circular; and there is a linear relationship between Ψ and PV in the interior. Thus linear exterior and interior equations for the total streamfunction, including the cy term, are written:

$$\left(\nabla^2 + \frac{1}{f} \frac{\partial}{\partial z} \bar{\rho} \frac{\partial}{\partial z} \right) \Psi + \beta y = A + B \Psi \quad (7)$$

In the exterior region where $\Psi = \bar{\Psi} + cy$, we have $A = 0$, $B = \bar{q}_\Psi$ and we need $\bar{q}_\Psi$ to be constant (or at most a function of z). The exterior

solution is just $\Psi = \bar{\Psi} + cy + \left(\frac{N}{f \bar{\rho}} \right) \phi_e(r, \theta, z)$

with $\nabla^2 \phi_e = \nu^2 \phi_e$. In the interior, $A = A(z)$, $B = k^2$ and the

solution is $\Psi = a(z) \frac{\beta y}{k^2} + \left(\frac{N}{f \bar{\rho}} \right)^{1/2} \phi_i(r, \theta, z)$

where

$$\left(\frac{1}{f} \frac{\partial}{\partial z} \bar{\rho} \frac{\partial}{\partial z} + k^2 \right) a = A$$

and

$$\nabla^2 \phi_i = -k^2 \phi_i$$

The Helmholtz equations for the ϕ 's are deceptively simple; the difficulties come in determining the location of the free boundary

$$r = a(\theta, z)$$

where $\bar{z}(a(\theta, z), \theta, z) = \bar{z}(z)$

and $\bar{z}_r(a(\theta, z), \theta, z)$ is continuous.

Radiating Modons

There are two difficulties with the formalism presented previously which make application to geophysical situations difficult. First, the requirement that $\nabla^2 \psi > 0$ in the far field will rarely hold over the whole region of interest. Use of a normal mode structure in the vertical may be able to offset negative $\nabla^2 \psi$ regions (positive index of refraction zones), but it is in general more likely that the y, z modes will be "leaky", with some regions of negative $\nabla^2 \psi$. Secondly, the integral theorem requiring no net angular momentum is quite severe. Understanding of the coupling between nonlinear coherent structures and waves seems both an important and fruitful direction for research.

Some understanding of this coupling has been obtained by using a two layer model with the coherent structure in one layer and the waves confined to the other (Flierl, 1984). In the deep layer, radiating Rossby waves will be found far from the eddy. The interaction between the upper layer **modon** structure and the wave field drains energy out of the coherent feature. Meiss and Horton (1983) provide another approach to a radiating problem: they assume that $\nabla^2 \psi$ varies slowly becoming negative far from the dipole structure. The radiation occurs in the same horizontal strata as the **modon**. The problem is solved by expanding the

linear waves near the turning point and matching to the far field solution of the **modon**. They find that the energy loss is exponentially small unless the speed of the **modon** approaches the long linear wave limit. In contrast, Shutts (1983) and Farrell (priv. comm.) indicate that small scale eddies impinging upon a split flow may result in enhancement of the eddy structure; the relative importance of the radiative losses to such energy inputs is not yet understood.

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EXPERIMENTS ON INTERNAL WAVE CRITICAL LAYERS

Donald B. Altman

A series of laboratory experiments on internal wave critical layers is described. The experiments are performed in a tilting tank similar to Thorpe's (1968). When tilted, the tank produces an accelerating shear layer flowing over a fixed sinusoidal bottom, forcing internal waves. A critical layer occurs at the point in the shear layer where the mean flow vanishes with respect to the bottom.

In the case of two-layer stratification, a complicated phenomenon occurs, consisting of a slowly-growing "primary vortex" which remains phase locked to the topography, followed by a smaller, rapidly-growing "secondary vortex". The primary vortex occurs at a mean flow bulk Richardson number, Ri , of $0(0.5)$. The secondary vortex occurs at Ri $0(0.15)$. Growth of the primary vortex can be stopped by stopping the mean flow acceleration.

The initial stages of growth of the primary vortex are well-modelled by a linear, two-layer **inviscid** theory. If one assumes a single mode solution, then in a frame of reference moving with a weighted average velocity of the two layers, the equation for the time evolution of the interface is

$$\ddot{\xi} + \omega_n^2(F_1^2)\xi = \sigma C [i\dot{F}_1 - F_1^2] \exp\left\{D \int_0^{\tau} F_1 d\tau\right\}$$

Here ξ is the interface amplitude, C , D , and ω_n^2 are real functions of the flow geometry, F_1 is a measure of the mean flow velocity of the lower layer, σ the steepness of the forcing, $(\cdot)$ indicates functional

dependence and $\{\}$ indicates $\frac{d}{dt}$.

The phase shift of the interface with respect to the forcing, caused by non-zero $\hat{F}_1$, causes a closed-streamline region on the leading face of the interface, corresponding to the observed primary vortex. This recirculating region is observed at apparently small mean flow accelerations ($\hat{F}_1 \leq 0.05$) due to the quadratic dependence of the phase shift on $\hat{F}_1$.

At F , greater than that needed for **dynamical** instability of the linear model ($0.3 \leq F \leq 41.2$, depending on ω_1) observed growth rates are lower than those predicted by the model, presumably due to non-linear effects. Observed phase is well predicted in this non-linear region.

The secondary vortex is shown to be a shear instability occurring in a region of locally reduced Ri which is induced by the primary vortex.

This mechanism of convective instability due to forcing of a weakly transient flow is of interest to oceanographers since it provides another possible source of mixing in the range $0.25 < Ri < 1.0$.

At values of ω_1 sufficiently low, the experiment is dominated by Kelvin-Helmholtz instability. For sufficiently large shear layer thickness, behavior similar to the experiments described below was obtained. The exact boundaries of the regimes in which each behavior dominates are not yet fully defined.

Experiments in accelerating linearly stratified flows over topography, similar to **Thorpe's (1981)**, were also described. An attempt was made to measure internal wave-induced contributions to the mean flow using particle streak photography. Although detailed measurement of

wavelength averaged Reynolds stresses failed due to undersampling, the leading edge of internal wave energy, as predicted by WKBJ theory, was observed.

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SIMPLE MODELS OF THE GENERAL CIRCULATION OF THE OCEAN

Rick Salmon

The theory of the wind-driven ocean circulation simplifies considerably if inertia is neglected, and if a line or drag friction replaces the conventional Laplacian viscosity (in all directions). The resulting equations are useful in two ways. First, if the density equation is also linearized, then the simplified equations admit boundary layer solutions which are much easier to analyze than the corresponding solutions of the conventional linear equations.

Second, if the density equation is retained in its fully nonlinear form, then it can be used to step the density forward in time. Then the remaining equations determine the new velocity by a linear elliptic equation for the pressure. The solution contains nonhydrostatic coastal upwelling layers that depend on the vertical friction. These upwelling layers exist wherever the Ekman transport or the geostrophic "thermal wind" impinges on a coastline. Numerical experiments suggest that cross-isopycnal flow within these upwelling layers is an important component of the large-scale circulation.

CHAOS AND TURBULENCE

Alan C. **Newell**

I want to thank the organizers of the 1986 **summer** program on Geophysical Fluid Dynamics on "Turbulent Shear **Flows**" for their generous hospitality during **my** week long stay. What follows is an essay on possible connections between shear flow **turbulence** and the ideas of modern dynamics in which I have **indulged** in much speculation. Nevertheless, I hope the reader finds **some** of the ideas stimulating.

The realization that finite dimensional dissipative systems can have attractors ("**strange attractors**") on which the motion is everywhere unstable has brought a new perspective to nonlinear dynamics. Motion on the attractor depends sensitively on initial conditions (nearby orbits diverge locally at **an** exponential rate on the average) and this sensitive behavior leads to **an** apparently stochastic time signal with a broadband power spectrum. The finite amount of information contained in the finite accuracy specification of the initial state is eroded **by** the flow and once sufficient time has elapsed to uncover the **unknown** part of the initial data, the state of the **dynamical** system is unpredictable. Although **Poincaré** was fully aware that **Hamiltonian** systems could exhibit such behavior, it was not widely appreciated until the early seventies, following the pioneering work of Lorenz [1] on weather prediction models and the bold and imaginative ideas of **Ruelle** and Takens [2], that dissipative systems could have unstable

asymptotic behavior. We will refer to this long time sensitive behavior of a finite order system of ordinary differential equations and associated maps as chaos. Its **signatures** are a broadband power spectrum and a fractal dimension.

It is of course tempting to speculate that turbulence can also be explained on the grounds that the apparent stochastic time dependence of a fluid is the **manifestation** of sensitive dependence on a relatively low dimensional strange attractor. Appealing as the idea may sound, one must realize that a fully developed turbulent flow has a complicated spatial as well as temporal character and that any theory which purports to explain turbulence in terms of a finite set of ordinary differential or difference equations must come to grips with this fact. Chaos has broadband temporal behavior; fully developed turbulence has broadband spatial and temporal behavior. In Table 1, we list some properties of and comparisons between the two. First, however, let us make some operational definitions. Chaos is the unpredictable **behaviour** (broadband power **spectrum** and at least one positive Lyapunov exponent) of systems of ordinary differential equations and associated maps. Turbulence is 'the unpredictable behavior of partial differential equations. It may have an ordered or disordered spatial structure. In situations where the effective **stress** Parameter (e.g., Reynolds **number**) is moderate and there are severe geometrical constraints, there is ample evidence (e.g., the experiments of Libchaber [3] on convective flow in small aspect ratio systems) that the turbulence is **purely** temporal, that the spatial structure is ordered and that the long time

that the transport is carried by the coherent eddies does not imply that no transport is carried by small scales.

In summary, then, the first idea worth exploring involves a decomposition of the dynamics into a few coherent modes and many, statistically governed, fluctuating modes. The corresponding Lyapunov spectrum λ_j ; $\lambda_1 > \lambda_2 > \lambda_3 \dots$ (which is the natural analogue to the linear stability spectrum at bifurcation points) would consist of a few order one positive values and many small, positive and negative values. The reader will recall that the degree of freedom of a system can be associated with the number of Lyapunov exponents whose sum is positive. It gives the maximum growing subvolume in the tangent space to the **system's** trajectory in phase space. Therefore the dimension of the coherent states could be N where $\sum_{j=1}^N \lambda_j > \text{order one}$, the "inertial spectrum" would be characterized by the dimension n , where $\sum_{j=1}^n \lambda_j > 0$, $n \gg N$, and the heavily damped viscous **subrange** would correspond to the larger negative exponents. (Note: It would be worthwhile determining **if** the Lyapunov exponent **spectrum** of turbulent flows has this structure; unfortunately, the techniques for determining the spectrum from a time signal are not sufficiently advanced in practice for this calculation to be made at this time.)

It is clear that if the picture I have described is going to be useful, one will have to have some way of determining the "coherent structures". To give the reader some idea about how difficult this will be in general, let me remind him that even in those situations in which the signal analysis tells us that the attractor is low dimensional, we have no way presently of

identifying which spatial structures are active in the dynamics. Modern dynamicists can tell you all about the topology and information content of the attractor, but cannot tell you the heat **flux** across a convecting cell! In some cases, for example, when the system is a perturbation of one of the soliton equations, the choice of natural basis is fairly clear. What we need to develop, through a combination of physical insight and **signal** analysis, is a way of identifying the spatial **structures** of the components of the appropriate basis in which to project the field variable. The goal would be to find a basis for the coherent structures which would require the least number of parameters. (One wants to avoid a situation in which one describes the dynamics of a four parameter soliton by its Fourier decomposition which would require many more parameters.)

In shear flows, there has been some new evidence (see the abstract of Wygnanski, this **volume**) suggesting a means of getting at what may be the coherent structures. The basic idea is that the mean **turbulent** profile in mixing layers and wakes is always unstable and this leads to the continuous formation of modes whose shapes may be calculated from a linear stability analysis of the mean profile. This idea gains further credence when one looks at the forced situation in which the turbulent mixing layer, jet or wake is forced at a fixed location with a frequency which lies under the **gain** curve of the linearly unstable mean profile. In this way, only one of the modes under the gain curve is excited. What is remarkable is that this mode not only grows in accordance with the linear stability theory of the mean profile but its nonlinear development appears to be well described by a weakly nonlinear **theory** up to that point where its amplitude is **saturated** and

dynamics of the **potentially** infinite dimensional system is governed by a low dimensional attractor.

There are also many other situations involving partial differential equations, like the two dimensional Navier-Stokes equations, the one dimensional complex Ginzburg-Landau (CGL) or Kuramoto-Sivashinsky (KS) equations where the turbulence is purely temporal and the lack of predictability is associated with the loss of phase information concerning the exact location of well defined lumps (e.g., solitary waves, vortices) which are the principal components of the flow field. In the CGL and KS situations [4, 51, the dimension scales with the length of the system and is essentially given by the number of unstable modes of the linearly unstable spatially uniform profile. Whereas the time signal exhibits broadband behavior, the spatial structures are coherent and readily identified. We will refer to such turbulence as phase or wimpy turbulence.

On the other hand, the challenge of turbulent shear flows, and the challenge that faces this audience, lies in the fact that the flow field exhibits both temporal and spatial disorder. This state we can call macho turbulence. Indeed, assuming that global solutions to the Navier-Stokes equations exist, it is highly unlikely that for large Reynolds numbers any attractor will have low dimension. There are theorems (Ruelle [6]; Foias, Manley, Teman and Treve [7]; Hymn and Nicolaenko [5]) which place an upper bound on the dimension of attractors for partial differential equations in finite geometries. For the Navier-Stokes equations, this bound is equivalent to the intuitive notion of Landau who argued that it would be sufficient to resolve the flow field down to the viscous dissipation scale

of Kolmogorov $(\nu^3/\epsilon)^{1/4}$ (ν is the molecular viscosity, ϵ the dissipation rate). This would require $1^3(\nu^3\epsilon^{-1})^{-3}$ modes which is proportional to the nine-fourths power of the Reynolds number. The energy in smaller scales is assumed to be immediately dissipated. This means that for Reynolds numbers of between 10^3 and 10^4 , the range in which turbulence is first seen, one would require up to a billion modes to resolve the system. It is hardly likely that replacing the Navier-Stokes equations by a system of a billion ordinary differential equations for the Fourier mode amplitudes will lead to much simplification or to an increased understanding of turbulent processes.

Moreover, for the open geometries associated with shear flows, there are no known bounds on the attractor dimension at all. In fact, there is no guarantee that there is any set, fractal or otherwise, in the phase space of the system (which is also not well defined) which acts as an attractor. It is clear therefore, that if we want to understand the transport and drag properties of flows in pipes, of boundary layers around submarines, of the wake behind the Martha's Vineyard Perry, some new ideas will be needed.

One line of inquiry worth exploring is the idea that whereas the flow itself has many active degrees of freedom, its transport properties depend only on a few, which we might call the coherent structures. One might decompose such a flow field into a superposition (not necessarily a linear combination) of coherent eddies, a mean flow, and fluctuating eddies,

$$\vec{u}(x,t) = \vec{u}_M + \vec{u}_c + \vec{u}_f. \quad (1)$$

This decomposition is a **natural** extension of writing the field as a **sum** of mean and fluctuating eddies, a decomposition which leads to the Reynolds equations. The crucial difference here is that we add a low dimensional component corresponding to the **coherent eddies** which, of course, we hope will play a dominant role in the transport properties.

Such decompositions **naturally** arise in the study of situations which **are**, to a good approximation, described by equations with soliton properties. For **example**, in nonlinear optical cavities [8], the dynamics are well described by a perturbed nonlinear **Schrodinger** equation

$$2iu_t + u_{xx} + 2u^2u^* = F(u, x, t) \quad (2)$$

and for a large class of perturbations F , the long time dynamics is dominated by a superposition of solitons. In this case, the **natural** phase space would be the space of periodic solutions on some interval, the appropriate basis **in which** to study the motion would be the inverse scattering basis, consisting of a "mean", "solitons" and "radiation modes" which we would identify with u_M , u_C and u_F of (1). **Assuming** the effect of the perturbation F to be dissipative, we might seek to capture the low dimensional dynamics by looking at the time behavior of the soliton parameters (amplitude, **position** etc.) which, for the unperturbed flow, would either be constant or vary linearly with time. The soliton dynamics could lead to fixed point, limit cycle, quasiperiodic or strange attractor behavior. The effects of the 'radiation modes' would be small. In fact, their main role **may** be to act as the source of energy loss for the coherent

component and their influence might well be captured by allowing the coefficients in the system of ordinary differential equations for the soliton parameters to be stochastic quantities. The governing equations for the system would then be a set of **o.d.e.'s** with stochastic coefficients. Let us imagine that, giving the stochastic coefficients their mean values, the solution of these equations relaxes to an attractor whose dimension, Lyapunov exponent spectrum and probability **measure** can be found. The effect of the noisy coefficients will be minimal and, if the noise is small, even beneficial. In particular, it should not change the key properties of the attractor from which the important transport quantities like fluxes of mass, **momentum** etc. can be worked out.

Further, one might even be able to argue the nature of the statistics of the fluctuating modes. In the turbulent shear flow context, they may very well have certain universal properties. There they would consist of the inertial range eddies which, from usual lore, have universal behavior. One might surmise that the mechanism which cascades energy to these scales is the reoccurrence of local, inflexional instabilities. A word of warning might be inserted here. I do not intend to suggest that the coherent eddies and fluctuations can be divided on the basis of scale alone. Indeed the traditional **Fourrier** basis for decomposing **turbulent** quantities can lead to misleading conclusions. The governing equations are nonlinear and it may very well be that a coherent or soliton mode will contain many length scales, both large and small. In fact I will argue in the next few paragraphs that, in certain situations, this may be the case. So, to say

the broadening of the **momentum** thickness due to its Reynolds stresses ceases.

I want to stress that all the influence of the other fluctuations is captured by how, through the effect of their Reynolds stresses, they shape the mean profile. No strong, direct interaction seems to take place between the forced and fluctuating modes. The fact that this must be the case is also evidenced by the narrowing of the **turbulent** spectrum about the forced frequency. One might explain this with a quasilinear theory as follows. If there is no initial forcing at a given frequency, any fluctuations present in the flow at a given location x_0 will grow according to a rate given by the linear stability gain curve at that particular frequency. Therefore an initial, white noise **spectrum** will develop a power spectrum in x approximately proportional to the local gain curve. On the other hand, if one of the frequencies under the gain curve is strongly forced and quickly reaches a finite, perhaps its saturation, amplitude, then its presence will inhibit modes in that frequency neighborhood **from** growing, thus creating a depression in the background spectrum about a sharp spike at the forcing frequency. **As** an exercise the reader might look at the linear growth of an initial field $u(0,t) = f(t)$ associated with the equation

$$u_x - u_{tt} = u - u^2 u^* \quad (3)$$

in the two cases

$$(1) \quad f(t) = \varepsilon \delta(t) \quad \text{and}$$

$$(11) \quad f(t) = [(1 - \omega_0^2)^{1/2} + \varepsilon \delta(t)] e^{-i\omega_0 t}, \quad \omega_0^2 < \frac{1}{3}.$$

However, it is important to point out, that remarkable as it may be that in the initial stages a forced mode will behave according to weakly nonlinear theory, **eventually** it does grow to an amplitude at which its direct interaction with the turbulent fluctuations cannot be ignored. Nevertheless, the weakly nonlinear theory of the mean turbulent profile may be a good starting point and, at the very least, provides candidates for the shapes of the coherent structures.

I have previously mentioned that in open **flows** there is no guarantee the long time dynamics is governed by an attractor at all. Indeed, because the flow evolves in one direction, x say, the influence of the downstream state of the system on the upstream should be minimal. Therefore in a sense x is more like a time coordinate and one would really like to follow evolutions of **(y,z) structures** in **(x,t)** space. In this coordinate decomposition it is difficult to define a suitable phase space. Nevertheless, one might argue that low dimensional dynamics might still be applicable in a sense perhaps best made clear by a consideration of boundary layer dynamics. In that situation, it is **arguable** that the momentum deficit is carried from the wall to the outer flow by coherent plumes which have the following components:

- (i) A large scale, three dimensional, **Tollmien-Schlichting** (TS) wave with phase velocity c which, under the influence of its own vortex forces and the stretch of its downstream vorticity by the mean shear, has a finite angle to the vertical.
- (ii) A rapidly fluctuating, small scale, wavetrain which is the result of an inflexional instability of the mean profile, the inflexion being

created by the distortion of the mean flow by the TS wave, and which moves with a group velocity C_g which is the same as the phase speed of the TS wave. In this way, the **unstable** short scale wave moves with the conditions that create it and therefore can continue to draw on its source of energy.

The idea that the momentum deficit is carried by a series of random eruptions rather than being a continuous flux parallels the ideas of Howard about how heat is transported by convection at very large Rayleigh numbers. The picture of a phase locking of a long (TS) wave and a short (inflexional) wave is due to Landahl (see this volume). If the notion that the momentum deficit is carried by random (or semiperiodic in x) plumes is accurate, then the corresponding dynamical picture is that, over a certain range of the time and downstream spatial coordinate, the dynamics is well described by a "temporary" attractor whose (y,z) shape is given by that of the TS wave and its short scale partner but whose (x,t) evolution and slow modulation in the (y,z) directions is given by an envelope equation of Ginzburg-Landau type for $A(x_1,t)$ ($x_1 = z, x_2 = y, x_3 = z$), the envelope of the "short" wave, and $n(x_1,t)$ the "long" wave:

$$\frac{\partial A}{\partial t} + \underbrace{\left[\frac{\partial \omega}{\partial k_1} \frac{\partial A}{\partial x_1} \right]}_{\text{group velocity}} - \underbrace{\frac{1}{2} \left[\left(\frac{\partial^2 v}{\partial k_1 \partial k_j} + i \frac{\partial^2 \omega}{\partial k_1 \partial k_j} \right) \frac{\partial^2 A}{\partial x_1 \partial x_j} \right]}_{\text{diffusion}} = \underbrace{CA}_{\text{linear growth}} - \underbrace{(\beta_r + i\beta_i) A^2 A^*}_{\text{nonlinear self interaction}} - \underbrace{(\delta_r + i\delta_i) A}_{\text{interaction of long and short wave}} \quad (4)$$

$$\frac{\partial \mathbf{n}}{\partial t} + \vec{c} \cdot \nabla \mathbf{n} = \vec{a} \cdot \nabla \mathbf{AA}^*, \quad |\vec{c}| = c. \quad (5)$$

"ponderomotive" force

In (4), $\omega(\vec{k})$ is the dispersion relation of the phase locked inflexional wave and the projection of its group velocity in the direction of propagation of the TS wave should be $\vec{c}$. One might also have to add mean drift effects $(-\vec{k} \cdot \mathbf{U})$ to equation (4) with an equation for $\mathbf{U}$ in terms of spatial gradients of $\mathbf{AA}^*$). The coherent mode solutions, the "temporary" attractors, would be the solitary wave solutions of (4), (5). One could, in principle, then compute the momentum deficit carried by one of these waves.

Finally, in searching for "coherent structures" to play the role of "temporary" attractors, it is important not to forget the finite time singular solutions of the 3-D Euler equations (nobody knows they exist; but it is widely believed that they do). (Indeed the fast distorting TS wave discussed in the last paragraph would more accurately be described by a shape which develops finite time singularities.) Moreover, we know already in Langmuir turbulence that collapsing Langmuir wavepackets (described by the Zakharov equation, very close in structure to (4), (5)) are responsible for the energy cascade. We also know that defects, the singular solutions in the evolution of patterns in solid state and fluid convection situations, play an important role in the long time dynamics. It is not too far fetched, then, to suggest that singular solutions of the Euler equations may play the role of local and "temporary" attractors in turbulent shear flows.

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TABLE 1

CHAOS	TURBULENCE
Unpredictable behavior of $v = \vec{F}(\vec{v}, R), \quad v \in \mathbb{R}^m$	Unpredictable behavior of solutions of NLPDE's
Dissipative e.g., Lorenz equations Henon and logistic maps	e.g., Navier-Stokes Maxwell-Bloch Envelope Equations
One independent variable (time)	Space and time independent variables
Finite number of degrees of freedom	Potentially and infinite number of degrees of freedom
Broadband temporal power spectrum	Onset of turbulence
Exponential decay of autocorrelation function	Period doubling Ruelle-Takens Intermittency
Sensitive dependence strange attractors	Onset of spatial disorder, gateway to fully developed turbulence
Fractal dimension	Broadband spatial and temporal power spectra
Positive Lyapunov exponents	Exponential decay of time and space correlations
Universal (self similar) behavior,	High levels of fluctuation vorticity and rapid stretching of vortex filaments
Routes to chaos	Diffusivity and rapid mixing
Period doubling Ruelle-Takens	Dynamic similarity, self similar ranges of spectrum
Intermittent behavior	Statistical closure problem

INSTABILITIES IN CHANNEL FLOWS: THREE-DIMENSIONALITY, GEOMETRY, AND RESONANCE

Anthony T. Patera

The instability and transition process in plane Poiseuille flow between infinite parallel plates is often proposed as a simple, yet generic, scenario relevant to a much larger class of flows. In this talk we first describe the transition process in plane Poiseuille flow, and then consider instability phenomena in geometrically complex channels such as grooved channels and channels with periodic cylinder arrays. Of interest is **determining** to what extent general statements can be made about the transition process in channels of arbitrary shape.

Numerical investigations of plane Poiseuille flow have isolated the following facts: above a Reynolds number of $R = 5772$ the flow is linearly unstable; above a Reynolds number of $R = 2900$ the flow is unstable to finite-amplitude quasi-equilibria persist due to "memory" of equilibria at higher Reynolds numbers; the two-dimensional equilibria and quasi-equilibria are explosively (**convectively**) unstable to three-dimensional disturbances. These facts can be combined to form a reasonable picture of transition which agrees in many regards with available experimental evidence.

For channel flows which represent geometric perturbations of plane Poiseuille flow (**e.g.**, grooved channels, channels with periodic arrays of small cylinders), certain aspects of the straight-channel physics remain remarkably invariant: the instability away from the immediate vicinity of the geometric variation takes on the form of (channel) Tollmien-Schlichting waves; the dispersion relation for these waves is approximately the same as that for the straight channel; and the ultimate three-dimensional breakdown of these waves results from a secondary instability not dissimilar to that found in plane Poiseuille flow.

Despite these **similarities**, there are also certain key differences between the stability characteristics of simple and complex channels. In particular, in complex-geometry channels: the instability is typically of "**inviscid**" shear layer origin, and occurs at relatively low Reynolds number (**e.g.**, $R \leq 1000$); the primary bifurcation is typically supercritical; the linear modes are susceptible to energy input from periodic flow modulation, unlike the corresponding translation-invariant modes of plane Poiseuille flow. This last point concerning resonance is a good example of why simple geometry results can not be taken as generic.

The differences in stability behavior induced by geometric variation have numerous ramifications, practical as well as fundamental. For instance, the fact that grooved-channel modes are susceptible to oscillatory forcing suggests that importance of free-stream disturbances in the transition process in rough-walled systems. Similarly, the fact that laminar two-dimensional unsteady

flows can be maintained (passively or by oscillatory forcing) at low Reynolds numbers suggests the possibility of controlled laminar transport enhancement as an alternative to turbulent transport. Experimental and numerical confirmation of the validity of these conjectures is currently underway.

STABILITY BOUNDS ON TURBULENT POISEUILLE FLOW

Willem V. R. Malkus (in collaboration with Glenn R. Ierly)

For steady state turbulent flows with unique mean properties, we establish that the local mean velocity is linearly supercritical. The shear turbulence literature on this point is ambiguous. As an example, we reassess the stability of mean profiles in turbulent Poiseuille flow. The W. C. Reynolds and W. G. Tiederman (J.F.M. **#27**, 253, 1967) numerical study is used as a starting point. They had constructed a class of one-dimensional flows which included, within experimental error, the observed profile. Their numerical solutions of the resulting Orr-Sommerfeld problems led them to conclude that the observed mean flow was very stable and also that the first non-linear corrections were stabilizing. In the realized flow this latter conclusion appears incompatible with the former. Hence, we have sought a more complete set of velocity profiles which could exhibit linear instability, retaining the requirement that the observed velocity profile is included in the set. We have added two dynamically generated modifications of the mean. The first addition is a fluctuation in the curvature of the mean flow generated by a Reynolds stress whose form is determined by the neutrally stable Orr-Sommerfeld solution. We find that this reduces the stability of the observed flow by more than a factor of two. The second addition is the zero-average **downstream** wave associated with the above Reynolds stress. The three-dimensional linear instability of this modification reduces the stability of the observed flow by yet another factor of

five. Those wave amplitudes which just barely will assure instability of the observed flow are determined. The relation of these particular amplitudes to the limiting conditions admitted by the absolute stability criterion for the mean flow is found. These quantitative results from stability theory lie in the observationally determined Reynolds-Tiederman similarity scheme, and hence are insensitive to changes in Reynolds number. A further finding is that the observed mean flow is within fifty percent of that flow which maximizes the turbulent fluctuation-dissipation rate, subject to the absolute stability constraints.

FORMATION AND DESTRUCTION OF INVISCID LARGE EDDIES

Raymond T. Pierrehumbert

Organized two-dimensional eddies occur in a wide variety of turbulent flows, ranging in scale from laboratory shear layers to the Great Red Spot of Jupiter. We review mechanisms for the creation and destruction of large eddies.

First, we examine the dynamics of two-dimensional vorticity distributions with an eye to identifying vortex interactions that are compatible with the classical 2-D **energy/enstrophy** cascade. It is remarked that the enstrophy cascade implies that formation of large eddies must generally be accompanied by generation of fine-scale vorticity structures, clumped into large, diffuse clouds. Several specific examples are discussed, based on contour dynamics simulations by D. Dritschel. Specifically, we discuss: (A) reversible transition between a 2-vortex state and an ellipse, (B) detrainment of fluid from an unstable ellipse, (C) irreversible collapse of a 3-vortex state into a disk, and (D) collapse of an annular vortex into a triangular V-state.

Next, we take up the question of the stability of two-dimensional vortices to three-dimensional perturbations. It is shown that 2-D instabilities have a shortwave cutoff in perturbation length scale, in contrast, 3-D instabilities have no such restriction, though the growth rate is nevertheless bounded above. A number of features of 3-D instability are discussed in terms of the stability problem for shear layer vortices first considered by Pierrehumbert and **Widnall** (JFM 1982).

It is emphasized that (A) the growth rate asymptotes to a positive constant at short waves, (B) only slight deviations from parallel flow are needed to produce considerable shortwave growth rates, and (C) for short axial wavelengths, the eigenmode becomes self-similar, reducing in scale in proportion to wavelength, without change of shape.

A shortwave asymptotic theory explaining these features is presented. The theory **shows** that the almost-circular limit is a regular perturbation, while the almost-parallel limit is a singular perturbation in which the growth rate is of order unity, owing to the continuum spectrum of plane Couette flow. It is shown that such shortwave instabilities are universal and occur for any **inviscid** flow with a non-circular center of rotation. The growth rate depends only on the eccentricity and vorticity at the center of rotation. It is argued that the properties of the secondary instability of wave-bounded shear flow, discussed in **Orszag** and **Patera** (JFM 1983) are consistent with the theory.

From these results we infer that energetic large eddies exert a profound influence on 3-D turbulence. They provide a means whereby energy can be transferred directly to the dissipation range, without the need for an intervening cascade.

VORTEX DYNAMICS - COMPUTER SIMULATIONS AND MATHEMATICAL MODELS

Mogens V. Melander

We discuss the most fundamental vortex interactions in 2D nearly **inviscid** fluid. These interactions include the **axisymmetrizations** of a single isolated monopole, merger of two like signed vortices, formation of dipolar pairs (binding) and **collisions/scattering** of dipoles with monopoles or dipoles. We discuss these fundamental vortex interactions in the context of the creation or growth of coherent vortex structures observed in a numerical simulation of **McWilliams (1984)**.

First we show that a spatially smooth vorticity distribution relaxes inviscidly towards axisymmetry on a circulation timescale as the result of filament generation. Heuristically, we derive a simple geometric formula relating the rate of change of the aspect ratio of a particular vorticity contour to its orientation relative to the streamlines (where the orientation is defined through second-order moments). Computational evidence validates the formula. By considering streamlines in a carotating frame and applying the new formula, we obtain a detailed kinematic understanding of the vortex's decay to its final state through a primary and a secondary breaking. The circulation transported into the filaments although a small fraction of the total breaks the symmetry and is the chief cause of **axisymmetrization**.

Two like signed vorticity regions can pair or merge into one vortex. This phenomenon occurs if the original two vortices are sufficiently close together, that is if the distance between the vorticity centroids is smaller than a certain critical merger distance, which depends on the initial shape of the vortices. We present the first description of the cause and mechanism behind the merger process. Two complementary methods are used to investigate the merger of identical vorticity regions. One method is based on a low order model of the **2D-Euler** equations. This model is a **Hamiltonian** system of ordinary differential equations for the evolution of the centroid position, aspect ratio and orientation of each region. By imposing **symmetry** this system is made integrable and we obtain a necessary and sufficient condition for merger. This condition involves only the initial conditions and the conserved quantities. The other method is a high resolution pseudo spectral algorithm governing the flow of a slightly viscous fluid in a box with periodic boundary conditions. When the results obtained by both methods are put together we obtain a detailed kinematic insight into the merger process.

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NORMAL FORMS FOR PARTIAL DIFFERENTIAL EQUATIONS NEAR TO THE ONSET OF INSTABILITIES

E. A. Spiegel

When a single mode goes unstable in a system — say a layer of fluid heated from below — we can describe the evolution of the system as the instability develops by an amplitude equation provided that the instability is slight. Even if several instabilities occur, amplitude equations governing the temporal evolution may be formulated for the nearly marginal modes, providing all the other modes are quite stable. The order of the equation is equal to the number of slow modes.

It is possible also to ask and answer a simpler question about all this, one that is less explicit. We can seek the general form of the amplitude equation without requiring the coefficients of the nonlinear terms. Then we can try to simplify the forms of these equations as much as possible by transformations of variables; these simplified versions of the amplitude equations are called normal forms. Their derivation parallels derivations of amplitude equations but avoids many of the complicating details of the original problem. Since it is possible to know the normal form without calculating the full amplitude equation, it is possible to decide in advance whether you need to bother to get the amplitude equation in any given circumstance.

When the system is extended in one or two directions, the modal spectrum usually becomes continuous. Then instead of

ordinary amplitude equations, we get amplitude-evolution equations. These are partial differential equations, something like those encountered in the theory of nonlinear waves. There are ways to calculate these that parallel the more familiar methods for getting amplitude equations, though the partial differential counterparts are less well documented and not possessed of the same kind of mathematical credentials. Nevertheless, the formal procedures are clear and the results are useful, much as the analogous Navier-Stokes equations are.

One of the procedures for getting the amplitude equations is to apply the method of Bogoliubov. P.H. Coullet and I have used this procedure to get amplitude equations, as we explained in the 1981 GFD notes. We have generalized our approach to the case of continuous spectra; an account should appear in the proceedings of the workshop on Energy Stability and Convection organized in Capri by **Galdi**, Herron and Straughan. It is clear once one looks at these calculations that a procedure like normal form theory can be used on the **p.d.e.** case that simply parallels the method we use. Whether a mathematical justification can be given for such procedures is a problem that has worried some people in statistical mechanics, and it is clearly quite subtle. Let me just proceed to do what comes naturally.

In the simplest case, with one band of wavenumbers, we have single amplitude function $a(\mathbf{x}, t)$ where, for the convective case, $\mathbf{x}$ is a horizontal coordinate vector. Let

$$a(\mathbf{x}, t) = \int e^{i\mathbf{k} \cdot \mathbf{x}} f_{\mathbf{k}}(t) d\mathbf{k}. \quad (1)$$

Suppose that that amplitude-evolution equation is

$$\partial_t f_{\mathbf{k}} = \sigma_{\mathbf{k}} f_{\mathbf{k}} + \iint \mathcal{I}_{\mathbf{k}}^{(2)}(\mathbf{p}, \mathbf{q}) f_{\mathbf{p}} f_{\mathbf{q}} d\mathbf{p} d\mathbf{q} + \dots \quad (2)$$

where the dots indicate higher order terms. The double integral is written for convenience. In typical problems, the integration is over a restricted region of the $\mathbf{p}$ - $\mathbf{q}$ plane. Most commonly, the double integral is a disguised convolution because $\mathcal{I}_{\mathbf{k}}^{(2)}(\mathbf{p}, \mathbf{q})$ has a factor $\delta(\mathbf{k} - \mathbf{p} - \mathbf{q})$ in it. So we have the selection rule

$$\mathbf{p} + \mathbf{q} = \mathbf{k}. \quad (3)$$

The quantity $\sigma_{\mathbf{k}}$ is a growthrate that I assume depends only on the magnitude of the wave vector: $k = |\mathbf{k}|$. As usual, $\sigma_{\mathbf{k}}$ depends also on a stability parameter that we shall call R , for no special reason. We assume that there is a value R_c of R such that when $R < R_c$, $\sigma_{\mathbf{k}}$ is negative for all $\mathbf{k}$. For $R = R_c$, $\sigma_{\mathbf{k}}$ is zero at some value (or values) of k called k_c . The procedures used here are valid only for R very close to R_c . We assume for this discussion that $R = R_c$. To get off R_c by perturbation theory is a standard operation that we need not repeat here.

