An Inertial Model of the Interaction of Ekman Layers and Planetary Islands

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ABSTRACT

An adiabatic, inertial, and quasigeostrophic model is used to discuss the interaction of surface Ekman transport with an island. The theory extends the recent work of Spall and Pedlosky to include an analytical and nonlinear model for the interaction. The presence of an island that interrupts a uniform Ekman layer transport raises interesting questions about the resulting circulation. The consequential upwelling around the island can lead to a local intake of fluid from the geostrophic region beneath the Ekman layer or to a more complex flow around the island in which the fluid entering the Ekman layer on one portion of the island's perimeter is replaced by a flow along the island's boundary from a downwelling region located elsewhere on the island. This becomes especially pertinent when the flow is quasigeostrophic and adiabatic. The oncoming geostrophic flow that balances the offshore Ekman flux is largely diverted around the island, and the Ekman flux is fed by a transfer of fluid from the western to the eastern side of the island. As opposed to the linear, dissipative model described earlier, this transfer takes place even in the absence of a topographic skirt around the island. The principal effect of topography in the inertial model is to introduce an asymmetry between the circulation on the northern and southern sides of the island. The quasigeostrophic model allows a simple solution to the model problem with topography and yet the resulting three-dimensional circulation is surprisingly complex with streamlines connecting each side of the island.

1. Introduction

In recent paper [Spall and Pedlosky (2013), hereafter SP], the problem of the interaction of an Ekman layer and a circular island is studied. They describe the nature of the resulting circulation induced by the upwelling around the island and, in particular, to what extent the Ekman flow on one side of the island leads to an induced geostrophic circulation around the island. In the simplest case of an island immersed in a uniform Ekman flow driven by a uniform and constant northward wind stress in an ocean of constant depth, the flow on the eastern side of the island produces an upwelling while the western side of the island experiences downwelling. In linear theory, the upwelling on the east is locally fed by an oncoming barotropic zonal flow that just balances the Ekman flux offshore. A similar but reversed circulation occurs on the western side of the island where the impinging Ekman flux drives a westward barotropic zonal flow beneath the Ekman layer. The two sides of the island are, in that sense, isolated from one another, and the island interrupts the eastward Ekman fluid flux. Fluid in the Ekman layer from the west returns to the west and the eastward Ekman flux on the eastern side of the island is ultimately fed by the upwelling of eastern-side water from the geostrophic region below. The upwelling is closed by the vertical circulation in thin boundary layers in which friction and thermal dissipation are important. The reader is referred to that paper for a discussion of earlier work on the upwelling problem and pertinent references.

On the other hand, when the island is girdled by a topographic skirt the geostrophic flow beneath the Ekman layer is at least partially diverted by the topography, which bends the geostrophic contours away from the island so that the feeding of the upwelling on the eastern side of the island must, instead, come from fluid downwelled on the western side of the island. In SP that process was described numerically and the analytical model, limited to a heuristic linear formulation, could only partially describe the dynamics in the case with topography.

In this study, I take up the same problem in the quasigeostrophic framework and, ignoring dissipation

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completely, study the nonlinear and adiabatic circulation induced by the interaction of the surface Ekman transport with the island. The principal result is that even in the absence of topography the adiabatic and inertial model requires the upwelling circulation around the island to be closed by a horizontal flux of fluid from one side of the island to the other in the upper levels of the fluid. Topography introduces a north–south asymmetry in the circulation around the island and also in the upper level fluxes connecting the upwelling and downwelling regions, but the qualitative picture of the flux connecting the upwelling and downwelling regions is unchanged.

The problem formulation is given in section 2, its analytical solution is discussed in section 3, and in section 4 the resulting circulation is shown and discussed.

2. The model

A circular island of dimensional radius r_i^* is girdled by a topographic skirt, which smoothly declines from an elevation h_0^* to a flat ocean bottom at a distance r_T from the island's center. The ocean has an initially uniform stratification with buoyancy frequency N. I will scale all horizontal lengths by a length scale L on the order of the island's radius and all vertical distances by the undisturbed, constant ocean depth beyond the island's topography H^*, and henceforth all variables lacking an asterisk are nondimensional.

The flow is driven by a dimensional wind stress τ^*, which is spatially uniform, time independent, and oriented in the northward direction. The northward direction has been chosen so that the Ekman flux is completely nondivergent, allowing us to focus on the dynamical effects of the interaction of the Ekman flux with the island. That is, it produces an eastward Ekman flux but no Ekman pumping because the curl $\boldsymbol{\tau}^*/f^*$ is zero. An oceanic eastern boundary returns the impinging Ekman flow as a uniform, barotropic, and geostrophic zonal flow directed westward with magnitude $U_{\text{scale}} = \tau^*/\rho f_0 H^*$. We scale all velocities by U_{scale}, where the density ρ is the mean density in the Boussinesq approximation, f_0 is the value of the (dimensional) Coriolis parameter at the island's center and, in this system of units, the return zonal barotropic flow will be U, an order one nondimensional parameter.

