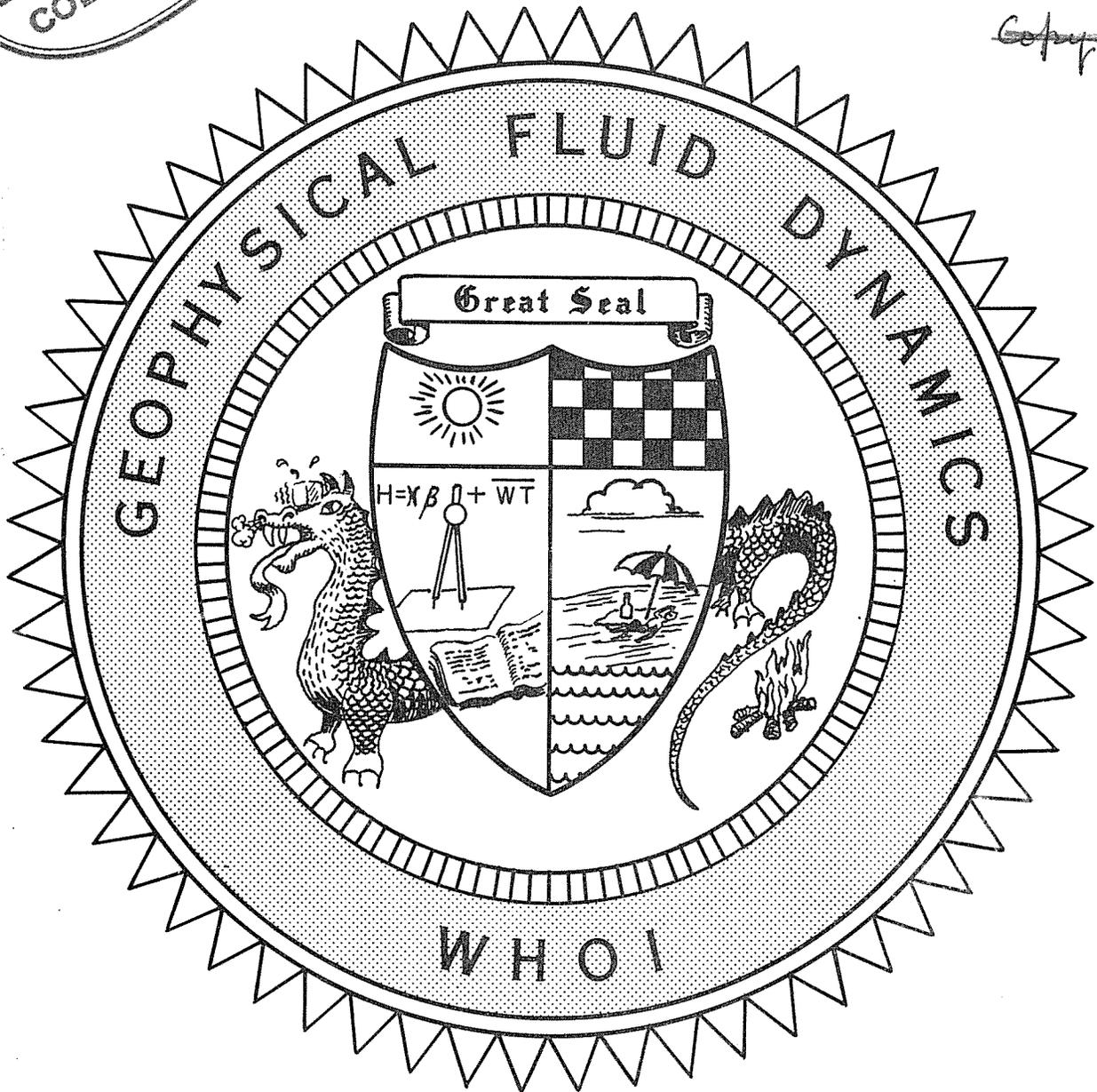


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LECTURES ON FLUID DYNAMICS BY L.N. HOWARD

Notes on the 1960  
Summer Study Program  
in  
GEOPHYSICAL FLUID DYNAMICS  
at  
The WOODS HOLE OCEANOGRAPHIC INSTITUTION

Edited by  
E. A. Spiegel  
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Contents of the Volumes

Volume I. Lectures on Fluid Dynamics - L.N. Howard

Volume II. Special Topics

Turbulence - W.V.R. Malkus

Oceanic Circulation - H. Stommel

Baroclinic Instability - M. Stern

Energy Transports

in the Tropical Atmosphere - J.S. Malkus

Volume III. Student Lectures.

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## Editor's Preface

The past decade has brought an exciting upsurge of interest and research in geophysical fluid dynamics. This development has been particularly manifested by the activities and enthusiasms of a growing number on the staff of the Woods Hole Oceanographic Institution, with the result that many scientists interested in fluid dynamics have become frequent visitors there. In summer especially, the regular staff of the Institution has provided a nucleus for lively gatherings of oceanographers, meteorologists, physicists, mathematicians, and even astrophysicists.