If we happen not to like (2) or, in any case would like to get it into a standard form, we can try to transform it. However, since the linear term does seem to be quite sensible, we would do well to leave it in tact. So we make a transformation like this:

$$f_k = g_k + \iint \psi_k(p, q) g_p g_q dp dq + \dots, \quad (4)$$

The result ought to be an equation for g_k of the form

$$\partial_t g_k = \sigma_k g_k + \iint G_k^{(2)}(p, q) g_p g_q dp dq + \dots \quad (5)$$

The question is this. Suppose you choose a certain Q that suits you. Can you have it? You can, if you can find the Ψ that does the job. That is normally a matter of solvability conditions on the equation for Y , since that equation has Q on the right hand side. The way to get that equation is to put (4) and (5) into (2). If you do this term by term, you get conditions on the $\Psi_k^{(n)}$. The easiest way to carry out the manipulations is to use squiggles like those the physicists call Feynman diagrams. If you only go to second order, it is an easy matter, however you do it, so let us stop there. We get,

$$\iint (\sigma_p + \sigma_q - \sigma_k) \psi_k^{(2)}(p, q) - \tau_k^{(2)}(p, q) + G_k^{(2)}(p, q) g_p g_q dp dq = 0 \quad (6)$$

We can satisfy this condition by requiring that

$$\psi_{\mathbf{k}}^{(2)}(\mathbf{p}, \mathbf{q}) = \frac{f_{\mathbf{k}}^{(2)} - g_{\mathbf{k}}^{(2)}}{\sigma_{\mathbf{p}} + \sigma_{\mathbf{q}} - \sigma_{\mathbf{k}}} . \quad (7)$$

But, in general, even with the constraint (5), there will be zeros in the denominator of expression (7). So we need to require that where (5) holds and wherever

$$\sigma_{\mathbf{k}} = \sigma_{\mathbf{p}} + \sigma_{\mathbf{q}}, \quad (8)$$

we need to have

$$g_{\mathbf{k}}^{(2)} = f_{\mathbf{k}}^{(2)}. \quad (9)$$

Otherwise, $g_{\mathbf{k}}^{(2)}$ is arbitrary. However, the wavevectors over which (3) and (8) are fulfilled may appear in diverse and inconvenient ways, so people in the amplitude-evolution equation business make a further approximation.

Instead of satisfying (6) by requiring that the **inter-**grand vanishes everywhere we can recall that, near to marginality, only wavevectors with magnitude near to k_c are **significantly** excited. Suppose then that the support of $g_{\mathbf{k}}$ is restricted to annuli of vanishing width around the critical circles $|\mathbf{k}| = k_c$. Then condition (7) need be met only on those

critical circles.

(Couillet and I were led to a similar condition on the completely stable bands of modes in order to arrive at a form like (2) or (5).) Then $G_{\mathbf{k}}^{(2)}$ is independent of the magnitude of its wavevector arguments and is a function only of the angles between them. These functions of angles are like the undetermined coefficients in the nonlinear terms of the ordinary amplitude equations. Model equations in which these functions of angle are maximally simplified are now in vogue in convection theory and other fields as in the Ginzburg-Landau equation.

Of course, the method generalizes in an evident way to cases where there are several bands of wavenumbers going unstable at once.

This work was supported by the NSF under grant PHY80-23721 to Columbia University.

WEIR FLOWS, SINK FLOWS, AND POURING FLOWS

Joseph B. Keller and Jean-Marc Vanden-Broeck

Flows of fluids with free surfaces are difficult to analyze because the location of the surface must be found along with the flow. This complicates the analysis considerably, introducing an additional source of nonlinearity. Nevertheless such flows are among the most interesting and important ones, as the theory of water waves illustrates.

We have analyzed three kinds of free surfaces flows by using some mathematical transformations followed by numerical calculation. They are flows in channels containing barriers called weirs, flows in reservoirs with submerged sinks, and flows which occur when a liquid is poured from a container. In all three cases we have considered the steady two-dimensional flow of an **inviscid** incompressible fluid with one or two free surfaces, taking account of gravity.

In the absence of gravity, such flows can be treated by the complex variable methods introduced by Kirchhoff and Helmholtz, but those methods do not suffice when gravity is present. However we use their method to **transform** the problems into forms which are convenient for calculation. In particular we take explicit account of the singularities of the flows, leaving a regular function to be found numerically in each case. Then we represent this function by a power series with unknown coefficients, truncate the series and use collocation to obtain equations for the coefficients. Finally we solve these equations numerically by Newton's method.

Abstracts of three papers describing these results are given below. The first two are to appear in the Journal of Fluid Mechanics and the third in the Physics of Fluids.

1) Weir Flows

The flow of a liquid with a free surface over a weir in a channel is calculated numerically for thin weirs in channels of various depths, and for broad crested weirs in channels of infinite depth. The results show that the upstream velocity, as well as the entire flow, are determined by the height of the free surface far upstream and by the geometry of the weir and channel, in agreement with observation. The discharge coefficient is computed for a thin weir, and a formula for it is given which applies when the height of the weir is large compared to the height of the upstream free surface above the top of the weir. The coefficients in this formula are close to those found empirically.

2) Free Surface Flow Due to a Sink

Two-dimensional free surface flows without waves, produced by a submerged sink in a reservoir, are computed **numerically** for various configurations. For a sink above the horizontal bottom of a layer of fluid there are solutions for all values of the Froude number F greater than some particular value. However, when the fluid is sufficiently

deep, there is an additional solution for one special value of $F \ll 1$. The results for a sink at the vertex of a sloping bottom, treated by Craya and by Hocking, and for a sink in fluid of infinite depth, treated by Tuck and Vanden-Broeck, are confirmed and extended. In particular it is shown that as the bottom tends to the horizontal, the solution for a sink at the vertex of a sloping bottom approaches a solution for a horizontal bottom with $F = 1$. However solutions are found for all values of the Froude number $F \geq 1$ for a sink on a horizontal bottom.

3) Pouring Flows

Free surface flows of a liquid poured from a container are calculated numerically for various configurations of the lip. The flow is assumed to be steady, two dimensional and irrotational, the liquid is treated as **inviscid** and incompressible, and gravity is taken into account. It is shown that there are flows which follow along the under side of the lip or spout, as in the well-known "**teapot** effect", which was treated previously without including gravity. Some of the results are applicable also to flows over weirs and spillways.

NUMERICAL EXPERIMENTS IN FORCED STABLY STRATIFIED TURBULENCE

Jackson R. Herring

We present results of numerical simulations of stably stratified turbulence, randomly forced at small scales. The selection of forcing and damping are designed to give insight into the question of whether cascade of energy to large scales is possible for strongly stratified three-dimensional turbulence, in a manner similar to two-dimensional turbulence. We consider random forcing at a sharp wave-number, whose angular distribution ranges from two-dimensionally to three-dimensionally isotropic. Our principle results are as follows: for two-dimensional forcing--and for a sufficiently small Froude number--the statistically steady state is characterized by a weakly inverse-cascading horizontal velocity variance field. Its vertical variability is pronounced, with vertical scale determined by the forcing scale, and the Brunt-Vaisala frequency N . If, on the other hand the Froude number exceeds a critical value, the vertical variability is weak, and the statics of the scales larger than the forcing wave number is near that predicted by **inviscid** equipartitioning. For all forcing functions considered the vertical motion and temperature field **(w,T)--centered** at smaller scales--are more three-dimensionally isotropic. At large N , (small Froude number) the w - T fields scale as $1/N$, with the magnitude of the horizontal field independent of N . In addition, as $N \rightarrow \infty$, the strong vertical variability tends to dynamically reduce the forcing of the wave-component. At large N , the vertical variability of the horizontal motion field is consistent with the condition that a substantial fraction of the total dissipation is attributable to it. In simple terms, this implies a drag action on all horizontal scales of motion, which in turn flattens the slope of the energy spectrum in the inverse cascade range, and increases it in the enstrophy-cascade range. We further estimate -- via this mechanism and the numerical results -- that increasing resolution decreases the effects of the drag.

A REVIEW OF LARGE BAROCLINIC LENSES IN THE OCEAN

Thomas Rossby

The properties of large baroclinic lenses known as 'meddies' for Mediterranean eddies are reviewed. These have been found on numerous occasions in the Eastern North Atlantic, but the first one was observed off the Bahamas in the fall of 1975, some 5000 km from its probable place of origin. These lenses are entirely subsurface: centered at 1 km depth they may be nearly **1000** m thick. With a radius of 30-50 km, the lenses are large, but very thin. The Mediterranean origin of the Bahamas lens was inferred from its enormous positive salt anomaly.

This discovery was so striking that a second cruise, a half year later, was organized to conduct a more complete chemical investigation. Unfortunately, the original lens was lost, but another one with nearly the same dimensions was found. It had only a weak positive salt anomaly, and the dissolved oxygen and tritium levels were very high. From this it was evident that the lens must have been formed at the surface in the central North Atlantic. The fact that both eddies were centered on the same density surface has raised some doubt recently as to whether the first lens did originate in the Eastern Atlantic. However, the discovery and description of large saline lenses in the Eastern Atlantic shortly after the original finding was welcome confirmation of the hypothesized origin of the first meddy observation.

It was quickly established that these lenses are not all that uncommon, at least in the Eastern North Atlantic. This **immediately raised** very obvious questions about what kind of role they might play in the transport and maintenance of the distribution of salt throughout the North Atlantic basin. It is easy to show that only a few of these a year can provide a salt **flux** comparable to what the mean flow (only poorly known) could provide. Thus, how common are the meddies, how stable are they, how and where do they migrate became questions of considerable interest and relevance.

In the fall of 1984 and 1985, Larry **Armi** and I organized two cruises in the Eastern Atlantic to study the lenses; their structure, evolution and movement. These studies were part of a major **SOFAR** float program organized by Jim Price and Phil Richardson to study mean flow and dispersion in the region of the Mediterranean outflow. CTD studies of the density field and **Pegasus** profiles of the velocity field were planned. Moreover, in order to study the evolution and decay of these eddies, they were to be tagged with **SOFAR** floats so that they could be relocated at a later date.

One lens, named 'Sharon', was observed on both cruises. Centered on the 27.5 sigma-theta surface, it had a solid body core nearly 40 km in diameter and an overall diameter of at least 100 km. The interior of the lens was characterized by a region of low stratification, above and below which were transition layers of high static stability. The general shape of the pressure anomaly field can be described by a Gaussian function of the **form**.

The inner core rotated approximately as a solid body with a period of about 5 days, which is equivalent to a relative vorticity of about -30% of the local Coriolis parameter. The maximum orbital speeds were **.25 cm/s** at 25 km radius.

At the time of the second survey, a year later, the extrema of temperature and salinity in the core of the lens were unchanged, but its vertical extent was significantly reduced, particularly from below. The lower edge rose 200 m from 1400 to 1200 m whereas the upper edge dropped at most 50 m. Measurements of static stability and velocity shear suggest that the upper and lower boundaries were stable to shear flow instability. On the other hand the density ratio, R , was about 1.5 in the lower boundary of the lens whereas on the top side the stratification was stable in both temperature and salinity. This suggests that salt fingering is an important erosional process operating on the lower part of lens. The frequent observatoin of very well defined steps in the density field in lower parts of the lens can be taken as evidence of double-diffusive convection. In as much as 'Sharon' drifted south into increasingly fresh Antarctic Intermediate Waters, it is possible that the lens was loosing its potential energy more rapidly than it would have had it remained in the more saline waters where it was first observed. A simple calculation of salt flux based on the 413-salt flux law and the amount of salt lost in the year's time leads to an estimate for the step size, **=0.15 ppt**, which is quite comparable to what was observed.

This rate of decay argues forcefully that 'Sharon' should not be able to survive a transoceanic journey: it had moved south 5 degrees in one year, the distance to the site of the old meddy is anothr 40 degrees west! Whether or not this means no lens can travel that far remains to be considered.

On the second cruise, a new, even larger lens, 'Io', was located farther north. From the first six months of tracking--two SOFAR floats are trapped in it--this lens seems to be on a more westward heading. It will be very interesting to see how this lens evolves. The fact that floats remain trapped in a lens for months if not years makes this task much easier.

This coming fall and perhaps in the future as well, we hope to revisit the lenses to examine how they change and ultimately disappear. Tracking these eddies is essential for another reason: if a lens passively advected by the waters around it or is it able to propel itself through the surrounding fluid? In the latter case they would provide mesoscale mechanism for the redistribution of salt on a large scale to a degree independent of what the large scale circulation looks like. In fact, even if they are passively carried by the surrounding velocity field, their ability to contain and displace salt over large distances may lead to a distribution of salt that is different than would be the case for a classical mean flow - eddy dispersion type of salt balance. Without knowing what kind of physics is operating one risks misinterpreting the observed fields. Thus it is very important to understand the mechanisms maintaining the property distributions in the ocean.

AMPLITUDE PROPAGATION IN TRAINS OF ORR-SOMMERFELD WAVES

John M. Russell

The work reported concerns the evolution of local amplitude in a train of instability waves in a shear flow of a viscous incompressible uniform-density fluid. The work addresses, in particular, the problem of separating the effects of exponential growth of individual normal modes in linear stability theory from the effects of constructive and destructive interference between distinct modes.

Two small parameters are introduced. The first, denoted ϵ_f , measures the **streamwise** and **spanwise** nonuniformity and unsteadiness of the reference flow velocity field. The second, denoted ϵ_w , measures the spatial and temporal nonuniformity of the parameters of the disturbance wave train such as local frequency, local wavenumber, and the nonexponential part of the local amplitude.

The dependent variable $\mathcal{L}$ is introduced whose linearized substantial derivative is the cross-stream disturbance velocity component and the small disturbance equations of motion are manipulated to yield a single partial differential equation for this "liftup" variable. A bilinear variational principle is introduced whose two Euler **Lagrange** equations are the equation for the liftup $\mathcal{L}$ and an **adjoint** equation for a corresponding **adjoint liftup** variable m .

Substitution of trial solutions for $\mathcal{L}$ and m of the form

$$\mathcal{L} = 2 \operatorname{Re} \{ \tilde{\mathcal{L}} \exp(i\theta) \} \qquad m = 2 \operatorname{Re} \{ \tilde{m} \exp(-i\theta) \}$$

and use of the assumption ~~Eq. (1)~~ leads to a phase averaged variational principle. The three Euler Lagrange equations of the latter are the "liftup" form of the Orr-Sommerfeld equation (for $\tilde{u}$), its adjoint (for $\tilde{m}$), and a conservation law for a local wave action density.

Imposition of homogeneous boundary conditions leads to a dispersion relation and to a law of conservation of modal wave action density. Propagation equations for the wavenumber vector, the wavenumber gradient matrix, and the modal wave action density are derived. Singularities of the latter are shown to be possible in two special cases, namely steady parameter wave trains in a steady nonuniform medium (cf. Landahl, 1972), and steady wave trains in a stationary horizontally uniform background medium (cf. Russell, 1986). The latter class of singularities include far-field and near field "caustics".

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MOVEMENTS AND INTERACTIONS OF ISOLATED EDDIES

Doron Nof

I) Migration of isolated eddies on a 0-plane and the behavior of joint vortices:

An analytical model describing the R-induced drift of isolated non-linear eddies such as the cold- and warm-core rings observed in the Atlantic Ocean is proposed. The ocean is approximated by two layers and attention is focused on frictionless upper ocean eddies whose surface area is finite. These isolated eddies are nonlinear in the sense that (a) the corresponding Rossby number is relatively large and (b) the interface vertical displacements ("amplitudes") are comparable to the upper layer undisturbed depth. Solutions for steadily translating eddies which carry their entire mass as they move are sought. Examination of the problem in a moving coordinate system enables one to construct such solutions analytically by using the equations of motion in an integrated form and a power series expansion.

Significant differences between the behavior of cyclonic and anti-cyclonic eddies are found. Although both cyclonic and anticyclonic eddies drift to the west due to β , their speeds and dynamical behavior are very different. For some range of parameters the β -induced drift of an anticyclonic eddy differs by as much as 400% from the drift of a cyclonic eddy with similar characteristics. Furthermore, the R-induced translation of cyclonic eddies increases with size and decreases with "amplitude" whereas the speed of anticyclonic eddies decreases with the size and increases with increasing amplitude. In addition, the translation of anticyclonic eddies is larger than the long wave speed (based on the undisturbed depth) whereas the translation of cyclonic eddies is smaller than the long wave speed. Since such a dynamical behavior is not revealed by quasi-geostrophic theory (which does not distinguish between cyclonic and anticyclonic eddies) it is suggested that nonlinearity plays an important role in the dynamics of some isolated rings.

With the aid of the above information, the behavior of an isolated pair of vortices consisting of two eddies situated on top of each other in a three-layer ocean is also examined. The amplitudes of both eddies are high and, consequently, the two eddies behave as one unit and migrate together in the ocean. For this reason, it is proposed to call the system joint vortices. The eddies are of equal or opposite sign; each vortex is situated in a different layer so that there are two active layers and one passive layer.

Attention is focused on the behavior of joint vortices on a R-plane in the upper ocean. Special attention is given to the cases where one of the vortices is a lens-like eddy. Approximate solutions for β -induced drifts in the east-west direction are obtained.

It is found that because of the high amplitudes and the resulting nonlinear coupling, the joint eddies have a mutual drift which is very different from the drift that each individual vortex would have. For

example, while each individual vortex translates to the west in the absence of a conjugate vortex, the combined vortices may drift steadily to the east. This bizarre behavior stems from the presence of a "planetary lift" which is the oceanic equivalent of the side pressure force associated with the so-called Magnus effect. It is directed at 90 to the left of the drifting eddies.

Other results of interest are: (i) Under some conditions, the westward drift of joint eddies consisting of two cyclonic vortices is much faster than the long-wave speed. (ii) As it translates westward, an anticyclonic lens-like eddy can carry a Taylor column on top of it.

II) Merging of isolated lens-like eddies:

The interaction of two isolated lens-like eddies is examined with the aid of an **inviscid** nonlinear model. The barotropic layer in which the lenses are embedded is infinitely deep so that there is no interaction between the eddies unless their edges touch each other. It is assumed that the latter is brought about by a mean flow which relaxes after pushing the eddies against each other and forming a "figure eight" structure.

Using arguments based on continuity and conservation of energy along the eddies edge it is shown that, once a "figure eight" shape is established, intrusions along the eddies' peripheries are generated. These intrusions resemble "arms" or "tentacles" and their structure gives the impression that one vortex is "hugging" the other. As time goes on the tentacles become longer and longer and, ultimately, the eddies are entirely converted into very long spiral-like tentacles. These spiraled tentacles are adjacent to each other so that the final result is a single vortex containing the fluid of the two parent eddies.

Because of the inherent nonlinearity and the fact that the problem is three-dimensional(x, y, t), the complete details of the above process cannot be described analytically. It is, however, possible to show analytically that the intrusions and tentacles are inevitable. For this purpose, one of the interacting eddies **is** conceptually replaced by a solid cylinder. Initially, the cylinder drifts toward the eddy; subsequently, it is pushed slightly into the eddy and is then held fixed. The subsequent events are examined in detail.

It is found that as the cylinder is forced into the eddy, a band of eddy water starts enveloping the cylinder in a clockwise manner. This tentacle continues to intrude along the cylinder perimeter until it ultimately reattaches itself to the eddy, forming a "padlock" flow. Using the details of this process it is argued that, in the actual eddy-eddy interaction case, intrusions must be established and that, consequently, merging of the two eddies is inevitable.

The mutual interaction of the eddies is then examined with the aid of a laboratory experiment on a rotating table. The isolated anti-cyclonic eddies are formed by withdrawing two **cy**linders containing a mixture of Freon (with a **density** of 1.53 gm cm^{-3}), and silicone oil (with a density of 0.853 gm cm^{-3}). The cylinders are embedded in water

and the collapse of the mixture forces two identical lens-like eddies (with an anticyclonic circulation) on the bottom of the tank. Initially, the lens-like eddies are completely separated from each other so that one vortex does not "know" about the presence of the other.

Due to small bottom friction, the vortices spin down slowly so that after some time their edges meet and they touch each other, forming a "figure eight" structure. After this happens there is a rapid (i.e., within 20 revolutions) interleaving of the two eddies. Arms are extended from one vortex to the other and the vortices become one unit consisting of two main lenses. As the interaction continues, the two lenses become less distinct and, ultimately, a single lens-like vortex is formed.

A total of about 20 experiments were performed and all showed that merging takes place after the eddies touch each other. Experiments with vortices whose densities are not identical were also performed and these also resulted in vortices that merged. All the experiments suggest that the potential vorticity of the eddies is altered during their interaction and that no external source of energy is needed for the merging.

It is suggested that the change in potential vorticity is achieved by the action of shock waves, (i.e., organized depth discontinuities involving a balance between steepening and dissipation). This is qualitatively shown by using an analytical "reduced gravity" model which examines a special kind of shock wave. The wave under study is a depth discontinuity associated with a transition between a supercritical and subcritical flow in a channel. Even though the wave itself is highly nonlinear, the adjacent upstream and downstream fields are exactly geostrophic in the cross-stream direction. For this reason we term the wave a geostrophic shock wave. We focus on a stationary shock wave whose horizontal projection is a straight line perpendicular to the side walls. Solutions for the entire field are constructed analytically using power series expansions and shock conditions equivalent to the so-called Rankine-Hugoniot constraints.

It is found that, for particular upstream conditions, a geostrophic shock wave can be formed if the particle speed exceeds the surface gravity wave speed (i.e., the flow is "supercritical"). Specifically, in addition to supercriticality, a stationary geostrophic wave requires the upstream velocity to have a particular structure which depends on the strength of the shock and the channel width.

Being the only known analytical solution for the entire field of shock waves on a rotating earth, the geostrophic shock provides useful information on the wave structure. It is shown that even though momentum is conserved across the shocks, relatively large changes in potential vorticity take place. For depth discontinuity of $O(1)$ (i.e., high "amplitudes"), there is a generation of potential vorticity that is also of $O(1)$.

It is suggested that such shock waves are active in the intruding tentacles which are involved in the merging process.

INSTABILITIES OF SALT FINGERS

George Veronis

The stability problem for salt fingers is considered for three configurations. The first (Holyer, 1984) is a fluid with constant vertical gradients of salt and temperature (S_z and T_z , respectively) in an infinite fluid, where the basic fingers are independent of the vertical coordinate, z , and have the form $\sin kx$ in the horizontal. The second (Veronis, 1986) is the same as the first except that the fluid is contained in a Hele Shaw cell. The third (Howard and Veronis, 1986) is an array of fingers with finite depth and with zero salt diffusivity. The basic fingers of the first two experiments constitute exact solutions to the equations. In the third, independence of the vertical coordinate is only an approximation since it neglects processes near the top and bottom of the finger zone.

Because the preferred width of salt fingers is proportional to $(\text{gravity})^{1/3}$, the Hele Shaw set-up enables one to produce wide fingers by inclining the Hele Shaw cell to the horizontal so that the component along the facing plates is the only pertinent contribution to g . If the cell is raised upright after the wide fingers are established, the preferred finger width is smaller. Experimental observations (Taylor and Veronis, 1986) show that fingers of smaller width are generated by perturbations that protrude laterally into the interior of each finger from the finger boundary. The perturbations have a vertical length scale of the order of the buoyancy layer thickness. The wider the initial fingers, the closer the perturbations are inclined to the horizontal.

A Floquet theory analysis for the stability problem leads to results that agree with the observations. The most unstable (non-oscillatory) modes penetrate into the finger interior so that the protrusions from each side alternate in the vertical, as is observed in the experiments. For the Hele Shaw cell this is the only instability. There is no "collective instability" (Stern, 1969). Fingers of "optimum" width also become unstable to the same type of disturbance. The conclusion is that salt fingers in a fluid with uniform gradients should take on a wiggly form in the vertical. Such wiggly fingers are observed in some experiments. Where they are not, the system is normally transient and the basic fingers are not steady.

The same type of analysis for a regular fluid (Holyer, 1984) shows that the same kind of direct instability is the preferred one in the heat-salt system. For salt-sugar (with a much larger diffusivity ratio) a collective instability is also possible though it has a smaller growth rate than the small scale, direct instability described above.

A stability study of the Howard-Veronis model concludes that strongly driven fingers of optimum scale are also unstable to a non-oscillatory mode with a vertical scale of the buoyancy layer thickness. This mode is the only one to occur when the vertical

velocity is sufficiently large to override viscous effects. For slower flows, where effects of viscosity and advection are comparable, a collective instability occurs first. This study complements that of Holyer for the case in which salt diffusion is not important. It supplements the Hele Shaw study by showing the effects of inertia.

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VORTEX METHODS FOR STRATIFIED FLOWS

Gretar Tryggvason

Recent progress in application of vortex methods to sharply stratified flows was discussed. Most of the work described has focused on simulations of mixing of two fluids due to a Rayleigh-Taylor instability. When the fluids are incompressible, inviscid, and each of constant density, the interface constitutes a generalized vortex sheet with vorticity generated on the interface as prescribed by **Bjerkness'** theorem. Previous simulations of this problem have only been successful for the single fluid case, where the density of one fluid vanishes. In other cases the simulations have eventually run into unresolvable difficulties, and refinement of the simulations. For finite density differences the interface rolls up into concentrated vortices, and the spontaneous appearance of a singularity on the interface causes difficulties akin to those encountered for "ordinary" vortex sheets across which there is no density difference. This has been identified as the main source of trouble in simulations in the past. Regularizations to prevent the formation of this singularity were discussed at length. For homogenous flows vortex blobs provide one type of regularization, and we have successfully extended this regularization to stratified flows. Several implementations are possible, and a fully **Lagrangian** method, as well as a very efficient grid based method were described.

Large amplitude Rayleigh-Taylor instability simulations, including (relatively) long time simulations of multi-mode initial conditions, were presented. These "large" calculations have led to the identification of various important large-amplitude mixing mechanisms, such as reduction of large-amplitude stage of these subharmonic instabilities may be described as vortex interactions or bubble competition, depending on the density ratio.

DEEP-OCEAN CIRCULATION AND TOPOGRAPHY

Peter Rhines

The deep and abyssal circulation of the oceans is, at the planetary scale, driven by buoyancy forcing at the sea surface, and density mixing effects within the water column. Wind-driven circulation and eddy activity 'overhead' also drive elements of deep circulation, which tend to be smaller in scale, but they also may modify the flow paths of the great planetary convection cell. Unlike the rectangular boxes we use in GFD, however, the oceans have significant topography at all length scales. Even after rescaling the **horizontal/vertical** aspect ratio (essentially by f/N , the ratio of Coriolis to buoyancy frequencies) to account for the dynamics the oceans seem to have no sides: only a bottom. Topography with the scale greater than, say, 250 km acts to reshape the mean potential vorticity field, f/H , where H is an effective layer thickness for the vertical mode in question. Smaller scale topography is a scattering agent which can induce systematic forces on the large-scale circulation, by generalized Rossby-wave drag. **McCartney** (1975) and **Charney and Flierl** (1981) have looked at various linear and weakly nonlinear aspects of 'lee Rossby waves'.

Baroclinic flows of scale larger than the appropriate Rossby radius of deformation (say 10-50 km depending on vertical mode) obey planetary geostrophic equations conserving f/H . The simplest such problem has an active deep layer underlying a thick, essentially motionless layer the equation for the interface height η is

$$\eta_t + [U_b - c_0(\eta, h) - (g'/f)h_y]\eta_x + (g'/f)h_x\eta_y - U_b h_x = 0$$

where the active layer has thickness $d = d_0 + \eta - h$ with $h(x, y)$ the bottom topography and d_0 the mean layer thickness. U_b is the barotropic zonal velocity, which may be specified as a function of latitude, y . The nonlinearity comes in through the baroclinic Rossby wave speed $c_0 = g'\beta d/f^2$ ($\beta = df/dy$).

The equation has a deceptively simple appearance. With small amplitude, and ignoring the secular variation in $f(y)$, the equation becomes just the quasi-geostrophic equation on a mid-latitude β -plane. Rhines (1983) describes the response of an eastward current encountering topography, in which there is sharp resonance at $U_b = c_0$. The subcritical/supercritical transition is analogous to classic open-channel flow, with η taking on the inverted shape of the topography for $U_b < c_0$, and the direct shape for $U_b > c_0$. The relevant wavespeed c_0 is 1 to 10. With nonlinearity, if y -variations of h , f , and η are ignored it can be cast as a Korteweg de Vries problem with amplitude steepening of long Rossby waves competing with wavenumber-dispersion. But we find that meridional (y -) variations of f alone (not to mention h) lead to complex and interesting structure. Solutions may be found by integrating along characteristics, or by grid-point simulation.

The dominant properties to emerge are: (i) the 'softening' of resonance by nonlinearity; (ii) Taylor 'caldera' occur, which are β -plane analogues of the Taylor column, regions of closed characteristics impenetrable to the

upstream boundary conditions (see figures). The simple QG solutions which are images of the topography are truncated by an interior solution with autonomous dynamics. In the time-dependent evolution η winds into a spiral with steepness growing like t , the time. Since the slope of η is proportional to velocity, intense jets form in this region. The mechanical energy increases greatly, at the expense of the large-scale flow. In a full-physics calculation these jets will be strongly unstable to mesoscale disturbances.;(iii) in the **time-** dependent evolution the 'starting vortex' of anomalous potential vorticity moves away (upstream if subcritical, downstream if supercritical), leaving behind the vortex bound to the topography. Yet due to a saddle point in the characteristics the two are connected by trough or ridge of intense flow (figures). Fluid (and potential vorticity) from a great distance is pulled into this region and wound into the interior spiral where it comes to diffusive equilibrium;(iv) blocking occurs, even in the linearized problem, due to variations in criticality with latitude (due to variation in U_b , f or h with latitude). For example, a jet of eastward flow encountering a ridge topography may be supercritical at its maximum, yet will be subcritical where the velocity becomes small. This means that characteristics directed eastward (downstream) in the jet core will be deflected **poleward** by the topography, where they will connect with characteristics directed zonally westward. An intense ridge of η penetrates infinitely far upstream, in the **inviscid** model, at a speed c_r , and flow directed at the ridge is turned back westward, where it tries to round the advancing ridge.

These flows, though posed as idealized '**circumpolar**' zonal flows, are relevant to the greater question of the global deep circulation. The effects of even small topographies on flat-bottom dynamics are catastrophic. In particular fast topographic waves replace Kelvin waves as the relevant '**boundary**' process for setup of the deep circulation, and the pathways and blocking defined by these waves will be intricate indeed.

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1a. Evolution of a planetary-geostrophic eastward flow over a **seafloor** ridge (northern hemisphere, 2500 km sq., 100x100 grid). An initially barotropic flow of speed 7 cm/sec is switched on at the start. The thin lines are the interface elevation, $\eta(x,y)$, plotted as a surface (hidden lines not removed). The bands are potential vorticity (Q-) contours, which are initially just the f/h contours appropriate to the topography. Q is advected with the flow. Although there are initially no closed geostrophic contours, the flow generates them, as the 'starting vortex' is swept downstream from the topography: the flow is supercritical, faster than the internal Rossby wave speed. **Linear** theory would predict a simple hump in the interface at large time, but the closed characteristics wind up Q and η , liberating kinetic energy, and leading to shear dispersion of Q. The gyre moving downstream is connected to the topographic region by a long trough, which steepens and draws in a tongue of high potential vorticity water from a great distance.

1b. Evolution from rest of the interface $\eta(x,y)$ in a subcritical (i.e., slower than the baroclinic Rossby wave speed) **flow** over a rise with circular depth contours. A 2 cm/sec eastward barotropic flow is started impulsively at $t=0$. The spiral of shear dispersion is evident over the topography, and the transient Rossby wave moves upstream. There is a ridge connecting the free and bound vortices. In the final panel the time-longitude history is shown, for the middle latitude, and a superposition of particle trajectories (daily positions plotted). The linear solution would show a simple dip in the thermocline.

On Large Scale Structures in Free Shear layers

Israel Wygnanski

The flows considered are the classical free shear layers obeying in the mean the similarity laws defined by Townsend. Experiments were carried out on small deficit wakes, plane and **axisymmetric** jets and a plane mixing layer. Although each of the flows obeys the similarity laws and all measurable quantities in it scale by a single velocity scale (U_0) and a single length scale (L_0) there is mounting experimental evidence suggesting that the description is not universal. For example: in a small deficit wake

$$U_0 \propto x^{-1/2} \quad \text{and} \quad L_0 \propto x^{1/2}$$

(where x is the distance from the body, or more precisely, from the virtual origin defined by the mean velocities measured far away from the body)

then the dimensionless velocity and length scale could be described by two constants W and Δ defined by:

$$W = \left(\frac{U_0}{U_\infty} \right) \left(\frac{x}{\theta} \right)^{1/2}$$

and

$$\Delta = L_0 (x \theta)^{-1/2}$$

for which

$$\theta = \int_{-\infty}^{\infty} \frac{U}{U_\infty} \left(1 - \frac{U}{U_\infty} \right) dy = \frac{F}{\rho U_\infty^3}$$

where U_∞ is the velocity of the free stream θ is the momentum thickness related to the drag force per unit length F .

It was expected that data obtained in a given facility at a prescribed $R_\theta = U_\infty \theta / \nu$, U_∞ , θ will determine the values of W and Δ uniquely but this did not turn out to be the case.

W and Δ depend on the shape of the body generating the wake and therefore on the detailed nature of some oscillatory drag force f which does not appear in the Reynolds averaged equations. One may be forced to conclude that Reynolds averaging has limited usefulness in describing turbulent shear flows.

Linear **inviscid** stability analysis in two dimensions applied to local mean velocity profiles describes the large eddy motions in all these flows fairly accurately as may be seen from figure 1. This figure shows the evolution of amplitudes and phases of the streamwise component of velocity fluctuations which are phase-locked to an artificial excitation at the edge of a splitter plate in a plane mixing layer. The solid curves in this figure represent the prediction while the triangular symbols the experimental data. The results were obtained in a free mixing layer and are described in detail in an article by Gaster Kit and Wygnanski in the J. Fluid Dynamics, 1985.

The need to associate the spreading rate of the mixing layer with amalgamation of two-dimensional vortices is questioned in view of the fact that a slightly non-linear stability model applied to the mean flow field may also predict the rate of spread of the shear layer. On the other hand the "pairing" or amalgamation process of vortices is not quantitatively defined in the literature. It is mostly derived from flow visualization which differs from one experiment to another. It was observed that re-orientation or amalgamation of the **spanwise** component of vorticity is not synonymous with the amalgamation of streamlines used in many experiments.

Figure 1

NEGATIVE ENERGY WAVES

William R. Young

In collaboration with **Y.-Y. Hayashi**, I've analyzed the instabilities of rotating, shallow water, shear flows on an equatorial **β -plane**. Because of the free surface, the motion is horizontally divergent and the energy density is cubic in the field variables (*i.e.*, in standard notation the kinetic energy density is $h(u^2 + v^2)/2$). **Marinone** and Ripa (1984) observed that as a consequence of this the wave energy is no longer positive definite (there is a cross term $Uh'u'$). A wave with negative wave energy can grow by transferring energy to the mean flow. Of course total (mean plus wave) energy is conserved in this process. Further, when the basic state has constant potential vorticity, **we've** shown that there are no exchanges of energy and momentum between a growing wave and the mean flow. Consequently when the basic state has no potential vorticity gradients an unstable wave has zero wave energy and the mean flow is modified so that its energy is unchanged. This result strikingly shows that energy and momentum exchanges between a growing wave and the mean flow are not generally characteristic of, or essential to, instability.