The starting point for the analysis is the conservation of quasigeostrophic potential vorticity (Pedlosky 1987),

$$q = \nabla^2\psi + \frac{1}{S}\frac{\partial^2\psi}{\partial z^2} + by, \tag{2.1}$$

$b = \beta L^2/U_{\text{scale}}$, $S = N^2H^2/f_0^2L^2$, β is the dimensional planetary vorticity gradient, and the Laplacian in (2.1) is the two-dimensional operator in the horizontal plane. The geostrophic streamfunction ψ is the pressure divided by the density and the central value of the Coriolis parameter f_0 and is scaled by $U_{\text{scale}}(L)$.

For steady, frictionless, quasigeostrophic flow the potential vorticity is a constant along streamlines. That is,

$$q = Q(\psi). \tag{2.2}$$

Far from the island and its topography, the flow is uniform and independent of the vertical coordinate z. Thus, at great distances from the island the potential vorticity is simply by.

At the same time, the streamfunction far from the island is just given by $\psi = U(y)$. Thus along streamlines impinging on the island

$$q = by = \frac{b}{U}\psi = Q(\psi). \tag{2.3}$$

This relationship will continue to hold on all streamlines impinging on the island.

The condition that the vertical velocity at the bottom matches the vertical velocity induced by the flow over the topography is

$$w = J(\psi, h) = -\frac{1}{S}J\left(\psi, \frac{\partial\psi}{\partial z}\right) \quad \text{and} \quad z = 0. \tag{2.4}$$

The second equality derives from the expression for w from the adiabatic condition. Integrating (2.4) and using the fact that both $\partial\psi/\partial z$ and h vanish far from the island, a conservation statement analogous to (2.3) is obtained, namely,

$$\frac{\partial\psi}{\partial z} = -Sh \quad \text{and} \quad z = 0. \tag{2.5a}$$

The topographic variable h is given by $h = (f_0^2 h^*/HU)L$, that is, in analogy with b it is the contribution to the potential vorticity owing to the topography scaled by the characteristic relative vorticity of the flow (Pedlosky 1987).

On the upper boundary,

$$\frac{\partial\psi}{\partial z} = 0 \quad \text{and} \quad z = 1. \tag{2.5b}$$

The topography that will be used is chosen for solution convenience and is

$$h = \begin{cases} h_0 J_0(k_1 r/r_T)/J_0(k_1 r_i/r_T) & r_i \le r \le r_T \\ 0 & r_T \le r \end{cases}, \tag{2.6}$$