Against this background of meeting and discussion students, sponsored by the Institution's summer fellowship program, have come to take part in the various research programs that develop. As the size of the summer group has increased the danger has arisen that the summer fellowship students might become lost in milieu of high level discussion, and not profit adequately from their efforts. Accordingly, the possibility of providing tangible opportunities for the training of summer fellowship students was explored, and it was decided to institute a summer course in geophysical fluid dynamics.

The first course in geophysical fluid dynamics at Woods Hole was given in the summer of 1959 by staff members of the Institution and some of their summer colleagues. The participants numbered about twenty and included four graduate-student and two postdoctoral fellowship holders provided for by funds from the National Science Foundation. At that time the dragon

which adorns the cover of this volume was born. He was created by Prof. Henry Stommel in recognition of the efforts of Prof. Willem Malkus in organizing the course. The success which this first course enjoyed accounts for the reappearance of our dragon in its present position of prominence. For in the summer of 1960, a second course was given whose contents are outlined in the present notes.

These notes were prepared by the students, whose names are given above, with the capable assistance of Mrs. Mary Thayer. They were designed as working notes to be of use during the course. For each series of lectures, two students accepted the responsibility of preparing the notes and it was attempted (with surprising success) to have the notes typed, duplicated and distributed within four days after each lecture. Naturally, such a project could be completed only with rough edges, but the final collection of notes has succeeded very well in presenting the essential content and spirit of the course. They have therefore been assembled in limited number for use by interested persons.

It has seemed worthwhile to divide the notes into three volumes to avoid making them too cumbersome for easy reference. The division of material reflects the structure of the course. In Volume I we have an introduction to the subject as given by the invited lecturer, Professor L.N. Howard. The second volume contains notes on the more specialized lectures given by various staff members of the Institution. Finally, the manuscripts summarizing the student research lectures are reproduced in

Volume III. The topics discussed by the students were either selected by them or suggested by staff members.

Those of us from other institutions who have participated in this course have been treated to an abundant bill of fare, as a look at these notes will attest. For this, we can but express our gratitude to Dr. Willem Malkus and the other staff members of the Oceanographic Institution for their extensive efforts. We are also indebted to the Institution itself for its hospitalities and facilities. Finally, we should like to thank the National Science Foundation for providing funds for student fellowships and the support of an invited lecturer.

E. A. Spiegel  
Inst. for Mathematical Sciences  
Sept. 1960.

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Fluid Dynamics: Lecture #1, 6/22/60; Dr. Howard.

1. INTRODUCTION: In the mathematical formulation of fluid dynamics we shall use the following approach: A fluid will be viewed as a continuum; i.e., we assume the existence of "infinitesimal volume". In particular, we assume that fields such as velocity, density, temperature, etc. can be defined. We also introduce the concept of stress to describe the forces on "particles" of fluid, as will be developed below.

We shall use the Eulerian variables,  $\vec{x} = (x_1, x_2, x_3)$ , to be spatial position variables. At a particular instant of time,  $t = 0$ , we affix labels to the individual particles of our system; i.e.

$\vec{X} = \vec{X}(t, \vec{x}) = (X_1, X_2, X_3)$  are Lagrangean variables which label particles. Specifically,  $X_i = X_i(t, x_j)$  is a continuous transformation of the space  $(t, x_j)$  such that  $X_i|_{t=0} = x_i(0)$ . We then have the two sets of coordinates,  $(t, X_i)$  and  $(t, x_i)$ , describing the fluid.

Define the velocity,  $\vec{u} = (u_1, u_2, u_3)$ , of a particle by

$$u_i \equiv \frac{DX_i}{Dt} = \left( \frac{\partial X_i}{\partial t} \right)_X, \quad X \text{ held constant (where we have } x_i = x_i(t, X_j)).$$

If  $f = f(t, x_i)$ , using the chain rule we have  $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} \frac{Dx_i}{Dt}$ .

If  $R(x)$  is some region in coordinate space, let  $J$  be the Jacobian

$$\left| \frac{\partial(x_i)}{\partial(X_i)} \right|; \text{ then we have}$$

$$\frac{D}{Dt} \int_{R(x)} f \rho d^3x = \frac{D}{Dt} \int_{R(X)} f \rho J d^3X.$$

Since  $R(X)$  is fixed with respect to  $t$ , if we assume  $R$  moves with the fluid,

$$\frac{D}{Dt} \int_{R(x)} f \rho d^3x = \int_{R(X)} \frac{D}{Dt} (f \rho J) d^3X. \quad (1)$$