A useful conceptual tool in understanding these counterintuitive results is that of disturbance energy or (pseudoenergy) of a shear mode. This is the **amount** of energy in the fluid when the mode is excited minus the amount in the unperturbed medium. Equivalently, the disturbance energy is the sum of the wave energy and that in the modified mean flow. The disturbance momentum (or pseudomomentum) is defined analogously.

For an unstable mode, which grows without external sources, the disturbance energy must be zero. On the other hand the wave energy may increase to plus infinity, remain zero, or decrease to minus infinity. Thus there is a tripartite classification of instabilities. We suggest that one **common** feature in all three cases is that the unstable shear mode is roughly a linear combination of resonating shear modes each of which would be stable if the other were somehow suppressed. The two resonating constituents must have opposite signed disturbance energies in order that the unstable alliance has zero disturbance energy. The instability is a transfer of disturbance energy from the member with negative disturbance energy to the one with positive disturbance energy.

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COHERENT STRUCTURES IN A IC ATMOSP TEORY FOR ATMOSPHERIC BLOCKING AND COMPARISON WITH DATA

Paola Malanotte **Rizzoli**

A nonlinear, local theory has been proposed as a model for atmospheric blocking by **Malguzzi** and Malanotte-Rizzoli in a series of papers (**JAS**, 41, 2620-2628, 1984; **JAS**, 42, 2463-2477, 1985). In the analytical theory, stationary coherent structures are found as asymptotic solutions of the inviscid, quasi-geostrophic potential vorticity equation with a mean zonal wind with vertical and horizontal shear, in the limit of weak dispersion and nonlinearity. Successively, the theory is extended to the time-dependent, highly nonlinear case by means of a truncated dynamical model. The steady solutions are **antisymmetric** dipoles, with the anticyclone north of the cyclone; they have an equivalent barotropic vertical structure and are meridionally as well as zonally trapped. An extensive series of numerical experiments is carried out to investigate the persistence of the steady solutions and their stability to different superimposed perturbations. The persistence time is of the order of 10-12 days, consistent with the persistence of blocking patterns. Some aspects of the theory have been checked against the experimental evidence. In particular, the theory predicts the function

$$v = \frac{1}{4} - \frac{q_y}{\bar{u}}$$

where q_y = meridional gradient of potential vorticity of the mean upstream wind $u(y, z)$ to be very different for the cases of blocking versus other meteorological patterns. The V-function must have a specific shape for the blocking **patten** to exist. This prediction is tested against data and the results seem to be very encouraging.

A MODEL FOR FASTEST-GROWING FINGERS IN A GRADIENT

Eric **Kunze**

A finger Richardson number [equivalent to the collective instability criterion of Stern (**1969**)] is used as a constraint to limit the length and **fluxes** of growing finite amplitude fingers. Unless the interface is thin enough that an individual finger extends through it (~ 10 cm), the **fluxes** do not obey the lab **flux law** but depend on the gradients within the interface. This may explain why **fluxes** almost two orders of magnitude below lab **flux law** predictions were found in the thermohaline staircases east of Barbados where interface thicknesses are typically 2-4 m. As the fluxes are more appropriately parameterized by their driving gradients, $\partial S / \partial z$, than the total step across the **interface, ΔS** , accurate estimation of salt-fingering fluxes in the ocean requires resolving the interface to 10 cm.

SHEAR-LAYER STRUCTURES IN NEAR-WALL TURBULENCE, THEIR EVOLUTION AND CONNECTION TO WALL-PRESSURE PEAKS

Arne Johansson

The formation and evolution of shear-layer like flow structures in the near-wall region of turbulent shear flows are intimately coupled to the process of turbulence production. Their formation is often initiated as a slow outward migration of a low-speed streak, followed by a rapid outward motion. The experimental work of the first part of this talk was carried out in the Gottingen oil channel, and focuses on the evolution of shear-layer structures by means of two-probe measurements in the buffer region of turbulent channel flow. Various separations in the streamwise and normal directions were used to obtain data on the inclination angles and propagation velocity of the shear-layers. They were found to retain their identity over a travelling distance of at least 500 viscous length units (L^*), while their own streamwise extent is of the order of 100. The propagation velocity was found to be approximately constant from the wall up through the buffer region, and the coherent motion was confined within a region of about 100 L^* from the wall.

It was also found (from experimental work in a wind tunnel at MIT) that high-amplitude wall-pressure peaks are predominantly generated by shear-layer structures in the buffer region. This generation mechanism was found to be dominated by the so called turbulence-mean shear interaction, which is a linear process. This is of importance, *e.g.*, for attempts of modelling near-wall turbulence.

OPTICAL MICROSTRUCTURE DURING C-SALT

Eric Kunze

Shadowgraph images collected in the thermohaline staircases east of Barbados reveal optical signatures very different from what was expected or what has been observed in fingering-favorable interfaces in other parts of the ocean.

In nonfingering segments of the profiles, the optical structure was more familiar. Isotropic structures were inferred to be due to turbulence. Sharp increases in temperature with depth were accompanied by strong isolated horizontal bands. Weaker, curving filaments occupy the several meters of water above and below them. These are thought to be diffusely-unstable interfaces with convecting **plumes** rising and falling from them. Very rarely, a rolled-up, spiral structure suggestive of an overturning billow was observed.

The big surprise was finding horizontal rather than vertical lamina with scales of 0.5 cm in the fingering-favorable interfaces between the layers. These bands would appear in over 30 images progressing through a 2-m thick interface. The same structures were found in over 200 interfaces. They tended to lie not exactly horizontal but to be tilted $\sim 5^\circ$.

STABILITY OF STREAMWISE VORTICES

C. Frederick Pearson

Streamwise vortical structures are observed both in transition from laminar to turbulent flow and prior to the bursting instability in fully developed turbulent flow. An idealized **streamwise** vortical structure imbedded in an infinite linear shear flow $U(y) = ky$ is analyzed, with a view to understanding the role of these structures in the dynamics of shear flows. For the idealized initial conditions, a similarity solution for the evolution of the flow is found, consistent with the full incompressible Navier-Stokes equations. The solution is parametrized completely by the ratio of the circulation Γ to the kinematic **viscosity** ν . The calculated streamwise velocity profiles are inflectional in both the span and cross stream directions, in good qualitative agreement with instantaneous velocity profiles observed prior to a bursting event using conditional sampling (VITA) techniques. The profiles are also similar to those observed experimentally in a trailing "**wingtip**" vortex formed by a shear flow over an obstacle.

The linearized (three dimensional) secondary instability of these calculated two dimensional flowfields is then considered. The flowfields are found to be unstable, with wavespeeds and wavelengths of the instabilities similar to classical results concerning the **inviscid** instabilities of inflectional **1-D** shear profiles to two dimensional disturbances. The secondary disturbances are found to be highly localized near the vortex centerline, again in accord with observations of the bursting event. Both the size and the strength of the vortex are found to be important parameters of the secondary instability; the initial shear parameter k serves only to set the time scale for the exponential growth of the secondary disturbance.

The conclusion is drawn that isolated streamwise vortical structures interacting with locally shearing flows will induce flowfields which are susceptible to strong secondary instabilities. Although evidently not the only possible secondary instability mechanism for linearly stable flows, this analysis shows that the heuristic picture that a coherent structure will induce locally inflectional profiles which are then subject to strong **inviscid** instabilities is mathematically consistent with the Navier-Stokes equations.

FLOW STRUCTURES IN THE CALIFORNIA CURRENT

Pierre Flament

The summer-time mesoscale flow off the Central California shelf consists of westward baroclinic jets emanating from the coastal upwelling region, embedded in a field of cyclonic and anti-cyclonic eddies. These jets transport cold coastal water offshore, and appear as cold filaments on satellite infrared images. On-shore flows are generally observed between the filaments. Some typical scales of the filaments are: width, 20-50 km, depth 100-200 m, speed 0.5-0.8 m s⁻¹, total flow 0.5-2 10⁶ m³ s⁻¹, along-shore spacing 50-500 km, offshore extension 50-500 km, horizontal shear $-f/2$ to more than $+f$ (Davis, 1985; Flament *et al*, 1985; Kosro and Huyer 1985; Rienecker *et al*, 1985).

Satellite data suggest a marked annual cycle: there is no evidence of the filaments from December to April, they appear in May and increase in size until the end of the upwelling season. Some seem to be rooted near capes where the coastal upwelling is strongest, but the roots of others have been observed to move northward or southward at up to 5 km/day.

Several hypotheses have been proposed for the formation of these filaments: interaction of the southward coastal jet with offshore mesoscale eddies, instability of the coastal jet triggered by bottom topography (Narimousa and Maxworthy, 1985), baroclinic instability of the combined coastal jet/northward undercurrent (Ikeda and Emery, 1984), and non-linear evolution of a coastal jet with initial alongshore variations (Stern, 1986).

The horizontal shear on the cyclonic side of the filaments can reach at least 3 f, whereas the shear on the anticyclonic side has not been observed to exceed $(-f)/2$. The cyclonic shear is frequently unstable to barotropic instabilities at a scale of 5 to 30 km. The asymmetry in the shear is also observed in temperature: the front limiting the filaments on the cyclonic side has a scale less than 300 m, but on the anticyclonic side the temperature change is generally spread over 5-20 km. These temperature fronts are usually confined to the mixed layer, and are shallower than the velocity field. There is some evidence of surface convergence near the cyclonic fronts, at a rate of 10⁻⁴ to 5 10⁻⁵ s⁻¹. T-S analyses suggest that water subduction may be occurring there.

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CHAOS OFF THE SHELF: TOPOGRAPHICALLY GENERATED MEAN FLOW ON THE CONTINENTAL SLOPE

Roger Samelson

This lecture was based on work with John Allen of **Oregon State University**.

Observations over the continental margin off central California indicate monthly mean mid-shelf mid-depth currents are northward all year, while mean wind stresses are southward. The phase of continental shelf waves propagates in one direction only, so the wave drag resulting from lee waves with these dynamics will depend asymmetrically on the current, and topography should generate a mean flow in response to an oscillating zero-mean wind stress. We investigate this mechanism with a simple theoretical model.

We specify barotropic quasigeostrophic f-plane channel flow over a sloping bottom with thin **sinusoidal** cross-channel (cross-shelf) ridges, bottom friction, and an along-channel wind stress. Following Hart (1979), who formulated a similar model to study atmospheric flow over mountains, we assume that the cross-shelf variation in the ridges is slow and the associated along-shore velocity negligible. Three ordinary differential equations result; they are identical to those of Hart (1979) except that the scaling is different and the forcing of interest is periodic rather than steady.

The **inviscid** unforced system is **Hamiltonian** and integrable, with a potential that is quartic like that of the Duffing equation (Duffing, 1918) but not symmetric. Its small amplitude oscillations have a simple physical interpretation. Along-shore flow across the ridges creates a topographic wave by vortex stretching. Wave drag decelerates the along-shore flow until the cross-shelf flow is uncorrelated with the topography, the wave drag vanishes, and the along-shore flow reaches its opposite maximum. The pattern repeats, with alternating sign.

The presence of time-dependent forcing complicates the problem. We restrict the analysis here to weak near-resonant forcing with weak friction and use the method of averaging. All steady solutions of the averaged equations have negative (poleward on an eastern boundary) mean current. Multiple equilibria occur. For parameter values near a Hopf bifurcation point, no stable steady solutions exist. **Numerical** integrations in these regions indicate period doubling sequences and chaotic behavior. We interpret some solutions in terms of the logistics map and a coordinate change.

Numerical integrations of the original equations indicate that mean current generation persists past the formal validity of the perturbation analysis. The irregular (chaotic) response to regular (periodic) forcing near resonance appears to be confined to very weak forcing,

Our results stimulate questions regarding the natural modes of oscillation and the response to forcing when the topography and wind stress are more realistic. The model robustly predicts negative mean current generation. It suggests that the response of slope flow to atmospheric forcing may be irregular even if the forcing is weak. Whether such chaos may in fact be found off the shelf is an open question.

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SIMPLE MODELS OF THE GENERAL CIRCULATION OF THE OCEAN

Rick Salmon

The theory of the wind-driven ocean circulation **simplifies** considerably if inertia is neglected, and if a line or drag friction replaces the conventional Laplacian viscosity (in all directions). The resulting equations are useful in two ways. First, if the density equation is also linearized, then the simplified equations admit boundary layer solutions which are much easier to analyze than the corresponding solutions of the conventional linear equations.

Second, if **the density** equation is retained in its fully nonlinear form, then it can be used to step the density forward in time. Then the remaining equations determine the new velocity by a linear elliptic equation for the pressure. The solution contains nonhydrostatic coastal upwelling layers that depend on the vertical friction. These upwelling layers exist wherever the **Ekman** transport or the geostrophic "thermal wind" impinges on a coastline. Numerical experiments suggest that cross-isopycnal flow within these upwelling layers is an important component of the large-scale circulation.

EFFECTS OF CONDENSATION AND WIND-INDUCED SURFACE HEAT FLUXES ON BAROCLINIC INSTABILITY

Kerry A. Emanuel

Cyclones and cyclogenesis have historically been, and remain central topics of investigation in geophysical fluid dynamics. Until very recently, however, investigations of tropical and extra-tropical cyclones have remained largely separate enterprises. I shall attempt to show that elements of the physical processes at work in both types of cyclones contribute to certain cases of maritime explosive cyclogenesis.

While baroclinic instability is well known to students of GFD, the dynamics of hurricanes are far less well understood. One generally accepted but probably incorrect view holds that tropical cyclones represent a particular mode of release of conditional instability stored in the tropical atmosphere. The concept of Conditional Instability of the Second Kind (CISK), advanced by Charney and Eliassen (1964), maintains that in tropical cyclones the release of conditional instability is made possible by the frictional inflow in the **Ekman-like** boundary layer associated with the vortex. Observations of the tropical atmosphere show, however, that there is very little available energy for convection what little exists is continuously released in the form of cumulus clouds.

Observations of tropical cyclones made prior to the advent of CISK clearly showed that anomalous heat fluxes from the ocean are necessary to maintain hurricanes. These and other considerations led Emanuel (1986) to propose that hurricanes can be regarded as consequences of a finite-amplitude air-sea interaction instability in which strong surface winds near the storm center induce large latent heat **fluxes** which, when distributed through a deep layer by convection, produce the radial buoyancy gradients which in turn intensify the surface winds and so on. This idea has received further support in a recent paper by **Totunno and Emanuel** (1987).

In the present work we apply some of these new concepts to the case of very rapid extratropical maritime cyclogenesis. We advance the hypothesis that wind-induced surface heat **fluxes** can serve to increase the growth rate of baroclinically unstable waves under the appropriate circumstances. First, we develop a semi-geostrophic two-level model which allows for effectively neutral stability in regions of ascent (saturation) and positive stability where the motion is downward (unsaturated). Analytic solutions show that growth rates are increased by a factor of 2.5 over the classical (dimensionless) dry value, with a wavelength of maximum growth of about 60% of the dry wavelength. This brings the classical theory of baroclinic instability into much closer agreement with observations of ordinary cyclogenesis events. Second, we allow for the effect of surface heat **fluxes** by adding a heating term proportional to the lower layer meridional velocity in the ascent region only. This further increases the growth rates to values typical of observed maritime explosive cyclogenesis. These early results suggest that the physical mechanisms operative in baroclinic cyclones and tropical storms may combine constructively in the case of certain oceanic extratropical cyclones.

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LECTURES OF FELLOWS

INTERACTIONS BETWEEN A POINT VORTEX AND A SHEAR-FLOW

Katherine Hedstrom

Introduction

This study was motivated by the interactions between the Gulf Stream and the Gulf Stream Rings. When a ring is very close to the Gulf Stream the interaction is quite complicated and non-linear. I am only going to consider the problem where the ring is farther from the stream and the linear dynamics are a good approximation. Rings which are farther from the stream move parallel to the stream, but in the opposite direction. Possible explanations include advection by a return flow and an effect due to beta- the variation of the Coriolis parameter with latitude. This paper gives the two-layer version of a theory presented by Flierl and Stern (1986) where the Gulf Stream Ring motion is possibly due to an interaction with the vorticity jump at the edge of the Gulf Stream.

The ocean resembles a two layer fluid where the lower layer is deeper and with lower velocities than the upper layer. The two-layer solutions with realistic parameters closely resemble the one-layer solutions except that the induced vortex motion is slower. The two-layer problem with non-zero potential vorticity in the lower layer has many other possible solutions which are quite interesting and shed light on the interactions between the two layers. Some of these solutions are also presented.

The Model

In the model the Gulf Stream Ring is idealized as a point vortex and the wall of the Stream is represented by a jump in the mean vorticity. The lower layer has a corresponding vortex and vorticity jump (weaker in a realistic model).

An equation obeyed by this system is the conservation of potential vorticity, where we have used the quasi-geostrophic, constant Coriolis parameter potential vorticity.

$$\frac{D p_i}{D t} = 0 \quad p_i = \nabla^2 \psi_i + \frac{f_i^2}{2 h_i} (-1)^{i+1} (\psi_2 - \psi_1)$$

The model assumes zero potential vorticity north of the interface except for a delta function of strength s which is the circulation around the point vortex. The potential vorticity to the south is constant in each layer.

$$p_i = s_i \delta(\vec{r} - \vec{r}_{0,i}) + Q_i H(L_i - y)$$

where $H(x)$ is the step function. One also needs an equation for the interface:

$$\frac{\partial L_i}{\partial t} + u_i \frac{\partial L_i}{\partial x} = v_i \quad @ \quad y = L_i$$

Scaling and non-dimensionalizing brings out the important parameters in the problem. I scaled the distance by the initial vortex-stream distance R_0 , so as to always have the same scale in the computer model. Time is normalized by the amplitude of the upper layer vorticity, which normalizes the speed of the interfacial Rossby waves.

$$x, y = R_0 (\tilde{x}, \tilde{y})$$

$$t = \theta \tilde{t} \quad \theta = |Q_1|$$

$$u, v = R_0 (\tilde{u}, \tilde{v})$$

$$\psi = \tilde{\psi} \theta / \theta$$

$$s = \tilde{s} \theta / \theta$$

The equations become:

$$\nabla^2 \tilde{\psi}_1 + \gamma_1^2 (\tilde{\psi}_2 - \tilde{\psi}_1) = \tilde{s}_1 \delta(\tilde{r} - \tilde{r}_0) + \tilde{Q}_1 H(\tilde{L}_1 - \tilde{y})$$

$$\nabla^2 \tilde{\psi}_2 + \gamma_2^2 (\tilde{\psi}_1 - \tilde{\psi}_2) = \tilde{s}_2 \delta(\tilde{r} - \tilde{r}_0) + \tilde{Q}_2 H(\tilde{L}_2 - \tilde{y})$$

$$\frac{\partial \tilde{L}_1}{\partial t} - \frac{\partial \tilde{\psi}_1}{\partial y} \frac{\partial \tilde{L}_1}{\partial x} = \frac{\partial \tilde{\psi}_1}{\partial x} \quad \text{at } y = L_1$$

where $\gamma_1^2 = \frac{R_1^2 \lambda^2}{g' L_1}$ $\tilde{Q}_1 = \pm 1$ $\tilde{Q}_2 = \frac{Q_2}{\theta}$

The important parameters are:

$$\lambda^2 = \gamma_1^2 + \gamma_2^2 = \frac{R_1^2}{R_2^2}$$

$$\alpha = \frac{\gamma_2^2}{\gamma_1^2} = \frac{L_1}{L_2}$$

$$\tilde{s} = s \frac{\theta}{R_1^2}$$

λ is the vortex-stream separation in units of the deformation radius. L is the ratio of the layer depths and s is the strength of the vortex as seen by the stream and depends on their separation. I am going to choose Q_1 to be positive. To solve the equations they were linearized as shown below.

Single Layer Solutions (Flierl and Stern)

In order to make sense of the two layer solutions it is necessary to understand the one layer solutions. The one layer problem has only one parameter. $\tilde{s} = s/R_1^2$. For large s (dropping the tildes) the vortex is strong and close to the stream and it will advect part of the stream around itself as seen in Stern's contour dynamical solutions.

$$R_0 = 0.8$$

When s is small the equations can be linearized. The linear problem has a steady solution where the vortex is moving at a constant speed parallel to the interface. When s is negative (or Q and s are of opposite sign) this solution is well behaved. There is a lump on the interface which is seen by the vortex as an image, or the other half of a modon, and the system propagates at the modon speed.

$$s = -1 \quad c = \frac{s}{4\pi}$$

When s changes sign the fully linear problem has an equal and opposite solution. In reality, the vortex moves in the direction of the phase velocity of the interfacial waves and it excites a resonance with the waves. As the waves grow they exert a force on the vortex, pulling it towards the stream.

$$s = 1$$

One can also look at the transient problem where the interface is initially flat and the vortex appears at time $t=0$. Once again the solution depends on the sign of s . For a negative s the initial motion is as seen in the figure. The region of positive L (interface displacement) corresponds to positive anomalous vorticity and likewise negative L has negative anomalous vorticity. These exert a force on the vortex to the north, away from the stream. At a later time the waves have moved the interface so that a lump appears under the vortex, pushing it to the west. The long term vortex motion is cycloidal, with net transport towards the west.

$$\uparrow \quad s = -1$$

$x=30$

For a positive vortex the motion is once again quite different. The initial motion of the interface is the opposite and the resulting force on the vortex is towards the stream. As the vortex approaches, the stream the interface deforms more and the vortex is pulled all the way into the stream.

$x=2$

$x=4$

$x=9$

Effect of Changing the Vortex Strength

In the fully linear problem the vortex strength only changes the amplitude of the motions and does not change the qualitative results. Therefore I used a relatively large value for s (one) in all my computation to have large motions which are easy to see. I had checked this assumption for a well-behaved case and found it to be valid.

$$s = -\frac{1}{2}$$

At the end of the summer I checked a different case and found that for $s = 1/2$ the solution is qualitatively different from $s = 1$. This is because the model is only partially linearized, as shown in the next section.

$$s = \frac{1}{2}$$

The Two-Layer Model

The velocity $\vec{v}$ is assumed to be a linear superposition of the mean flow, the circulation around the vortex, and the flow induced by the interface displacement. The interface displacement is taken infinitesimally small so that its effect is located at $y = 0$.

$$\psi = \bar{\psi} + \psi_v + \psi_L$$

$$\nabla^2 \bar{\psi} + \gamma_1^2 (\bar{\psi}_2 - \bar{\psi}_1) = Q_1 H(-y)$$

$$\nabla^2 \psi_v + \gamma_1^2 (\psi_{2v} - \psi_{1v}) = s_1 \delta(\vec{r} - \vec{r}_0)$$

$$\nabla^2 \psi_L + \gamma_1^2 (\psi_{2L} - \psi_{1L}) = Q_1 L_1(x) \delta(y)$$

$$\frac{\partial L_1}{\partial x} = v \quad @ \quad y = 0$$

With similar equations for ψ_2

The mean flow solution was chosen to have zero velocity at the interface and therefore also to the north. This was the simplest case to work with and leads to a flow which is not baroclinically unstable (following Pedlosky. 1985).

$$\bar{u}_1 = \frac{-1}{\alpha+1} \left(y (\alpha Q_1 + Q_2) + \frac{1}{\lambda} (Q_1 - Q_2) \sinh(\lambda y) \right)$$

$$\bar{u}_2 = \frac{-1}{\alpha+1} \left(y (\alpha Q_1 + Q_2) - \frac{1}{\lambda} (Q_1 - Q_2) \sinh(\lambda y) \right)$$

The flow around an upper vortex at x_0, y_0 is:

$$\vec{V}_1 = \frac{S_1}{2\pi(\alpha+1)} \left(\frac{1}{r^2} + \lambda \frac{K_1(\lambda r)}{r} \right) (- (y - y_0), x - x_0)$$

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$\vec{V}_2 = \frac{S_1}{2\pi(\alpha+1)} \left(\frac{1}{r^2} - \alpha \lambda \frac{K_1(\lambda r)}{r} \right) (- (y - y_0), x - x_0)$$

and similarly around a lower vortex:

$$\vec{V}_1 = \frac{S_2}{2\pi(\alpha+1)} \left(\frac{1}{r^2} - \lambda \frac{K_1(\lambda r)}{r} \right) (- (y - y_0), x - x_0)$$

$$\vec{V}_2 = \frac{S_2}{2\pi(\alpha+1)} \left(\frac{1}{r^2} + \alpha \lambda \frac{K_1(\lambda r)}{r} \right) (- (y - y_0), x - x_0)$$

The flow due to the interface displacements can be found using Green's functions.

$$u_1 = \frac{-1}{2\pi(\alpha+1)} \left[\int_{-\infty}^{\infty} y \frac{(\alpha Q_1 L_1 + Q_2 L_2) dx'}{(x-x')^2 + y^2} + \int_{-\infty}^{\infty} \lambda y \frac{K_1(\lambda \sqrt{(x-x')^2 + y^2})}{\sqrt{(x-x')^2 + y^2}} (Q_1 L_1 - Q_2 L_2) dx' \right]$$

$$u_2 = \frac{-1}{2\pi(\alpha+1)} \left[\int_{-\infty}^{\infty} y \frac{(\alpha Q_1 L_1 + Q_2 L_2) dx'}{(x-x')^2 + y^2} - \alpha \int_{-\infty}^{\infty} \lambda y \frac{K_1(\lambda \sqrt{(x-x')^2 + y^2})}{\sqrt{(x-x')^2 + y^2}} (Q_1 L_1 - Q_2 L_2) dx' \right]$$

$$v_1 = \frac{-1}{2\pi(\alpha+1)} \left[\int_{-\infty}^{\infty} \frac{(x-x')(\alpha Q_1 L_1 + Q_2 L_2) dx'}{(x-x')^2 + y^2} + \int_{-\infty}^{\infty} \lambda \frac{(x-x') K_1(\lambda \sqrt{(x-x')^2 + y^2})}{\sqrt{(x-x')^2 + y^2}} (Q_1 L_1 - Q_2 L_2) dx' \right]$$

$$v_2 = \frac{-1}{2\pi(\alpha+1)} \left[\int_{-\infty}^{\infty} \frac{(x-x')(\alpha Q_1 L_1 + Q_2 L_2) dx'}{(x-x')^2 + y^2} - \alpha \int_{-\infty}^{\infty} \lambda \frac{(x-x') K_1(\lambda \sqrt{(x-x')^2 + y^2})}{\sqrt{(x-x')^2 + y^2}} (Q_1 L_1 - Q_2 L_2) dx' \right]$$

These velocities were inserted into the interface equation and a steady solution was assumed for the problem where Q_1 , Q_2 , s_1 , and s_2 are all non-zero. A contradiction was encountered in that either one of the vortices would be pulled toward the stream or they would move parallel to the stream at different speeds, thereby separating and forcing a time dependant solution.

Since the problem is time dependant, it was decided that modeling it on a computer would be the way to solve it. The equations are the same except that the vortices were allowed to move finite distances, which is the only non-linear aspect of the model. A Runge-Kutta integration scheme was used.

The interface position was evaluated at a finite number of places separated by dx , over a finite length. By changing the length and spacing dx I came to the conclusion that although dx was large, the short length of the interface was a more serious problem. To get quantitative agreement with the steady solution a longer interface is needed, although the solutions agreed qualitatively. The short length of the interface also meant that the integrations had to be stopped after the time it took for the waves to reach the end and reflect back.

This problem has a large number of parameters and I chose to start by looking at the cases where $\alpha = 1$, $Q_1 = 1$, and s_1, s_2 , and Q_2 are each either -1, 0, or 1. This leaves eighteen different cases, each of which changes with λ .

The barotropic component of the velocity decreases as $1/r$ while the baroclinic component goes like $\lambda K_1(\lambda r)$. Plotting this for different values of λ one can see that at $r=1$ it changes from almost $1/r$ to 0 over the range $\lambda = 0.2$ to 5. These correspond to qualitatively very different solutions as the flow changes from strong baroclinicity to almost purely barotropic motions.

When there are two vortices' (one in each layer) one's force on the other is proportional to $1/r - \lambda K_1(\lambda r)$. If the vortices are equal and opposite they are a *heton* (Hogg and Stommel, 1985) which can self-propagate at a speed

$$c = \frac{g}{4\pi} \left(\frac{1}{r} - \lambda K_1(\lambda r) \right)$$

For small λ this is insignificant while for large λ it is a dominant effect. The transition is for $\lambda = O(1)$.

In the cases where the two vortices are of the same sign there is a similar transition, where the vortices become locked into spinning around each other. To demonstrate these effects let us look at the cases where

$$\begin{bmatrix} Q_1 \\ Q_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

In all three cases the upper layer is the same and there is no lower vorticity jump. Only the lower vortex changes.

When λ is small the motion is the same; the upper vortex motion is like the one-layer motion, while the lower layer has very little flow.

For $\lambda = 1$, the vortices still separate, but this is in the transition region where it is difficult to say exactly what is happening.

When $\lambda = 3$, unless the vortices are equal and opposite they will spin around each other and not become separated. If they are equal and opposite, as soon as the flow from the interface motion separates them, they will move as **heton**. This is the only case that is different when $\lambda = 5$, in which case the baroclinic vortex flow is damped out before it reaches the interface and so the interface does not move enough to separate the vortices.

The motion for the two negative vortices is larger than for just the upper one because the interface is responding to both of them and its motion is simply larger because the vortices remain together.

Another case where the motion is reasonably simple is with

$$\begin{bmatrix} a_1 \\ a_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

where there is only one vortex and one can observe how the lower interface responds as λ varies. For small λ there is once again very little motion in the lower layer while the upper layer motion resembles the one-layer solution. As λ increases, the baroclinic component of the interface motion decreases until at $\lambda=5$ the flow is essentially barotropic.

An interesting case is

$$\begin{bmatrix} a_1 \\ a_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

If the two layers are independent (λ small) the upper vortex will be drawn south towards the interface, while the lower one would move to the east.

At large λ the two vortices stay together so that the forces on the pair bring them south and east.

Comparison with Gulf Stream Rings

In the ocean α is closer to .2 than to 1 so I did a few cases for $\alpha=.2$ and the results are not qualitatively different from the $\alpha=1$ solution. A steady state solution exists for an upper layer vortex and stream over a passive lower layer. Using realistic numbers leads to the case:

$$\begin{bmatrix} Q_1 \\ Q_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -s \\ 0 \end{bmatrix}$$

where s is a function of the distance from the stream.

$$\begin{array}{ll} \lambda = 1 \rightarrow s = 2.5 & \text{strongly non-linear} \\ \lambda = 3 \rightarrow s = 0.3 & c = 2.5 \text{ cm/sec} \\ \lambda = 5 \rightarrow s = 0.1 & c = 1.2 \text{ cm/sec} \end{array}$$

The corresponding speeds when $\alpha = 1$ are 7.7 cm/sec and 4.4 cm/sec. The deeper lower layer tends to reduce the speeds.

Conclusions

Adding a lower layer to the one layer problem does not qualitatively change the results unless the lower layer also has changes in the potential vorticity. In that case a large variety of solutions are possible, the most interesting of which have interactions between an upper and a lower vortex. The ratio of the dominant scale to the deformation-radius is an important parameter in determining if the solution is primarily barotropic or baroclinic and has a dramatic affect on the character of the solution. In most cases, the vortices are drawn in toward the edge of the current, either by resonant interaction with waves on the interface or by "heton"-like interaction of vortices.

Not all of the possible solutions are relevant to the ocean. A problem similar to the Gulf Stream and a ring allows a steady solution where the ring moves parallel to the stream and in the opposite direction at a speed which is of the right order of magnitude.

Aknowledgements

I wish to thank Glenn Flierl for suggesting the problem and for all his help along the way. I would also like to thank Melvin Stern for his helpful comments. I am indebted to Peter Franks and Cabell Davis for allowing me the use of their computers for the calculations.

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STABILITY OF PARALLEL SHEAR FLOWS WITH MEAN SPANWISE VARIATION

Dan Henningson

I Introduction

In traditional stability analysis a wave disturbance is introduced in a mean flow field which varies with one coordinate, usually the vertical. In the **inviscid** case the waves can only grow if the mean flow has a local point of maximum vorticity. Viscosity can also, in some cases, destabilize the flow resulting in so called Tollmien-Schlichting waves. (For a text on stability analysis see eg. Drazin and Reid, 1981) However, stability analysis of more complicated flow fields have also been made, e.g. for flows with mean flow dependence on more than one coordinate or with velocity components in more than one direction. Usually one takes the mean flow to be a specific solution to the Navier-Stokes or Euler equations and thus the results may have limited general applicability. Some recent examples incorporating these additional features are the following.

Pierrehumbert and Widnall (1982) considered the **inviscid** stability of the rows of vortecies formed in a shear layer as analyzed by Stuart (1967). They found two types of instability, one where the **spanwise** periodic variation of two neighbouring vorticies was 180 degrees out of phase and another where the perturbation was in phase. The former instability, which corresponds to vortex pairing, had a preference for lower **spanwise** wavenumbers whereas the the latter did not have a high wave number cut off. Pierrehumbert and Widnall noted that this latter broad band instability, extending to large wave numbers, provided a mechanism for direct generation of small scales and could thus be important in turbulent flows or in the transition process.

Pearson and **Abernathy** (1986) also considered the stability of a vortex flow. Their mean field consisted of one single streamwise vortex embedded in a linear shear. They found an instability that was stronger with increasing vortex circulation and where the most unstable wave number scaled with the size of the vortex core. They related their findings to the breakup of low speed streaks found in turbulent boundary layers, which have been thought to be associated with streamwise vorticity.

A third type of basic field has been considered by, among others, **Orzag** and **Patera** (1983). They used a finite amplitude two-dimensional Tollmien-Schlichting wave and its background shear flow as the basic state. They found that three dimensional waves riding on the two dimensional one were unstable and were growing in a convective (**inviscid**) time scale. Even decaying two dimensional waves were found to be unstable to a broad band of three dimensional disturbances, possibly explaining subcritical transition in Poiseuille and Couette flow.

In this paper the stability of another type of basic flow will be considered: namely a mean velocity profile with flow in the streamwise direction only, but with shear in both the vertical and **spanwise** directions. Only the inviscid case will be addressed. If the **spanwise** variation is made periodic and limited to the wall region in a boundary layer this could model the low speed streaks found in turbulent boundary layers, (Kline et al. 1967). These have now been shown to mainly consist of semi periodic velocity defects in the **streamwise** velocity only (Moin and Kim 1982), thus their breakup might better be modelled with this mean flow than with the mean flow used by Pearson and Abernathy (1986). Various other flows with **spanwise** variation can also be found for which stability is an interesting question. The laminar but nonuniform velocity outside a turbulent spot for example.

The solution method used in this paper is similar to that of Orzag and Patera (1983) and Pearson and Abernathy (1986). The **spanwise** variation is expanded in a Fourier series which then results in a coupled set of Rayleigh-like equations, which are truncated and solved numerically.