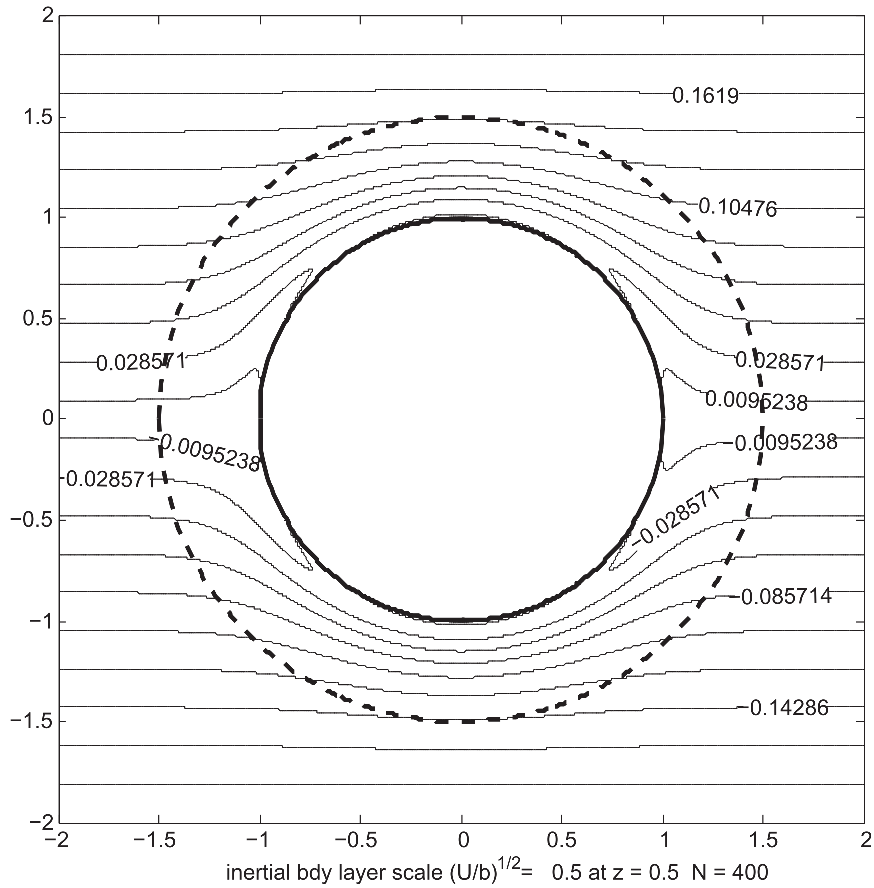

FIG. 1. Pressure field for flow past the island at $z = 0.5$ for a flat bottom ocean, $h_0 = 0$. In this calculation, $S = 1$, $b = 0.4$, and $U = 0.1$.

where k_1 is the first zero of the zero-order Bessel function. Thus, the topographic elevation (scaled) has the maximum value h_0 at the island's coast and smoothly falls to zero at the edge of the topography at the radius r_T. The ocean's bottom is flat beyond that.

The boundary conditions on the island at $r = r_i$ are related to the Ekman flux. The Ekman layer requires that fluid be sucked from the interior and enter the Ekman layer at a rate that depends on the total nondimensional flux U, multiplied by the inner product of the normal to the island with the direction of the fluid, that is, with the unit vector in the x direction. This will be represented as a Dirac delta function sink at $z = 1$ on the island perimeter. Thus, with the angle θ measured counter clockwise from the x axis, the radial velocity u is

$$u = -U \cos\theta\delta(z-1). \tag{2.7}$$

Note that for $\pi/2 \le \theta \le 3\pi/2$, that is, the western side of the island, the sink becomes a source of fluid and requires further discussion because (2.3) is only rigorously valid for streamlines originating at large distances from the island. That form is obviously correct for streamlines impinging on the island and that enter the sink at $(z, r) = (1, r_i)$. I argue that in the absence of dissipation and friction the solution should be east–west symmetric and so the form (2.3) should also hold for streamlines that leave the source on the western part of the island. Thus, I will apply (2.3) to the full problem, namely,

$$q = \nabla^2\psi + \frac{1}{S}\frac{\partial^2\psi}{\partial z^2} + by = \frac{b}{U}\psi, \tag{2.8a}$$

$$\frac{\partial\psi}{\partial z} = -Sh, \quad z = 0, \tag{2.8b}$$

$$\frac{\partial\psi}{\partial z} = 0, \quad z = 1, \quad \text{and} \tag{2.8c}$$

$$\frac{\partial\psi}{r\partial\theta} = U\delta(z-1)\cos\theta, \quad r = r_i. \tag{2.8d}$$