For the function  $f \equiv 1$ , we get

$$\frac{D}{Dt} \int_{R(x)} \rho d^3x = \int_{R(X)} \frac{D}{Dt} (\rho J) d^3X = 0;$$

by conservation of mass within  $R(X)$ . Hence, we have

$$\boxed{\frac{D}{Dt}(\rho J) = 0} \quad (2)$$

We now use the relation (2) in eq. (1), yielding

$$\begin{aligned} \frac{D}{Dt} \int_{R(x)} f \rho d^3x &= \int_{R(X)} \frac{D}{Dt} (f \rho J) d^3X = \int_{R(X)} \left\{ \frac{Df}{Dt} \rho J + f \frac{D}{Dt} (\rho J) \right\} d^3X \\ &= \int_{R(X)} \frac{Df}{Dt} \rho J d^3X = \int_{R(x)} \frac{Df}{Dt} \rho d^3x ; \end{aligned}$$

i.e.,

$$\boxed{\frac{D}{Dt} \int_{R(x)} f \rho d^3x = \int_{R(x)} \frac{Df}{Dt} \rho d^3x} \quad (3)$$

2. CONTINUITY EQUATION: If there are no "sources" or "sinks" in the region  $R$ ; we can equate the rate of accumulation of mass in  $R$  to the outward flux; i.e.,

$$\frac{\partial}{\partial t} \int_R \rho d^3x = - \int_{\partial R} \rho \vec{u} \cdot \vec{n} dS ,$$

$\vec{n}$  being the outward normal to the surface;  $\partial R$ , the surface enclosing  $R$ . Using the divergence theorem (and the fact that  $\frac{\partial}{\partial t}$  means

$\frac{\partial}{\partial t} \Big|_{x \text{ fixed}}$ ) we get

$$\int_R \frac{\partial \rho}{\partial t} d^3x = - \int_R \vec{\nabla} \cdot (\rho \vec{u}) d^3x ,$$

or since  $R$  is an arbitrary region, we have the continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0} \quad (4)$$

We now assert that (4) is equivalent to (2), which we show by the following argument: Rewrite (4) as

$$\left\{ \frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho \right\} + \rho \vec{\nabla} \cdot \vec{u} = \frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u}.$$

Using the substitution  $\frac{D\rho}{Dt} = \frac{1}{J} \frac{D(\rho J)}{Dt} - \rho \frac{DJ}{Dt}$  in this equation we obtain

$$\frac{1}{J} \frac{D(\rho J)}{Dt} - \left\{ \frac{DJ}{Dt} - \vec{\nabla} \cdot \vec{u} \right\} \rho = 0.$$

There is a straightforward argument which we omit which shows that, as a property of the transformation,

$$\frac{1}{J} \frac{DJ}{Dt} = \vec{\nabla} \cdot \vec{u}.$$

Hence, as asserted,

$$\frac{D}{Dt} (\rho J) = 0.$$

3. MOMENTUM EQUATION: Newton's law applied to the region R requires

$$\frac{D}{Dt} \int_R u_i \rho d^3x = \int_R f_i d^3x + \int_{\partial R} \sigma_{ij} n_j dS,$$

where the function  $f_i$  denotes force per unit mass;  $\sigma_{ij}$  is the stress tensor. Using (3) and the divergence theorem, we have

$$\int_R \frac{Du_i}{Dt} \rho d^3x = \int_R \rho f_i d^3x + \int_R \frac{\partial \sigma_{ij}}{\partial x_j} d^3x.$$

Hence, since R is arbitrary, we have the momentum equation

$$\boxed{\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j}} \quad (5)$$

We wish to show that the stress tensor is symmetric. Equating rate of change of angular momentum to torque, we get

$$\frac{D}{Dt} \int_R \vec{r} \times \rho \vec{u} d^3x = \int_R \rho \vec{r} \times \vec{f} d^3x + \int_{\partial R} \vec{r} \times \vec{\sigma}(\vec{n}) dS,$$

where  $\vec{\sigma}(\vec{n}) = \sigma_{ij} n_j \vec{e}_i$ . Using (3) and the divergence theorem,

$$\int_R \rho \frac{D}{Dt} (\vec{r} \times \vec{u}) d^3x = \int_R \rho \vec{r} \times \vec{f} d^3x + \int_R \vec{e}_i \frac{\partial}{\partial x_i} (\epsilon_{ijk} x_j \sigma_{kl}) d^3x.$$

This equation is valid under the assumptions of no intrinsic angular momentum or body torques. Using (5) in the **first term** on the right side of this equation, and expanding all the derivatives, we get

$$\begin{aligned} \int_R \rho \vec{e}_i \epsilon_{ijk} \left( \frac{Dx_j}{Dt} u_k + x_j \frac{Du_k}{Dt} \right) d^3x &= \int_R \vec{e}_i \epsilon_{ijk} x_j \left( \rho \frac{Du_k}{Dt} - \frac{\partial \sigma_{kl}}{\partial x_l} \right) d^3x \\ &+ \int_R \vec{e}_i \epsilon_{ijk} \left( \delta_{jl} \sigma_{kl} + x_j \frac{\partial \sigma_{kl}}{\partial x_l} \right) d^3x. \end{aligned}$$

Simplifying, 
$$\int_R \rho \vec{e}_i \epsilon_{ijk} u_j u_k = \int_R \vec{e}_i \epsilon_{ijk} \delta_{jl} \sigma_{kl};$$

where  $\delta_{jl}$  is the Kronecker symbol. But the left side of this equation

clearly vanishes; hence,  $\int_R \vec{e}_i \epsilon_{ijk} \delta_{jl} \sigma_{kl} = 0$ . Since  $R$  is arbitrary,

$\vec{e}_i \epsilon_{ijk} \delta_{jl} \sigma_{kl} = 0$ . In expanded form, this is

$$\vec{e}_1(\sigma_{32} - \sigma_{23}) + \vec{e}_2(\sigma_{13} - \sigma_{31}) + \vec{e}_3(\sigma_{12} - \sigma_{21}) = 0;$$

therefore, as we wanted to show,  $\sigma_{kl} = \sigma_{lk}$ , and the tensor is symmetric.