II Derivation of the stability equations

The starting point are the non-dimensional Euler equations, linearized around a mean field of the form $U = (U(y,z), 0, 0)$, we have:

$$u_t + Uu_x + U_y v + U_z w = -p_x \quad (1)$$

$$v_t + Uv_x = -p_y \quad (2)$$

$$w_t + Uw_x = -p_z \quad (3)$$

$$u_x + v_y + w_z = 0 \quad (4)$$

where $x = (x, y, z)$ and $u = (u, v, w)$ are the coordinates and perturbation velocities in the **streamwise**, vertical and **spanwise** directions respectively, and p is the pressure. The flows we consider will be of boundary layer type bounded by one horizontal surface at $y=0$, (the length scale is the boundary layer thickness and the velocity scale is the maximum mean velocity). With both a vertical and a **spanwise** gradient in the mean flow it is not possible to obtain an uncoupled equation for any of the velocity components alone. (This also made attempts for a generalization of Rayleigh's inflexion point criterion fail, so far). An equation for the pressure was instead sought. Such an equation can be found by first taking the divergence of the vector equation (1) - (3) with the use of (4). This gives

$$\nabla^2 p = -2v U_y - 2w U_z \quad (5)$$

Then, by taking the linearized substantial time derivative of (5) using (2) and (3) one finds

$$(\partial/\partial t + U\partial/\partial x)\nabla^2 p = 2U_y p_{xy} + 2U_z p_{xz} \quad (6)$$

We will now assume that the mean field can be expanded in a Fourier series in the spanwise direction, i.e.

$$U(y, z) = \sum_{n=-\infty}^{\infty} U_n(y) e^{i\beta_n z} \quad \beta_n = n\beta_1 \quad (7)$$

If we then assume the perturbation quantities to be single wave components, the pressure has to have the following form:

$$p(x, y, z) = \sum_{n=-\infty}^{\infty} p_n(y) e^{i\beta_n z} e^{i(\alpha x + \beta z - \omega t)} \quad (8)$$

The perturbation quantities also has to be expanded in a Fourier series. This can most easily be understood if we consider the spanwise part of the differential equation as a generalized Floquet problem. $U(y, z)$ is then a periodic coefficient in the spanwise direction. This means that the pressure also has to be a periodic function with the same basic periodicity multiplied with a characteristic exponent. Introducing (7) and (8) in (6) we find (for each n)

$$(p'' * U)_n - c p_n'' - k_n^2 ((p * U)_n - c p_n) = 2(U' * p')_n - 2((\beta + \beta_k) p * \beta_k U)_n \quad (9)$$

where prime indicates y -derivative, $k_n^2 = \alpha^2 + (\beta + \beta_n)^2$ is a generalized wave number and $(\cdot * \cdot)$ indicates a convolution sum, i.e.

$$(a * b)_n = \sum_{k+l=n} a_k b_l$$

The boundary conditions are that the normal derivative of the pressure should vanish at solid boundaries and go towards zero at infinity, thus

$$\begin{aligned} p'_n &= 0 & \text{solid boundary} \\ p'_n &\rightarrow 0 & \text{as } y \rightarrow \infty \end{aligned} \quad (10)$$

Equations (9) and (10) constitutes the eigenvalue problem we wish to solve. It consists of an infinite number of coupled Rayleigh-like equations where the eigenvalue will be taken as the complex phase speed c .

As a simple first case we will only retain the U_1 , U , U_{-1} terms in the mean velocity, i.e.

$$U(y, z) = U_0(y) + \epsilon U_1(y) e^{i\beta_1 z} + \epsilon U_{-1}(y) e^{-i\beta_1 z} \quad (11)$$

where U_1 is taken as real and equal to U_{-1} and a factor epsilon is multiplied into U_1 terms as a measure of the amplitude of the spanwise variation. If (11) is introduced into (9) we have

$$\begin{aligned} (U_0 - c) p''_n + \epsilon U_1 p''_{n+1} + \epsilon U_{-1} p''_{n-1} &= k_n^2 (U_0 - c) p_n + 2U'_0 p'_n \\ + \epsilon U_1 (k_n^2 + 2\beta_1 (\beta + \beta_{n+1})) p_{n+1} &+ 2\epsilon U'_1 p'_{n+1} \\ + \epsilon U_{-1} (k_n^2 - 2\beta_1 (\beta + \beta_{n-1})) p_{n-1} &+ 2\epsilon U'_{-1} p'_{n-1} \end{aligned} \quad (12)$$

If epsilon for the moment is assumed small and $p_0 = O(1)$ the coupling terms can be seen to come in at $O(\epsilon)$. If the problem is now considered a perturbation problem with epsilon as a small parameter we find that the following must hold:

$$\begin{aligned} p_0 &= p_{00} + \epsilon^2 p_{02} + \epsilon^4 p_{04} + O(\epsilon^6) \\ p_1 &= \epsilon p_{11} + \epsilon^3 p_{13} + \epsilon^5 p_{15} + O(\epsilon^7) \\ p_2 &= \epsilon^2 p_{22} + \epsilon^4 p_{24} + \epsilon^6 p_{26} + O(\epsilon^8) \end{aligned}$$

where p_{-1} has the same type of expansion as p_1 , etc. Thus we find that the smaller the amplitude of the **spanwise** variation the fewer equations need to be retained in the coupled system (12). Here we have chosen to retain five terms p_{-2} , p_{-1} , p_0 , p_1 and p_2 . One may note that this solution procedure is equivalent to that of both **Orzag** and **Patera** (1983) and **Pearson** and **Abernathy** (1986). **Orzag** and **Patera** go further in their simplification, however, and asserts that p_n and p_{-n} must be complex conjugates. This means that a term of the form

$$p_n e^{i\beta_n z} + p_{-n} e^{-i\beta_n z}$$

is real. There seems to be no a priori reason for this assertion since p_0 is inherently complex and that all of the terms are multiplied with the complex factor $\exp[i(\alpha_x + \beta_z - a ct)]$, which is complex. Instead p_n is in our case seen to equal p_{-n} , if β is equal to zero. This fact could be used to reduce the number of coupled equations to be solved if one is not interested in the case $\beta \neq 0$.

III Numerical solution methods

By retaining the terms p_n with $|n| \leq 2$ in (10) and (12), as indicated above, we obtain five coupled second order equations. These can be turned into a system of ten first order equations with the following form:

$$A \phi' = B \phi \quad (13)$$

$$\phi_n = 0 \quad \text{solid boundary} \quad n = 6, \dots, 10 \quad (14a)$$

$$\phi_n \rightarrow 0 \quad \text{as } y \rightarrow \infty \quad n = 6, \dots, 10 \quad (14b)$$

$$\phi = \begin{bmatrix} p_{-2} \\ p_{-1} \\ p_0 \\ p_1 \\ p_2 \\ p_{-2} \\ p_{-1} \\ p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \\ \phi_{10} \end{bmatrix}$$

$$A = \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & I \\ B_{21} & B_{22} \end{bmatrix}$$

where $\mathbf{I}$ is a 5×5 identity matrix and

$$A_{22} = \begin{vmatrix} U_0 - c & \epsilon U_1 & 0 & 0 & 0 \\ \epsilon U_1 & U_0 - c & \epsilon U_1 & 0 & 0 \\ 0 & \epsilon U_1 & U_0 - c & \epsilon U_1 & 0 \\ 0 & 0 & \epsilon U_1 & U_0 - c & \epsilon U_1 \\ 0 & 0 & 0 & \epsilon U_1 & U_0 - c \end{vmatrix}$$

$$B_{21}: b_{nn} = k_{n-3}^2 (U_0 - c)$$

$$b_{n(n+1)} = \epsilon U_1 (k_{n-3}^2 + 2\beta_1 (\beta + \beta_{n-2}))$$

$$b_{n(n-1)} = \epsilon U_1 (k_{n-3}^2 - 2\beta_1 (\beta + \beta_{n-4}))$$

$$B_{22} = \begin{vmatrix} 2U'_0 & 2\epsilon U'_1 & 0 & 0 & 0 \\ 2\epsilon U'_1 & 2U'_0 & 2\epsilon U'_1 & 0 & 0 \\ 0 & 2\epsilon U'_1 & 2U'_0 & 2\epsilon U'_1 & 0 \\ 0 & 0 & 2\epsilon U'_1 & 2U'_0 & 2\epsilon U'_1 \\ 0 & 0 & 0 & 2\epsilon U'_1 & 2U'_0 \end{vmatrix}$$

Equations (13) have the general solution

$$\phi = \sum_{n=1}^{10} \gamma_n \phi^{(n)} \quad (15)$$

where $\phi^{(n)}$ are one of ten linearly independent **homogeneous solution**. If the mean flow is a boundary layer profile such that $U \rightarrow 1$ as $y \rightarrow \infty$ and $U \rightarrow 0$ as $y \rightarrow 0$ then the equations (13) decouple and ten simple linearly independent **solutions** can be found. Five grow and five decay as y approaches infinity. The solutions that grow can immediately be disregarded while the others can be used as boundary conditions for large y .

$$\phi_n^{(n)}(y_{\max}) = e^{-k_{n-3} y_{\max}} \quad n = 1, \dots, 5$$

$$\phi_{n+5}^{(n)}(y_{\max}) = -k_{n-3} e^{-k_{n-3} y_{\max}} \quad n = 1, \dots, 5 \quad (16)$$

$$\gamma_n = 0 \quad n = 6, \dots, 10$$

where $y_{\max}$ is the starting point of the intergration. Equations (13) are then integrated down to $y = 0$ (the **solid** boundary) where the boundary conditions.

(14a) are applied. This reduces to

$$\begin{vmatrix} \phi_6^{(1)} & \phi_6^{(2)} & \phi_6^{(3)} & \phi_6^{(4)} & \phi_6^{(5)} \\ \phi_7^{(1)} & \phi_7^{(2)} & \phi_7^{(3)} & \phi_7^{(4)} & \phi_7^{(5)} \\ \phi_8^{(1)} & \phi_8^{(2)} & \phi_8^{(3)} & \phi_8^{(4)} & \phi_8^{(5)} \\ \phi_9^{(1)} & \phi_9^{(2)} & \phi_9^{(3)} & \phi_9^{(4)} & \phi_9^{(5)} \\ \phi_{10}^{(1)} & \phi_{10}^{(2)} & \phi_{10}^{(3)} & \phi_{10}^{(4)} & \phi_{10}^{(5)} \end{vmatrix} \begin{vmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{vmatrix} = 0 \quad (17)$$

If a solution is to exist to this system the determinant of the matrix has to be identically zero. This condition can only be satisfied for certain values of the phase speed c . This condition will thus determine the eigen values. Computationally the eigenvalues can be found by a shooting method: A particular value for c is chosen, c_1 say. Equations (13) are integrated and the determinant in (17) is evaluated, d_1 say. This is done for another value of c , c_2 say, resulting in another value of d , d_2 say. If these chosen values are sufficiently close to a correct eigenvalue a better approximation can be found by linear extrapolation in the complex c -plane, *i.e.*

$$c = c_2 - d_2(c_1 - c_2)/(d_1 - d_2)$$

This process can be repeated until desired accuracy is obtained.

The integration method used is a fourth order **Runge-Kutta** method, (see *e.g.* **Dahlquist** and Bjork, 1974). It is for a linear problem defined as follows:

$$k_1 = \bar{B}(y)\phi(y)h$$

$$k_2 = \bar{B}(y+h/2)(\phi(y) + k_1/2)h$$

$$k_3 = \bar{B}(y+h/2)(\phi(y) + k_2/2)h$$

$$k_4 = \bar{B}(y+h)(\phi(y) + k_3)h$$

$$\phi(y+h) = \phi(y) + (k_1 + 2k_2 + 2k_3 + k_4)/6$$

where $h=h(y)$ is the discretization step. Equations (13) has to be rewritten in the following form to fit the assumed form of the Runge-Kutta method

$$\phi' = \bar{B}\phi \quad (18)$$

This can be done as follows:

$$\bar{B} = \begin{vmatrix} 0 & I \\ A_{22}^{-1}B_{21} & A_{22}^{-1}B_{22} \end{vmatrix}$$

There are some singularities present in the above equations. They are found where the determinant of A_{22} is zero. Evaluation of the determinant gives the result

$$(U_0 - c)(U_0 - c + \epsilon U_1)(U_0 - c - \epsilon U_1)(U_0 - c + \sqrt{3}\epsilon U_1)(U_0 - c - \sqrt{3}\epsilon U_1) = 0$$

If the imaginary part of c is small these singularities are close to the real y -axis. They can then be avoided by choosing a complex integration path in the complex plane. According to Lin (1955) this path should be below the singularities in order to assure that the inviscid solution is a true limit of the viscous case. Such a path can be constructed by going in a circular arc from y_{max} to $y = 0$ under the real axis. In the numerical examples presented below such an integration path was used for cases where the eigenvalue was close or on the real axis. The mean velocity has to be an analytic function of y in the part of the complex plane covered by the circular path in order for this procedure to give correct results.

IV Numerical examples

As a simple model for a turbulent boundary layer type profile with inflexion we use for the mean field the hyperbolic tangent function. It is displaced so that its inflexion point is located at a positive y -value. This is done to insure that an unstable solution should exist even for the case of no spanwise variation. U_1 is taken equal to U_0 , multiplied by a function that makes it go to zero at infinity. The expressions used are as follows:

$$U_0 = (\tanh(y-h_1) + \tanh h_1) / (1 + \tanh h_1) \quad (19)$$

$$U_1 = \epsilon U_0 (1 - \tanh(a_1(y-b_1))) / 2$$

For the calculations presented here the following parameter combinations were used, if not other wise indicated: $h_1 = 0.5$, $a_1 = b_1 = 2.5$. A plot of these mean profiles can be seen in figure 1.

Equations (18) were then solved using the mean velocities (19) with the methods indicated in the previous section. In addition to the mentioned singularities coming from the zeroes of the determinat of A22 in (18) another singularity was found to cause some difficulty when the integration path was chosen too far below the real y-axis. U_1 as chosen has a singularity in the complex plane at $y = (b_1, \pi / 2a_1)$. The iteration path has to be taken above this point if a correct eigenvalue is to be found. Different parameter ranges were explored. The resulting eigenvalues are found in figures 2 and 3. Figure 2 shows how the eigenvalues depend on ϵ (the amplitude of the mean **spanwise** variation), on a (the period of the mean **spanwise** variation) and on h_1 (the position of the inflexion point) for $\beta = 0$. Figure 3 shows their variation with β for fixed a and ϵ . The results clearly indicate that the **spanwise** variation destabilizes the flow. The flow is also more unstable for shorter mean **spanwise** wave lengths. For even larger β_1 (not shown in any figure) the growth rate started to decline somewhat. However, this effect might be due to the truncation of the coupled system since the higher terms of the pressure series grew to large **amplitudes** when β_1 increased. The results for $\beta = 0$ does not increase the **instability**, instead the growth rate decreases except at positions where β is equal to β_1 . This resonance mechanism can be seen in figure 3. Another interesting result is that positive growth rates are found even though the original inflexion point has been taken out of the flow. This shows that the **spanwise** variation can in itself destabilize the flow.

Eigen functions were also calculated for certain parameter combinations. Figures 4 and 5 show the absolute value of two different sets. It is the vertical derivative of the pressure that is shown. The first figure shows the eigen functions for a case where $\beta = 0$ and $\beta_1 = 0$ and $\beta = 1.5$. There p_1 and p_{-1} , and p_2 and p_{-2} are equal, whereas in the next figure where $\beta = 1$ and $\beta_1 = 1$ they are all different. The general shape of the eigen functions were very similar for all parameter ranges. High vertical velocities are seen in the upper part of the boundary layer and only a small local maximum is found at the critical layer. The expected ordering is also seen to hold, with p_0 the largest, then p_1 with p_2 the smallest.

V Discussion and Conclusions

A system of coupled pressure equations governing the **inviscid** stability of a parallel mean flow with periodic **spanwise** variation has been derived. Using these equations the stability of a hyperbolic tangent boundary layer has been investigated. The boundary layer profile was at first such that there was an inflexion point in the flow field. This ensured that there existed some unstable solutions. The growth rate of this unstable mode increased substantially as the amplitude of the **spanwise** variation increased, and the interval of the unstable wave

numbers also grew larger. However, it did not have the broad band nature that **Orzag** and **Patera** (1983) and Pierrehumbert and Widnall (1982) found in their instabilities. It is also worth noting that the instability, although somewhat weaker, persisted when the original inflexion point was taken out of the flow. This shows that **spanwise** variation in itself is able to destabilize an otherwise stable flow. The instability also became weaker with $\beta = 0$ except for cases where $\beta = \beta_1$ where there was resonance and the magnitude of the growth rate was comparable to that of the $\beta = 0$ case.

The eigen functions provide additional information the role of the instability. First they show the amplitudes of the higher order terms in the pressure expansion indeed do drop off as expected, at least for low values of β_1 . They also show that an unusual amount of the vertical velocity is concentrated in the upper part of the boundary layer. This is, however, also the case for no **spanwise** variation. Thus it is more a property of the vertical variation than of the **spanwise** one.

Another interesting feature is the possibility of finding a preferred **spanwise** wave number for the mean variation. The results first showed an increase in growth rate with increasing β_1 then it seemed to go down somewhat. As mentioned in the previous section, this might be due to the truncation of the coupled system since large amplitudes of the higher pressure terms found for this case. This invalidates the assumption that higher order terms in the variation amplitude should be smaller than the lower order terms. The reason this is occurring for large β_1 can be seen in the coupled system of equations. There β_1 is found as a factor in some of the coupling terms. Extending the number of retained terms in the pressure series should resolve this question. It is plausible that this would stop the decline of the growth rate with β_1 and that the term with the highest amplitude in the pressure series is the one corresponding to the chosen β_1 value. In such a case the preferred wave number could only be found including viscosity in the equation.

The possibility of finding some preferred **spanwise** wave number of the mean variation is encouraging, however, since it might have some relation to the distance between the high and low speed streaks found in the wall region in a turbulent boundary layer. These streaks are seen to become unstable, breakup and produce a turbulent burst. This process could be initiated by an instability of the kind investigated presently. A comparison with the other instabilities mentioned in section one might suggest that the types investigated by **Orzag** and **Patera** (1983) and Pierrehumbert and Widnall (1982) could be important in the formation of the low speed streaks (they could produce substantial three dimensionality from a two dimensional flow), the present in their **liftup** and **Pearson & Abernathy's** (1986) in their breakup.

The limited calculations presented here are somewhat inconclusive as to the general role of the investigated instability mechanism. The need for further work naturally presents itself. A few suggestions are the following:

- 1) Some approximate analytical calculations, already started, should be **persued**. Such results might better expose general mechanism of the instability.
- 2) Other types of mean variations should be tried. The **spanwise** variation of the mean flow could, for example, be expanded in **Hermite** polynomials instead of exponentials. This would provide a way of investigating the effect of a local **spanwise** variation.
- 3) Viscosity should be included. This would provide a damping of the smallest scales allowing for a viscous cut off and a possibility of destabilizing Tollmien-Schlichting waves.

Acknowledgments

I would like to thank Marten Landahl for suggesting this problem to me and for all the help I have received working on its solution. It has been a pleasure and privilege to work together. I would also like to thank the organizers of the GFD summer program in Woods Hole for giving me the possibility to come here and to work in such a lovely setting. It has been a great summer in every way.

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Figure 1. Mean velocities U_0 and U_1 . $h_1=0.5$, $l=b_1=2.5$ and $\epsilon=0.1$.

Figure 2. a) Phase speeds and b) growth rates as a function of

wave number α for $\delta = 0$, and

1 : $\epsilon_1 = 0.0, \epsilon = 0.0, h_1 = 0.5$

2 : $\epsilon_1 = 1.0, \epsilon = 0.1, h_1 = 0.5$

3 : $\epsilon_1 = 1.5, \epsilon = 0.1, h_1 = 0.5$

4 : $\epsilon_1 = 1.5, \epsilon = 0.1, h_1 = 0.0$

Figure 3. a) Phase speed and b) growth rate as a function of wave number s for $a = 0.4$, $\beta_1 = 1.0$, $\epsilon = 0.1$, $h_1 = 0.5$.

Figure 4. Eigenfunctions p'_{-2} , p'_{-1} , p'_0 , p'_1 , p'_2 for $\alpha = 0.4$, $\beta = 0.0$, $\delta_1 = 1.5$, $\epsilon = 0.1$, $h_1 = 0.5$.

Figure 5. Eigenfunctions p'_{-2} , p'_{-1} , p'_0 , p'_1 , p'_2 for $\alpha = 0.4$, $\beta = 1.0$, $\delta_1 = 1.0$, $\epsilon = 0.1$, $h_1 = 0.5$.

CONVECTION WITH SPATIAL FORCING AND MEAN FLOW GENERATION

Ya Yan Lu

Abstract: A general scaling is given for a 2-dimensional convection with spatial forcing of one wave number so that the amplitude equations predicted by Coulet (1985) can actually be derived. Forcing at both boundaries with different wave numbers are also considered. More general amplitude equations and mean horizontal flows are found in these cases. The results may have some feature of the strong mean flow found experimentally at high Rayleigh number by Krishnamurti & Howard (1981).

I. Introduction

In the classical studies of convection, a layer of fluid is heated uniformly at the bottom and cooled at the top. After the Rayleigh number exceeds certain value (critical Rayleigh number R_c), free convection is set up at certain critical wave number k_c . In the experiments by Krishnamurti & Howard (1981), the fluid layer are kept, for example, in an annulus. They found a strong horizontal mean flow which could be in both directions at high Rayleigh numbers (about 2,000,000). To understand this, they (Krishnamurti & Howard (1981)) introduced a truncated model of Lorenz type. In this study, we allow certain spatial forcing at the boundaries and deduce a mean flow near the critical value. Of course, there is no forcing in the original experiments, but the mean flow we found can go either directions. Meanwhile, new experiments with forcing can be performed so that we should see mean flow near the critical Rayleigh number.

Convection problems with spatial forcing were studied by Kelly & Pal (1976, 1978) Travantzis & Matkowsky (1978) and more generally by Coulet to the strongest imperfect bifurcation -- the case when the wave number of spatial forcing k equals to the critical wave number. The work done by Coulet is based on a symmetry argument which leads to the resonance condition and a general amplitude equation. However, no actual calculations for convection are carried out. Particularly, there must be some relationship between the strength of forcing and the scale of amplitude of convection. This is the first thing we will study in this paper.

Meanwhile, all these works are limited to forcing with one wave number. No mean flow can be generated to this case. We study the convection problem with symmetric (with respect to space reflection) **forcings** of different wave numbers. One thing should be pointed out is that for forcing at both exists for any Rayleigh number. However, we would like to keep the symmetry, so that the mean flows can be in both directions. Furthermore, the mean flows are generated by the interaction between free convection and the forced convection caused by boundary conditions.

2. Forcing With One Wave Number

Consider the convection problem as follows:

The convection field is assumed to be **2-dimensional**. We introduce a stream function Ψ such that

$$u = -\Psi_z \quad v = \Psi_x \quad (1)$$

After a usual scaling, we get the following equations:

$$\begin{aligned} \nabla^4 \Psi + \theta_x &= \frac{1}{\sigma} \left[\frac{\partial}{\partial t} \nabla^2 \Psi + J(\Psi, \nabla^2 \Psi) \right] \\ \nabla^2 \theta + R \Psi_x &= \frac{\partial \theta}{\partial t} + J(\Psi, \theta) \end{aligned} \quad (2)$$

where θ is the scaled departure of temperature from the linear profile $T_1 + z/d (T_2 - T_1)$ which is the conductive temperature distribution without the forcing. The equations (1)-(2) must be solved with the following boundary conditions:

$$\begin{aligned} \Psi &= 0 & \text{at } z=0 \text{ and } 1 \\ D\Psi &= 0 & \text{for rigid boundary} \\ D^2\Psi &= 0 & \text{for free boundary} \\ \theta &= 0 & \text{at } z=1 \\ \theta &= \eta e^{ikx} + \text{c.c.} & \text{at } z=0 \end{aligned}$$

where η is the scaled strength of forcing.

In the usual convection problem without forcing, one let $R = R_c (1 + \epsilon^2)$ (where R_c is the critical Rayleigh number, ϵ is $O(1)$), and expand Ψ, θ as follows:

$$\Psi = \epsilon \Psi_1 + \epsilon^2 \Psi_2 + \epsilon^3 \Psi_3 + \dots \quad (3)$$

$$\theta = \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \dots \quad (4)$$

The $O(\epsilon)$ solution can be found as

$$\Psi_1 = i \alpha A e^{i \alpha x} g_0(z) + \text{c.c.} \quad (5)$$

$$\theta_1 = -A e^{i \alpha x} (D^2 - \alpha^2) g_0(z) + \text{c.c.} \quad (6)$$

The $O(\epsilon^2)$ solution can also be found. The solvability condition for Ψ_2, θ_2 gives the usual **Landau** equation:

$$\dot{A} = \mu A - \beta |A|^2 A$$

where $\tau = \epsilon^2 t$ is the slow time scale of the amplitude modulation.

When we have a forcing as stated above, the amplitude equation is possible to have another term for k satisfying the resonance condition:

$$mk = na_c \quad (8)$$

where m, n are positive intergers without common factor. Coulet (1985) got the amplitude equation for resonance cases:

$$A_T = \mu A + \alpha \bar{A}^{n-1} - \beta |A|^2 A \quad (9)$$

We find that the proper scaling between η and ϵ to have the above amplitude equation is

$$\eta \sim O(\epsilon^{\frac{1-n}{n}}) \quad \text{for } n=1, 2, 3 \quad (10)$$

Particularly, when $n=1$, the amplitude equation has a constant term in it. Thus we have imperfect bifurcations. When $n=2$, the critical value is shifted by $O(\epsilon^{\frac{1-n}{n}})$. When $n=3$, we have transcritical bifurcations.

It is easy to prove that above scaling results the right amplitude equation (9) if we notice the appearance of e^{ina} term with coefficient independent of A in the $O(\epsilon^{\frac{1-n}{n}})$ equations or boundary conditions. However, when $n \geq 4$, the scaling do not make much sense, so equation (9) probably will never appear for $n \geq 4$ in our problem here.

3. Forcing With Wave Number $a_c, 2a_c$

We consider convection problems with spatial forcing at both boundaries so that a mean flow is possible. The case of two wave numbers being $a_c, 2a_c$ is particularly favoured because of its simplicity. The scaling in previous section lead us to impose the following boundary conditions:

$$\theta = \lambda_3 e^{iax} \epsilon^3 + c.c \quad \text{at } z=1 \quad (11)$$

$$\theta = \lambda_2 e^{i2ax} \epsilon^2 + c.c \quad \text{at } z=0 \quad (12)$$

Expand ψ, θ as in (3), (4), we find ψ_1, θ_1 given by (5), (6). ψ_1, θ_1 are found as

$$\psi_1 = -ia^2 A^2 e^{i2ax} \psi_1(z) - i2a \lambda_2 e^{i2ax} g_1(z) + c.c. \quad (13)$$

$$\theta_1 = a^2 A^2 e^{i2ax} \bar{\psi}_1(z) + a^2 |A|^2 \bar{\psi}_1(z) + \lambda_2 e^{i2ax} (D^2 - 4a^2) g_1 + c.c. \quad (14)$$

where $\psi_1, \bar{\psi}_1, \bar{\psi}_1, \theta_1$ are all real and satisfy certain ordinary different equations. If we let

$$\psi_1 = e^{iax} \tilde{\psi}_1 + c.c. + \dots$$

$$\theta_1 = e^{iax} \tilde{\theta}_1 + c.c. + \dots$$

where ... means terms involving e^{i3ax} . We get

$$(D^2 - \alpha^2) \tilde{\varphi}_3 + \alpha \tilde{\theta}_3 = \frac{1}{\epsilon} \left[\frac{\partial A}{\partial t} \alpha (D^2 - \alpha^2) g_0 + |A|^2 A \mathcal{A}(z) + \lambda_1 \bar{A} \mathcal{B}(z) \right] \quad (15)$$

$$(D^2 - \alpha^2) \tilde{\theta}_3 - R \alpha \tilde{\varphi}_3 = \alpha^2 R \delta g_0 A - \frac{\partial A}{\partial t} (D^2 - \alpha^2) g_0 + |A|^2 A \mathcal{C}(z) + \lambda_2 \bar{A} \mathcal{D}(z) \quad (16)$$

where $\mathcal{A}(z)$, $\mathcal{B}(z)$, $\mathcal{C}(z)$, $\mathcal{D}(z)$ are real function of z which can be expressed in terms of $\gamma, \delta, \epsilon, \eta$. The solvability condition for $O(\epsilon^3)$ equations is the same as that for (15), (16), since only $e^{i\alpha x}$ term will affect it.

We get

$$\dot{A} = \mu_0 + \mu A + \alpha \bar{A} - \beta |A|^2 A \quad (17)$$

where $\mu_0 \propto \lambda_1$, $\mu \propto \delta$, $\alpha \propto \lambda_2$. All these coefficients can be written in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, g_0$ easily. We first analyse the stability of steady solutions of (17). Without loss of generality, we assume $\beta = 1$.

Let $A = R e^{i\theta}$, we get

$$\dot{R} = \mu_0 \cos \theta + \mu R + \alpha R \cos 3\theta - R^3 \quad (18)$$

$$R \dot{\theta} = -\mu_0 \sin \theta - \alpha R \sin 2\theta \quad (19)$$

There are two set of equations:

$$(I): \quad \theta = 0 \quad A = A_0 \quad \mu_0 + \mu A_0 + \alpha A_0 - A_0^3 = 0$$

$$(II): \quad R_0^2 = \mu - \alpha, \quad \cos \theta_0 = -\frac{\mu_0}{2\alpha R_0} \quad \text{for } A = R_0 e^{i\theta_0}$$

A linear stability analysis based on equation (17) reveals the following conclusion.

(1) $\alpha > 0$. (II) is always unstable. The stability of (I) is drawn in Fig. 1

Fig. 1, 2

We notice that when $\alpha \rightarrow 0^+$, the result obtained by Kelley & Pal (1978) is recovered.

(11). $\alpha < 0$. For $|R_0| \geq |\frac{\mu_1}{2\alpha}|$, (II) exist and is stable, when $|R| < |\frac{\mu_1}{2\alpha}|$, (I) is stable, we have the following figure.

Fig. 3

It is clear from Fig. 3 that when $\mu < K = \alpha + \frac{\mu_1^2}{4\alpha^2}$, solution (I) is stable. When $\mu > K = \alpha + \frac{\mu_1^2}{4\alpha^2}$, solution (I) becomes unstable, (II) becomes stable.

For steady solutions, the mean flow $\bar{u}$ is given by the following formula:

$$\frac{d\bar{u}}{dz} = \frac{1}{\sigma} \bar{w} \bar{u} = -\frac{1}{\sigma} \bar{\varphi}_x \bar{\varphi}_z \quad (20)$$

We already have

$$\varphi = \varepsilon i a A e^{i\alpha x} g_0 + \varepsilon^2 \varphi_2 + \varepsilon^3 i e^{i\alpha x} \tilde{\varphi}_3 + \varepsilon^3 (\dots) + o(\varepsilon^3) + c.c.$$

Therefore, the mean flow is found to be

$$\bar{u} = -\frac{1}{\varepsilon} \varepsilon^4 a^2 i \bar{A} \tilde{\varphi}_3 g_0 + c.c. + o(\varepsilon^4) \quad (21)$$

For A real (solution (I)), $\tilde{\varphi}_3$ will also be real, therefore, $\bar{u}$ is zero. However, for steady solution (II), we usually have nonzero mean flow.

4. Forcing With Wave Numbers $a_c, 3a_c$

Another situation which is relatively simple is the case with boundary forcing waves numbers are a_c and $3a_c$. A nonzero mean flow is also possible in certain ranges of parameters. The boundary conditions are set to be

$$\theta = \lambda_3 e^{i\alpha x} \varepsilon^3 + c.c. \quad \text{at } z = 1 \quad (22)$$

$$\theta = \lambda_1 e^{i3\alpha x} \varepsilon + c.c. \quad \text{at } z = 0 \quad (23)$$

Again, we expand φ, θ as in (3), (4). We find

$$\varphi_1 = \frac{1}{(1 + aI)} \sin \pi z - i3a \lambda_1 g_0 e^{i3\alpha x} + c.c. \quad (24)$$

$$\theta_1 = A e^{i\alpha x} \sin \pi z + \lambda_1 (D^2 - q a^2) g_0 e^{i3\alpha x} + c.c. \quad (25)$$

where $g(z)$ is easy to give. This solution is in fact the original free convection plus a new forced convection. For free-free boundaries, we have

$$\varphi_2 = i \chi_2 \bar{A} e^{i2\alpha x} + i \chi_4 \bar{A} e^{i4\alpha x} + i \chi_6 \bar{A} e^{i6\alpha x} + c.c. \quad (26)$$

$$\theta_2 = \gamma_0 + \gamma_0 |A|^2 + \gamma_2 \bar{A} e^{i2\alpha x} + \gamma_4 \bar{A} e^{i4\alpha x} + \gamma_6 \bar{A} e^{i6\alpha x} + c.c. \quad (27)$$

where X_i, Y_i, γ_i all are real functions satisfying certain ordinary differential equations. The solubility condition for ψ, θ equations gives the following amplitude equation:

$$A = \mu_0 + A + \frac{2}{3} \beta |A|^2 A \quad (28)$$

The exact formula for μ_0, β are messy, but we notice the following important feature:

$$\mu \propto \delta, \quad \mu_0 \propto \lambda_3, \quad \alpha \propto \lambda_1$$

As in section 3, we first consider the stability of steady solutions of (28). β is also assumed to be 1.

Let $A = R e^{i\theta}$, we get

$$\dot{R} = \mu_0 \cos \theta + \mu R + \alpha R^2 \cos 3\theta - R^3 \quad (29)$$

$$R \dot{\theta} = -\mu_0 \sin \theta - \alpha R^2 \sin 3\theta \quad (30)$$

The steady solutions are

$$(I) \quad \theta = 0, \quad \mu_0 + \mu R + \alpha R^2 - R^3 = 0$$

$$(II) \quad A = R_0 e^{i\theta_0}, \quad \mu - 2\alpha \cos \theta_0 R_0 - R_0^3 = 0 \quad \text{with} \\ \sin^2 \theta_0 = \frac{3}{4} + \frac{\mu_0}{4\alpha R_0^2}$$

The linear stability of (I) is easy to carry out. Without loss of generality, we assume $\alpha > 0$ (in fact, even $\alpha = -1$ is okay), and find

(1) When $\mu_0 > 0$, we have two cases

a $0 < \mu_0 < \frac{1}{27} \alpha^3$. The stability is shown below

Fig. 4

(1.b) $\mu_0 > \frac{1}{27} \alpha^3$

Fig. 5

(11). When $\mu_0 < 0$, there is a interval of μ (i.e. Rayleigh number) in which all (I) branches lose stability.

The interval is $[\mu_-, \mu_+]$ as shown in Fig. 6, where

$$\mu_{\pm} = -\frac{\mu_0}{3\alpha} \pm \frac{2}{3} \sqrt{-3\alpha\mu_0}$$

It is reasonable to think that solution (II) will have stable branch in this interval.

We have the stream function $\psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + \dots$ where

$$\psi_3 = i\tilde{\psi}_{31} e^{i\alpha x} + i\tilde{\psi}_{33} e^{i3\alpha x} + c.c.$$

thus the mean flow is found to be

$$\bar{u} = -\frac{1}{\pi} \epsilon^4 \left[\frac{i\alpha^3 \bar{A}}{(\pi^2 \alpha^2)^2} \sin \pi x \tilde{\psi}_{31} - 9\alpha^2 \lambda_1 g \tilde{\psi}_{33} \right] + c.c. + o(\epsilon^4) \quad (31)$$

It is likely that for stable solution (II), this mean flow is nonzero..

5. Forcing With Wave Numbers $2a_c, 3a_c$

We study forcing with such a pair of wave numbers in this section. The boundary conditions are

$$\begin{aligned} \theta &= \epsilon \lambda_1 e^{i3\alpha x} + c.c. & \text{at } z=0 \\ \theta &= \epsilon^2 \lambda_2 e^{i2\alpha x} + c.c. & \text{at } z=1 \end{aligned}$$

One might guess that we will get an amplitude equation with terms $\alpha_1 \bar{A}$ and $\alpha_2 \bar{A}^2$ only from the studies in previous sections. However, this is again an imperfect bifurcation, since the difference of two wave numbers is a_c . So this is in fact a quite rich situation. To be more precise, we expand ψ, θ again in ϵ ,

$$\begin{aligned} \psi &= \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + \dots \\ \theta &= \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \dots \end{aligned}$$

Plug these into the original equations (1)-(2), we get a set of equations at each order of ϵ :

$$O(\epsilon) \quad \nabla^4 \psi_1 + \theta_{1x} = 0 \quad (32)$$

$$\nabla^2 \theta_1 + R_c \psi_{1x} = 0 \quad (33)$$

The boundary conditions for these equations is homogeneous except

$$\theta_1 = \lambda_1 e^{i3\alpha x} + c.c. \quad \text{at } z=0$$

The solution for this linear problem is the summation of the homogeneous free convection (with amplitude A) solution and forced convection by the inhomogeneous boundary forcing. We get

$$\varphi_1 = -\frac{iaA}{(\pi^2 a^2)^2} e^{iax} \sin \pi z - i3a\lambda_1 g e^{i3ax} + c.c. \quad (34)$$

$$\theta_1 = Ae^{iax} \sin \pi z + \lambda_1 (D^2 - 9a^2)^2 g e^{i3ax} + c.c. \quad (35)$$

where g satisfies the following equation

$$(D^2 - 9a^2)^2 g + 9a^2 R_c g = 0$$

with an inhomogeneous boundary condition $D^4 g = 1$ at $z=0$. The other boundary conditions are the usual boundary condition for free-free boundaries. We therefore can write g as

$$g = \tilde{A} \sin 9_0(1-z) + B \sin 9(1-z) + B^* \sin 9^*(1-z)$$

so that the boundaries at $z=1$ already satisfied. A, B can be determined by the boundary conditions at $z=0$.