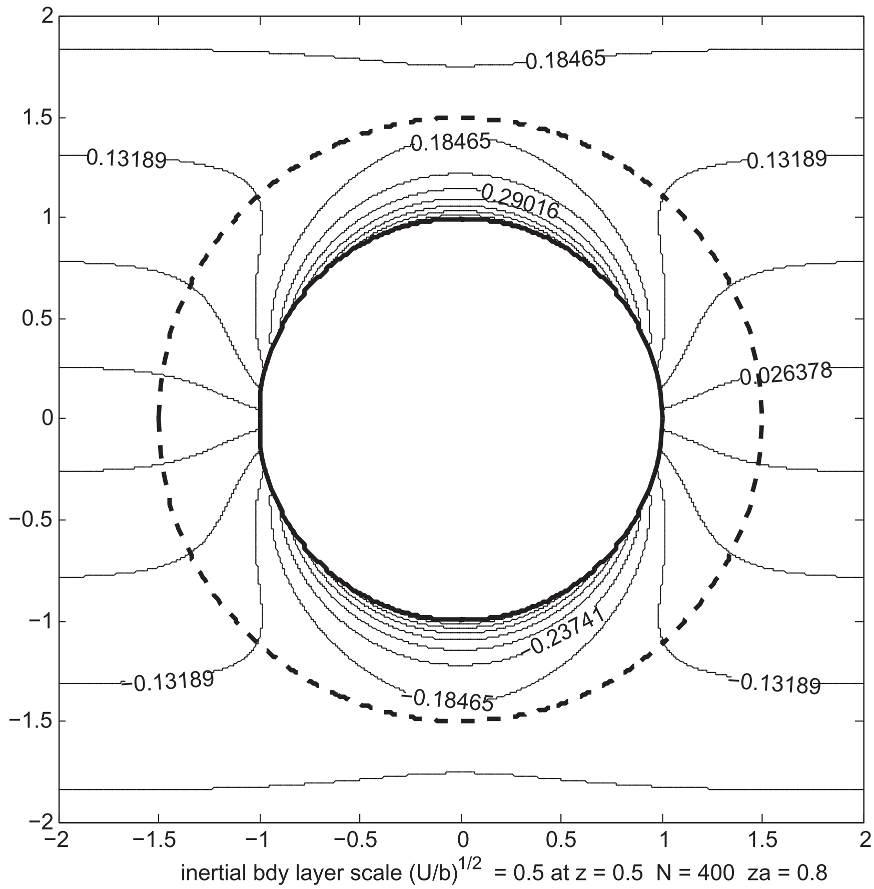

FIG. 2. As in Fig. 1, but what is plotted is the vertical average of the pressure between $z = 0.8$ and the upper surface, $z = 1$.

3. The solution

We first write

$$\psi = Ur\sin\theta + \varphi, \tag{3.1}$$

so that φ satisfies,

$$\nabla^2\varphi + \frac{1}{S}\frac{\partial^2\varphi}{\partial z^2} - \frac{b}{U}\varphi = 0, \tag{3.2a}$$

$$\frac{\partial\varphi}{\partial z} = -Sh(r), \quad z = 0, \tag{3.2b}$$

$$\frac{\partial\varphi}{\partial z} = 0, \quad z = 1, \quad \text{and} \tag{3.2c}$$

$$\varphi = Ur_i\sin\theta[\delta(z-1) - 1]. \tag{3.2d}$$

A particular solution of (3.2a), which satisfies (3.2b) and (3.2c), is

$$\varphi_p = \begin{cases} h_0\dfrac{S}{m}\dfrac{J_0(k_1 r/r_T)}{J_0(k_1 r_i/r_T)}\dfrac{\cosh m(z-1)}{\sinh(m)} & r_i \le r \le r_T \\ 0 & r_T \le r \end{cases} \quad \text{and} \tag{3.3a,b}$$

$$m = S^{1/2}\left(\frac{b}{U} + \frac{k_1^2}{r_T^2}\right)^{1/2}. \tag{3.3c}$$

The particular solution satisfies the boundary conditions in z but the homogeneous solutions, discussed below, are crucial in matching the lateral boundary conditions at the island's boundary.

The particular solution is a maximum at the lower boundary and decays upward. The larger the stratification, the more rapid the exponential decrease. As S tends to zero the solution becomes barotropic. The remaining streamfunction anomaly φ_h will satisfy the homogeneous (3.2a) with homogeneous boundary conditions at $z = 0$ and 1.

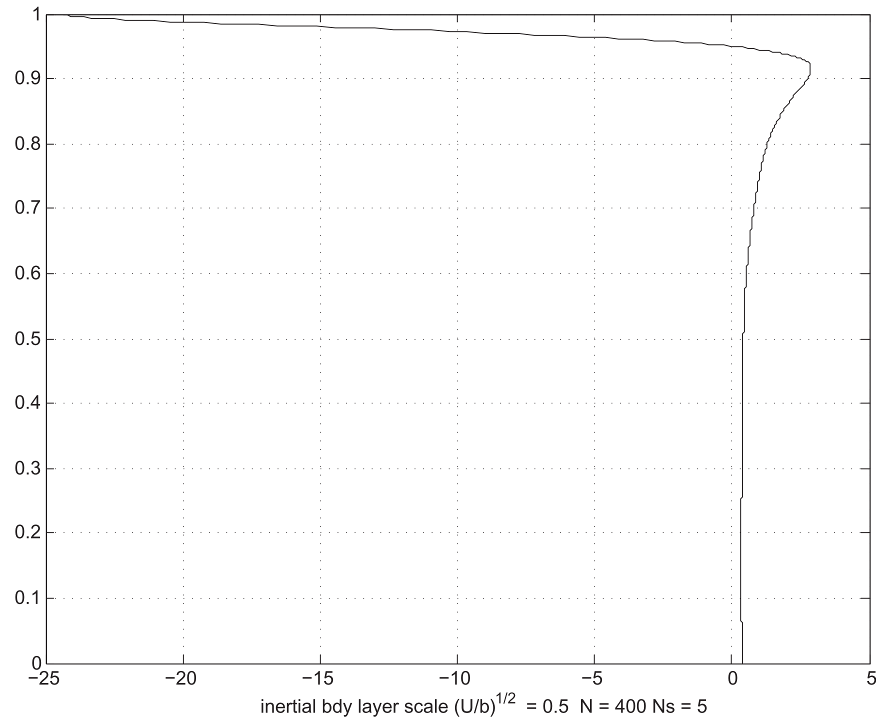

FIG. 3. The profile of the azimuthal velocity at $r = 1.05$, slightly off the island's coast at its most northern point $\theta = \pi/2$. Over most of the water column the velocity is counterclockwise. Near the surface the direction is reversed.

