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A note about Gaussian statistics on a sphere

Alan D. Chave

Department of Applied Ocean Physics and Engineering, Woods Hole Oceanographic Institution, Woods Hole, MA 02543*, USA. E-mail:* achave@whoi.edu

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SUMMARY

The statistics of directional data on a sphere can be modelled either using the Fisher distribution that is conditioned on the magnitude being unity, in which case the sample space is confined to the unit sphere, or using the latitude–longitude marginal distribution derived from a trivariate Gaussian model that places no constraint on the magnitude. These two distributions are derived from first principles and compared. The Fisher distribution more closely approximates the uniform distribution on a sphere for a given small value of the concentration parameter, while the latitude–longitude marginal distribution is always slightly larger than the Fisher distribution at small off-axis angles for large values of the concentration parameter. Asymptotic analysis shows that the two distributions only become equivalent in the limit of large concentration parameter and very small off-axis angle.

Key words: Numerical approximations and analysis; Probability distributions; Marine magnetics and palaeomagnetics.

1 INTRODUCTION

The statistics of directional data on a sphere are usually modelled using the Fisher (1953) distribution or one of its generalizations that incorporate, for example, antipodal bimodality (Bingham 1974) or variance inhomogeneity (Kent 1982). This class of distribution is an extension of the trivariate Gaussian distribution to the unit sphere. By contrast, Love & Constable (2003) derived the joint and a variety of marginal probability density functions for paleomagnetic vectors based on the same trivariate Gaussian model, but without the unit sphere constraint. There appears to be confusion about the distinction between the Fisher and their latitude–longitude marginal distributions, and it is the purpose of this note to clarify the difference.

The Fisher distribution and the latitude–longitude (or inclination–declination) marginal distributions are distinct, being based on different statistical assumptions, and neither is formally equivalent to the other over any part of parameter space. The Fisher distribution is the conditional distribution of latitude and longitude under the premise that the magnitude is unity, and hence is the distribution of direction on the unit sphere. The latitude–longitude marginal distribution does not include the unit sphere constraint, but rather is the distribution of latitude and longitude without reference to any characteristic of the magnitude, as that is the core definition of a marginal distribution. The only formal equality between these two distributions is shown to occur in the asymptotic limits of large concentration parameter and very small off-axis angle, where both are proportional to an angular Gaussian distribution after normalization by the uniform distribution on the sphere.

2 THE DISTRIBUTIONS

Consider a Cartesian three-component vector $\mathbf{x}$ that is modelled using a trivariate Gaussian distribution having a mean three-vector $\boldsymbol{\mu}$ and a common scalar variance σ^2:

$$f(\mathbf{x}; \boldsymbol{\mu}, \sigma) = \frac{1}{(2\pi)^{3/2}\sigma^3} \mathrm{e}^{-(\mathbf{x}-\boldsymbol{\mu})^T \cdot (\mathbf{x}-\boldsymbol{\mu})/(2\sigma^2)}, \tag{1}$$

where '•' denotes the inner product. By convention, x points north, y points east and z points down. The Cartesian coordinate system will be converted to one using magnitude or intensity F, latitude or inclination θ and longitude or declination ϕ, where $F = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}(z/\sqrt{x^2 + y^2})$ and $\phi = \tan^{-1}(x/y)$. Using a standard approach covered in elementary statistics books, the transformed joint distribution is

$$\begin{aligned} f(F, \theta, \phi; \mu_F, \mu_\theta, \mu_\phi, \sigma) &= \frac{F^2 \cos\theta}{(2\pi)^{3/2}\sigma^3} e^{-(F\cos\theta\cos\phi - \mu_F \cos\mu_\theta \cos\mu_\phi)^2/(2\sigma^2)} \\ &\times e^{-(F\cos\theta\sin\phi - \mu_F\cos\mu_\theta\sin\mu_\phi)^2/(2\sigma^2)} e^{-(F\sin\theta - \mu_F\sin\mu_\theta)^2/(2\sigma^2)} \\ &= \frac{F^2\cos\theta}{(2\pi)^{3/2}\sigma^3} e^{-(F^2+\mu_F^2)/(2\sigma^2)} e^{F\mu_F\cos\xi/\sigma^2} \end{aligned} \tag{2}$$

where $\cos\xi = \cos\theta\cos\mu_\theta\cos(\phi - \mu_\phi) + \sin\theta\sin\mu_\theta$ is the cosine of the off-axis angle ξ between a particular unit vector and the mean unit vector, and $(\mu_F, \mu_\theta, \mu_\phi)$ are the mean magnitude, latitude and longitude, respectively.