The equations for φ_2, θ_2 are just as usual, i.e.,

$$D^4 \varphi_2 + \theta_{2x} = \frac{1}{\sigma} J(\varphi_1, D^2 \varphi_1) \quad (36)$$

$$D^2 \theta_2 + R_c \varphi_{2x} = J(\varphi_1, \theta_1) \quad (37)$$

The only inhomogeneous boundary condition is

$$\theta_2 = \lambda_2 e^{2iax} + c.c. \quad \text{at } z=1$$

Solutions for φ_2, θ_2 can also be separated in two parts. One is the solution with homogeneous boundary condition but with inhomogeneous rhs (i.e., the linear equations for φ_1, θ_1) with forcing boundary condition. The first part is already considered before. We have

$$\varphi_2 = i\chi_3 \tilde{A} e^{i2ax} + i\chi_4 A e^{i4ax} + i\chi_6 e^{i6ax} - i2a\lambda_2 \tilde{g} e^{i2ax} + c.c.$$

$$\theta_2 = \gamma_0 + \gamma_1 |A|^2 + \gamma_2 \tilde{A} e^{i2ax} + \gamma_4 A e^{i4ax} + \gamma_6 e^{i6ax} + i\lambda_2 (D^2 - 4a^2)^2 \tilde{g} e^{i2ax} + c.c.$$

The $O(\epsilon^3)$ equations are

$$D^4 \varphi_3 + \theta_{3x} = \frac{1}{\sigma} \left[\frac{\partial}{\partial t} D^2 \varphi_1 + J(\varphi_1, D^2 \varphi_2) + J(\varphi_2, D^2 \varphi_1) \right] \quad (38)$$

$$D^2 \theta_3 + R_c \varphi_{3x} = -\delta R_c \varphi_{1x} + \frac{\partial}{\partial t} \theta_1 + J(\varphi_1, \theta_2) + J(\varphi_2, \theta_1) \quad (39)$$

Without going into the detail, we can see that terms like $J(\varphi_1, D^2 \varphi_2)$ in equation (38) will give a e^{iax} term from ge^{i3ax} term of φ_1 and $\tilde{g}e^{2ax}$ term of φ_2 . Therefore, we get the following amplitude equation:

$$A_T = \mu_0 + \mu A + \alpha_1 \tilde{A} + \alpha_2 \tilde{A}^2 - \beta |A|^2 A \quad (40)$$

It is easy to see that

$$\mu_0 \propto \lambda_1 \lambda_2, \quad \mu \propto \delta, \quad \alpha_1 \propto \lambda_2, \quad \alpha_2 \propto \lambda_1$$

In this problem, λ_1, λ_2 are free parameters, but μ_0 is not. However, if we choose boundary conditions like

$$\theta = \varepsilon^2 \lambda_2 e^{i2\alpha x} + \varepsilon^2 \lambda_3 e^{i\alpha x} + c.c. \quad \text{at } z=1$$

there will be another contribution to μ_0 from the α_c wave number forcing.

Consider the steady solution (40). Let $\Lambda = R e^{i\theta}$, the original equation becomes

$$\mu_0 \cos \theta + \mu R + \alpha_1 R \cos 2\theta + \alpha_2 R^2 \cos 3\theta - \beta R^3 \quad (41)$$

$$\dot{\theta} = -\frac{\mu_0}{R} \sin \theta - \alpha_1 \sin 2\theta - \alpha_2 R \sin 3\theta \quad (42)$$

The steady solutions are

$$(I) \quad \theta = 0. \quad \mu_0 + (\mu + \alpha_1) R_0 + \alpha_2 R_0^2 - \beta R_0^3 = 0 \quad (43)$$

$$(II) \quad \begin{cases} -\frac{\mu_0}{R_0} - 2\alpha_1 \cos \theta_0 + 4\alpha_2 R_0 \sin^2 \theta_0 - 3\alpha_2 R_0 \sin \theta_0 = 0 \\ \mu - \alpha_1 + (\cos \theta_0 - \frac{3}{2} \sin 2\theta_0) \alpha_2 R_0 - \beta R_0^2 = 0 \end{cases} \quad (44)$$

$$\mu - \alpha_1 + (\cos \theta_0 - \frac{3}{2} \sin 2\theta_0) \alpha_2 R_0 - \beta R_0^2 = 0 \quad (45)$$

We look at the linear stability of (I). Let

$$R = R_0 + R'$$

$$\theta = \theta'$$

The linearized equations for R', θ' are

$$\dot{R}' = (\mu + \alpha_1 + 2R_0 \alpha_2 - 3\beta R_0^2) R' \quad (46)$$

$$\dot{\theta}' = -\left(\frac{\mu_0}{R_0} + R_0\right) \theta' \quad (47)$$

From (46), we concluded that the steady solution (I) has an unstable branch if there are three branches of solutions for one value of μ , it is the one with medium value of R_0 . From (47), we conclude that a necessary condition for stability is

$$\frac{\mu_0}{R_0} + 2\alpha_1 + 3\alpha_2 R_0 \geq 0$$

We assume $\alpha_2 > 0$. The case of $\alpha_2 < 0$ is the same for $-A$. There are several cases:

$$(1) \mu_0 > 0$$

if $\alpha_1 < -\sqrt{3\alpha_2 \mu_0}$, the branch with positive sign may have a unstable piece. It is an interesting case, since all the branches of (I) may be unstable.

if $|\alpha_1| < \sqrt{3\alpha_2 \mu_0}$, then the positive branch is basically stable (except for the possible the case of three positive solution). The negative branches are unstable.

if $\alpha_1 > \sqrt{3\alpha_2 \mu_0}$, then the negative branch of large absolute value may have a stable piece.

(ii) $\mu_0 < 0$.

$$R_{0\pm} = \frac{-2\alpha_1 \pm \sqrt{4\alpha_1^2 - 12\alpha_1\mu_0}}{6\alpha_1}$$

We define $R_{0\pm}$. It is very similar to the $\alpha_1 = 0$ case where we have a interval $[\mu_-, \mu_+]$ in which no (I) solutions are stable.

6. Near Resonance and Slow Space Modulation

Up to now, we only studied amplitude equations as ordinary differential equations. The reason for us to **ignor** the slow space variable is the mean flow generated by convection can be seen clearly from the space independent amplitude equation. However, the spatial modulation is an important feature especially when the resonance is not exact. The amplitude equation for near resonance are also considered for spacial forcing of one wave length by **Coullet (1985)**. Here, we study the problem related to the $a_c, 2a_c$ forcing. Assume our forcing at the boundaries are

$$\begin{aligned}\theta &= \lambda_3 \varepsilon^3 e^{i(\alpha + \varepsilon \eta_1)x} + c.c. & \text{at } z=1 \\ \theta &= \lambda_2 \varepsilon^2 e^{2i(\alpha + \varepsilon \eta_1)x} + c.c. & \text{at } z=0\end{aligned}$$

The amplitude equation will be for $A=A(T, X)$, where $X=\varepsilon x$. We get

$$A_T - \gamma^2 A_{XX} = \mu_0 e^{i\eta_1 X} + \mu A + \alpha e^{2i\eta_1 X} - \beta |A|^2 A$$

We consider the stability of the solution which corresponding to the mean flow. Without lose of generality, we can assume $\gamma = \beta = 1$. For simplicity, we assume $\eta_1 = \eta_2 = \eta$. The basic steady solution we are considering is

$$A = R_0 e^{i\eta x + i\theta_0}$$

with

$$\begin{aligned}R_0^2 &= \mu - \alpha - \eta^2 \\ \cos \theta_0 &= -\frac{\mu_0}{2\alpha R_0}\end{aligned}$$

Let $A = R e^{i\theta}$, we get the equation for R and θ :

$$\begin{aligned}R_T &= \mu_0 \cos(\eta x - \theta) + \mu R + \alpha \cos 2(\eta x - \theta) R - R^3 + R_{XX} - R\theta_x \\ R\theta_T &= \mu_0 \sin(\eta x - \theta) + \alpha R \sin 2(\eta x - \theta) + 2R_x \theta_x + R\theta_{xx}\end{aligned}$$

Let the perturbations are $\tilde{R}$ and $\tilde{\theta}$, i.e., we let

$$\begin{aligned}\theta &= \eta x + \theta_0 + \tilde{\theta} \\ R &= R_0 + \tilde{R}\end{aligned}$$

we get the linearized equations

$$\begin{aligned}\tilde{R}_T &= (2\alpha \cos^2 \theta_0 - 2R_0^2) \tilde{R} + \tilde{R}_{XX} + \mu_0 \sin \theta_0 \tilde{\theta} - 2\eta R_0 \tilde{\theta}_x \\ \tilde{\theta}_T &= \frac{\mu_0}{R_0^2} \sin \theta_0 \tilde{R} + \frac{2\eta}{R_0} \tilde{R}_x + 2\alpha \sin^2 \theta_0 \tilde{\theta} + \tilde{\theta}_{xx}\end{aligned}$$

If the perturbations are assumed in following form:

$$\tilde{R}, \tilde{\theta} \propto e^{\sigma t + i k x}$$

We therefore have the linear **eigenvalue problem** for $\tilde{R}$.

$$\begin{bmatrix} 1 - \left(\frac{z \alpha \cos^2 \theta_0 - 2 R_0^2 - k^2}{R_0^2} \right) & \mu_0 \sin \theta_0 - 2 i g R_0 k \\ \frac{\mu_0}{R_0^2} \sin \theta_0 + \frac{2 g i k}{R_0} & 2 \alpha \sin^2 \theta_0 - k^2 \end{bmatrix} \begin{bmatrix} \tilde{R} \\ \tilde{\theta} \end{bmatrix} = 0$$

It is easy to find that σ must satisfy the following equation

$$\sigma^2 + 2(R_0^2 + k^2 + 1)\sigma + k^4 + 2(R_0^2 - 2g^2 + 1)k^2 + 4R_0^2 \sin^2 \theta_0 = 0$$

It is obvious that g has **destablizing** effect. Instability by the g effect occurs at the critical number of g

$$g_c^2 = (R_0^2 + 1)/2 - |R_0 \sin \theta_0|$$

for the critical wave number

$$k_c^2 = 2 |R_0 \sin \theta_0|$$

This analysis reveals the fact that if we want to find mean flow in experiments, we have to make the forcing wave length close to the resonance wave number to a certain degree described by g_c .

7. Discussion

We have studied some aspects of the two-dimensional convection problems with spatial forcing on the boundaries. For the spatial forcing with one wave number, we give a scaling for the strength of forcing to get the right amplitude equation. In the case of convection with two spatial forcing, we studied three examples: (a, **2a**), (a, **3a**) and (**2a**, **3a**). In all these cases there seems to be a mean flow in certain parameter ranges. The amplitude equations for two forcing problems are also derived which may include a constant term (imperfect bifurcation) A , of A^2 or both.

The study for these amplitude equations is still not deep. Particularly, an understanding of the nonlinear aspect of the amplitude **equations** including slow space and near resonance modulation will be of interest. A better understanding of the resulting amplitude equations for general spatial forcing also would be very valuable.

The main feature of this work is of course the mean flow induced by the convection. However, mean flow found in experiments by Krishnamurti & Howard can maintain itself, while we have to provide a stable phase lock by symmetric ($x \rightarrow x$) forcing on the boundaries. In that sense, the mean flow here need help, while that in experiments is totally working by itself.

Acknowledgement:

I would like to thank Professor W. V. R. Malkus for suggesting this problem to me and for his continued advice and encouragement. Much of the work is based on his thought and intuition. Conversations with Dr. Ed Spiegel were greatly appreciated as well.

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THE LIVES OF AN EDDY-EXPERIMENTAL STUDY OF PLANETARY EDDIES

Yehuda Agnon

An intense cyclonic eddy was generated in a rotating tank with a sloping bottom. The eddy translates towards a western meridional wall until it reaches the wall. Fluid leaves the eddy at its northern part and flows eastward forming a wake. When the eddy is near the wall its wake is trapped in a narrow jet that flows northward along the wall. Several eddies are formed in the process. The flow patterns observed for various values of the parameters and some of the mechanisms involved are discussed.

1. Introduction

Mesoscale eddies are an important component of the general circulation. Gulf stream rings are believed to be a major transport agent, and filaments of fluid are observed, that are associated with them. It is of interest to study the factors determining their fate. These factors include the β effect, nonlinearity and the western boundary.

Several studies of the evolution of a barotropic **axisymmetric** eddy on the 0-plane have been carried out. Firing and Beardsley (1976) have performed experiments with a moderate amplitude eddy that had a cyclonic core, as well as linear calculations. They found that the eddy had dispersed rapidly while translating to the west, generating (**primarily**) an anticyclone to the east. Nonlinear effects produce a northward motion component of a cyclonic eddy. Flierl (1977) has studied the linear dynamics and McWilliams and Flierl (1977) have studied the nonlinear evolution of quasigeostrophic eddies nonlinearity acts to preserve the coherence. It can nearly balance the dispersing effect of the potential vorticity gradient. The imbalance sustains the propagation of the eddy through the emission of weak Rossby waves which resemble the aforementioned anticyclone.

The study of the interaction of an eddy with a wall has been limited. Kamenkowitch (1975) has analyzed the linear response to a single Fourier component on a western boundary. Mied and Lindemann (1979) have computed the motion of an eddy in a 1000 km square box. Their results show the initial stages of ~~interaction with~~ the boundary.

In order to study the interaction of eddies with a wall, we have generated intense cyclonic eddies in a rotating **tank** with a sloping bottom. The eddy translated to west by northwest, as described

above. Fluid leaves the eddy on its northern side in a rate that decreases with increasing eddy intensity. The wake formed by this fluid is observed in the absence of the wall as well. When an intense eddy reaches the wall, the wake gets channeled into a narrow jet flowing northward along the wall. Finally the jet departs from the wall and flows to the south, where it forms an anticyclone. A new cyclone is generated between the jet and the wall. Another anticyclone is generated near the wall, to the south of the original eddy forming a modon-like structure.

In section 2 the experimental procedure is described. In section 3 the results are presented and compared to the numerical results of Mied and Lindemann (1979), and in the appendix an approximate analytical model for separation of the jet from the wall, due to Nof (1980) is suggested.

2. The Experimental Procedure

A 2m diameter rotating tank was filled with homogenous water and spun up at an angular frequency $f/2$. The depth h the sloping bottom is given by

$$h(y) = H - ay \quad (1)$$

where ($H = 15 \text{ cm}$) and ($a = 0.2$).

Fluid is pumped from the upper surface through a funnel of radius ($R = 3.5 \text{ cm}$) at the speed ($W = 1.1 \text{ cm/sec}$) for a period of $T \text{ sec}$. (see Figure 1).

Neglecting viscosity, the dimensional quasigeostrophic equation is

$$\nabla^2 \psi_t + \mathcal{J}(\nabla^2 \psi + \beta y, \psi) = F \quad (2)$$

where the free surface effect is neglected since the scale of motion is much smaller than the Rossby radius of deformation.

ψ is the stream function and F the forcing:

$$F = (Wf/H) U(R-r) U(T-t) \quad (3)$$

Where U is the Heaviside step function, and

$$\beta = \alpha f / H \quad (4)$$

Following Firing and Beardsley (1976), we choose R as the length scale, fW/H as the vorticity scale and $H/(f\alpha R)$ as the time scale. In nondimensionalized variables, (3) becomes:

$$\nabla^2 \psi_t + \psi_x + Ro \mathcal{J}(\nabla^2 \psi, \psi) = F \quad (5)$$

where ($Ro = Wf/\alpha R$) is the Rossby number ($= 40$ in our experiment).

$$F = U(1-r) U(\tau-t) / \tau \quad (6)$$

and
$$\tau = T (f\alpha R / H)$$

a)

b)

Figure 1: The experimental set up. a) top view. b) Side view.

is the nondimensional pumping time assumed small. The velocity profile for the initial eddy (V) is found by assuming that the first term dominates the left hand side of (5) during the pumping. We get

$$\frac{1}{r} (r v_e)_r = F \quad (7)$$

Hence

$$V = \begin{cases} r/2 & 0 \leq r \leq 1 \\ 1/2r & 1 \leq r \end{cases} \quad (8)$$

In the experiment, T is not small and the eddy is more diffuse.

The funnel penetration is about 5 mm and has no noticeable effect on the flow. The flow was visualized by squirting dye around the funnel, thus colouring the water of the eddy. In some experiments dye was spread around the eddy, to see the motion around it. Some experiments were videotaped and others photographed at short intervals. In addition, runs with other configurations - such as: flat bottom, no wall and walls at various azimuths - were performed in order to separately observe different features of the flow.

We note that the quasigeostrophic equation, (5), is antisymmetric with respect to y . By this we mean that by reflecting x with respect to, say ($x = 0$) and reversing its sign, we still obtain a solution to (5). Hence if we started with an anticyclone in place of our cyclone, we would expect, at the present approximation, to observe the same pattern with north and south reversed. However, we were not successful in producing an anticyclone by pumping in fluid. We also note that the Ekman layer was visualized using Potassium Permanganate, and the flow was directed to the cyclone's center, as expected.

We turn to the specific experiments.

3. Observation and discussion

The physical parameters that we chose to vary were f , the Coriolis frequency, and T the pumping time. T determines the nonlinearity of the flow, while f drops out of Equation 5. When viscosity is accounted for, the spindown time scale is found to be proportional to $f^{1/2}$ (after nondimensionalization). We see that bottom friction affects the details of the flow observed.

In experiment I (plate I, a-d) the parameters are ($T=25$ sec., $4\pi/f = 25$ sec.). There is no wall. The eddy is seen to maintain its coherence over a long time while leaking fluid in a thin tail to its east. This picture compares well with figure 2 from Dewar (1983) reversing north and south (he studied an anticyclone).

In experiment II (plate II, a-e) the parameters are ($T=25$ sec., $4\pi/f=17$ sec.). A meridional wall is placed 10 cm to the west of the

Plate I: Experiment I -
Pictures a-d; a) $t = 1:12$ min
b) 2:59; c) 4:11; d) 6:07

Experiment II; Pictures e-g:
e) $t = 2:04$ min; f) 3:05 g) 4:07

Figure 2. Dynamic Streakfunctions (Dewar 1983).

Here we have plotted the dynamic streakfunctions appropriate to days (a) 40, (b) 60, and (c) 80, as computed by an application of the kinematic formula to the dynamic Ring, and using $\underline{c} = (-2.4 \text{ km/day}, -.5 \text{ km/day})$. Notice the nearly permanent form of the Ring. For comparison, in (d) we have included the kinematic streakfunction as computed from Olson's streamfunction, with $\underline{c}$ as above.

Plate II: Experiment II. a) $t=0:25$ min.; b) $0:54$; c) $1:19$
d) $3:22$; e) $4:16$

funnel. Initially, the eddy translates the west by northwest, as it would in the absence of a wall. Fluid leaves the eddy near its stagnation point at its northwest side and flows eastward. The amount of fluid leaking decreases with increased nonlinearity. When the eddy reathes the wall the leakage intensifies (for intense eddies the leakage only becomes visible near the wall). The wake is now flowing northward along the wall in a thin jet. The trapping of fluid in a western boundary current resembles the well-known visco-inertial boundary layers.

An anticyclone (that we shall call **B**) can be seen forming next to the wall to the south of the original cyclone (hereafter called **A**). The flow tends to a quasi-steady state, with a modon-like **structure** (**A** and **B**) **next** to the wall and fluid flowing northward along the wall. At this stage there is fluid flowing southward along the wall from **B**. The northward jet is deflected to the east and Separated **from** the **wall** turning around and going south. This feature will be discussed in the appendix. The final picture has a north-south **symmetry** (or antisymmetry of the stream function, ψ).

In experiment III we have ($T=25$ sec., $4\pi/f=25$ sec.). The first stages are similar to those of the previous case, and we only show the later stages (Plate I, e-g). **B** is apparently weaker than **A** and the two rotate along a concave arc to the northeast until **B** is east of **A**, but then move back with **B** again to the south of **A**.

Additional experiments were done varying the parameters. In weaker eddies ($T=10$ sec) the trapping of the jet near the wall was not as pronounced and the eddy **dispersed** quickly with Rossby waves reflected eastward from the wall. **Experiments** with a wall in different orientations, and on the f plane revealed that a substantial jet formed only on the **β -plane** and with a western or northwestern wall. In the latter, the eddy **moved** along the wall to the southwest while the jet was moving slower than before to the northeast.

The only available numerical calculations were those of Mied and **Lindemann (1979)**. They did not **wish** to study the wall interaction and **thus** only the initial **stages** of it are seen in their plots. Figure 3 **is** taken from their paper, and the **stream** lines that are extending northward are seen as well as an anticyclone to the south corresponding to our **B**. We plan to make further comparisons with their results.

Smith and O'Brien (1985) who **studied numerically** the propagation of an eddy up a slope, **argue** that the **meridional component of motion** of an eddy **on** the **β -plane** **arises** from the initial dispersion tendencies, which **prevent** the eddy from maintaining circular

Figure 3. Contours of constant pressure at 10 days intervals
(Mid and Lindemann 1979).

symmetry. As the longer wavelengths Rossby components travel faster, there is a weakening of gradients on the leading western side of the eddy and sharpening of gradients on the trailing eastern side. This assymetry forces advection in the direction of the flow on the trailing side (northward for a cyclone). In the present problem, when the eddy reaches the wall, the gradients on the wall side steepen, resulting in a southward advection tendency that balances and may overcome the **previous** northward motion. The balance is now between nonlinearity and dispersion that includes reflected waves. As the eddy stops, or moves slowly to the south, the stagnation point is now near the wall, on the western side of the eddy. The leaking fluid is trapped in a familiar viso-inertial boundary layer and flows to the north. This is of course just a schematic description. In the various experiments, we observed a tendency to generate vorticity in front of an eddy or a current that is moving along a wall. The generated vortex is such that the current deflects from the wall. Thus, in front of the southward moving cyclone A, we found the anticyclone B, and in front of the jet there forms a new cyclone as the jet departs from the wall. Alternatively, B could be the trailing anticyclone of A that propagated to the southwest and settled south of A as Figure 3 suggests.

The motion of A and B in experiment III is compatible with our understanding of the motion of two opposite sign vortices of different strength. As A leaks through the jet it **becomes** weaker than B and the eddies rotate back, with B overcoming their eastward advection. They settle near the wall that blocks westward motion in a modon-like configuration. ~~The~~ disturbance due to the wall results in a continuing leakage along the wall. **This** structure resembles the results of Haidvogel and Rhines (1983) for orccillatory **antisymmetric** forcing.

Finally we note that Flierl, Stern and Whitehead (1983) have derived an integral theorem that states that for an isolated disturbance in the ψ -plane:

$$\oint \gamma \, dx \, dy = 0 \quad (9)$$

This means that any cyclonic vortex ~~must~~ be accompanied by equal amounts of anticyclonic flow, if the structure is to remain isolated. The theorem is derived by studying the **y-momentum** balance, and hence applies to a domain with a western wall, **since** such a wall does **not** impart any y momentum on the fluid. The **observed** anticyclone B is in agreement with this theorem.

In our **experiment** the northward flow of the inertial jet seems to indicate anticyclonic motion to the north of A. **This** could be associated with the trailing anticyclone or with the vorticity gradient at the edge of the eddy and **needs** to be **studied** further.

We also note that the eddy acquires northward momentum before it reaches the wall. It seems that the northward **jet** carries that momentum when the eddy's northward motion ceases.

4. Conclusion

We have carried out rotation tank experiments on the interaction of intense cyclonic barotropic eddies on a B plane with a western wall.

1) The propagation of a nonlinear cyclone on a **β -plane** in a rotating tank has been observed and found to be **coherent** for long times. The observation was limited by spin down due to bottom friction. A thin tail was observed in the eddy's wake.

2) When the cyclone encounters a western boundary, its west by northwest motion is replaced by a slow southward motion along the wall. The leaking fluid is trapped in a thin jet flowing along the wall to the north. The pairing of an anticyclone to the original cyclone has been observed. This is in agreement with the balance of vorticity **in β -plane** dynamics. Separation of the current from the wall is observed and a cyclone appears in between. The separation agrees with existing theory. Future work will include developing a quantitative model that can be compared with the observations.

ACKNOWLEDGEMENTS

It is a pleasure to thank Glenn Flierl, Bob **Frazel**, Karl Helfrich, Abbie Jackson, Jane **Larsen**, Richard Mied, Doron Nof, Jake Peirson, Paola **Rizzoli**, Melvin Stern, Jack Whitehead and the GFD program **for making** it all possible. Thanks **go** to **Rotem** who was born in time **for me to make** it to the seminar, and to Efrat, who shared it with me.

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APPENDIX

A Simple Model for Jet Deflection

Nof (1980) has studied the flow of inertial boundary currents over topography. He also considered a southward current near an eastern wall on a flat bottom: ~~plane~~ and showed that it detaches from the wall. We sketch here how his theory can apply to the present case. At the zeroth order, the potential vorticity equation can be written (for small y variation) as:

$$\psi_{xx} + \beta y = 0 \quad (A1)$$

the boundary condition on the wall is

$$\psi = 0 \quad (x=0) \quad (A2)$$

and on the free stream line at the edge of the current (in normalized form):

$$\psi = 1 \quad (x = X) \quad (A3)$$

$$\psi_x = 1 \quad (x = X) \quad (A4)$$

the solution of A1 that satisfies A2 is:

$$\psi = -\beta y \frac{x^2}{2} + A(y)x \quad (A5)$$

substituting A5 in A3 and A4 and eliminating $A(y)$ we get a quadratic for X . Applying the upstream condition

$$X = 1 \quad (y=0) \quad (A6)$$

we find

$$X = \frac{-1 + [1 + 2\beta y]^{1/2}}{\beta y} \quad (A7)$$

the **velocity** near the wall is:

$$v_0 = A(y) = 1 + \beta y X \quad (A8)$$

when the term in square brackets in A7 becomes zero the solution breaks down. At that point v_0 is zero **which** indicates separation from the wall. Higher order approximation is required to determine the exact behavior at that point.

SELF-PROPAGATION OF A BAROTROPIC CIRCULAR EDDY

Gavin L. Biebuyck

INTRODUCTION

I consider a model for an oceanographic eddy having a velocity profile as in figure 1. To approximate such an eddy I examine the motion of two concentric circles, with constant positive vorticity in the inner circle and constant negative vorticity in the annulus. The velocity profile is given in figure 2. The case where the velocity at the outer boundary is zero is of particular interest as it represents a so-called 'compact' eddy, and is thus representative of the localized structures found in the ocean such as the Gulf Stream rings. Of course, the Gulf Stream eddies are very complicated dynamically, they are baroclinic and beta effects and other smaller scale eddies surely play a role in their motion. Nonetheless the 'self-propagating' mode I examine in the barotropic case might be an important contribution to the rings' self motion.

The motion of a compact barotropic circular eddy with piecewise uniform vorticity is studied. The linear stability of a two region model is presented first. It is found that a system consisting of two concentric circles with positive vorticity in the inner circle and negative vorticity in the annulus is unstable to infinitesimal disturbances if the ratio of the radii is less than two. The most unstable mode is $m=2$. If the integrated vorticity is zero one finds a degenerate root for the $m=1$ mode. Stern has demonstrated that one of these roots corresponds to the circles being shifted and that the system will then propagate at a constant velocity perpendicular to the displacement of the circles.

I consider the more general case where the circulation does not vanish. I find that in general the system moves in a circle, and the previous result is the limit where the radius of the circle goes to infinity. The two region model is somewhat special in the sense that one of the $m=1$ frequencies is zero. In general one would expect two different frequencies for $m=1$. To test this I considered a three circle model with constant vorticity. Indeed two different frequencies are found from the $m=1$ linear analysis.

To see if these results have any general validity I review some contour **dynamical** numerical work by Stern. He finds that for finite amplitude shifts of the two circle system, the motion is close to a straight line at the velocity predicted by the linear theory. The vorticity pattern reveals entrainment of irrotational fluid into the system through wave-breaking at the outer boundary. In the case of net vorticity one sees the expected circular motion of the system, although the runs are not long enough to be able to compare the radius with the linear theory. I then used contour dynamics to examine the finite amplitude motion of the three circle model.

LINEAR STABILITY ANALYSIS

A. TWO CIRCLES

The basic state for the case of two circles of radius 1 and radius a is as such

$$\bar{v}_\theta = \begin{cases} r & 0 < r < 1 \\ \frac{-r}{a^2-1} + \frac{a^2}{a^2-1} \frac{1}{r} & 1 < r < a \\ 0 & r > a \end{cases}$$

where the azimuthal velocity is normalized to 1 at $r=1$, and we have assumed that the velocity vanishes at $r=a$. Such an undisturbed state has vorticity 2 in the inner circle and negative vorticity $-2/a - 1$ in the annular region.

Consider the stream function $\psi = \bar{\psi}(r) + \psi'(r, \theta, t)$ where the perturbed stream function has the form

$$\psi'(r, \theta, t) = R(r) \psi(r) e^{im\theta - i\omega t}$$

and

$$\hat{\psi}(r) = \begin{cases} A r^m & 0 < r < 1 \\ B r^m + C r^{-m} & 1 < r < a \\ D r^{-m} & r > a \end{cases}$$

Demanding that the radial velocity and the pressure at $r=1$ and $r=a$ be continuous we get the dispersion relation

$$m^2 \omega^2 + \omega(m - m^2) + \frac{m + \frac{a^2}{a^2-1}(-1 + a^{-2m})}{a^2-1} = 0$$

For $m=1$ we see that $\omega^2 = 0$ which means we have a degenerate root. For $m=2$

$$4\omega^2 - 2\omega + \frac{1}{a^2} = 0$$

$$\text{So } \omega \text{ real if } a > 2$$

So we get linear stability only when the ratio of the radii is greater than two. If the outer radius is less than two the shear in the annulus will be too great and a Kelvin-Helmholtz instability results. It can also be seen that the $m=2$ mode is the fastest growing linear mode.

Let us examine the $m=1$ degeneracy. One of the roots corresponds to a displacement of the whole circle. The other, however, is a shift of one circle with respect to the other circle by a distance Δ . Clearly the system cannot remain stationary and satisfy continuity of pressure with such a configuration. Consider the system moving at a constant velocity perpendicular to Δ . In a reference frame moving with the eddy one can match radial velocities at $r=1$ and a , and azimuthal velocities at $r=a$ and the disturbed boundary $r=1-4\cos\theta$, assuming a constant velocity U , to obtain

$$U = \frac{4}{a^2-1}$$

This is a kind of 'self-propagation' of the eddy which is proportional to the displacement of the two circles. In figure 3 there is a heuristic way of understanding such propagation: if one examines the area between the initial and final positions of the outer circle then

the vorticity anomalies of these two areas will move the center of the large circle perpendicular to the displacement .

Now I consider the case where there is a finite velocity y at $r=a$. The basic state now looks like

$$\bar{V}_\theta = \begin{cases} r & 0 < r < 1 \\ -\frac{(1-y)}{a^2-1} r + \frac{(ay-a^2)}{1-a^2} \frac{1}{r} & 1 < r < a \\ 0 & r > a \end{cases}$$

and the vorticity is given by

$$Q_1 = 2 \quad Q_2 = -\frac{2(1-y)}{a^2-1} \quad Q_3 = 0$$

Assuming a general time-dependent velocity of the outer circle as such

$$U = U_e(t) e^{i\theta(t)}$$

and letting the small circle be at position

$$r = 1 + \Delta(t) e^{i\theta}$$

I obtain

$$\frac{\partial U^*}{\partial t} + i \frac{y}{a} U^* = 0$$

as the differential equation for U^* . Δ now appears as an initial condition. We can recover the $y=0$ case easily since U^* will then be a constant and for $\Delta = \frac{a^2-1}{a^2}$ we get $U = \frac{\Delta t}{a^2-1}$

In general for $\Delta = \frac{a^2-1}{a^2}$ (displacement in the $\theta = \pi/2$ direction) we get

$$x = \frac{a}{y} \frac{(1-y/a) \Delta_0}{a^2-1} e^{i(y/a)t}$$

which is the equation of a circle of radius

$$R = \frac{a(1-y/a) \Delta_0}{y(a^2-1)}$$

Therefore the motion of the eddy is in general a circle, and if $y=0$ then the motion is the limit where the radius goes to infinity, i.e. a straight line.

B. THREE CIRCLES

I now consider three concentric circles. I have two additional parameters: the velocity y at the middle circle and the radius b of the outer circle. The basic state velocity and vorticity now looks like

$$\bar{V}_\theta = \begin{cases} r & 0 < r < 1 \\ -\frac{(1-y)}{a^2-1} r + \frac{a^2-ya}{a^2-1} \frac{1}{r} & 1 < r < a \\ -\frac{ya}{b^2-a^2} r + \frac{b^2-a^2}{(b^2-a^2)} \frac{1}{r} & a < r < b \\ 0 & r > b \end{cases}$$

$$Q_1 = 2 \quad Q_2 = -\frac{2(1-ya)}{a^2-1} \quad Q_3 = -\frac{2ya}{b^2-a^2} \quad Q_4 = 0$$

I now perform an $m=1$ perturbation analysis assuming the outer circle moves with time-dependent velocity

$$U = U_0(t) e^{i\theta(t)}$$

and that the two inner circles are at

$$r_1 = 1 + \Delta_1(t) e^{i\theta}$$

and

$$r_2 = a + \Delta_2(t) e^{i\theta}$$

respectively. The perturbed velocity field for $m=1$ is given by

$$V' = \begin{cases} B_1(t) r^{-1} e^{i\theta} & 0 < r < 1 \\ (B_2(t)r + B_3(t)(r)) e^{i\theta} & 1 < r < a \\ (B_4(t)r + B_5(t)(r)) e^{i\theta} & a < r < b \\ B_6(t)(r) & r > b \end{cases}$$

matching boundary conditions as in the two circle case we arrive at

$$\begin{aligned} \frac{\partial \Delta_1}{\partial t} &= c_1 \Delta_1 + i c_2 \Delta_2 \\ \frac{\partial \Delta_2}{\partial t} &= i c_3 \Delta_1 + i c_4 \Delta_2 \end{aligned} \quad U = A_1 \Delta_1 + A_2 \Delta_2$$

(are functions of a, b, t)

There are two coupled first order differential equations for Δ_1 and Δ_2 . And U is a linear combination of the two displacements. The solution will then have two frequencies associated with it ω_1 and ω_2 . The slower frequency will correspond to motion in a large circle as in the two circle eddy, but the fast frequency will now be superimposed. The net motion will thus be epicyclic. If we average over the fast frequency we see motion in a circle of radius

$$R = \frac{\Delta_0 (1 + 1/\omega_1)}{b^2/a^2 - 1}$$

For values

$$\begin{aligned} a &= 1.5 & \gamma &= 0.5 \\ b &= 0.25 & \gamma &= 0 \end{aligned}$$

we find $\omega_2/\omega_1 = 16$. Motion in a straight line is now possible in certain circumstances. If the middle circle only is shifted then straight line motion is not possible. But if both the inner two are shifted the same amount in the same direction then motion in a straight line is possible for a particular value of the velocity on the middle circle.

CONTOUR DYNAMICS

The numerical method of contour dynamics is ideally suited for constant vorticity regions in two dimensions. At any point in time the velocity of a point on the boundary of a constant vorticity region may be found by integrating over the area of the region. This area can then be replaced by a contour integral. So at a given time the vorticity is a linear superposition of two circular constant vorticity regions. The time evolution of the contours can then be followed numerically by integrating around the contours. The expressions for the velocity at a point (x_2, y_2) on the outer circle are

$$\begin{aligned} \left\{ \begin{array}{l} \frac{dy_2}{dt} \\ \frac{dx_2}{dt} \end{array} \right\} &= \frac{Q_2}{4\pi} \oint \left\{ \frac{dn_2}{d\xi_2} \right\} \ln((x_2 - \xi_2)^2 + (y_2 - \eta_2)^2) + \\ &\quad \frac{(Q_1 - Q_2)}{4\pi} \oint \left\{ \frac{dn_1}{d\xi_1} \right\} \ln((x_2 - \xi_1)^2 + (y_2 - \eta_1)^2) \end{aligned}$$

and similarly for points on the inner circle. A second order Runge-Kutta scheme is then used to compute the time evolution.