The boundary condition on the island, rewritten in terms of the streamfunction anomaly is then

$$Ur_i \sin\theta + \varphi_p + \varphi_h = Ur_i \sin\theta\delta(z-1), \tag{3.4}$$

that is, except at $z = 1$ the perimeter of the island is a streamline. At the intersection of the island perimeter and the upper boundary a source or sink (depending on azimuth) connects fluid from the geostrophic region to the Ekman layer. It is clear from (3.2a) and (3.4) that the homogeneous solution must have two, mutually orthogonal contributions. The topographic forcing, as manifested by the particular solution, is independent of azimuthal angle θ while the source/sink forcing requires a solution proportional to $\sin\theta$. These solutions will be labeled φ_h^0 and φ_h^1, respectively, reminding us of the dependence on θ.

It is important to note that (3.4) implies that the solution proportional to $\sin\theta$ has no barotropic component because the difference of the vertical integral of the first term on the left-hand side is equal to the same integral of the term on the right-hand side. It follows directly that the homogeneous solution satisfying the forcing by the source/sink is

$$\varphi_h^1 = 2Ur_i \sin\theta \sum_{n=1}^{\infty} (-1)^n \frac{K_1(\mu_n r)}{K_1(\mu_n r_i)} \cos(n\pi z) \quad \text{and}$$

$$\mu_n = \frac{n\pi}{S^{1/2}}\left(1 + \frac{bS}{Un^2\pi^2}\right)^{1/2}. \tag{3.5}$$

In (3.5), $K_1(\mu_n r)$ is the second modified Bessel function of order one and decays exponentially for large values of its argument. The solution (3.5) is valid for all $r \ge r_i$.

The solution forced by topography is more complex. It is broken into two parts, reflecting the differing forms of the particular solution above and beyond the topography. Thus,

$$\varphi_h^0 = \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n \frac{K_0(\mu_n r)}{K_0(\mu_n r_T)} \cos(n\pi z), & r_T \le r \\ \displaystyle\sum_{n=0}^{\infty}\left[B_n \frac{K_0(\mu_n r)}{K_0(\mu_n r_T)} + C_n \frac{I_0(\mu_n r)}{I_0(\mu_n r_T)}\right] \cos(n\pi z), & r_T \ge r \end{cases} \tag{3.6a,b}$$

and where $K_0(\mu_n r)$ is the second modified Bessel function of order zero while $I_0(\mu_n r)$ is the first modified Bessel function of order zero. It exponentially grows

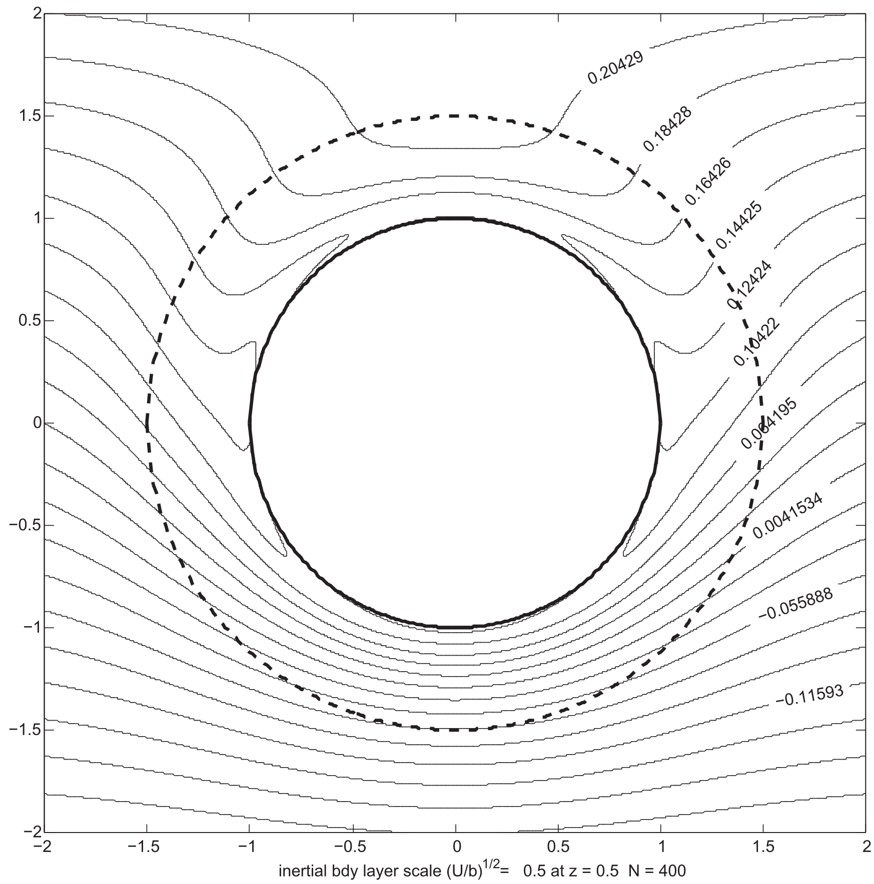

FIG. 4. As in Fig. 1, but for $h_0 = 1$. Note the deflection of the oncoming geostrophic flow by the topography.