4. MECHANICAL ENERGY EQUATION: From (5) we develop the energy equation as follows:

$$\rho \frac{D}{Dt} \left( \frac{1}{2} u_i u_i \right) = \rho f_i u_i + u_i \frac{\partial \sigma_{ij}}{\partial x_j} \quad (6)$$

Or, in the region R, this becomes

$$\begin{aligned} \frac{D}{Dt} \int_R \rho \cdot \frac{1}{2} q^2 d^3x &= \int_R \rho f_i u_i d^3x + \int_R u_i \frac{\partial \sigma_{ij}}{\partial x_j} d^3x \\ &= \int_R \rho f_i u_i d^3x + \int_R \frac{\partial}{\partial x_j} (u_i \sigma_{ij}) d^3x - \int_R \frac{\partial u_i}{\partial x_j} \sigma_{ij} d^3x, \end{aligned}$$

where  $q = |\vec{v}|$ ,

Using the divergence theorem, we have the mechanical energy equation

$$\boxed{\frac{D}{Dt} \int_R \rho \frac{1}{2} q^2 d^3x = \int_R \rho f_i u_i d^3x + \int_{\partial R} u_i \sigma_{ij} n_j dS - \int_R \frac{\partial u_i}{\partial x_j} \sigma_{ij} d^3x} \quad (7)$$

The left side is the rate of change of kinetic energy within R; the first term on the right is the work done by body forces; the second term on the right is the work done by surface forces; the third term on the right is not easily interpreted without specifying the stress tensor, but is related to the work done by dissipative forces, as will be seen in Lecture #2.

5. ENERGY CONSIDERATIONS: We now shall develop the energy equation from a different point of view than that of lecture #1; viz., we consider internal molecular energy, heat conduction and sources, etc.

(1) In addition to the kinetic energy per unit mass,  $\frac{1}{2}q^2$ , we introduce  $e$  as the internal (molecular) energy per unit mass. Then the rate of change of total energy in the region  $R$  is

$$\frac{D}{Dt} \int_R \rho \left( \frac{1}{2}q^2 + e \right) d^3x.$$

(2) As in lecture #1, we have  $\int_R \rho f_i u_i d^3x$  as the work done by body forces in  $R$ ; also we have  $\int_{\partial R} u_i \sigma_{ij} n_j dS$  as the work done by surface forces on this region.

(3) We introduce the vector  $\vec{K} = (K_1, K_2, K_3)$  where  $K_i$  represents the heat conducted per unit area per unit time. Then  $-\int_{\partial R} \vec{K} \cdot \vec{n} dS$  is the total heat conduction through the surface bounding  $R$ . Letting  $Q$  denote the external heat sources per unit mass,  $\int_R \rho Q d^3x$  is the heat energy due to external sources in  $R$ .

Combining the energy contributions (1), (2), and (3) we can write the total energy equation in  $R$ ,

$$\frac{D}{Dt} \int_R \rho \left( \frac{1}{2}q^2 + e \right) d^3x = \int_R \rho f_i u_i d^3x + \int_{\partial R} u_i \sigma_{ij} n_j dS + \int_{\partial R} -\vec{K} \cdot \vec{n} dS + \int_R \rho Q d^3x.$$

By the divergence theorem and the arbitrariness of  $R$  (and eq(3) of lecture #1)

$$\rho \frac{D}{Dt} \left( \frac{1}{2}q^2 + e \right) = \rho f_i u_i + \frac{\partial}{\partial x_j} (\sigma_{ij} u_i) - \vec{\nabla} \cdot \vec{K} + \rho Q.$$

Comparing this equation with (6) of lecture #1, we have the differential

equation of internal energy,

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial u_i}{\partial x_j} - \vec{\nabla} \cdot \vec{K} + \rho Q \quad (1)$$

The first term on the right of this equation represents energy contributions from shearing forces; i.e.,

$$\int_R u_i \sigma_{ij} n_j dS = \int_R \frac{\partial}{\partial x_j} (\sigma_{ij} u_i) d^3x = \int_R u_i \frac{\partial \sigma_{ij}}{\partial x_j} d^3x + \int_R \sigma_{ij} \frac{\partial u_i}{\partial x_j} d^3x,$$

and even if the net force per unit volume  $\frac{\partial \sigma_{ij}}{\partial x_j} u_i = 0$ , there can be an energy source (from shear) due to the remaining term.