Using this technique Stern has computed the finite amplitude behavior of the two circle eddy. His results for $a=2.5$ and are **summarized** in figure 4 in which we see that the motion of the inner circle is close to the linear result. This may seem surprising when examining the extent of the nonlinearity as shown by the wave breaking on the outer boundary. Figure 5 is a graph of displacement versus time for different amplitudes. It is seen that departures from the linear result (the straight line) increase with time and with increasing amplitude.

I looked at the time evolution of three contours but the results are as of yet inconclusive. It appears as though, for small shifts of one of the inner circles, the circles become increasingly elliptical and then throw off two spiral arms (see figure 6). This behavior seems qualitatively similar to some of Zabusky's work on the evolution of Kirchoff ellipses, except in this case the arms have negative vorticity and the inner core of positive vorticity seems stable- it is a rotating ellipse. This work is still in progress.

CONCLUSIONS

In conclusion it appears that the motion of piecewise uniform barotropic circular eddies is quite well accounted for by the linear $m=1$ self-propagation velocity. Of course the finite amplitude effects such as entrainment and detrainment are very interesting from the point of view of mixing, especially in oceanographic situations. But the motion of the eddy as a whole is predicted very well by the linear theory.

The question then arises as to what mechanism could be responsible for a shift of the positive and negative regions of vorticity in an eddy. The candidates include smaller eddies at the edge of the eddy, the ambient currents, beta and topographic effects. Stern has investigated the effect of a point vortex placed outside the eddy, and has shown that it produces an $m=1$ perturbation. Thus, while I have examined a very idealized situation, its implications can be tested in oceanographic eddies once data is available for them.

Acknowledgements

I would like to thank Melvin Stern for suggesting this problem to me and for his encouragement, and Glenn **Flierl** for numerous helpful discussions.

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Figure 1

Figure 2

Figure 3

Figure 4

Coordinate ($\bar{L}_1$) of the C.G. of Inner Circle Normalized as a Function of Time

Figure

$$a=2$$

$$b=3 \quad \gamma=1/4$$

Figure 6

STABILITY OF STATIONARY-ENSTROPY BETA-CHANNEL FLOW

Joseph Barsugli

I. Introduction

Barotropic, quasi-geostrophic flow in a **periodic** channel **is commonly used** as a **model** of the atmosphere between **two** latitude circles, and has been **the** basis of theoretical work and numerical simulations. If we **non-dimensionalize** the problem (see appendix I) so that $\beta = 1$ and so that the channel extends in the north-south direction from $y = -1$ to $y = 1$ we get the following form of the quasi-geostrophic equation:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0 \quad q = \nabla^2 \psi + Y \quad (1)$$

with boundary conditions:

$$\psi(x, y) = \psi(x+a, y) \quad \psi_x = 0 \text{ at } y = \pm 1 \quad \bar{U}(y=1) = U_0 \quad \bar{U}(y=-1) = -U_0 \quad (2)$$

Since equation (1) is invariant under Galilean transformations we can choose a reference frame in which the **x-averaged** velocity at the two boundaries are of equal in magnitude and opposite in sign. For this **inviscid** system the total integrated **x-momentum** (M), energy (E), circulation (C), and enstrophy (2) are conserved.

$$M = \int U \, dA \quad E = \frac{1}{2} \int \nabla \psi \cdot \nabla \psi \, dA \quad Z = \frac{1}{2} \int (\nabla^2 \psi)^2 \, dA \quad C = \int \zeta \, dA = -2aU_0$$

This list is not exhaustive, as there are in fact an infinite number of conserved quantities.

The qualitative behavior of two-dimensional turbulent flow is well known. Enstrophy cascades to small scales, while energy, momentum, and circulation remain at large scales. If viscosity **is** present at small scales the enstrophy will be dissipated much faster than the other quantities. This process continues until the fluid approaches a state in which the potential vorticity q is a function of ψ , at which time the enstrophy cascade is slowed. Schematically one might picture the time evolution of the inviscidly conserved quantities in the **viscous system** as follows:

One might **say** that E, M , and C are more **'robust'** than Z in the presence of small scale **dissipation**.

This motivates the **following** question: For a given amount of momentum, energy, and circulation in **the** channel, **what is** the minimum possible value of **enstrophy**, and **what** does **this** state look like? It **is** worth noting that it might **seem** more physically relevant to **minimize** the 'potential enstrophy' $Q = \int q^2 \, dA$. It can be **shown** that, in the chosen frame of reference, $Q = 2 + M + \frac{1}{2} Z$, so that the two **minimization problems** are equivalent.

Bretherton and Haidvogel (1976) applied a minimum enstrophy principle to two-dimensional turbulence above topography in a square basin. Their numerical simulation approached a state resembling the theoretically calculated minimum-enstrophy **state**, though differing in some ways. Leith (1984) has applied this minimum principle to a circular geometry to get what he calls 'minimum enstrophy vortices'.

In choosing a channel geometry I am restricting the possible solutions from the start. I will further restrict the problem to consider only solutions in the class of purely zonal flow, of waves, or of some combination. This eliminates the possibility of obtaining coherent structures such as modons. The motivation for this restriction lies partially in the prevalence of zonal and wavy flow in atmospheres -- such as the banded structure of Jupiter or the wavy structure of atmospheric blocking -- and partially in the interest of tractability. The following results help support such a notion. A numerical simulation in a periodic domain by Matthaeus and Montgomery (1984) showed that enstrophy decayed as predicted and that energy tended toward the largest scale compatible with the geometric constraints of the domain. Williams' (1978) barotropic simulations of Jupiter's atmosphere strongly suggested that the two-dimensional turbulent cascade is central to the production of the observed banded (zonal) structure, though he also found 'ovals' between the bands. Hou and Farrell's (1986) numerical simulation in a periodic β -channel found the flow tended to the gravest mode wave, though this may not be the final state of the system.

II. Variational Problem

To solve the minimization problem stated above we use the calculus of variations. Consider the following functional:

$$F = Z + \delta_0 E + \epsilon_0 M \quad (3)$$

The channel boundary conditions, equation (2), are applied, which ensures that the circulation is held fixed.

Varying with respect to ψ we get the Euler-Lagrange equation:

$$\nabla^2 \psi - \delta_0 \psi + \epsilon_0 y = 0 \quad (4)$$

or equivalently,

$$q = \delta_0 (\psi + \mu y) \quad \mu = \frac{1 - \epsilon_0}{\delta_0} \quad (5)$$

Where the parameters δ_0 and ϵ_0 depend on E and M . These stationary-enstrophy flows ("stationary", as they are not necessarily minima) are exact, steadily translating solutions of the equations of motion (1) if the phase speed is equal to μ .

The purely zonal solutions are:

$$U(y) = \begin{cases} -\frac{\epsilon_0}{\alpha^2} \left(1 - \frac{\cosh \frac{yY}{\gamma}}{\cosh Y} \right) + U_0 \frac{\sinh \frac{yY}{\gamma}}{\sinh Y} & \delta_0 = \alpha^2 \\ \frac{\epsilon_0}{\gamma^2} \left(1 - \frac{\cos \frac{yY}{\gamma}}{\cos Y} \right) - U_0 \frac{\sin \frac{yY}{\gamma}}{\sin Y} & \delta_0 = -\gamma^2 \end{cases}$$

Fig. 1. z vs. E for the zonal solutions to the variational problem. γ can be thought of as parametrizing this curve. Values of γ are plotted along the curve.

Fig. 2. z vs. E as in Fig. 1, including the first wave mode. Known stability properties are shown.

The momentum, energy and enstrophy can be calculated directly. For the trigonometric solutions ($\delta_0 \ll 0$) these are:

$$\begin{aligned} M &= \frac{3\varepsilon_0}{\gamma^2} \left(1 - \frac{\tan \gamma}{\gamma} \right) \\ E &= \left(\frac{\varepsilon_0}{\gamma^2} \right)^2 \left(3 - \frac{3 \tan \gamma}{\gamma} + \tan^2 \gamma \right) + U_0^2 \left(1 - \frac{\cot \gamma}{\gamma} + \cot^2 \gamma \right) \\ Z &= \gamma^2 \left(\frac{\varepsilon_0}{\gamma^2} \right)^2 \left(1 - \frac{\tan \gamma}{\gamma} + \tan^2 \gamma \right) + U_0^2 \left(1 - \frac{\cot \gamma}{\gamma} + \cot^2 \gamma \right) \end{aligned}$$

From this point on I limit the discussion to the trigonometric solutions in the case where $U = 0$, that is, no circulation in the channel. It is convenient to form the quantities $\beta = Z/M^2$ and $\mathcal{E} = E/M^2$ in order to eliminate the parameter ε_0 . It can be shown that $d\gamma/d\mathcal{E} = \gamma^2$. As $M \rightarrow 0$ both and blow up. This will occur at all values of γ where $\gamma = \tan \gamma$, call this value γ_c . The lowest non-zero value of γ_c is 4.4943.... In the limit as $\gamma \rightarrow \gamma_c$ the following relation holds:

$$\beta = \gamma_c^2 \mathcal{E} \pm \gamma_c \sqrt{2\mathcal{E}} + \dots \quad (6)$$

The graph of γ vs. $\mathcal{E}$ is shown in figure 1.

Let $\gamma_{11} = (k^2 + l^2)^{1/2}$, the total wavenumber of the gravest mode in the channel. γ_{11} is determined solely by the aspect ratio of the channel, For $\gamma = \gamma_{11}$ a wave of the form:

$$\psi = A e^{ikx} \cos ly \quad k = \frac{2\pi}{a} \quad l = \frac{\pi}{2}$$

is also admissible, superimposed on a zonal flow. The zonal component is necessary to have non-zero momentum in the channel. The Euler-Lagrange equation have only one value of the total wavenumber γ and thus requires that the value of γ in the zonal solution equal γ_{11} . The total solution looks like:

$$\psi(x, y) = \psi_{\text{zonal}}(\gamma; \gamma_{11}) + A e^{ikx} \cos ly$$

Individual higher modes are also allowable solutions for different γ and have analogous behavior.

On the γ - $\mathcal{E}$ diagram the wavy solutions are straight lines of slope γ^2 which are tangent to the zonal curve at $\gamma = \gamma_{11}$. (Figure 2) The position along the zonal curve where the wavy solution branches off is dependent on the aspect ratio of the channel. Only for a very short channel, shorter even than a square channel, does the first wavy solution lie on an upper branch of the zonal curve. Thus for all reasonable geometries the minimum enstrophy flow can be purely zonal or wavelike in character, depending on the value of $\mathcal{E}$.

If $\gamma = \gamma_c$ then the momentum is zero, and it is appropriate to look at a graph of Z vs. $\mathcal{E}$:

The relationship is linear for all the solutions, with slope equal to γ^2 . For $M = 0$ a **modon** of the type described by Stern (1975) is a solution to the variational problem. The relation between energy and enstrophy for a group of non-interacting modons is $Z = (5.136)^2 R^{-2} E$. For the largest **modon** which will fit in the channel we find that the enstrophy falls above the minimum. Smaller modons fall even higher on the graph. This is reassuring in light of elimination of these structures from consideration in the more general case.

These diagrams indicate very clearly that there are usually many stationary-enstrophy solutions for a given energy and momentum. If we are to believe that the flow tends toward the minimum enstrophy state then it is reasonable to guess that some, if not all, the upper branch solutions are unstable. To this end I will explore the stability of the zonal solutions.

III. Stability of Zonal Solutions

Enstrophy is conserved exactly in the **inviscid** system. For the flow to be unstable there must be a path in phase space away from the stationary point on which Z does not change. In terms of the variational problem this means that the stationary point is a saddle point. Thus the second variation of (3)

$$\delta^2 F = \int [(\nabla^2 \eta)^2 + \delta \cdot \nabla \eta \cdot \nabla \eta] dA$$

must be zero for instability to be possible. For $\delta^2 F = 0$ the flow is stable. For $\delta > 0$ the integral is positive definite and the flow is stable by this criterion. Young (pers. comm., 1986) has shown that for $\delta < 0$ the flow is stable for $\gamma < \gamma_n$. At $\gamma = \gamma_n$ a neutral perturbation exists, namely the gravest mode wave, which makes $\delta^2 F = 0$.

The stability of zonal flow just above γ_n can be investigated by looking in the neighborhood of the neutral perturbation and asking whether the real phase speed of the neutral evolution acquires an imaginary part. The procedure closely follows that of Lin (1955).

The Rayleigh equation for this system can be written in a very nice form due to equation (5):

$$\phi'' - k^2 \phi + \frac{\gamma^2 (U - \mu)}{U - c} \phi = 0$$

This is satisfied by the wave if $c = \mu$ and $\gamma = \gamma_n$. The Rayleigh equation for a nearby value of γ is:

$$\tilde{\phi}'' - k^2 \tilde{\phi} + \frac{\tilde{\gamma}^2 (\tilde{U} - \mu)}{\tilde{U} - \tilde{c}} \tilde{\phi} = 0$$

Multiplying the first by $\tilde{\psi}$ and the second by ϕ , subtracting, and integrating results in the following:

$$\int_{-1}^1 \left(\gamma'' - \tilde{\gamma}'' \frac{\tilde{U} - \tilde{\mu}}{\tilde{U} - \tilde{c}} \right) \phi \tilde{\phi} dy = 0$$

Expand about $\tilde{\psi}$ as follows

$$\tilde{\gamma}'' = \gamma'' + \eta \quad \tilde{c} = \mu + \eta c_1 \quad \tilde{\phi} = \phi + \eta \phi_1$$

Keeping only the terms to $O(\eta)$ and solving for c_1 yields:

$$c_1 = - \left(\int_{-1}^1 \phi^2 dy \right) \left(\gamma'' \int_{-1}^1 \frac{\phi^2}{\tilde{U} - \tilde{c}} dy \right)^{-1}$$

Taking the limit as $\eta \rightarrow 0$, $\tilde{c} \rightarrow c$, $\tilde{U} \rightarrow U$, the calculation proceeds exactly as in Lin (1957) and yields the result that as γ is perturbed in the positive direction the flow becomes unstable to the gravest mode wave, providing there is a critical layer where $U = \mu$ somewhere in the channel. If it is perturbed in the negative direction it remains stable. The condition on the existence of the critical layer is equivalent in the re-dimensionalized problem too a condition on β versus the parameter ϵ_0 which is a function of E and M . Large values of ϵ_0 are able to stabilize the flow by removing the critical layer.

IV. Stability of a Severely Truncated System

In order to gain some insight into the stability of the zonal stationary-entropy flows I choose to investigate a severely truncated version of the quasi-geostrophic equation. In addition to the mean flow I have kept one wave in x and two in y . We first expand the streamfunction in x :

$$\psi = \psi_0(y, t) + \psi_1(y, t) e^{ikx} + \psi_1^*(y, t) e^{-ikx}, \quad k = \frac{2\pi}{a}$$

ψ_0 and ψ_1 are then expanded in terms of the following shape functions in y :

$$\psi_1 = a_{11}(t) f_1(y) + a_{13}(t) f_3(y)$$

$$U_0 = -\psi_{0y} = M + U(t) g_2(y) + (U-M) g_4$$

$$\text{where } f_1 = \cos \frac{\pi y}{2} \quad f_3 = \cos \frac{3\pi y}{2} \\ g_2 = \cos \pi y \quad g_4 = \cos 2\pi y$$

Plugging **this** expansion into equation (1) the following equations for the **amplitudes** of the various modes are obtained:

$$\begin{aligned}
 \dot{U} &= 4ikl^2 [a_{11} a_{11}^* - a_{13}^* a_{13}] = -8kl^2 AB \sin \chi \\
 \dot{a}_{11} &= \frac{ik}{k^2 + l^2} \left\{ a_{11} - (k^2 + l^2) a_{11} M + \frac{1}{2} (3l^2 - k^2) a_{11} U \right. \\
 &\quad \left. + (l^2 - k^2) a_{13} U - \frac{1}{2} (7l^2 - k^2) a_{13} M \right\} \\
 \dot{a}_{11}^* &= \text{complex conjugate of above} \\
 \dot{a}_{13} &= \frac{ik}{k^2 + 9l^2} \left\{ a_{13} - (k^2 + 9l^2) a_{13} M \right. \\
 &\quad \left. + (9l^2 - k^2) a_{11} U - \frac{1}{2} (15l^2 - k^2) a_{11} M \right\} \\
 \dot{a}_{13}^* &= \text{complex conjugate of above}
 \end{aligned} \tag{7}$$

$$a_{11} = Re^{i\theta} \quad a_{13} = Be^{i\phi} \quad \chi = \phi - \theta$$

As in the continuous case the following are conserved exactly by this system:

$$E = M^2 + U^2 - UM + \gamma_{11}^2 A^2 + \gamma_{13}^2 B^2$$

$$Z = 8l^2 M^2 + 10l^2 U^2 - 16l^2 UM + \gamma_{11}^4 A^2 + \gamma_{13}^4 B^2$$

(where $\gamma_{11}^2 = k^2 + l^2$, $\gamma_{13}^2 = k^2 + 9l^2$)

are C and M.

The variational problem **now** becomes a problem in differential calculus. Using **Lagrange multipliers** we wish to find the values of the amplitudes which minimize enstrophy for a given momentum and energy.

$$\frac{\partial}{\partial U} (Z + \delta_0 E) = 0 \Rightarrow 20l^2 U - 16l^2 M + \delta_0 (2U - M) = 0$$

$$\frac{\partial}{\partial A^2} (Z + \delta_0 E) = 0 \Rightarrow (k^2 + l^2 + \delta_0) A^2 = 0$$

$$\frac{\partial}{\partial B^2} (Z + \delta_0 E) = 0 \Rightarrow (k^2 + 9l^2 + \delta_0) B^2 = 0$$

The first of these conditions **requires** that

$$\frac{U}{M} = \frac{16l^2 + \delta_0}{2(10l^2 + \delta_0)} = \alpha$$

where I have introduced the parameter α . In terms of α , the zonal minimum enstrophy flows have the following values of $\mathcal{E}$ and $\mathcal{Z}$.

$$\mathcal{E} = \alpha^2 - \alpha + 1$$

$$\mathcal{Z} = (10\alpha^2 - 16\alpha + 8) l^2$$

so

$$\mathcal{Z} = (10\mathcal{E} \pm 6\sqrt{\mathcal{E} - 3/4} - 5) l^2$$

The $\mathcal{Z}$ - $\mathcal{E}$ diagram is a tilted parabola as shown in figure 3. One can think of this curve as parametrized by either α or γ . The solutions for $\delta > 0$ are now collapsed onto the small section of negative slope. Note the similarity between this and the asymptotic behavior in the continuous case as $M \rightarrow 0$. Another similarity can be seen by computing the derivative:

$\frac{d\mathcal{Z}}{d\mathcal{E}} = (10 \pm 3(\mathcal{E} - 3/4)^{-1/2}) l^2 = \gamma^2$
as before. In addition, the value at which $M \rightarrow 0$ is 3.162, compared to 4.4934 in the continuous case.

For $\gamma = \gamma_0$ we get the gravest wave mode plus the zonal solution, For $\gamma = \gamma_1$ we get the other wave. As in the continuous case, the Lagrange multiplier selects only one length scale. The wavy solution is tangent to the purely zonal solution, as in the continuous case.

It can be shown that the class of steady zonal and steadily-translating wavy solutions is equivalent to the class of minimum-enstrophy solutions. In this fact the severity of the truncation is quite apparent, as in the continuous system the stationary enstrophy solutions are only a small subset of the steady flows.

The primary benefit of the truncated system is that the linear stability of the entire system is easily determined. For U held fixed, the amplitude equations can be written symbolically in matrix form:

$$\frac{d\mathbf{A}}{dt} = \mathbf{P} \mathbf{A} \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \quad \mathbf{P} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where P, Q, R , and S are obviously related to the coefficients in equations (7). Taking the determinant of $\mathbf{P} - \sigma \mathbf{I}$ we get a quadratic equation for the complex phase

Fig. 3. z vs. ϵ for the truncated system, showing both zonal and wave solutions, and including a schematic representation of the linear stability of the zonal solutions.

Fig. 4. The neutral stability curves for the truncated system for various values of $(M^2)^{1/2}$ as indicated. The two diagonal lines represent the positions where the two wavy solutions join the zonal curve and are always either neutral or stable.

speed σ of a linear perturbation

$$\sigma^2 - (S+P)\sigma + PS - QR = 0$$

If the discriminant of this equation is less than zero then the flow is linearly unstable. For $\gamma = \gamma_{1,2}$ the corresponding relation between γ and implies that $Q = 0$. For $\gamma = \gamma_{1,2}$, $R = 0$. For these two values of γ , therefore, the flow is not unstable. The stability depends on the aspect ratio of the channel, the value of δ , and on M^{-1} . In the dimensional problem the latter parameter becomes $\frac{gH}{M^2}$. The line of neutral stability for various values of M is plotted in figure 4. Figure 5 shows a schematic example of the stability calculation plotted on the γ - E diagram. The parameter M^{-1} seems to act to stabilize the flow, particularly for long channels.

V. Conclusions

The variational principle of minimizing the enstrophy while holding energy, momentum, and circulation fixed picks out certain flows from the class of solutions to the quasi-geostrophic equations. The truncated system shows remarkable similarity to the continuous system and warrants further investigation. However, it appears that meaningful information on the stability of the continuous system, particularly for the upper branches of the γ - E curve, will require a higher order truncation. Numerical integration of this and other low order truncated systems might illuminate the behavior of the continuous system. It is also worth considering results from a large numerical simulation of channel flow in light of the minimum enstrophy problem, perhaps by plotting the trajectory a model follows on the diagram. Careful simulations could demonstrate whether the minimum enstrophy states are really approached, and if there are stable or quasi stable states on higher branches of the zonal curve.

I gratefully acknowledge Bill Young for suggesting this area of inquiry, and guiding me through the specific problems. I also wish to thank the GFD program for granting me the fellowship.

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Non-hysteretic Non-intermittent Transitions Directly to Chaos

R. Van Buskirk

1 Introduction

What I will **discuss** in **this paper falls under the** general category of **dynamical systems** theory, bifurcation theory or **transitions to chaos**.

One of the major **accomplishments** of **this field** is the delineation and **characterisation** of a finite **number** of ways in which a **finite-dimensional dynamical** system can make a transition to chaotic behavior. The presently known **transitions to chaos** are as **follows**:

A) Period Doubling

B) Quasi-Periodicity

C) Intermittency

D) Subcritical Bifurcations

E) High **Codimension** Bifurcations or Multiple **Instabilities**

Let me briefly describe each of **these routes** and their **properties**.

In period doubling, a **periodic system** **develops** as one **changes** a parameter in the **system**—a **series** of **subharmonics** of the fundamental frequency. **First** a **subharmonic** of one half the fundamental **frequency develops**; then, **subharmonics** of $1/4$ and $3/4$, then of $1/8$... **ad infinitum** until **at a certain point, there are subharmonics** of $p/2^n$ for all n and p . At **this** critical point, the motion is chaotic regions of **subharmonic** behavior.

In the **quasi-periodic route to chaos**, a **periodic system undergoes a hopf** bifurcation **and develops a second** frequency. The **resultant** motion on a torus

*GFD participant supported by an NSF graduate student fellowship.

can develop instabilities, that can lead to its bifurcation to a chaotic attractor. Several properties and characteristics of this bifurcation have been described.

In the intermittency route to chaos, a periodic or quasiperiodic system experiences chaotic bursts of motion whose average frequency of appearance grows from zero to effectively once per period as the transition **proceeds**.

In a subcritical bifurcation to chaos, the periodic or quasi-periodic state becomes unstable and the system converges to some stable chaotic attractor in another part of phase space. Except in very special situations, such a system shows a hysteretic sudden jump to chaos.

In a high **codimension** bifurcation, several **instabilities** occur at once and it is possible for a stationary or periodic solution to bifurcate directly ~~to chaos without intermittent bursts~~, hysteretic jumps or intermediate period doubling or hopf bifurcations. This type of bifurcation has been studied by several **authors**,¹ and has been shown to lead directly to chaos in triple **convection**,² but to my knowledge has not been treated in detail as a possible route to chaos for a periodic system. The properties of these bifurcations directly to chaos in periodic systems will be the subject of this paper which will examine possible non-hysteretic non-intermittent bifurcations directly to chaos and their distinguishing **characteristics**.

2 Experimental/Physical Motivation

The main motivation for this study is the experimental observation of a **non-hysteretic non-intermittent bifurcation** from periodic motion to weakly turbulent motion in the flow between **differentially** rotating concentric cylinders as mentioned by Alan **Newell** in one of the **earlier** lectures of the GFD program³.

The scenario for this transition is as follows: If we define a dimensionless Reynolds number for the flow as $R = a(b-a)\Omega/\nu$, where a is the radius of the inner cylinder, b is the radius of the outer **cylinder**, Ω is the angular velocity of the inner cylinder, and ν is the viscosity, then this **first** bifurcation to Taylor vortices occurs at $R_c = 125^4$. At $R = 150$ there is a bifurcation to wavy rolls, where the waves travel **around** the cylinder at the mean azimuthal flow speed. In the proper **rotating** reference frame, this is a steady flow. At $R = 1000$ the wavy rolls develop a modulation. And at $R = 11R/R_c$, the flow develops higher spacial harmonics, whose time evolution becomes chaotic at about $R = 11.7R/R_c$. At this value of R there is an increase in the previously constant noise level in the power spectrum, and **nearby** paths in an experimentally

¹See for example references (5), (4) and (9)

²See reference (4)

³for a complete description of this transition to chaos see ref.(7). Other discussion of this transition can be found in references (1), (5), (6), (8), and (12).

⁴See reference (1).

reconstructed phase space begin to diverge exponentially.

Note that in the **proper** reference frame, the transition is a transition from a periodic **non-axisymmetric** flow **directly** to an aperiodic flow. This transition is found to be non-hysteretic non-intermittent and without any intermediate period doubling or hopf **bifurcations**.

There **are** two reasonable explanations for the bifurcation of modulated wavy Taylor vortices: one is modeled by **an** iterative map which has **discontinuous** derivatives and **thus can** undergo the **period** doubling transition or hopf transition **all** at once; the **second** is that several **directions** in **phase** space become unstable **all** once and a high **codimension** bifurcation **directly** to chaos occurs.

3 Possible Mechanisms

In the **following** analysis, I **will analyse** the dynamics of the Taylor-Codlet system in **terms** of iterative maps. The **reason this** can be done is that one can in general factor out the **periodic** behavior of a **dynamical system** by taking a **Poincaré section**. That is, if a system has an orbit in some (**possibly** high-dimensional) phase space, one can **take** a hyperplane **transversal** to the orbit and **define** the map:

$$\vec{x}^{n+1} = \vec{F}(\vec{x}^n)$$

which takes a point in the hyperplane, $\vec{x}^n$, and maps it to the point $\vec{x}^{n+1}$ **where** the **trajectory next crosses** the hyperplane. Then the periodic motion is factored out and the **dynamics** is reduced to the **dynamics** of **an iterative** map. By **analysing** the **dynamics** of **particular iterative maps**, we **will** examine the two **possible mechanisms** for the bifurcation of **modulated** wavy Taylor vortices.

3.1 Mechanism I: Quick Period Doubling Transition

3.1.1 Discussion/Analysis

If one **has** a **one-dimensional** map:

$$x^{n+1} = F_\epsilon(x^n)$$

(where ϵ is some parameter **analogous** to $(R - R_c)$) if it **has continuous** derivatives, it **can** go to chaos only through period doubling, a **subcritical bifurcation**, or **intermittency**. But if it **has a very large** change in derivative, the range of **parameter** in which the period doubling occurs **can be very small**. We can explicitly **construct** a map which behaves in **this manner**:

$$F_\epsilon(x) = (1 + \epsilon)F_0(x) \tag{1}$$

where $F_0(x) = 3 - 2x$ for $x > 1$ and $F_0(x) = 2 + \frac{1}{20}x(1 - x)$ for $x < 1$

For this map, for $c > .05$ there are no stable periodic orbits and no period doubling orbits are stable. Effectively for $c > .05$, the iterates of this map are randomly distributed in the interval $1 - \epsilon$ to $1 + \epsilon$.

Throughout the rest of this discussion, we will shift the origin of our coordinates to the point $x = 1$ so that the iterates of the map are effectively randomly distributed in the interval $[-\epsilon, \epsilon]$.

3.1.2 Experimental Implications

Prediction I: Deterministic noise in the power spectrum develops as a gradual increase in an approximately uniform background.

We expect the spectrum for this map to be similar to the power spectrum for maps with continuous derivatives when these maps have aperiodic orbits. These spectra may have a peak at $\omega = 0$ but then are mostly uniform broadband noise. Figure (1) shows a spectrum for the map described in the previous section for $\epsilon = .05$ and it is seen that this is a constant noisy spectrum.

Prediction II: There is a discontinuous jump in the Lyapunov exponent at $\epsilon = 0$ and then it increases approximately proportional to ϵ . By adjusting the map, the jump can be made arbitrarily small.

The Lyapunov exponent measures the rate of exponential divergence of nearby orbits in time. Since the geometric average value of the slope in the range $-\epsilon$ to $+\epsilon$ is $1.33(1 + \epsilon)$, and since we might expect the iterates to be randomly distributed in this interval, we would expect the Lyapunov exponent to have a $c_1 + c_2\epsilon$ functional dependence for small $\epsilon > 0$ where c_1 and c_2 are constants.

Prediction III: The integrated noise should scale as ϵ^2 .

We expect the integrated noise to scale as the mean squared deviation of the iterates, x^n , which if they are randomly distributed in the interval $[-\epsilon, +\epsilon]$ should be proportional to ϵ^2 .

Prediction IV: The dimension of the attractor should increase by a number between zero and one.

Since the dimension of the iterative map attractor increases from 0 (a fixed point) to something between 0 and 1, we expect the same increase of the dimension of the attractor in an experiment.

⁴for $\epsilon < 0$ the Lyapunov exponent is less than 0 and in an experiment on a periodic system, the Lyapunov exponent will be 0.

3.2 Mechanism II: High Codimension Bifurcation

3.2.1 Discussion/Analysis

The properties of the bifurcation in this case are not nearly so straightforward as the previous one-dimensional case and are only partially worked out. Therefore, in this section I will derive the properties of a specific high codimension (read multiple instabilities) bifurcation to chaos which I believe may be generalizable to a large class of high codimension bifurcations to chaos.

One possibility in the Taylor-Coulet system is that either due to symmetries of the system or chance, several modes (a directions in phase space) are becoming unstable in a small enough range of parameter so that what is effectively seen is a high codimension bifurcation.

So let's do some math to analyse such a bifurcation. Again, the dynamics of the system can be reduced to the dynamics of an iterative map by taking a Poincaré section, so we will want to study the dynamics of an iterative map:

$$\vec{x}^{n+1} = \vec{F}_\epsilon(\vec{x}) \quad (2)$$

Where a fixed point (which we can choose to be at $\vec{x} = 0$) goes unstable at $\epsilon = 0$. The fixed point is equivalent to the periodic orbit for $\epsilon = R - R'_0$ where R'_0 is the value of R where the modulated wavy Taylor vortices bifurcate to weak turbulence. In dynamical systems theory, there is a theorem called the center manifold theorem that says if one has a bifurcating fixed point, with m directions becoming unstable, the dynamics can be fully described by an m -dimensional submap from an m -dimensional subspace onto itself. Therefore, we can consider $\vec{F}_\epsilon(\vec{x})$ to be only m -dimensional.

We want to examine bifurcations directly to a chaotic attractor which as $\epsilon \rightarrow 0$, shrinks to zero size; i.e., to a fixed point.

If we Taylor series expand around the fixed point, we get

$$\vec{x}^{n+1} = M(\epsilon)\vec{x}^n + O(x^2 \text{ or } x^3) \quad (3)$$

where $M(\epsilon)$ is a matrix function of ϵ . We are assuming m unstable directions at once. Therefore, at $\epsilon = 0$, M has m eigenvalues of magnitude one. I am here going to restrict myself to the case where all such eigenvalues are real. In this case the eigenvalues are ± 1 and it can be shown⁶ that the dynamics are the equivalent to those for a map with all eigenvalues ± 1 . Then there are two possibilities for the form of M at $\epsilon = 0$. In one case M is the identity matrix and in the other case, M in its simplest representation has ones along the diagonal with some of the places just above the diagonal also occupied by ones.

⁶ since the second iterate of the function, $\vec{F}(\vec{F}(\vec{x}))$, will have all eigenvalues ± 1 , by considering the second iterate of the map, we can analyse the dynamics in terms of a map of all ± 1 linearised eigenvalues.

Let's consider the case where $M(\epsilon=0) = I$, I = the identity matrix. Then we have for the iterative map near $\bar{x} = \bar{0}$ and for small ϵ :

$$\bar{x}^{n+1} = \bar{x}^n + \epsilon M'(\epsilon) \bar{x}^n + \bar{x}^n A(\epsilon) \bar{x}^n + \bar{x}^n B(\epsilon) \bar{x}^n \bar{x}^n \quad (4)$$

Then suppose that the chaotic attractor scales as ϵ^a as $\epsilon \rightarrow 0$. Then if we make the coordinate transformation $\bar{x}^n = \epsilon^a \bar{\chi}^n$ we have:

$$\epsilon^a (\bar{\chi}^{n+1} - \bar{\chi}^n) = \epsilon^{a+1} M'(\epsilon) \bar{\chi}^n + \epsilon^{2a} \bar{\chi}^n A(\epsilon) \bar{\chi}^n + \epsilon^{3a} B(\epsilon) \bar{\chi}^n \bar{\chi}^n \bar{\chi}^n \quad (5)$$

and

$$\frac{(\bar{\chi}^{n+1} - \bar{\chi}^n)}{\epsilon} = M'(\epsilon) \bar{\chi}^n + \epsilon^{a-1} \bar{\chi}^n A(\epsilon) \bar{\chi}^n + \epsilon^{2a-1} B(\epsilon) \bar{\chi}^n \bar{\chi}^n \bar{\chi}^n \quad (6)$$

where A and B are tensors of rank 3 and 4 respectively. Therefore if we require that there be a chaotic attractor in the limit $\epsilon \rightarrow 0$, then we have in this limit for $A(0) \neq 0$, $a = 1$ and

$$\frac{d\bar{\chi}(\tau)}{d\tau} = M'(0) \bar{\chi}(\tau) + \bar{\chi}(\tau) A(0) \bar{\chi}(\tau) \quad (7)$$

where $\tau = n\epsilon$ and $\bar{\chi}(n\epsilon) = \bar{\chi}^n$. While for $A(0) = 0$ and $B(0) \neq 0$ we have

$$\frac{d\bar{\chi}(\tau)}{d\tau} = M'(0) \bar{\chi}(\tau) + \bar{\chi}(\tau) B(0) \bar{\chi}(\tau) \bar{\chi}(\tau) \quad (8)$$

If the solutions of these differential equations are chaotic, the iterates of the iterative map will also be chaotic in the limit $\epsilon \rightarrow 0$.⁷ Note that since one needs at least three coupled ODE's to get chaos, one needs at least three instabilities to get chaos through this mechanism. Also note that these equations are independent of ϵ and are related to the iterates of the map by simple scalings:

$$\bar{x}^n = \epsilon^a \bar{\chi}(n\epsilon) \quad (9)$$

and if the position of the fixed point of the iterative map is $\bar{x}_0$ instead of $\bar{0}$ we have:

$$\bar{x}^n = \bar{x}_0 + \epsilon^a \bar{\chi}(n\epsilon) \quad (10)$$

Notice also that if the fixed point is to be stable before the bifurcation, we need $M'(0)$ to be positive.