for large values of its argument and its contribution to (3.6) is valid only in the region over the topography. Both the streamfunction and its first derivative must be continuous at r_T. That yields two equations relating A_n, B_n, and C_n, namely,

$$A_n = B_n + C_n, \tag{3.7a}$$

$$A_n = B_n - C_n\frac{I_1(\mu_n r_T)K_0(\mu_n r_T)}{I_0(\mu_n r_T)K_1(\mu_n r_T)} + \frac{2\varepsilon_n}{(m^2+n^2\pi^2)}\frac{h_0 S}{(\mu_n r_T)}k_1\frac{J_1(k_1)}{J_0(k_1 r_i/r_T)}\frac{K_0(\mu_n r_T)}{K_1(\mu_n r_T)},$$

and

(3.7b)

$$m = \left[S\left(\frac{b}{U}+\frac{k_1^2}{r_T^2}\right)\right]^{1/2}, \quad \varepsilon_n = \begin{cases}1/2 & n=0\\ 1 & n\geq 1\end{cases}. \tag{3.7c,d}$$

From which it follows that

$$C_n = \frac{\dfrac{2\varepsilon_n}{\mu_n r_T}\dfrac{h_0 S}{(m^2+n^2\pi^2)}\dfrac{J_1(k_1)}{J_0(k_1 r_i/r_T)}\dfrac{K_0(\mu_n r_T)}{K_1(\mu_n r_T)}}{\left[1+\dfrac{I_1(\mu_n r_T)K_0(\mu_n r_T)}{I_0(\mu_n r_T)K_1(\mu_n r_T)}\right]}. \tag{3.8}$$

Another condition is required to determine the remaining constants A_n and B_n. The inviscid and adiabatic model must relax the no-slip boundary condition or an equivalent condition on the density gradient on $r = r_i$. Instead, a condition on the circulation around the island is required. The part of the solution forced by the source/sink forcing has zero circulation so it is the topographically induced flow that must satisfy the circulation condition. The condition that I choose is to make the circulation zero and for the following heuristic argument. Assuming that the circulation is zero before the onset of the wind stress, the circulation would remain zero because the stress field has zero circulation around the island. The frictionless region below the Ekman layer also lacks a mechanism to change the circulation.