To utilize equation (1) we must make additional assumptions:

(1) Assumptions Concerning Heat Conduction. We assume that a temperature field can be defined for the fluid. If  $T$  denotes absolute temperature, we assume  $\vec{K} = \vec{K}(\rho, e, \frac{\partial T}{\partial x_i})$  - i.e.,  $\vec{K}$  depends only upon  $\rho$ ,  $e$ ,  $T$  and, in fact, the dependence on  $T$  is one of linearity in  $\frac{\partial T}{\partial x_i}$ . In particular,

$$K_i = K_i^0(\rho, e) + A_{ij}(\rho, e) \frac{\partial T}{\partial x_j}$$

(This is equivalent to neglecting higher order terms in the Taylor expansion for  $K_i = K_i(\rho, e, \frac{\partial T}{\partial x_i})$ .)

We now make the additional assumption that this expression for  $K_i$  is isotropic (i.e., independent of orthogonal coordinate system). This means that  $K_i^0(\rho, e) = 0$  (since 0 is the only isotropic vector) and  $A_{ij}(\rho, e)$  is an isotropic 2nd order tensor. Since multiples of  $\delta_{ij}$  are the only isotropic 2nd order tensors,

we have  $A_{ij} = -k(\rho, e) \delta_{ij}$ .

Then

$$\vec{K} = -k \vec{\nabla} T \quad (2)$$

It is important to remember that (2) is not an empirical result, but an assumption.

(2) Assumptions Concerning the Stress Tensor. In lecture #1 it was shown that the stress tensor  $\sigma_{ij}$  is symmetric (under the assumptions of no intrinsic angular momentum or body torques). We now make the assumption that  $\sigma_{ij} = \sigma_{ij}(\rho, e, \frac{\partial u_k}{\partial x_l})$ ; i.e., we assume the dependence of  $\sigma_{ij}$  on  $\vec{u}$  is one only of linearity in  $\frac{\partial u_k}{\partial x_l}$ . Then we write

$$\sigma_{ij} = B_{ij}(\rho, e) + C_{ijkl}(\rho, e) \frac{\partial u_k}{\partial x_l}$$

and we further assume that  $B_{ij}$  is an isotropic 2nd order tensor,  $C_{ijkl}$  is an isotropic 4th order tensor. Since  $\delta_{ij}$  is the only isotropic 2nd order tensor; and  $(\delta_{ij} \delta_{kl})$ ,  $(\delta_{ij} \delta_{jl} + \delta_{jk} \delta_{il})$ ,  $(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$  are the only isotropic 4th order tensors, we may write

$$B_{ij} = -p \delta_{ij}$$

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \underbrace{(\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il})}_{\text{symmetric in } i, j} + \nu \underbrace{(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})}_{\text{anti-symmetric in } i, j}$$

Since we require that  $\sigma_{ij}$  be symmetric, we must have  $\nu = 0$ ; hence,

we get

$$\sigma_{ij} = -p \delta_{ij} + \lambda \delta_{ij} (\vec{\nabla} \cdot \vec{u}) + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (3)$$

This formula is called the Navier-Stokes expression for the stress tensor.

The function  $p(\rho, e)$  is known as pressure; and

$\tau_{ij} \equiv \lambda \delta_{ij} (\vec{\nabla} \cdot \vec{u}) + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$  is called the viscous stress

tensor,  $\mu(\rho, e)$  is called the 1st viscosity,  $\lambda(\rho, e)$  the second viscosity.

We can now substitute the values (2) and (3) into eq.(1),

$$\text{getting } \rho \frac{De}{Dt} = \vec{\nabla} \cdot (k \vec{\nabla} T) + \rho Q - p \vec{\nabla} \cdot \vec{u} + \phi \quad (4)$$

$$\text{where } \phi \equiv \lambda (\vec{\nabla} \cdot \vec{u})^2 + \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = \tau_{ij} \frac{\partial u_i}{\partial x_j} \quad (5)$$

Note that the 2nd term in (5) is actually  $\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j}$ ,

but since  $\sigma_{ij}$  is symmetric, we have  $\left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i} = \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \frac{\partial u_i}{\partial x_j}$ ,

and hence we have the appearance as in (5).

By assuming the average of the normal viscous stresses is zero

(assuming  $\vec{\nabla} \cdot \vec{u} \neq 0$ ) we have Stokes Assumption,  $\lambda + \frac{2}{3} \mu = 0$ . (For an incompressible fluid,  $\vec{\nabla} \cdot \vec{u} = 0$ ,  $\lambda$  is arbitrary and hence this assumption reduces to a triviality.) As an exercise, Dr. Howard asks that one show the following:  $\phi \geq 0 \Leftrightarrow \mu \geq 0$  and  $3\lambda + 2\mu \geq 0$ .