Now I will illustrate all of this by explicitly constructing an iterative map which undergoes this bifurcation. I do this first by looking for an ODE of the form of equation (8), which has a robust bounded chaotic attractor and has M' positive definite. I pick M' to be

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}$$

⁷Strictly speaking this statement is false but practically speaking it is true. See the cautionary note on page 11.

which results near the origin in an unstable oscillation and a simple expansion. Then, in order that the system not run off to infinity, cubic terms are chosen that will push the system back towards the origin at large distances. Strong nonlinear coupling between two of the components of $\vec{x}$ is added to get the stretching and folding that will give chaos. The result is:

$$\begin{aligned}x_1' &= x_2 - (x_1^2 - \frac{1}{2}x_2^2 + x_3^2)x_1 \\x_2' &= x_2 - x_1 - (x_1^2 + x_2^2 + 10x_3^2)x_2 \\x_3' &= \frac{1}{10}x_3 - (3x_1^2 + \frac{1}{2}x_2^2 + 2x_3^2)x_1 + 6x_2^2\end{aligned}\tag{11}$$

I calculated the solution of this ODE on a computer using a fourth-order Runge-Kutta algorithm. Figure (2a) shows the results of such a calculation for a time step of .05 and $t=30$ to 530 and plots the projection of the path in phase space onto the x_1 - x_2 plane. The picture is sort of the classic picture of the lowest-dimensional chaotic attractor that one can have in a continuous system.⁸ There are two motions: a periodic oscillation, and a lateral stretching and folding.

The iterative map that I constructed from this ODE is thus as follows:

$$\begin{aligned}x_1^{n+1} &= x_1^n + \epsilon x_2^n - (x_1^{n^2} - \frac{1}{2}x_2^{n^2} + x_3^{n^2})x_1^n \\x_2^{n+1} &= x_2^n + \epsilon(x_2^n - x_1^n) - (x_1^{n^2} + x_2^{n^2} + 10x_3^{n^2})x_2^n \\x_3^{n+1} &= x_3^n + \frac{\epsilon}{10}x_3^n - (3x_1^{n^2} + \frac{1}{2}x_2^{n^2} + 2x_3^{n^2})x_1^n + 6x_2^{n^2}\end{aligned}\tag{12}$$

where $x^{n^2} = x^n x^n$

I calculated 55000 iterates of this map and plotted them for $\epsilon = .01$ connecting every eleventh iterate in order to get figure (2b) which is a very similar looking chaotic orbit of precisely $\epsilon^{1/2} = .1$ times the size of the attractor for the differential equation as is predictable.

3.2.3 Experimental Implications

Now that we have the iterative map solution in terms of the solution $\vec{x}(t)$ of the differential equations (7) and (8) which we are assuming to be chaotic. We can predict several experimentally measurable features of the chaos. The predictions are as follows:

Prediction I: Deterministic noise develop in the power spectrum through a noisy broadening of sharp peaks which scales as ϵ .

⁸My particular choice of parameters is not special. For a wide range of similar choices, the same qualitative behavior is found.

To see this, consider the finite fourier transform of the iterates of the iterative map which are given by equation (10). The b:

$$\bar{g}(\omega) = \frac{1}{N} \sum_{n=1}^N \bar{x}^n e^{i\omega n} \quad (13)$$

Now let $\bar{h}(\nu)$ be the fourier transform of $\bar{x}(\tau)$, then

$$\bar{x}(\tau) = \int d\nu \bar{h}(\nu) e^{-i\nu\tau} \quad (14)$$

and

$$\bar{g}(\omega) = \frac{\bar{x}_0}{N} \sum_{n=1}^N e^{i\omega n} + \frac{\epsilon^a}{N} \sum_{n=1}^N \int d\nu \bar{h}(\nu) e^{i(\omega - \nu\epsilon)n} \quad (15)$$

But if we consider the case of large N so that $\sum_{n=1}^N e^{i\omega n} = \sum_{j=-\infty}^{\infty} \delta(\omega - 2\pi j)$ and if we consider only the range ϵ small and $\omega < \pi$ so that there is no contributions from $\bar{h}(\nu)$ for $j \neq 0$ we get:

$$\bar{g}(\omega) = \bar{x}_0 \delta(\omega) + \epsilon^{a-1} \bar{h}(\omega/\epsilon) \quad (16)$$

Figures (3) and (4) illustrate the evolution of the attractor and the power spectrum respectively for the iterative map given in equation (12). The iterative map is iterated 500 times and then x_2^n vs x_1^n is plotted for $n = 500$ to 10,500 for $a = .03, .1$, and $.25$. It may at first seem that we have very different attractors for $a = .03$ and $a = .1$. But if x_1^n, x_2^n , and x_3^n are solutions to the iterative map equations, so are $-x_1^n, -x_2^n$, and $-x_3^n$ when there are no second order terms. Modulus this parity transformation the attractors for $a = .03$ and $.1$ are very similar except for the fact that for $a = .1$, successive iterates appear farther apart due to the time waling. For $a = .25$ on the other hand, the attractor has undergone further bifurcations, and no ODE approximation of the iterates is valid.

In figure (4) and we set between fig.(4a) and fig.(4b), a simple scaling relationship where the only apparent difference between the pictures is an apparent stretching of the horizontal axis by a factor of $.1/.03 = 3\frac{1}{3}$. In contrast, figure (K) is qualitatively different and is noisier.

Therefore in an experiment we expect (for chaos, i.e. $h(\nu)$ broadband) a broadband component to grow out of the base of the various sharp frequency components with a waling of ϵ .

Prediction II: The Lyapunw exponent increases as a rate proportional to c .

The Lyapunw exponent measures the rate of exponential divergence of nearby orbits when there is chaos. The distance between two nearby trajectories diverge as $e^{\lambda t}$ where λ is the largest Lyapunw exponent. If in our model differential

equation nearby trajectories diverge as $e^{\lambda r}$, we have that for the iterative map, nearby trajectories diverge as $e^{\lambda n \epsilon} = e^{(\lambda \epsilon)n}$ and the Lyapunov exponent for the iterative map is therefore $\lambda \epsilon$, i.e. it is proportional to ϵ .

Prediction III: The integrated noise of the spectrum scales as ϵ^a for $a = 1$ and has a discontinuous jump for $a = 1/2$.

This result is simply found by integrating the $h(\nu)$ contribution to the square of the power spectrum given by equation (16).

Prediction N: The dimension of the attractor increases by one during the bifurcation.

Since the minimum dimension of a chaotic attractor of a differential equation system is over 1 (assuming no resonances, phaselockings, etc. in the iterative map) the dimension of the attractor of the iterative map will go from 0 (a point) to over 1 during the bifurcation and the dimension of the system will show the same increase.

Cautionary Note: All of the previous arguments assume in some sense that the properties of the model differential equations (7) and (8) persist under small perturbations, but in actuality, for such equations, quantities such as Lyapunov exponent, dimension, power spectra, and integrated noise may undergo infinitely many discontinuous changes in a finite parameter range. But the addition of random noise may smooth out these changes when one makes measurements on a real system. For the iterative map that I studied, a bifurcation to periodic behavior occurred for $a = .04 - .07$, but there were often long chaotic transients which could possibly be continually excited by noise since they lasted for as long as 3000 iterates. To understand these secondary bifurcations, it would be instructive to examine the properties of the model differential equations for first order changes in ϵ which are:

$$\frac{d\bar{x}(r)}{dr} + \frac{\epsilon}{2} \frac{d^2 \bar{x}(r)}{dr^2} = M'(\epsilon) \bar{x}(r) + \bar{x}(r) A(\epsilon) \bar{x}(r) + \epsilon B(0) \bar{x}(r)^3 \quad (17)$$

or

$$\begin{aligned} \frac{d\bar{x}(r)}{dr} + \frac{\epsilon}{2} \frac{d^2 \bar{x}(r)}{dr^2} = & M'(\epsilon) \bar{x}(r) + \bar{x}(r) B(\epsilon) \bar{x}(r) \bar{x}(r) \\ & + \epsilon^{1/2} C(0) \bar{x}(r)^4 + \epsilon D(0) \bar{x}(r)^6 \\ & + \epsilon^{1/2} \bar{x}(r) A'(0) \bar{x}(r) \end{aligned} \quad (18)$$

where A, A', B, C and D are tensors of rank 3, 3, 4, 5 and 6 respectively, to get a more precise theory on the nature of the bifurcation.

	Experiment	Mechanism I	Mechanism II
Prediction I: Power Spectra	Increase in background noise (?)	Increase in background noise	Broadening of peaks
Prediction II: Lyapunov exp.	$\propto \epsilon$ (?)	$c_1 + c_2 \epsilon$ dependence	$\propto \epsilon$
Prediction III: Noise Scaling	$\propto \epsilon$ (?)	$\propto \epsilon^2$	$\propto \epsilon$ or jump at $\epsilon = 0$
Prediction IV: Dim. Increase	about .5	0 to 1	more than 1

Table 1: Comparison of experimental **measurements** and theoretical predictions for the transition to turbulence in **modulated** wavy Taylor vorticity flow. The **experimental** statements are **inferred** from the Ph.D. thesis of **Anke Fraser**. Prediction I is **inferred** from the **power spectra** shown in figure (5) and the text of the thesis. Prediction II and III are **inferred** from the plots of **these** quantities in **Fraser's** thesis, and seem to hold for $R/R_c = 11.7$ to 13.6 . Prediction IV is explicitly **stated** in the text of the **thesis**.

4 Comparison with the Experiment

So now let's compare the two **sets** of theoretical predictions with the apparent **experimental results**. Table 1 shows the predictions of the two mechanisms and the **experimental results** for power spectra, **Lyapunov** exponent, noise scaling, and dimension.

We see from the table that of the four experimental **measurements**, the **one-dimensional** model is consistent with **three** out of four and that the high **codimension** model is consistent with two out of four of the **measurements** for the $a = 1$ case and only one out of four for the $a = 1/2$ case.

In the following I would like to illustrate in detail the **different** predictions of the two models and **determine** which can be **seen** clearly under present experimental conditions of background **noise** and **limited** data acquisition, and which are **less robust**.

I will do this by using the two iterative map models to **simulate** experimental data, then I will **repeat** the experimental calculations on the **simulated** data.

For the high **codimension** model, I could not **find** a model ODE with the **necessary** properties of a positive definite M' and a **chaotic** attractor for the $a = 1$ case (in fact I **suspect** such ODE's to be **extremely** rare). Whereas, for the $a = 1/2$ case, **such** an ODE was **very easy** to **construct**. Thus, for this model, I will use the iterative **map** model given by equation (16) though **theoretically** it gives the **worse** answer for the **scaling** of the integrated power of the noise.

To simulate experimental data, I added a **sinusoidal oscillation** at one frequency, $\sqrt{2}$, to a **sinusoidal** oscillation at another frequency, $\sqrt{3}$, to get a basic

quasiperiodic motion. On top of the oscillation, I added iterates of the iterative map model—iterating the maps at a frequency $2\sqrt{2} + 3\sqrt{3}$. I interpolated between iterates with a cosine function.

I sampled this motion at a fixed time interval and added a small random number of instrumental noise in the data. Then this was fed into a nonlinear function⁹ to generate harmonica and the output was analysed as pseudoexperimental data.

In the following, I will look at the different experimental measurements and discuss in which of the cases one of the models is distinctly better at simulating the experimental results in a robust fashion and for which cases a distinction cannot be made.

4.1 Power Spectra

Figure (5) shows the evolution of the power spectrum for the experiment. Before the bifurcation occurs, the spectrum has two fundamental frequencies whose harmonica produce all of the sharp components in the spectrum, and a fairly constant level of background noise. After the bifurcation, the background noise level increases and various of the harmonica decrease in amplitude.

Figures (6) and (7) show the analogous power spectra mechanism I and mechanism II respectively. For both the models, before the bifurcation, we have sharp components from the harmonics of the two fundamental frequencies and a constant background noise level, as shown in fig. (6a).

For the one dimensional model, we see after the bifurcation, an increase in the nearly constant background noise level as the bifurcation parameter is increased.

For the three-dimensional model, we see noisy and sharp components grow out of the base of the larger peaks and broaden the smaller peaks in the spectrum. For other choices of parameters, the sharp components in the spectrum are not as evident and the peaks broaden in a more noisy manner.

Thus for the two mechanisms there is a very robust qualitative difference in the evolution of the power spectrum: one gives an increase in background noise while the other gives a noisy broadening of peaks in the spectrum. Unfortunately the resolution in the spectra in Fraser's thesis is high enough in neither frequency or parameter dependence for me to clearly distinguish which mechanism is responsible. For example figure (5b) looks qualitatively similar to figures (7a) and (7b) but the text in Fraser's thesis states that the 'spectrum shows an increasing number of higher harmonics only.'

⁹this function is $f(r) = \sin(\frac{1}{2}r) - \frac{1}{10}\sin(r-2) + \frac{1}{25}\cos(3r)$

4.2 Lyapunov Exponent Measurements

Lyapunov exponent **measurement**¹⁰ from the pseudo-experimental data is a formidable **calculation** that I did not attempt, but **figure** (8) shows the **experimental** plot of Lyapunov exponent vs R/R_c from **Fraser's** thesis and **corresponding** plots of estimates of the Lyapunov exponent vs ϵ for the two models **based** on calculation on the iterative **maps**. Note that for the **1-d** model there is a jump in the Lyapunov **exponent** to very large **values** at the bifurcation point but for the 3-d model then is no **jump** though the **values** of the Lyapunov **exponent** are rather small. By altering the **1-d** model it is possible to make this jump in the Lyapunov exponent **very** small. So in this case the experimental **measurements** cannot reliably **distinguish between** the two mechanisms.

4.3 Dimension

I attempted to repeat this calculation on the **pseudoexperimental** data but could not get **the** calculation to converge though I did get the **correct** answer for **before** the bifurcation. It may simply be that **Anke Fraser** is much better at calculating dimension than I am. **If this is** the case then **since theoretically** the 3-d model cannot give a fractional increase of dimension, the experimental calculation would be a very strong piece of evidence for the 1-d model

5 Conclusion/Further Work

In **this paper** I have **examined two possible mechanisms** for the transition to **chaos** in modulated wavy Taylor vortex flow: a **quick** period doubling **bifurcation** or a high codimension bifurcation **where** the **linearisation** of the **instability** of the underlying iterative map at the **bifurcation point** is the identity matrix. **These two mechanisms should be** experimentally **distinguishable** and I have **outlined** in fair detail **how to make** the **distinction** for higher **resolution** experimental data **which** probably **exists** but which **at this** writing I do not have at my disposal.

The **next** steps in **this investigation** is to **re-examine** the **experimental** data in detail and to **generalize** the above **results** of high **codimension bifurcations** to **other types** of multiple **instabilities** (i.e. **different forms** of M').

Acknowledgements

I **would** like to thank Phil **Marcus** for initially bringing this problem to my attention **and** would like to **thank** Ed **Spiegel** for **many** **useful** **suggestions** and friendly encouragement this **summer**. **Also** I would like to **thank** the **GFD** program for **providing** the **stimulating** environment in which much of this work was done (**e.g.** all of the computer **calculations**).

¹⁰See ref.(11) to see how this is done.

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Figure (1): Power spectrum of long time iterates of equation (1) with shifted origin (to eliminate the spike at frequency = 0), showing flat noisy or very fine spikey spectrum.

Figure (2): Two dimensional projections of orbits of a) the model differential equation given by equation (1) and b) a corresponding iterative map equation given by equation (1). For b) successive iterates are connected by straight lines. For the differential equation, $t = 30$ to 530 , and for the iterative map $n = 500$ to 55500 .

Figure (3): Long time iterates of the iterative map given by equation (16).
 For $n = 500$ to $10,500$, X_1 vs X_2 is plotted. a) $\epsilon = .03$, b) $\epsilon = .1$, c) $\epsilon = .25$
 See text for further discussion.

Figure (4): Power spectra for the iterates of equation (19). Spectra are of 16.384 iterates with no averaging. a) $\epsilon = .03$, b) $\epsilon = .1$, c) $\epsilon = .25$. See text for further discussion.

Figure (5): The evolution of the power spectrum for the experiment (again from ref. (7)) 32,768 points transformed with 8 points taken per faster period (f_2 in the spectra) a) $R/R_c = 10.1$, b) $R/R_c = 11.4$, c) $R/R_c = 12.0$, d) $R/R_c = 12.8$.

Figure (6): The evolution of the pseudo-experimental power spectrum for mechanism I. 16,384 points were sampled (12 times per faster period), were fourier transformed, squared, averaged over 4 frequency steps and the natural log was plotted vs frequency. a) At $\epsilon = 0$ there are sharp peaks from the harmonics of the two fundamental frequencies and a nearly constant background noise level. b) At $\epsilon = .07$ there is an increase in the noise level, but it is not concentrated at the peaks; it is broadband. c) At $\epsilon = .2$ there is a very large background noise level.

Figure (7): The evolution of the pseudo-experimental power spectrum for mechanism II, calculated in the same manner as for mechanism I. a) $\epsilon = .03$, b) $\epsilon = .051$, c) $\epsilon = .1$, d) $\epsilon = .25$. Note the gradual broadening of the peaks in a) and b). In c) the sharp components of the iterative map spectrum dominate the general appearance of the spectrum. In d), the spectrum broadens due to secondary bifurcations of the Iterative map.

Figure (8): Plots of the Lyapunov exponent vs control parameter for the experiment and the pseudo-experimental data. The Lyapunov exponent is in units of bits per orbit, where for the pseudo-experimental data and orbit is the length of the shorter of the two fundamental periods. a) experimental plot taken directly from ref. (7), b) plot for mechanism I which tends to have a very large exponent. By altering how the data is generated, the exponent can be decreased by a factor of four. c) Plot for mechanism II shows points calculated from the iterative map and a line resulting from an estimate inferred from the behavior of the model differential equation. It is possible to make changes in the pseudo-data that would increase the exponent to that seen in the experiment, but such a model would be analogous to a mode 15 times the frequency of the fundamental becoming unstable. This seems a bit artificial of a construction to the author.

INSTABILITY WAVES IN GALACTIC SLAB

Zhongshan Qian

Abstract:

Throughout this paper, we use a polypotropic gas model for a very thin, infinite disc of galaxy with $\gamma = 2$. The linear dispersion relation is:

$$s^2 = c^2 k^2 - \mu k^4$$

Where μ is the surface density of the disc. s is the small parameter defined by the thickness of the disc multiplying 'Jeans' wave number. Then the nonlinear partial differential equations for the amplitude are derived.

Introduction

Since the discovery of the self-gravitational instability by Jeans, much work has been done in this problem. In 1951, **Ledoux** used an isothermal model, that is-to study the instability wave in galactic disc. He found the disc is unstable with the critical wave number

where ρ_c is the central density.

There are two factors which can stabilize the system. One is the **uniform** rotation of the galaxies. The other one is the external force imposed by the halo.

Chandrasekhar proved the uniform rotation stabilizes the modes with wave vectors exactly perpendicular to the rotation axis for Jeans' simple homogeneous model. Changed critical wavenumber is,

$$k_c^2 c^2 = 4\pi G \rho - 4\Omega^2$$

In **Ledoux's** sheet the same result should ensue.

In our problem, we choose polytropic gas model for the disc with $n = 2$. In this case, the disc will be restricted within a finite region. Then we put the vertical external force which can be adjusted later on to near marginal instability so that we can do nonlinear problem. The full equation are as follows:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \\ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\partial p}{\rho} - \nabla \varphi - \nabla \Phi \\ \nabla^2 \varphi = 4\pi G \rho \end{cases}$$

Where $\rho, \vec{u}, p, \varphi, \Phi$ are mass density, velocity, pressure, self-gravitational potential and external potential separately. The assumption is:

$$p = K \rho^2, \quad \Phi = \frac{1}{2} \alpha \left(\frac{z}{z_0} \right)^2$$

z_0 is half thickness of the layer. Using the velocity potential χ defined by

$$\vec{u} = \nabla \chi$$

we get

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \chi) = 0 \\ \frac{\partial \chi}{\partial t} + \frac{1}{2} (\nabla \chi)^2 + 2K\rho + \varphi + \frac{1}{2} \alpha \left(\frac{z}{z_0} \right)^2 \\ \nabla^2 \varphi = 4\pi G \rho \end{cases}$$

The next step is to nondimensionalize the equations. Let's start at the dimension of the variables.

$$[\varphi] = \alpha, \quad [\alpha] = [z] = z_0$$

$$[\rho] = \frac{\alpha}{2K}, \quad [\chi] = \sqrt{\alpha} z_0$$

$$[t] = z_0 / \sqrt{\alpha}$$

The nondimensionized equations are,

$$\begin{cases} \frac{\partial p}{\partial t} + \nabla \cdot (p \nabla \chi) = 0 \\ \frac{\partial \chi}{\partial t} + \frac{1}{2} (\nabla \chi)^2 + p + \varphi + \frac{1}{2} z^2 = 0 \\ \nabla^2 \varphi = (\nabla z_0)^2 p \end{cases}$$

define $\epsilon = \frac{1}{2} z_0^2$ and eliminate φ from the equations, we get the two-variable equations.

$$\begin{cases} D(p \nabla \chi) + \frac{\partial}{\partial t} (p \frac{\partial \chi}{\partial t}) + \frac{\partial p}{\partial t} = 0 \\ (D^2 + \epsilon^2) p + 1 + \frac{\partial}{\partial t} p = -\nabla^2 [\frac{\partial \chi}{\partial t} + \frac{1}{2} (\nabla \chi)^2] \\ D \equiv \frac{\partial}{\partial t} \end{cases}$$

1. Steady Solution

Using symmetric solution, the mathematical problem becomes

$$(D^2 + \epsilon^2) p + 1 = 3$$

with boundary condition.

$$\begin{cases} Dp = 0 & z = 0 \\ p = 0 & z = 1 \end{cases}$$

The result is

$$p = \frac{1}{\epsilon^2} \left(\frac{\cos \epsilon z}{\cos \epsilon} - 1 \right) = \frac{1}{2} (1 - z^2) + \epsilon^2 R_2(z) + \dots$$

2. Linear Solution

If we set perturbation

$$\begin{aligned} p &\sim p(z) e^{st+ikx} \\ \chi &\sim \chi(z) e^{st+ikx} \end{aligned}$$

then the equations for p, χ

$$\begin{cases} D(p \nabla \chi) = -s p + k^2 p \chi \\ (D^2 + \epsilon^2) p = k^2 p - (D^2 - k^2) \chi \end{cases}$$

$$\text{with } \varphi = -p - s \chi$$

The B. C. for p is must be smooth at boundary. That is

$$\varphi_{in}|_{\text{boundary}} = \varphi_{out}|_{\text{boundary}}, \quad D\varphi_{in}|_{\text{boundary}} = D\varphi_{out}|_{\text{boundary}}$$

at boundary. In linear case, we can move the boundary to $z = 1$ outside the slab.

1 - order B. C. (2.4) gives

$$f_0 - \epsilon A \sin \epsilon = 0 \quad (2.6)$$

And combining (2.5), (2.6) we get

$$S_1 = m \epsilon^2$$

The second order B.C. (2.4) gives

$$Df_1|_{z=1} + D\rho_1|_{z=1} + \rho_1|_{z=1} = 0$$

So we can express

$$\int_0^1 \rho_1 dz \quad \text{in terms of } f_0 \text{ and } \rho_1.$$

The third order of (2.1) yields

$$D(PDf_1)$$

The solvability condition yields

$$S_2 = -S_1 \left(1 + \frac{\cos \epsilon}{\epsilon \sin \epsilon} \right)$$

So, the dispersion relation is

$$S^2 = m \epsilon^2 |k| - m \left(1 + \frac{1}{3} \epsilon^2 \right) k^2$$

When we make the ϵ go to zero, the modes will approach to neutral ones. Nearby the marginal case. We can do nonlinear analysis. This results in another small parameter ϵ . We have to use distinguished limit.

From the dispersion relation, we can see that in order to make every term tie together k^2 should have the same order as ϵ^2 .

$$\text{So } k \sim \epsilon^2$$

3. A distinguished limit.

Assume $k = \lambda \epsilon^2$, $s^2 = \sigma \epsilon^4 + \dots$

Then the equations are,

(3.1)

$$\begin{cases} D(PDf) = \lambda^2 \cdot PDf - s^2 P \\ D^2 P = -\epsilon^2 P + \epsilon^4 \lambda^2 P - D^2 f + \lambda^2 \epsilon^4 f \end{cases} \quad (3.2)$$

with B. C. $\begin{cases} Df = DP = 0 & \text{at } z=0 \\ Df \text{ regular} & \text{at } z=1 \\ Df + DP + |\lambda| \epsilon^2 (f+P) = 0 & \text{at } z=1 \end{cases} \quad (3.3)$

(3.4)

Make the expansion

$$\begin{aligned} f &= \epsilon^2 f_1 + \epsilon^4 f_2 + \dots \\ P &= \epsilon^2 P_1 + \epsilon^4 P_2 + \dots \end{aligned}$$

The second order of (3.1) (3.2) is

$$\begin{cases} D(P_0 Df_1) = 0 \\ D^2 P_1 = 0 \end{cases}$$

$$\therefore \begin{aligned} f_1 &= \text{const} \\ P_1 &= \text{const} \end{aligned}$$

The fourth order of (3.1) ((3.2) is

$$\begin{aligned} D(P_0 Df_2) &= \\ D^2 P_2 &= -P_1 \end{aligned}$$

We then get, $DP_2 = -P_1 z$

From the sixth order of (3.1) we obtain

$$D(P_0 Df_2) = \lambda^2 P_1 f_1 - \sigma P_1$$

We use the solvability condition and get

$$\lambda^2 P_1 f_1 - \sigma P_1 = 0 \quad (3.5)$$

The fourth order B. C. (3.4) gives

$$DP_2|_{z=1} + |\lambda| (P_1 + f_1) = 0 \quad (3.6)$$

(3.5) . (3.6) gives

$$S^2 = m_0 \epsilon' / k - m_0 k^2$$

which has the same form as we got before.

4. Nonlinear equations

In order to get Non-linear amplitude equation, we get to scale it. From the linear dispersion relation we get the scaling line

$$T = \epsilon^2 t, \quad X = \epsilon' x$$

equation becomes,

$$D(PDX) = -\epsilon^2 \frac{\partial^2 P}{\partial T^2} - \epsilon^4 \frac{\partial^2}{\partial X^2} (P \frac{\partial^2 X}{\partial X^2}) - D(PDX) \epsilon^4 \frac{\partial^2}{\partial X^2} (P \frac{\partial^2 X}{\partial X^2}) \quad (4.1)$$

$$\left\{ D^2 P = -\epsilon^2 P - \epsilon^4 \frac{\partial^2 P}{\partial X^2} - (D^2 + \epsilon^4 \frac{\partial^2}{\partial X^2}) \left\{ \epsilon^2 \frac{\partial^2 X}{\partial T^2} + \frac{1}{2} [(DX)^2 + \epsilon^4 (\frac{\partial^2 X}{\partial X^2})^2] \right\} \right\} \quad (4.2)$$

Boundary condition for P are as before

$$\begin{cases} \varphi_{in}|_q = \varphi_{out}|_q \\ \nabla_n \varphi_{in}|_q = \nabla_n \varphi_{out}|_q \end{cases}$$

Where q is the perturbed boundary. ∇_n is vertical to the boundary.

$$\begin{aligned} \nabla_n &= (1 - \frac{\epsilon^2}{2} \eta_x^2) D - \epsilon^4 \eta_x (1 - \epsilon^2 \eta_x^2) \frac{\partial^2}{\partial X^2} \\ &= D + O(\epsilon^4) \end{aligned}$$

So we take D as ∇_n since the boundary conditions that I will use later are below ϵ^4 . Next we derive the relation between φ_{out} and $D \varphi_{out}$.

Generally,

- 238 -

$$\varphi_{out} = \int_{-\infty}^{\infty} e^{ikx - |k|z} \varphi_k dk$$

So,

$$\begin{aligned} \varphi_k &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{out}(\xi, z) e^{-ik\xi + |k|z} d\xi \\ D \varphi_{out} &= \int_{-\infty}^{\infty} -|k| e^{ikx - |k|z} \varphi_k dk = -\frac{1}{2\pi} \int_{-\infty}^{\infty} |k| e^{ik(x-\xi)} \varphi_{out}(\xi, z) d\xi dk \\ &= \frac{\partial}{\partial x} H(\varphi_{out}) = e^z \frac{\partial}{\partial x} H(\varphi_{out}) \end{aligned}$$

where Hilbert transform is defined by

$$H(f(x)) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi$$

So boundary conditions are

$$D\varphi_{in}|_{z=1} + D^2\varphi_{in}|_{z=1} \eta + \dots = e^z \frac{\partial}{\partial x} H[\varphi_{in} + D\varphi_{in}|_{z=1} \eta + \dots] \quad (4.3)$$

$$\varphi_{in} = -\rho - e^z \frac{\partial \chi}{\partial T} - \frac{1}{2} [(DX)^2 + e^z \frac{\partial \chi}{\partial x}]$$

$$DX = D\rho = 0 \quad (4.4)$$

$$DX|_{z=1} + D^2\chi|_{z=1} \eta + \dots = e^z [\eta_T + e^z \eta_x (\frac{\partial \chi}{\partial x} + \frac{\partial}{\partial x} DX \eta + \dots)] \quad (4.5)$$

(4.5) is kinematic B.C.

Using the expansion as before-

$$\begin{cases} \eta = e^z \eta_1 + e^{2z} \eta_2 + \dots \\ \rho = e^z \rho_1 + e^{2z} \rho_2 + \dots \\ \chi = \chi_0 + e^z \chi_1 + \dots \end{cases}$$

We get the zero order of the equation (4.1) is

$$D(P_0 DX_0) = 0 \quad \therefore \chi_0 = f_0(x, T)$$

The second order of the equations (4.1) (4.2) are,

$$\begin{cases} D(P_0 DX_2) = 0 \\ D^2\rho_2 = 0 \end{cases}$$

$$\begin{aligned} \therefore \chi_2 &= f_2(x, T) \\ \rho_2 &= g_2(x, T) \end{aligned}$$

The fourth order of equations (4.1), (4.2) are,

$$\begin{cases} D(P_0 DX_0) = -g_{0T} - P_0 f_{0xx} \\ D^2 P_4 = -P_0 \end{cases}$$

The solvability condition gives

$$\begin{cases} g_{0T} + m_0 f_{0xx} = 0 \\ DX_0|_{x=1} = g_{0T} \end{cases} \quad (4.6)$$

$$P_4 = -\frac{1}{2} g_0 x^2 + g_1(x, T) \quad (4.7)$$

The fourth order B.C. (4.3) (4.) gives (4.8)

$$\begin{cases} g_{1T} = DX_0|_{x=1} \\ g_0 = \frac{\partial}{\partial x} H[-g_0 - f_{0T}(x, T)] \end{cases} \quad (4.9)$$

$$g_0 = \frac{\partial}{\partial x} H[-g_0 - f_{0T}(x, T)] \quad (4.10)$$

Combining (4.6), (4. 10), we get the linear amplitude equation

$$g_{0TT} + m_0 H(g_{0x}) - m_0 g_{0xx} = 0 \quad (4.11)$$

To get the non-linear equation, we have to go to the higher order and use the reconstitution method.

The sixth order of (4.1) (4.2) are

$$\begin{cases} D(P_0 DX_0) + D(P_1 DX_0) = -\frac{\partial P_2}{\partial T} - P_0 f_{1xx} - P_1 f_{0xx} \\ \quad - D(P_1 DX_0) - (f_{0x} g_0)_x \\ D^2 P_6 = -P_0 - g_{0xx} - \frac{\partial}{\partial T} D^2 X_0 - f_{0xxT} \end{cases} \quad (4.12)$$

The solvability condition gives

$$g_{1T} + m_0 f_{1xx} = \frac{1}{2} g_{0T} - m_0 f_{1xx} - g_0 g_{0T} - (g_0 f_{0x})_x \quad (4.14)$$

The sixth order B.C. (4.3) yields with (4.13)

$$g_2 + H(g_{1x} + f_{1xT}) = \frac{1}{2} g_0 - g_{0xx} - f_{0xxT} - g_0^2 + \frac{1}{2} H[g_{0x} - (f_{0x})_x] \quad (4.15)$$

Combining (4.14), (4.15), we get;

$$g_{,TT} + m_0 H(g_{,x}) - m_0 g_{,xx} = \frac{2}{5} m_0 g_{,xxx} - m_0 H(g_{,x}) + m_0 g_{,xxx} - (g_0 f_{,xx})_{,T} \\ - m_0 H[(g_0)_{,x}] + \frac{m_0}{2} (f_{,xx})_{,xx} - (g_0 g_{,TT})_T \quad (4.16)$$

We define $\begin{cases} f = f_0 \\ g = J_0 + \epsilon^2 g_2 \end{cases}$

Then combining (4.11) . (4.16), we get the nonlinear equation for amplitude, f.g.

$$g_{,TT} + m H(g_{,x}) - m(1 + \frac{1}{2} \epsilon^2) g_{,xx} = - \epsilon^2 [m H((g_0)_{,x}) - \frac{m}{2} (f_0)_{,xx} \\ + (g_0 g_{,TT})_T + (g f_{,xx})_{,TT}] \\ g_T + m f_{,xx} = 0$$

5. Future Work

The problem is not completely solved. Later on, we would try to reduce this PDEs into a ODEs and then study the structure of the equations. This can be done by putting the relevant parameters close to the marginal case.

Acknowledgement

This paper work is directed by Professor E. A. Spiegel. He gave me much help and encouragement. I would like to thank him and other professors and students with whom I discussed this paper.

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BAROTROPIC SHEAR FLOW INSTABILITY OVER THE CONTINENTAL SHELF

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Abstract

Based on observations of a vortex roll-up reminiscent of shear instabilities, linear theories relevant to barotropic instability on the continental shelf are derived. A generalized Rayleigh condition, in terms of potential vorticity, as well as explicit solutions for idealized cases are presented, all of which demonstrate the stabilizing effect of topography and rotation. The condition applies very well to the observations, but the existence of only one vortex is difficult to explain from instability theory. To that aim, a number of experiments with contour dynamics are performed, pointing towards some possible explanations.

I. Introduction

This study was inspired by observations from the Coastal Ocean Dynamics Experiment (CODE) conducted off the northern California coast in spring/summer of 1981 and 1982 (see e.g., Winant et al., 1986, for a presentation of the moored observations). On May 26, 1982, an IR satellite image of the region [Plate 1] showed a structure on mid-shelf of approximately 10 km size which

Plate I

which looked very similar to a vortex roll-up resulting from typical shear instabilities. In situ flow observations during that period from moored current meters (CODE Group, 1985) and current-following drifters (Davis, 1983) suggest that the IR temperature structure was, in fact, representative of actual fluid motion. This study will attempt to explain the observed feature as a developed instability. For that purpose, the simplifying assumption will be made that the flow is essentially barotropic. This can be justified with the moored observations at least over the mid- and inner-shelf, which is the main region of large horizontal shear during that period and where the structure was observed.

II. Generalized Rayleigh Criterion for Instability

The assumptions made for the subsequent theoretical considerations are: barotropic (one-layer), f-plane flow, rigid lid, absence of forcing, large variations in bottom topography.

The relevant equations then become

$$u_t + \vec{u} \cdot (\vec{\nabla} u) - fv = -\frac{1}{\rho} p_x \quad (1)$$

$$v_t + \vec{u} \cdot (\vec{\nabla} v) - fu = -\frac{1}{\rho} p_y$$

$$\frac{DH}{Dt} + H \vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot (H \vec{u}) = 0 \quad (2)$$

where u, v are the horizontal component of the flow $\vec{u}$, $H(x, y)$ the fluid depth (bottom topography), and subscripts denote derivatives. Introducing the relative vorticity $\zeta = v_x - u_y$, the potential vorticity (PV) equation

$$\frac{D}{Dt} \left(\frac{\zeta + f}{H} \right) = \frac{D}{Dt} q = 0 \quad (3)$$

may be obtained from (1), (2).