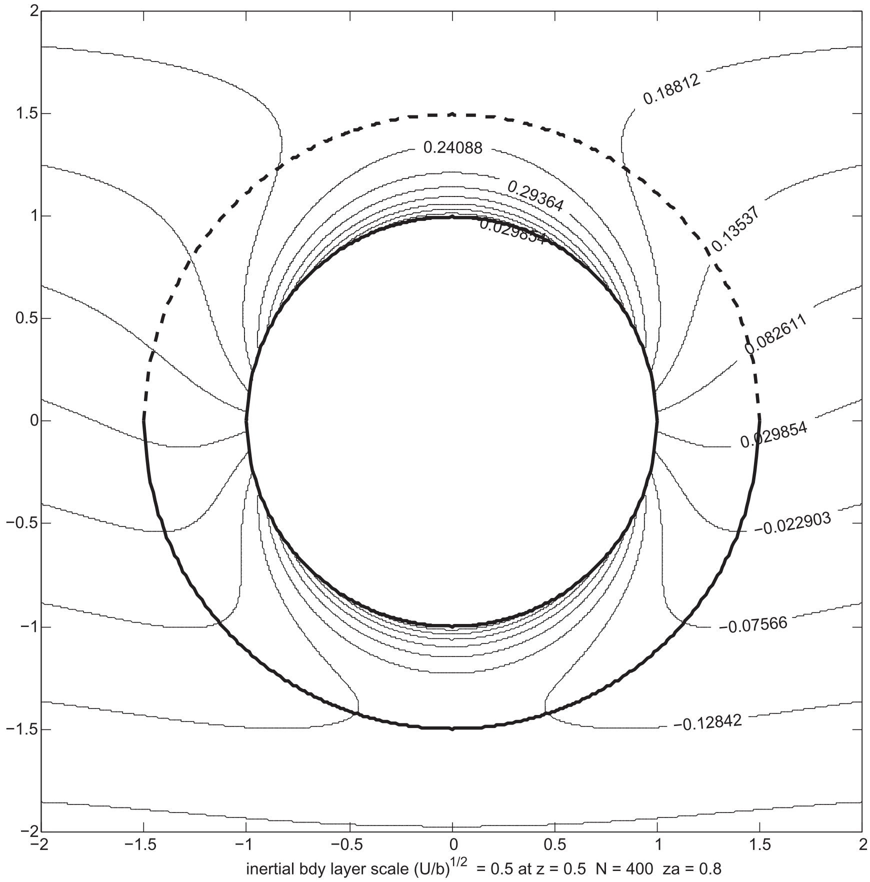

FIG. 5. The pressure field averaged between $z = 0.8$ and 1.0 as in Fig. 2, but for $h_0 = 2$.

Adopting this admittedly heuristic argument and applying it to (3.6b) yields

$$B_n = C_n \frac{K_0(\mu_n r_T) I_1(\mu_n r_i)}{K_1(\mu_n r_i) I_0(\mu_n r_T)} - 2\varepsilon_n \frac{h_0 S}{\mu_n r_T} \frac{k_1}{(m^2 + n^2\pi^2)} \frac{J_1(k_1 r_i/r_T) K_0(\mu_n r_T)}{J_0(k_1 r_i/r_T) K_1(\mu_n r_T)} \tag{3.9}$$

which, with (3.7a) and (3.8), completes the solution.

The evaluation of the series numerically can become problematic for large values of $\mu_n r$. The results below are obtained by using the explicit forms given above for the first five terms of the series and the remaining terms are evaluated using the standard asymptotic forms of the modified Bessel function. Normally between 200 and 400 terms are retained and are more than adequate to represent the solutions in the vicinity of the source and sink.

4. Results and discussion

First let us examine the solution for a flat bottom, that is, $h_0 = 0$. In these solutions, $r_i = 1$, $r_T = 1.5$, $S = 1$, and $b = 0.4$. Figure 1a shows the pressure field half way up the water column at $z = 0.5$, and it illustrates the pathways of the geostrophic flow. At this level, well below the direct effect of the source and sink associated with the upwelling on the eastern side of the island and downwelling on the western side of the island, the flow sweeps around in the island in a symmetric pattern flowing from east to west in an inertial boundary layer. Its width depends on depth and stratification as can be inferred from the definition of μ_n in (3.5). The flow will naturally have a singular structure near the island's rim at the upper boundary of the flow reflecting the idealized delta function forcing of the flow there. It is useful, therefore, to present the vertical average of the pressure field over a narrow but nonzero range of z near the surface. Figure 2 shows the pressure field averaged over the interval $0.8 \le z \le 1$. The pressure field clearly shows the emergence of fluid from the western side of the island and its flow around the island to be absorbed on the eastern side. A few contours are labeled to show the direction of the flow. The purely baroclinic nature of the geostrophic flow is demonstrated in Fig. 3 that shows the profile of the azimuthal velocity at a radius just a bit