Love & Constable (2003) derived the marginal distribution for magnitude by a heroic analytic integration of eq. (2) over latitude and longitude, yielding

$$f_F(F;\mu_F,\sigma)=\sqrt{\frac{2}{\pi}}\frac{F}{\mu_F\sigma}e^{-(F^2+\mu_F^2)/(2\sigma^2)}\sinh\left(\frac{\mu_F F}{\sigma^2}\right). \quad (3)$$

The conditional distribution for latitude and longitude given a particular value for the magnitude F^* is the joint distribution (2) divided by the magnitude marginal distribution (3). Defining the concentration parameter $\kappa=\mu_F^2/\sigma^2$, the result is

$$f(\theta,\phi;\kappa,\mu_\theta,\mu_\phi,\mu_F|F=F^*)=\frac{\kappa F^*/\mu_F\cos\theta}{4\pi\sinh(\kappa F^*/\mu_F)}e^{\kappa F^*\cos\xi/\mu_F}. \quad (4)$$

Setting the dimensionless magnitude F^*/μ_F to unity yields the Fisher distribution

$$f_{\text{Fisher}}(\theta,\phi;\kappa,\mu_\theta,\mu_\phi)=f(\theta,\phi;\kappa,\mu_\theta,\mu_\phi|F/\mu_F=1)$$
$$=\frac{\kappa\cos\theta}{4\pi\sinh\kappa}e^{\kappa\cos\xi}. \quad (5)$$

Eq. (5) is exact, commensurate with the original statistical model (1), and in particular holds for all values of κ.

The latitude–longitude marginal distribution is easily obtained by integrating eq. (2) over all possible values of the magnitude. Using Mathematica 10 (with numerical verification of the result) gives

$$\int_0^\infty F^2 e^{-\alpha F^2}e^{\beta F}\,dF$$
$$=\frac{2\sqrt{\alpha}\beta+\sqrt{\pi}e^{\beta^2/(4\alpha)}(2\alpha+\beta^2)\{1+\mathrm{erf}[\beta/(2\sqrt{\alpha})]\}}{8\alpha^{5/2}}, \quad (6)$$

where $\mathrm{erf}(x)$ is the error function, so that the latitude–longitude marginal distribution is

$$f_{\theta,\phi}(\theta,\phi;\kappa,\mu_\theta,\mu_\phi)$$
$$=\frac{\cos\theta}{4\pi}e^{-\kappa/2}\left\{e^{\kappa\cos^2(\xi)/2}(\kappa\cos^2\xi+1)\left[1+\mathrm{erf}\left(\sqrt{\frac{\kappa}{2}}\cos\xi\right)\right]+\sqrt{\frac{2\kappa}{\pi}}\cos\xi\right\}. \quad (7)$$

Eq. (7) is identical to eq. (A1) in Love & Constable (2003). Eqs (5) and (7) both become the uniform distribution on a sphere whose probability density function is $\cos\theta/(4\pi)$ as $\kappa\to 0$.

3 DISCUSSION

Fig. 1 compares eqs (5) and (7) normalized to the uniform distribution on a sphere for κ values of 0.1, 1, 10 and 100. For a given small value of κ, the Fisher distribution more closely approximates the uniform distribution as compared to the latitude–longitude marginal distribution. For large values of κ, the latitude–longitude marginal distribution is larger than the Fisher distribution for small values of the off-axis angle, but the two distributions appear to be quite similar, and it might appear reasonable to conclude that the latitude–longitude marginal distribution is identical to the Fisher distribution in the large κ limit.

However, a closer look reveals this to not quite be the case. Fig. 2 shows the difference between the latitude–longitude marginal and Fisher distributions for κ values of 1, 10 and 100. The latitude–longitude marginal distribution has an excess of probability over the Fisher distribution for small values of the off-axis angle followed by a weak deficit at intermediate values, and then becomes nearly identical to the Fisher distribution for larger off-axis angles. As κ increases, these differences shift to a smaller range of off-axis angles, but the magnitude of the difference becomes nearly constant, and the two distributions will never become identical.