Consider a basic state with parallel shear flow in the y-direction, $U = V(x)$, and topography parallel to that flow $H = H(x)$. Then the background PV is

$$\bar{q} = \bar{q}(x) = \frac{V_x + f}{H(x)} \quad (4)$$

Now introduce small perturbations $u', v', \zeta', q' = (\zeta'/H)$, and linearize (3) about the basic state, then the perturbation PV equation becomes

$$q'_t + v q'_y + u' \bar{q}_x = 0 \quad (5)$$

Introducing a volume transport streamfunction ψ (because of (2)) with $u'H = -\psi_y$, $v'H = \psi_x$, applying it to (5) and making the normal mode ansatz

$$\psi = \phi(x) \exp i(ky - \omega t), \quad (6)$$

the eigenvalue problem becomes

$$\phi_{xx} - \frac{H_x}{H} \phi_x + \left[-k^2 - \frac{\bar{q}_x H}{V - c} \right] \phi = 0 \quad (7)$$

The first derivative may be removed by the substitution $\phi = H^{1/2} \Phi$, upon which the familiar trick may be applied of multiplying by Φ^* , integrating over the domain, taking the imaginary part, dropping all purely real terms (see e.g., Drazin and Reid, 1981), to obtain

$$\int c_1 \frac{H|\phi|^2}{|V-c|^2} \bar{q}_x = 0. \quad (8)$$

which implies that instability ($c_i = 0$) can only occur if q changes sign somewhere, i.e., if q has a non-monotonic profile. This result, which is a generalization of the Rayleigh inflection point theorem, has also been noted by Collings and Grimshaw (1984), and for the quasi-geostrophic limit (for barotropic and baroclinic instability) by Pedlosky (1979). The effect of topography becomes very apparent, if the condition is re-written as

$$\underbrace{(\zeta + f)_x}_{(A)} - \underbrace{(\zeta + f)}_{(B)} \frac{H_x}{H} = 0 \quad (9)$$

which, with (A) alone, would be the stability criterion in the absence of topography. A generic shear flow profile like

Figure 1

could be unstable without topography ($[(\zeta+f)_x]$ has a zero crossing), but as long as $(\zeta+f)=0$ everywhere, a sufficiently large topography (H_x/H) can always stabilize the flow by shifting the curve in Figure 1b, up/down by the amount of (B) until no zero-crossing occurs.

III. Explicit Linear Solution for Idealized Cases

The eigenvalue problem (7) may be solved analytically if the basic state has regions of piecewise constant potential vorticity, since then in each region the singular term of (7) vanishes (it also means that at any point $q'=0$, which can be used instead of (5) to obtain the same equation). If, in addition, the function H_x/H is constant, i.e., H is exponential, the solution is particularly simple. Therefore the problem is set up as follows

$$H(x) = \begin{cases} H_0 \exp(aL) & x > L & \text{region (I)} \\ H_0 \exp(ax) & -L < x < L & \text{region (II)} \\ H_0 \exp(-aL) & x < -L & \text{region (III)} \end{cases} \quad (10)$$

$$V(x) = \begin{cases} V_0 & x > L & \text{(I)} \\ \frac{V_0 + fL}{\sinh(aL)} \exp(ax) - fx - \frac{V_0 + fL}{\tanh(aL)} & -L < x < L & \text{(II)} \\ -V_0 & x < -L & \text{(III)} \end{cases} \quad (11)$$

where the **form** of V in region **II** is determined by its values at $x = \pm L$ and by the constraint of constant potential vorticity. Note that $a < 0$. The free parameters in this dimensional problem are L , V_0 , H_0 , and the topographic steepness $H_x/H = a$. Scaling the variables like

$$x = Lx' \quad a = 1/L \quad a' \quad V = V_0 V' \quad H = H_0 H' = H_0 \left(1 + \frac{\Delta H}{H_0} h'\right) \quad \text{where } \Delta H \equiv H_x(0)L \quad \left(\Rightarrow \frac{\Delta H}{H_0} \equiv a'\right) \quad (12)$$

$$q = \frac{-f}{H} = \frac{f}{H_0} \frac{R_0 V' + 1}{H'} = \frac{f}{H_0} q' \quad \text{where } R_0 = \frac{V_0}{fL},$$

only two free parameters remain: the Rossby number R_0 and the steepness $(\Delta H/H_0) \equiv a$. (Hereafter, the primes for non-dimensionalized quantities will be dropped.) The resulting types of profiles for $H(x)$ and $V(x)$, and their dependence on the two parameters is shown in Figures 2 and 3. Note the change

FLOW PROFILES FOR CONSTANT PV MODELS (exponential)

Figure 2

Figure 3

of V from a monotonic to a "peaked" profile for small values of R_0 and large values of $\Delta H/H$, which is an artificial consequence of the constraints in the problem. The non-dimensional equation to solve in all three regions then is

$$\phi_{xx} - a \phi_x - k^2 \phi = 0 \quad (13)$$

with the boundary condition

$$\phi \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm \infty. \quad (14)$$

The solutions in each region have to be matched at $x = \pm 1$. Matching the volume flow across the regions on each side of the boundaries, and also the pressure on each side of $x = \pm 1$, the matching conditions become

$$[\phi]_{\pm}^{\pm} = 0 \quad \text{at} \quad x = \pm 1 \quad (15)$$

$$[(V-c) \phi_x - V_x \phi]_{\pm}^{\pm} = 0 \quad \text{at} \quad x = \pm 1$$

The general solution to (13) and (14) is

$$\phi = \begin{cases} A \exp(-kx) & x > 1 \\ \exp(\frac{a}{2}x) [D \exp(k'x) + E \exp(-k'x)] & -1 < x < 1 \\ B \exp(kx) & x < -1 \end{cases} \quad (16)$$

where $k' = (k^2 + a^2/4)^{1/2}$.

Applying the matching conditions, the dispersion relation is obtained, a quadratic equation for the phase speed c (yielding two roots since two boundaries are being matched):

$$\begin{aligned} & c^2 [e^{2k'} k_1 k_4 - e^{-2k'} k_2 k_3] + \\ & c^2 [e^{2k'} (\beta k_1 + \alpha k_4) - e^{-2k'} (\beta k_3 + \alpha k_2)] + \\ & [e^{2k'} (-k_1 k_4 - \beta k_1 + \alpha k_4 + \alpha \beta) - e^{-2k'} (-k_2 k_3 - \beta k_3 + \alpha k_2 + \alpha \beta)] = 0 \end{aligned} \quad (17)$$

where

$$\begin{aligned} k_1 &= \frac{a}{2} + k + k' & k_2 &= \frac{a}{2} - k + k' \\ k_3 &= \frac{a}{2} + k - k' & k_4 &= \frac{a}{2} - k - k' & a < 0 \\ \alpha &= \frac{2a}{1 - \exp(-2a)} \left(1 + \frac{1}{R_0}\right) - \frac{1}{R_0}, & \beta &= \frac{2a}{\exp(2a) - 1} \left(1 + \frac{1}{R_0}\right) - \frac{1}{R_0} \end{aligned}$$

This equation has been solved numerically for the imaginary parts of the roots of c , the results of which are plotted in Figures 4 and 5 as a function of k and the two parameters R_0 and a .

Topography $A=0,0.3,1,2$, Rossby # large

(a)

Topography $A=0.5$, $R_0 = \text{large}, 2, 1, 0.9, 0.8, 0.75$

(b)

Figure 4

STABILITY REGIMES FOR PIECEWISE CONSTANT PV MODEL (exponential)

Figure 5

In Figure 4a, the $a = 0$ case corresponds, of course, to the classical Kelvin-Helmholtz problem (no f , no topography), with maximum growth at $k = 0.4$ and a cut-off at $k = 0.64$ (e.g., Drazin and Reid, 1981). As the topography is increased, growth rates are reduced and less wavenumbers become unstable. Figure 4b shows the behavior for some average topography, and varying R_0 . As rotation becomes more important (or the shear is reduced), the growth rates decay again although the last unstable wavenumbers this time are not around $k = 0$, but somewhere between 0.4 and 0.5. The regime diagram, Figure 5, shows lines for several wavenumbers k in R_0 - a -space, above which that k is stable and below which it is unstable. The most unstable wavenumber is approximately $k = 0.45$. This transition line appears to occur at those parameter combinations

where the flow profiles (in Figure 3) change from the monotonic to the "peaked" shape. This is demonstrated by the large dots plotted in Figure 5, which represent the $a = a(R_0)$ solution for which the flow profile $V(x)$ has zero derivative at $x = 1$. In order to show that this qualitative change in shape is not the cause for the transition to stability, and to test the effect of the assumptions of the discontinuous PV distribution, a model without these constraints will now be applied for the quasi-geostrophic (QG) limit.

Consider the case of small Rossby number $R_0 \ll 1$ and steepness parameter $S = (\Delta H/H_0) \ll 1$ and scale the equations. To leading order, the potential vorticity derivative becomes

$$\overline{q}_x = R_0 V_{xx} - S h_x \quad (S \equiv \frac{\Delta H}{H_0}) \quad (18)$$

The eigenvalue problem (7) reduces to a particularly simple form to leading order, if $h_x = \text{const}$, i.e., a uniform bottom slope everywhere. Setting $h_x = 1$ then, one obtains

$$\phi_{xx} - k^2 \phi + \frac{S_R - V_{xx}}{V - C} \phi = 0 \quad (19)$$

where S_R is a 'rotational steepness parameter' $S_R = S/R_0$.

Equation (19) may also be derived the same way (7) was, but starting from the QG potential vorticity equation instead of (3). An analytic solution for (19) was demonstrated by Lipps (1965) for the smooth shear flow profile $V(x) = \tanh(x)$ and with the planetary β in place of S_R . By finding the neutral solutions ($C_1 = 0$) and perturbing about them, he showed that for $\beta > 4/3^{3/2} \approx 0.77$, the flow is stable at all k . Applying this result to the topographic problem (19), the conclusion is that the shear flow over topography is stabilized if $(\Delta H/H_0)(1/R_0) > (S_R)_{\text{crit}} = 0.77$. This solution does not have the problems mentioned above for the piecewise constant PV model. Comparing the results of the two models - the heavy line labelled 'QG' in Figure 5 is the curve $(\Delta H/H_0) = 0.77 R_0$ - it appears that the piecewise model does very well, at least in the parameter range where the comparison is valid $[(\Delta H/H_0) \ll 1, R_0 \ll 1]$.

It is easy to show that in both models the transition to the first unstable wavenumbers occurs where the generalized Rayleigh condition [(8) or (9)] admit the possibility of instability. Thus, under the assumptions made for the two models, the Rayleigh condition is sufficient for instability to occur at some wavelengths.

IV. The Observations Revisited

The generalized Rayleigh condition will now be applied to in-situ data from the CODE experiment in the areas of the observed vortex roll-up. With a section of 4 current meter moorings across the shelf, time series of the barotropic PV can be estimated at the three points in between. The result is plotted in Figure 6, which shows that the sampled profile of PV was monotonic (i.e., flow stable) most of the time, except for the three days (shaded) preceding the date of the satellite image which showed the eddy. Also, among 20-30 images spanning the measured time series, no similar structure was detected at any other time. Thus it would appear that there is an extremely satisfying agreement between the observations and the previous theories.

Figure 6

A severe problem, however, lies in the fact that only one isolated roll-up is observed, while all the previous theories assume infinitely long, uniform conditions in y , and would thus produce an infinite train of vortices. Profiles of PV similar to Figure 6 but further north and south along the coast suggest, that in those locations the flow was either stable or only marginally unstable. Thus the question arises whether a finite region of unstable PV profile, *i.e.*, a "patch" of vorticity maximum rather than an infinite strip, can lead to fewer or one instability roll-up. Some attempts to approach this problem are made in the final section.

V. The "One-Eddy" Problem

Some aspects of this problem were explored with contour dynamics. This is a particular numerical method (see, *e.g.*, Stern, 1985) for calculating the evolution of a two-dimensional flow with the simplifying assumption that the PV is uniform in various regions which are bounded by the contour(s). Since at any time the location/shape of the contour determines the entire PV distribution and thus the flow field, the problem reduces to one of evaluating the time evolution of the contour itself. For example, for one or several closed contours and barotropic flow, the flow field (u, v) everywhere, and in particular, on the contour, can be expressed as

$$\begin{pmatrix} u \\ v \end{pmatrix} (x, y, t) = \frac{q}{4\pi} \oint_{(z, \eta)} \left(\frac{d\zeta}{d\eta} \right) \ln [(x - \zeta)^2 + (y - \eta)^2] \quad (20)$$

where the integral is taken along the contour(s) and q is the vorticity inside them. The subsequent results have been obtained for the simplest case of barotropic flow, closed contours, and no boundaries.

Since by the earlier condition, instability is expected if a local "profile" of vorticity has a maximum, at least a long thin contour (vorticity = 1 inside and 0 outside), is expected to become unstable if perturbed appropriately (an infinitely long contour, *i.e.*, a strip of vorticity bounded by two straight lines, would be the Kelvin-Helmholtz limit, see also Pozrikidis and Higdon, 1985). The result of perturbing a contour with intermediate aspect ratio is shown in Figure 7, where the instability generates three isolated vortices, each of which would rotate and form roll-ups of streaklines or some tracer in the fluid, Making the contour nearly "round," however, in order to

Figure 7: Simulation for contour of vorticity 1, in zero-vorticity background. At $t=0$, the flow field is also shown.

Figure 8: Same as Figure 7, but in background of vorticity 1 ($\zeta=2$ in contour).

generate only one eddy, is not satisfactory because **it** already **is** one vortex, which would form roll-ups similar to the ones observed. Even though locally a section of vorticity across such a round initial state would have a maximum and satisfy the condition for instability, instability would **also** not be expected from the classical result that (elliptical) eddies of short aspect ratios ($< 3:1$) are stable.

In a background flow with significant shear though, as is the case for the conditions in the environment of the observed vortex, **it** appears that the behavior can be quite different. Figure 8 shows a simulation of a contour which is the same as in Figure 7, but in a background shear flow of vorticity equal to the anomaly within the contour. **It** can be seen that rather than forming three eddies as before, in a shear flow the contour and, correspondingly, a tracer line across **it**, may roll-up into ~~one~~ compact structure.

Another mechanism for generating an **isolated** vortex has been **investi-**gated, by using two equal and opposite contours which do not insert any net rotational motion into the field, but the results are too premature to be presented here.

VI. Conclusions

At first sight, simple linear instability theories appear to **apply** **sur-**prisingly well to the observed vortex roll-up on the California shelf. **How-**ever, the presence of only one eddy poses a challenging problem which needs further study than the initial experiments presented here. **Also**, a more thorough analysis of the observations and linear theories will be necessary, in terms of scales, growth rates, effects of the sidewall boundary, friction, QG approximation, etc. The contour dynamics approach will be extended to include the coastal boundary and some bottom topography.

Acknowledgments

Initial attention to this feature was drawn by R. Beardsley. Most of the work was done in continuous interaction with M. Stern and with helpful discussions with G. Flierl, W. Young, and others. The author is grateful for the opportunity to spend a summer in the stimulating environment of the GFD program.

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HEXAGONS, HARMONICS AND HELMOLTZ

Nicholas H. Brummell

INTRODUCTION

Symmetry, loosely defined as a pattern formed by regular repetition, is ever present in our world today, from the mirror symmetry of homo sapien himself to the actual matter that he is made up of, and as such, is a very familiar principle.

Symmetry implies some sort of order and because of this was followed by many early astronomers and other scientists almost as a faith. Keplers observation that the orbit of Mars was not in fact circular came as somewhat of a shock to these devotees as this contradicted their ideas of a nice, spherical system, and it was not until Kepler realised that although the orbit was not circular, it did possess a very strong regularity in that it was elliptical, that a major advance in astronomy was possible. The flood gates then opened and Newton's Laws came into being.

So the idea of order through symmetry is a powerful one, but it is also dangerous because it can confine and restrict thought, in the same way that too much order in a work of art makes it sterile and boring. But science cannot ignore symmetry because it is far too common in nature and far too powerful an analytical tool to throw away, yet the divide between illuminating and suffocating must always be sought in a dialogue between theory and experiment.

Asymmetry in a symmetrical world is not a frightening discovery. The laws that Newton developed after Keplers discovery are beautifully symmetric but the lower order symmetry of the elliptical orbit reflected information carried about the initial states of the planets. Just as the flight of the G.F.D. softball over the outfield reflects the strike of the bat, so the orbit reflects its formation, and thus the asymmetry is important, and may lead to a newer and deeper symmetry of a different order all together.

Another example of this is the breaking of the most common symmetry, that of mirror symmetry. It was impossible to completely distinguish left from right until 1956 when Lee and Yang discovered that neutrinos, which are radioactive particles from atomic decay, always spin in a left handed sense. Similarly to Kepler earlier, Lee and Yang only came to this conclusion reluctantly after theory failed to keep up with observational data, and once again with the eventual acceptance of the symmetry breaking, a whole new field opened up. This time the idea of chiral symmetry, where it is the sum of spins that is preserved in collisions, was uncovered and once again a new and deeper symmetry had been discovered.

Fig. 27. Testis of mouth-breeder fish (Marlow 1974, Horiguchi and Tatematsu 1980).

Fig. 28. Cell structure in the skin of a cucumber (Lewis 1928), $\mu_p = 0.6$.

Fig. 30. Cells in the stem of a plant (after Krammerbach et al. 1939).

summation of functions which fit into the planform structure and which solve the equation gained from looking for a separable solution, namely Helmholtz's equation,

$$\text{grad}^2 f = -k^2 f$$

and then solve the vertical ordinary differential equations which result from the original partial differential equations. For example, specifying an expansion of the two dimensional problem variables as a summation of sines and cosines i.e. a Fourier expansion, satisfies the boundary conditions and sets up a repeating solution as a set of rolls in the horizontal, which may be completed by solving the vertical two point boundary value problem for the vertical dependences. The Fourier series expansion is terminated at some order to make the problem tractable, and this termination point in some sense specifies the number of higher harmonics that you are allowing in the solution.

So what do we do for the equivalent problem in three dimensions? If we assume an initially hexagonal horizontal planform, what are the higher harmonics and how many must we take into account before we get a reasonable solution with the rest of the harmonics irrelevant? These are the simple questions that are now approached.

Chandrasekhar quotes the basic hexagonal planform as the Christopherson hexagon, or

$$f(x,y) = 2 \cos(rx)\cos(y) + \cos(2y)$$

where r is the square root of 3, so let us take this form with the convection equation and see what happens.

The conservation of momentum equation for convection is as follows:

$$\frac{du}{dt} + (u \cdot \text{grad})u = -\frac{1}{\rho} \text{grad} p + g \hat{z} + \text{grad}^2 u$$

Let us examine the effect of the non-linear $(u \cdot \text{grad}) u$ terms on a velocity field which has the hexagonal planform i.e. take

$$u = \text{curl}(\text{curl}(X e_3))$$

TABLE 1

Arbitrary name	Analytic form	Interaction produced by	Equivalent F_i level
Basic hexagon	$2\cos rx \cos y + \cos 2y$	-	F
Rotated hexagon 1/3 size	$2\cos 3y \cos rx + \cos 2rx$	Basic, basic	F2
Complex 1	$\cos rx \cos 5y + \cos 2rx \cos 4y + \cos 3rx \cos y$	Basic, 1/2 basic	F4
Complex 2	$\cos rx \cos 7y + \cos 3rx \cos 5y + \cos 4rx \cos 2y$	1/2 basic, rotated 1/3	F4
Complex 4	$\cos rx \cos 9y + \cos 4rx \cos 6y + \cos 5rx \cos 3y$	complex1, complex1	F8
Complex 5	$\cos 2rx \cos 12y + \cos 5rx \cos 9y + \cos 7rx \cos 3y$	complex2, complex2	F8
Complex 6	$\cos 3rx \cos 11y + \cos 4rx \cos 10y + \cos 7rx \cos y$	complex1, complex2	F8
Complex 7	$\cos rx \cos 11y + \cos 5rx \cos 7y + \cos 6rx \cos 4y$	complex1, complex2	F8
Complex 8	$\cos 2rx \cos 8y + \cos 3rx \cos 7y + \cos 5rx \cos y$	complex1, complex2	F8

Fig.3

Fig. 4

FIG. 13.—Solar granulation, pores, and small spots, July 5, 1885 (Jansxn, 1896), f mm = 0.5.
(This may be the ~~last~~ photograph of solar granulation in existence. Janssen wrote about it: "obtenue
sans aucune intervention de la main humaine.")

NORMAL HEXAGON, 1/1 SIZE
PLANFORM CONTOURS

ROTATED HEXAGON, 1/1 SIZE
PLANFORM CONTOURS

Fig.5

COMPLEX MODE 1
PLANFORM CONTOURS

COMPLEX MODE 2
PLANFORM CONTOUR3

COMPLEX MODE 4
PLANFORM CONTOURS

- 259 -

COMPLEX MODE 5
PLANFORM CONTOURS

Fig.6

COMPLEX MODE 6
PLANFORM CONTOURS

COMPLEX MODE 7
PLANFORM CONTOURS

COMPLEX MODE 8
PLANFORM CONTOURS

where

$$X = g(z)f(x,y)$$

and f is the Christopherson hexagon. The form of $(u \cdot \text{grad})u$ was examined using the CAMAL algebraic manipulation package on a VAX 8600 and the analytic harmonics that could be picked out of the results are shown in as functions in Table 1 and as line drawings in figures 1 and 2.

Simply using X as the Christopherson hexagon reproduces itself and two other harmonics, namely one of the same shape and form but half the size, and one of the same shape but rotated by 90 degrees and reduced to a third the size.

Sums of these harmonics and the basic Christopherson were then used as the function X and more, higher harmonics were produced in a similar fashion. The interactions and results are summarised in table 1 and figures 2 and 3.

The cellular structures exhibited were drawn based on an algorithm that decided the position of cell walls as those straight lines which have $f(x,y)$ all of one sign, with no horizontal flow of fluid across them, so that a cell is an isolated unit in the sense of Stuart (1961). In this manner, a cell wall is seen as purely falling fluid, and the matter of whether convection cells rise in the middle or fall is not resolved but decided.

Thus it can be seen that simple relatively low harmonic interactions can produce remarkable and very complicated structures. Notice in particular the appearance of a pentagonal shaped cell in the modes arbitrarily named complex 2 and its relation complex 5. The appearance of this structure is remarkable in that it has a five fold symmetry and yet was produced by interactions of shapes with purely six fold symmetry.

Note that all these patterns are still periodic in the same sense as the original Christopherson hexagon was i.e. periodic in a translation of $2\pi/r$ in the x direction, and 2π in the y , still tile the plane and of course are still perfectly valid solutions of the convective problem.

One of the main areas affected by such observations of cellular structures is the sun. A cellular-like pattern called solar granulation is observed on the surface of the sun as is shown in the photograph of fig.4. The interesting point is that cellular patterns observed on the sun are not purely hexagonal. Pentagons are seen, and the mean number of sides of the pattern cells is measured as 5.7 which implies that there are enough of these shapes to make a difference. This number also implies that the pentagons are not balanced by heptagons and so their appearance does not appear to be simply a dislocation of a hexagonal pattern.

The cellular line drawings of figs 2 and 3 could be drawn so that the location of a cell wall was not quite so precise, and then more than one

wall would be drawn where a wall may actually be, and then the fractal style nature of these patterns may be seen. This affect has already appeared on some of the drawings exhibited. Such pictures may more accurately represent what the eye and camera see of the sun as intensities modulated by the upward and downward flows within the cells.

To get even closer to the sort of picture that might be seen, contour diagrams of these shapes were drawn and are included as figures 5 and 6. The **quality** of these is not too great because they are reproduced by Xerox from nice colour drawings, but then, how good is a photograph of the sun.

So far we have only examined the pure harmonics created. In reality these harmonics would superpose so contour diagrams were created of various superpositions of the modes with varying relative strengths to try and simulate the way interactions may occur. A few of these interactions are included as figures 7, 8, 9, 10 in an attempt to give the reader an idea of the wide diversity of the patterns that emerge, from flower like patterns with globules, through very definite spoked patterns to very rounded structures.

Packets of such structures were also superposed and contoured in an attempt to smear the too definite recognition of a cell. The same shapes of similar sizes are added together with various strengths to form a wave packet (see diagram below)

and these are contoured as well in figures 11, 12.

Notice now that in these diagrams shapes are created that **fit** to such astronomical buzzwords as 'doughnut cells', 'spoked patterns' and 'clusters of cells'. Thus by simply taking a few harmonics of the simple hexagonal planform, patterns have been produced which create a collection of very diverse and often asymmetrical shapes which in some way may be considered as models of the solar granulation patterns.

But we still do not know how to truncate our **planform** function expansion in 3-D and indeed we still do not know the complete set of **planform** functions, so maybe we must look a little deeper into the problem.

PART III - HELMHOLTZ'S EQUATION.

A simpler approach to the above problem, without think in terms of cosine functions all the time, is to consider the basic hexagon as defined by three vectors and their complex conjugates as follows:

SUPERPOSITION - 262 -
PLANFORM CONTOURS

MODES SUPERPOSED:

COMPLEX MODE 7
STRENGTH = 1
COMPLEX MODE 8
STRENGTH = 1

SUPERPOSITION
PLANFORM CONTOURS

MODES SUPERPOSED:

COMPLEX MODE 4
STRENGTH = 5
COMPLEX MODE 5
STRENGTH = 1

SUPERPOSITION PLANFORM CONTOURS - 263 -

MODES SUPERPOSED:

COMPLEX MODE 6
STRENGTH = 5
COMPLEX MODE 5
STRENGTH = 3
ROTATED HEXAGON, 1/1 SIZE
STRENGTH = 3

SUPERPOSITION PLANFORM CONTOURS

MODES SUPERPOSED:

COMPLEX MODE 1
STRENGTH = 3
COMPLEX MODE 5
STRENGTH = 3

SUPERPOSITION PLANFORM CONTOURS - 264 -

Fig.9

MODES SUPERPOSED

NORMAL HEXAGON, 1/2 SIZE
STRENGTH = 1
ROTATED HEXAGON, 1/2 SIZE
STRENGTH = 1

SUPERPOSITION PLANFORM CONTOURS

MODES SUPERPOSED:

COMPLEX MODE 6
STRENGTH = 4
COMPLEX MODE 5
STRENGTH = 2

SUPERPOSITION
PLANFORM CONTOUR, 265

Fig.10

MODES SUPERPOSED:

COMPLEX MODE: 6
STRENGTH: 2
COMPLEX MODE: 7
STRENGTH: 4

SUPERPOSITION
PLANFORM CONTOURS

MODES SUPERPOSED:

COMPLEX MODE: 4
STRENGTH: 1
COMPLEX MODE: 5
STRENGTH: 1

SUPERPOSITION - 266
PLANFORM CONTOUR3

MODES SUPERPOSED:

NORMAL HEXAGON, 1/1 SIZE
STRENGTH = 2
COMPLEX MODE 5
STRENGTH = 4

SUPERPOSITION
PLANFORM CONTOURS

MODES SUPERPOSED:

COMPLEX MODE 8
STRENGTH = 2
COMPLEX MODE 5
STRENGTH = 4

PACKETS - 26.7 PLANFORM CONTOURS.

Fig.12

DATA BY RECORD, 1/1 2/1		
SLICE NUMBER =		0.0000
PACKETS = 1	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 2	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 3	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 4	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 5	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 6	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 7	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 8	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 9	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 10	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 11	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 12	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 13	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 14	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 15	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 16	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 17	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 18	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 19	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 20	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 21	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 22	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 23	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 24	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 25	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 26	0	0.0000
SLICE NUMBER =		0.0000

ABOVE	26
21 -	26
16 -	21
4 -	16
-3 -	4
-9 -	-3
-13 -	-9
-19 -	-13
-26 -	-19
HEMV	-26

PACKETS PLANFORM CONTOURS

DATA BY RECORD, 1/1 2/1		
SLICE NUMBER =		0.0000
PACKETS = 1	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 2	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 3	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 4	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 5	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 6	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 7	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 8	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 9	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 10	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 11	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 12	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 13	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 14	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 15	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 16	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 17	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 18	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 19	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 20	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 21	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 22	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 23	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 24	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 25	0	0.0000
SLICE NUMBER =		0.0000
PACKETS = 26	0	0.0000
SLICE NUMBER =		0.0000

ABOVE	32
28 -	32
8 -	28
-2 -	8
-5 -	-2
-8 -	-5
-12 -	-8
-15 -	-12
-21 -	-15
BELOW	-21

Fig.13

PACKETS PLANFORM CONTOURS

The more general problem may be specified as follows:

$$\text{grad}^2 f = -k^2 f \quad (\text{Helmholtz's equation})$$

$$f = \text{real part}(F)$$

$$F(x,y) = N \sum_j e^{ik_j \cdot x}$$

It is simple to verify in the case where $n=3$ and the vectors are evenly spaced that f turns out to be the Christofferson hexagon. Thus to examine all the **harmonics**, it is now simply necessary to take F and square it a few times to get the sums and differences of the wave vectors and then examine the real part. Notice now that in the previous work, all the harmonic examined are produced by F^8 or less, so we have not examined a very extensive range of **harmonics**.

The obvious question is now what are the more general solutions to the problem where n is not equal to **three**. This is asking as to what are the general solutions of Helmholtz's **equation**. What we are **actually** looking at is all those patterns produced by taking wave vectors all of the same magnitude but in different directions, so we are looking at superpositins of wave trains with the front of equal seperation but at different angles, **with** some concurrent origin or source somewhere. There are two cases that we may examine.

a) Symmetrically constrained solutions.

First, consider the wave vectors that define the solutions distributed evenly around the circle, along with the superposition of their complex conjugates, so the the real part of F is assured. **i.e.** consider

$$F = \sum_j e^{ik_j \cdot x}$$

such that $\theta_i = (i-1)\pi/n$ where θ_i is the angle of the i th wave vector from the horizontal axis **i.e.**

$$\theta_i = \tan^{-1}(k_{ix}/k_{iy})$$

Take $n=1$:

Then say $k = (k_x, k_y)$, and the following picture describes $k_x=1, k_y=0$

$$\begin{aligned} \text{Then } F &= N e^{(k_x + k_y)} \\ f &= \text{real part}(F) \\ &= N (\cos(k_x x + k_y y)) \\ &= N \cos x \end{aligned}$$

Where N is a normalisation factor.
Thus, this is a roll.

Similarly for k with $k_x=0$ and $k_y=1$, $f = \cos(y)$, which represents a roll aligned perpendicular to the first.

Notice now that $n=2$ implies that $f = \cos(x)+\cos(y)$ so that this planform is a square.

Taking $n=3$ reveals the usual hexagon pattern as explained earlier, and so it is possible to continue on producing the planform functions. The solution fields for $n=2,3,4,5,6,7,8$ and 9 are shown in figures 13-16.

As expected, two vectors produce a nice regular square pattern, and the three vectors produce the usual hexagonal tiling of the plane, but the interesting pictures are those of the other numbers of equally spaced vectors. Each has its own preordained symmetry about the n radial lines, but also show some very unusual patterns as well.

The four vector pattern has an octagonal centre with triangles and some outer shapes which are getting to be like pentagons. The five vector case with 10-fold symmetry has five-pointed stars which probably disguise underlying pentagonal cells. The six vector case is particularly exciting because it shows very obvious pentagonal shapes. All these patterns show that even a small move up in the Helmholtz equations breaks the symmetry and create asymmetrical shapes, of which the most common is the pentagon. This seems remarkably close to the sort of observational data that has been retrieved from the sun.

As the number of vectors increases, the symmetry becomes close to circular and the scales get smaller, but note the almost circular centres and the increase in the number of rings around these centres. This demonstrates the way the pattern approaches a Bessel function, $J_0(kr)$.

Unfortunately it is impossible to see here that simply changing the colour palette can radically change the scales that you see within these patterns. Some representations emphasize the radial nature, some the small scales between the large centres, some the large centres and some the straight lines that are actually the wave trains which produce the pattern. But note that all these different scales that you may see are in actual fact only one scale. This seems an important point to make because it needs to be said that all these things, as in all observations, are open to interpretation, and that interpretation is often tainted by preconceived notions and beliefs.

Notice that the well known tilings of the plane are the only totally repetitive patterns. For n greater than 3, the pattern has an infinitely spreading nature, and will not exactly repeat itself anywhere. However, the pattern becomes arbitrarily close to repeating itself. This is easily seen from the graphical procedure demonstrated for the four vector case in figure 17a.

The four vectors define four sets of wave trains travelling at intervals of 45 degree angles outward from one point where the lines are all concurrent. To find the next position where the lines are all four concurrent the condition is that

$$m = n\sqrt{2} \quad \text{where } m \text{ and } n \text{ are integers,}$$

i.e. we need a rational approximation to the square root of 2. How close you want to come in this, represented by the thickness of the lines in the diagram, is dependent on how good an approximation of the irrational number you can make with a rational number.

The distance of the n^{th} wave front from the nearest intersection of three vectors is given by

$$d_n = n - 2 \text{ int}(n/\sqrt{2})$$

where int means take the integer part. This needs to be generalised to any

TWO VECTORS + C.C.
REGULAR, -10 TO 10

THREE VECTORS + C.C.
REGULAR, -10 TO 10

FOUR VECTORS + C.C.
REGULAR, -10 TO 10

FIVE VECTORS+C.C
REGULAR, -10 TO 10

SIX VECTORS
REGULAR, -20 TO 20

Fig.16

SEVEN VECTORS + c.c
REGULAR, -20 TO 20

EIGHT VECTORS + C.C.
REGULAR, -20 TO 20

Fig.17

NINE VECTORS + C.C.
REGULAR, -20 TO 20

Finding the coherence for the eight vector case.

THREE VECTORS+C.C.
IRREGULAR, -20 PI TO 20 PI

Fig.18

SIX VECTORS+C.C.
IRREGULAR, -50 PI TO 50 PI

number of vectors and extended into an iterative map so that the process of finding the centres could be useful to the astronomers. The behaviour of the map itself would also be of interest.

Using the above method of prediction for repetition for the eight vector case, near repetitions are predicted for 7π and a good repetition is predicted for 82π and it can be seen that this is true.

b) Asymmetric solutions.

Solutions were also sought were not equally spaced but perturbed from the regular, or chosen arbitrarily. Examples of the three vector, irregular case are shown in figures 17, 18. Notice the banded like structure that now appears. The pattern is actually modulated hexagons, but the modulation appears to be almost roll-like.

Unfortunately, due to lack of time, this side of the subject has not been investigated further.

CONCLUSIONS

Cellular structures appear everywhere and not just in convection as we have seen, and many theories rely on their symmetries, periodicity and spacial homogeneity. It seems surprising that so little is actually known about the structures that underly these ideas and the simple solutions of the equations that describe the patterns. Just how much is really known about the solutions of Helmholtz's equation, it is not for me to say that the pictures that I have shown here are models of the sun's behaviour, but, on the other hand, is there a structure found in the sun so far that cannot be exhibited as a simple solution of Helmholtz's equation?

There is much work to be done to see how such bizarre cells behave with **vorticity** and time dependence so that these **pattens** grow and decay as the granulation does, but these must be saved for a later date!

I would like to express my appreciation for the kindness of everybody involved with G.F.D. and to express my special thanks to Bill Young, Glenn Flierl and Ed Spiegel for being interested, guiding me and not letting me get any sleep!

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REPORT DOCUMENTATION PAGE		1. REPORT NO. WHOI-96-45	2.	3. Recipient's Accession No.
4. Title and Subtitle Summer Study Program in Geophysical Fluid Dynamics, The Woods Hole Oceanographic Institution, Order and Disorder in Turbulent Shear Flow				5. Report Date December 1986
7. Author(s) Melvin E. Stern				6. Performing Organization Rept. No. WHOI-86-45
9. Performing Organization Name and Address Woods Hole Oceanographic Institution Woods Hole, MA 02543				10. Project/Task/Work Unit No.
12. Sponsoring Organization Name and Address Office of Naval Research and National Science Foundation.				11. Contract(G) or Grant(G) No. (C) N00014-82-G-0079 (G) DMS-85-04166
				13. Type of Report & Period Covered
14.				
15. Supplementary Notes This report should be cited as: Woods Hole Oceanog. Inst. Tech. Rept., WHOI-86-45				
16. Abstract (Limit: 200 words) This volume contains the lectures of E. Martin Landahl (as interpreted and reported by the fellows) concentrating on recent developments in laboratory scale turbulent flow. He covers coherent structure, statistical methods, equations of motion and instability in flow, Euler descriptions and numerical simulations, all as part of efforts at theoretical modelling. Landahl's lectures are followed by extended abstracts of seminars covering the experimental, analytical, and numerical point of view. Seminars on two-dimensional coherent structures provided a connecting link for subsequent lectures in large-scale ocean eddy dynamics. The final section contains the fellows' reports of their own research activity on problems of turbulent shear flow.				
17. Document Analysis a. Descriptors 1. Geophysical Fluid Dynamics 2. Turbulent Shear Flow b. Identifiers/Open-Ended Terms c. COSATI Field/Group				
18. Availability Statement: Approved for publication; distribution unlimited		19. Security Class (This Report) UNCLASSIFIED		21. NO. of Pages 277
		20. Security Class (This Page)		22. Price