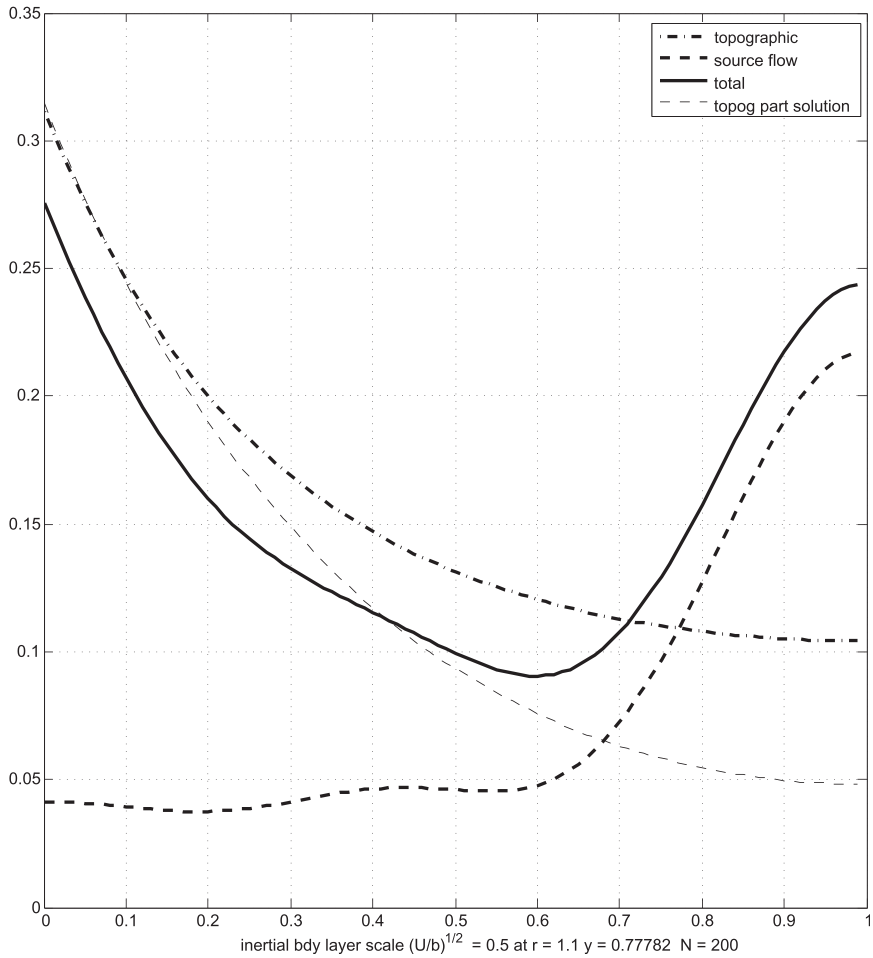

FIG. 6. The components of the solution for the pressure as a function of z at $r = 1.1$.

off the island at the northern extremity of the island, that is, at $r = 1.05$ and $\theta = \pi/2$. The flow near the surface is negative (clockwise) reflecting the feeding of the upwelling on the east by fluid downwelled on the west while the deeper azimuthal velocity is positive (counterclockwise) as the fluid arriving from the east moves around the island to continue to flow west of the island. Thus, in distinction with the linear solution in SP, even the flat bottom configuration closes the mass balance between by the upwelling and downwelling regions by lateral advection of water from one side of the island to the other.

A glance at Fig. 2 shows that the story is a bit more complicated. Not all the flow feeding the upwelling in the east comes from the western side of the island. Some of the streamlines feed the eastern upwelling directly from the interior in the east. The contour labeled 0.13189 indicates the most northeastern source in the lower fluid that enters the upwelling region. When that is added (for brevity the calculation is not given here) to the flow advected from the west of the island, the sum adds up to the total Ekman flux eastward in the range $0 \le y \le r_i$ with the same amount coming from the region $-r_i \le y \le 0$.

The presence of topography introduces a north–south asymmetry to the solution, as the oncoming geostrophic flow below the Ekman layer tends to flow along f/h contours. Figure 4 shows the flow at the midlevel, as in Fig. 1 except that here, $h_0 = 2$. The flow at the midpoint of the fluid, when over the topography, is diverted strongly southward introducing the north–south asymmetry. Nonetheless, as Fig. 5 shows, the average pressure near the upper boundary of the fluid is qualitatively similar to the flat bottom ocean. The solution near the surface is dominated by the effect of the source/sink forcing in both cases. Figure 6 makes this point by plotting the various components of the solution as a function of depth, in this figure at $r = 1.1$ and $\theta = 45°$ for the same parameters as Fig. 4 but for a weaker topography, for example, $h_0 = 1$. Each of these solutions corresponds to the solution in the presence of only one aspect of the forcing, for example, the topography or the sink at the island and so each also includes the oncoming zonal flow. Hence, the total solution is not the sum of these special solutions, although the form of each solution gives a clear picture of the structure each effect produces. The heavy, solid line is the profile with depth of the total solution. The dashed–dot line is the contribution to the flow profile that would be obtained with only the topographic forcing. We see that the total solution tracks this solution quite well for small z [as well as the particular solution (3.3a)]. The dashed curve is the solution due entirely to the source/sink forcing and it clearly dominates in the upper portion of the water column and it is this solution, in the quasigeostrophic dynamics that provides the advection of fluid from one side of the island to the other to feed most of the upwelling.

There are a number of caveats in discussing the solution presented here. Beyond the heuristic choice of the potential vorticity/streamfunction relation applied to the flow issuing from the western downwelling source, the basic assumption that quasigeostrophy is appropriate requires qualification. As shown in SP, a large enough island, for which the velocity normal to the boundary becomes predominantly ageostrophic, is also a consideration that would apply here. In the linear analysis of SP, the contribution of the ageostrophic velocity normal to the boundary could be included in a straight forward if complicated analysis. It is not so clear how to proceed analytically here but an examination of the nonlinear, adiabatic planetary equations would be an interesting challenge for future work.

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REFERENCES

Pedlosky, J., 1987: *Geophysical Fluid Dynamics.* Springer Verlag, 710 pp.

Spall, M. A., and J. Pedlosky, 2013: Interaction of Ekman layers and islands. *J. Phys. Oceanogr.,* **43,** 1028–1041.