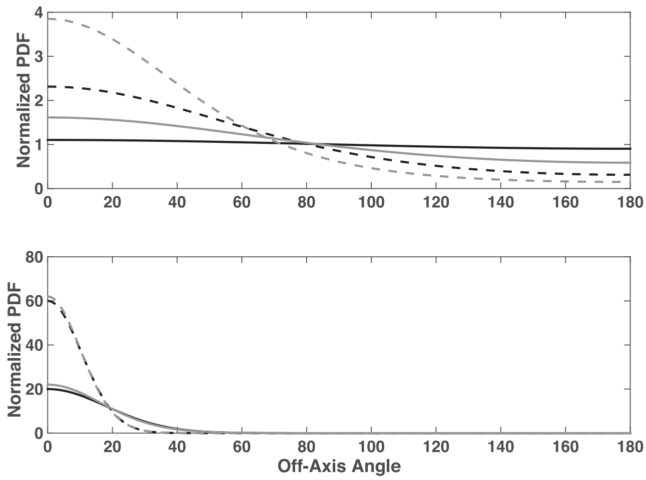

Figure 1. The Fisher (black lines) and latitude–longitude marginal (grey lines) distributions normalized by the uniform distribution on a sphere for a variety of values of the concentration parameter. The top panel shows the distributions for $\kappa = 0.1$ (solid lines) and 1 (dashed lines), while the bottom panel shows the distributions for $\kappa = 10$ (solid lines) and 100 (dashed lines).

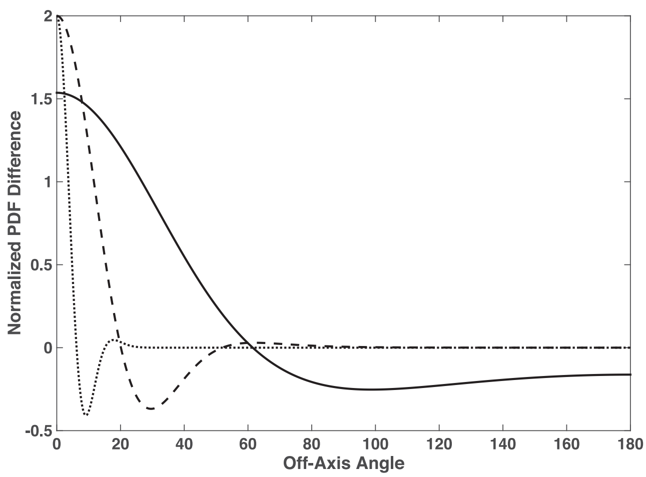

Figure 2. The difference between the latitude–longitude marginal and Fisher distributions, in both cases normalized by the uniform distribution on a sphere, for concentration parameters of 1 (solid), 10 (dashed) and 100 (dotted).

This distinction can be highlighted by comparing the asymptotic forms of the two distributions for large values of the concentration parameter. Since $\sinh\kappa\sim e^{\kappa/2}$ for large κ, the Fisher distribution (5) becomes

$$f_{\text{Fisher}}\sim\frac{\cos\theta}{2\pi}\kappa e^{\kappa(\cos\xi-1)}. \quad (8)$$

The asymptotic expansion for the error function is $\mathrm{erf}(x)\sim 1-e^{-x^2}/(\sqrt{\pi}x)$, hence the latitude–longitude marginal distribution becomes

$$f_{\theta,\phi}\sim\frac{\cos\theta}{2\pi}\kappa\cos^2\xi\ e^{-\kappa\sin^2(\xi)/2} \quad (9)$$

and is quite different from eq. (8).

For small values of the off-axis angle, $\sin\xi \approx \xi$ and $\cos\xi \approx 1 - \xi^2/2$. In this limit, the Fisher distribution becomes

$$f_{\mathrm{Fisher}} \sim \frac{\cos\theta}{2\pi}\kappa\,\mathrm{e}^{-\kappa\xi^2/2} \tag{10}$$

and the ratio of eq. (10) to the uniform distribution on a sphere is proportional to an angular Gaussian distribution. To the same level of approximation, the latitude–longitude marginal distribution becomes

$$f_{\theta,\phi} \sim \frac{\cos\theta}{2\pi}\kappa\left(1 - \frac{\xi^2}{2}\right)^2 \mathrm{e}^{-\kappa\,\xi^2/2}. \tag{11}$$

This reduces to eq. (10) only when the term in brackets is expanded to terms of $O(\xi^2)$ and the limit $1 - \xi^2 \approx 1$ is taken, which is a more stringent constraint on the size of ξ than that used to obtain eq. (10).

If latitude and longitude must be characterized without *a priori* information or assumptions about the magnitude, the expected value and variance using the latitude-longitude marginal distribution will provide the best estimate. However, if it is desirable to set the magnitude to unity and ignore information it may contain, focusing only on directional information so that the sample space is the unit sphere, then the expected values using the Fisher distribution will be more efficient estimates for latitude and longitude, hence will more closely approach the Crámer–Rao lower bound and have a lower variance.

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