Considering the entropy equation  $de = Td\delta - p d\left(\frac{1}{\rho}\right)$ , where

$\delta =$  entropy per unit mass, we have that

$$T \frac{D\lambda}{Dt} = \frac{De}{Dt} + p \frac{D(1/\rho)}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} .$$

Using the continuity equation (Lecture #1, eq (4) ), this becomes

$$T \frac{D\lambda}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2} (-\rho \vec{\nabla} \cdot \vec{u}) .$$

Substituting for  $\frac{De}{Dt}$  from (4) into the above equation, we get

$$\rho T \frac{D\lambda}{Dt} = \vec{\nabla} \cdot (k \vec{\nabla} T) + \rho Q + \Phi . \quad (6)$$

[Note: Dr. Malkus commented at this point, that in applications it can usually be shown that  $\Phi$  is negligibly small.]

6. STANDARD APPROXIMATIONS IN ABOVE WORK: I. Ideal Fluid Approximation:

Assumptions here are  $\lambda = \mu = k = 0$ , and either  $\rho = \text{constant}$  in time and space or  $\frac{D\rho}{Dt} = 0$  (Note that these are not equivalent; i.e.

$\frac{D\rho}{Dt} = 0 \not\Rightarrow \rho = \text{constant}$ . Note also that one could replace the assumption  $k = 0$  by  $T = \text{constant}$  or  $\frac{DT}{Dt} = 0$  and derive the equations that

follow.) Rewritten, the continuity equation states that

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u} = 0 . \quad \therefore \frac{D\rho}{Dt} = 0 \Leftrightarrow \vec{\nabla} \cdot \vec{u} = 0 . \quad \text{However, for the assumption}$$

$\rho = \text{constant}$ , the implication goes only one way - viz.,

$\rho = \text{constant in time and space} \Rightarrow \vec{\nabla} \cdot \vec{u} = 0$  (i.e., even though

$\vec{\nabla} \cdot \vec{u} = 0$ , we may not have  $\vec{u} \cdot \vec{\nabla} \rho = 0$  and, hence,  $\rho \neq \text{const.}$ ). Under

these assumptions, inserting (3) in the momentum equation, we have

Euler's equations,

$$\boxed{\begin{aligned} \rho \frac{D\vec{u}}{Dt} + \vec{\nabla} p &= \rho \vec{f} \\ \vec{\nabla} \cdot \vec{u} &= 0 \end{aligned}} \quad (7)$$

Flows with  $\vec{\nabla} \cdot \vec{u} = 0$  are sometimes called isochoric or solenoidal.

II. Viscous Incompressible Approximation. Assumptions here are

$\lambda = k = 0$ ,  $\mu = \text{constant}$ ,  $\frac{D\rho}{Dt} = 0$ . The momentum equation is then

$$\boxed{\begin{aligned} \rho \frac{D\vec{u}}{Dt} + \vec{\nabla} p &= \rho \vec{f} + \mu \vec{\nabla}^2 \vec{u} \\ \vec{\nabla} \cdot \vec{u} &= 0 \end{aligned}} \quad (8)$$

since for this case  $\frac{\partial \tau_{ij}}{\partial x_j} = \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_j} \right)$

$$\text{or } \frac{\partial \tau_{ij}}{\partial x_j} = \mu \nabla^2 u_i + \frac{\partial}{\partial x_i} (\vec{\nabla} \cdot \vec{u}) = \mu \nabla^2 u_i.$$

III. Irrotational Flow: Assumptions are  $\vec{f} = \vec{\nabla} F$ ,  $\rho = \text{constant}$ ,  $\vec{\Omega} = \vec{\nabla} \times \vec{u} = 0$ . Then we have  $\vec{u} = \vec{\nabla} \phi$  for some function  $\phi$ , and hence have Laplace's equation for  $\phi$ ,  $\nabla^2 \phi = 0$ .

IV. Under the assumptions of II, if we in addition assume  $\vec{u}$  is small,

we write  $\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + \vec{u} \cdot \vec{\nabla} u_i \approx \frac{\partial u_i}{\partial t}$ . Then (8) yields the approxi-

mation

$$\boxed{\begin{aligned} \rho \frac{\partial \vec{u}}{\partial t} + \vec{\nabla} p &= \rho \vec{f} + \mu \vec{\nabla}^2 \vec{u} \\ \vec{\nabla} \cdot \vec{u} &= 0 \end{aligned}} \quad (9)$$

called "Stokes' approximation".

V. Oseen Approximation: For steady flow around an obstacle, under the assumptions of II with  $\vec{f} = 0$ , Oseen's approximation is

$$\rho \vec{V} \cdot \nabla \vec{u} + \vec{\nabla} p = \mu \nabla^2 \vec{u}. \quad (10)$$

where  $\vec{V}$  is the (constant) velocity at  $\infty$ .

Boussinesq Approximation In 1903 Boussinesq presented a useful approximation to the hydrodynamical equations for the situation when gravitational forces drive the motion. This approximation, first used by Rayleigh in 1916, consists in neglecting density fluctuations except when they interact with gravity.

For convenience let us write the pressure and density which solve the hydrodynamic equations in the form

$$\begin{aligned} p &= p_0 + p' \\ \rho &= \rho_0 + \rho' \end{aligned} \quad (1)$$

where  $p_0$  and  $\rho_0$  are solutions to the hydrostatic equations below. The quantities  $p'$  and  $\rho'$  are not necessarily small.

The hydrostatic equations, obtained by specializing equation (5) of lecture #1 and equation (2) of lecture #2, are

$$\begin{aligned} -\nabla p - g \rho \vec{e}_3 &= 0 \\ k \nabla^2 T + Q &= 0 \end{aligned} \quad (2)$$

where  $k$  is the conductivity of the fluid

$Q$  includes radiative and chemical heating.

Mihaljan (1960) in a technical report gives a rigorous derivation of the Boussinesq approximations for a liquid. For a study of Boussinesq approximation for a compressible fluid, the reader is referred to a paper by Spiegel and Veronis (1960). The Boussinesq equations, for a compressible fluid, are

$$\left. \begin{aligned}
 \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} &= -\frac{1}{\rho_m} \nabla p' + g \alpha T' \vec{e}_3 + \nu \nabla^2 \vec{u} \\
 \nabla \cdot \vec{u} &= 0 \\
 \frac{\partial T'}{\partial t} + \vec{u} \cdot \nabla T' + \left( \frac{dT_o}{dt} + \frac{g}{C_p} \right) w &= \chi \nabla^2 T'
 \end{aligned} \right\} (3)$$

and for the eq. of state

$$\rho' = -\rho_o \alpha T'$$

where

$$\nu = \mu / \rho_m$$

$$\chi = k / (\rho_m C_p)$$

$\rho_m$  is the average density taken over the entire fluid.

If we denote the maximum density fluctuation between any two points in the fluid by  $\Delta \rho$ , it is sufficient that

$$\frac{\Delta \rho}{\rho_o} \ll 1 \quad (4)$$

in order that the Boussinesq approximation be valid. The quantity  $\Delta \rho$  depends on two others,  $\Delta \rho_o$  and  $\rho'$ . The static density variation across the fluid,  $\Delta \rho_o$ , may be found from the solution of (2). If we are interested in convection between flat parallel plates separated a distance  $d$ , in order that (4) be satisfied two conditions are needed. The first is that

$$\frac{\Delta \rho_o}{\rho_o} \ll 1$$

This will be satisfied provided

$$d \ll H$$

where  $H = \left( \frac{1}{\rho_0} \frac{d\rho_0}{dz} \right)^{-1}$  and is called the scale height of density.

The second sufficient condition is that

$$\frac{|e'|}{\rho} \ll 1$$

This condition must generally be checked a posteriori. If these two conditions are satisfied, then (4) will also be satisfied.

7. Acoustical Equations We wish to derive the acoustical equations, starting with the continuity and momentum equations for an ideal fluid (lecture #2):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad (5)$$

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_\alpha} (\rho u_i u_\alpha) + \frac{\partial p}{\partial x_i} = f_i$$

where  $f_i$  are external body forces.

Terrestrial acoustics is characterized by the physical situation in which

$$f_i \equiv 0 \quad (6)$$

Linearized equations are those in which we neglect products of velocities, and derivatives of these velocities. The linearized equations which result from (5) are

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u} &= 0 \\ \rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p_1 &= 0 \end{aligned} \quad (7)$$

where the subscript unity now refers to the linearized quantities and (6) has been used.

If the fluid changes adiabatically (or  $p_1 = c^2 \rho_1$ ) then equations (7) reduce to the system

$$\begin{aligned} \frac{1}{c} \frac{\partial \sigma}{\partial t} + \frac{\partial v_\alpha}{\partial x_\alpha} &= 0 \\ \frac{1}{c} \frac{\partial v_i}{\partial t} + \frac{\partial \sigma}{\partial x_i} &= 0 \end{aligned} \quad (8)$$

where

$$\begin{aligned} v_i &= \frac{u_i}{c} \\ \sigma &= \frac{p_1}{\rho_0} \\ \gamma &= \frac{c_p}{c_v} \\ c^2 &= \frac{\gamma p_0}{\rho_0} \end{aligned}$$

The quantity  $c$  is, of course, the local speed of sound in the hydrostatic medium.

Wave Equations Either  $\sigma$ , or  $v_i$ , may be eliminated from (8), which results in a single equation for  $v_i$ , or  $\sigma$ , respectively.

In terms of  $\sigma$ ,

$$\frac{\partial^2 \sigma}{\partial t^2} - c^2 \nabla^2 \sigma = 0 \quad (9)$$

In terms of  $v_i$ ,

$$\frac{\partial^2 v_i}{\partial t^2} - c^2 \frac{\partial^2 v_\alpha}{\partial x_i \partial x_\alpha} = 0 \quad (10)$$

If the fluid is initially irrotational, and is non-dissipative, then  $\nabla \times \vec{v} = 0$ , and the second term in (10) may be written simply as

$$\frac{\partial^2 v_\alpha}{\partial x_i \partial x_\alpha} = \frac{\partial^2 v_i}{\partial x_\alpha \partial x_\alpha} \quad (11)$$

Equation (9) is a wave equation in  $\sigma$ . Using result (11) for irrotational fluids, (10) may be written as a wave equation in  $\vec{v}$ . To explore further the analogy with e.m. waves, let us multiply through the first equation of (8) by  $\sigma$ , the second by  $\vec{v}$ , and add to obtain

$$\frac{1}{c} \frac{\partial}{\partial t} (W) + \nabla \cdot \vec{S} = 0 \quad (12)$$

where

$$W = \frac{\sigma^2}{2} + \frac{|\vec{v}|^2}{2}$$

$$\vec{S} = \sigma \vec{v}$$

We now consider the analogy with electrical waves. A quantity very similar to  $W$  occurs in electromagnetic theory and is called the energy density. Except for scaling factors it is simply

$$|\vec{E}|^2 + |\vec{H}|^2$$

which suggests the analogy to  $W$ . If we further replace  $\sigma$  by  $\vec{E}$ ,  $\vec{v}$  by  $\vec{H}$ , and the divergence operator by the curl operator, the equations (8) are strikingly similar to the e.m. equations. The Poynting vector  $\vec{S}$  of energy flow for electrical waves is given by

(in appropriate units):

$$\vec{S} = \vec{E} \times \vec{H}$$

Since  $\vec{S}$  is perpendicular to both  $\vec{E}$  and  $\vec{H}$ , e.m. energy is carried in transverse waves. From the expression for  $\vec{S}$  in acoustical waves it is clear the energy propagates in the same direction as  $\vec{v}$ , hence acoustical waves are longitudinal rather than transverse.

Gravitational Instability: To consider the gravitational situation, it is pertinent to write  $f_i$  in terms of a gravitational potential  $V$ , hence

$$f_i = -\rho \frac{\partial V}{\partial x_i}$$

The gravitational potential satisfies Poisson's equation,

$$\nabla^2 V = -4\pi G\rho$$

Substitution for  $f_i$  into the initial equations (5) results in the linearized equation governing  $\rho_1$ ,

$$\frac{\partial^2 \rho_1}{\partial t^2} = c^2 \nabla^2 \rho_1 + 4\pi G \rho_0 \rho_1 \quad (13)$$

Let us seek plane wave solutions of (13). In particular we may write

$$\rho_1 \propto e^{i(\omega t - \vec{k} \cdot \vec{x})} \quad (14)$$

Seeking solutions of the type (14), we find the equation (13) places a condition on  $\omega$ , which is the dispersion relation

$$\omega^2 = c^2 k^2 - 4\pi G \rho_0 \quad (15)$$

or 
$$v = \frac{\omega}{k} = \sqrt{c^2 - \frac{4\pi G \rho_0}{k^2}}$$

where  $V$  is the propagation velocity.

When  $V$  becomes imaginary, waves are no longer able to propagate, and the form of (14) suggests that the medium will then start to condense or contract if

$$\lambda = \frac{2\pi}{k} \geq \frac{c}{\sqrt{\frac{G\rho_0}{\pi}}}$$

#### References:

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3. Rayleigh, Lord, 1916, *Phil.Mag.*, 32, 529; *Collected Papers*, 6, 432.
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Notes submitted by

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1. Bernoulli equation (Non-viscous, barotropic fluid)

The momentum equation in the case of a conservative force field (see lecture #2, eq. 7) may be written

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho} \nabla p = \nabla \chi \quad (1)$$

where  $\chi$  is the force potential.

If we multiply through (1) by  $\vec{u}$  and assume the motion is steady, and the fluid barotropic, i.e.  $\rho = \rho(p)$ , we have Bernoulli's equation for steady motion. In integrated form, it is

$$\left( \frac{1}{2} q^2 + \int \frac{dp}{\rho(p)} - \chi \right) = k \quad (2)$$

where  $k$  is a constant on each streamline (not necessarily the same  $k$  for all streamlines). Equation (2) is an energy conservation relation. If the fluid originates from a homogeneous energy source (say at infinity) then the constant  $k$  will be the same throughout the flow field, i.e. on all streamlines.

For the case of irrotational ( $\nabla \times \vec{u} = 0$ , hence  $\vec{u} = \nabla \phi$ ), unsteady flow, the integrated form of Bernoulli's equation corresponding to (2) is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + \int \frac{dp}{\rho} - \chi = f(t) \quad (3)$$

where  $\phi$  is the velocity potential. If desired one may assume  $f(t) = 0$  by re-defining  $\phi$  so as to include  $f(t)$ . In (3) the fluid is assumed to be barotropic.

If we are interested in a locally isentropic fluid, i.e.

$\frac{Ds}{Dt} = 0$ , we first consider the energy equation of lecture #2,

$$\frac{Dh}{Dt} = \frac{De}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) + \frac{1}{\rho} \frac{Dp}{Dt} = T \frac{Ds}{Dt} + \frac{1}{\rho} \frac{Dp}{Dt} \quad (4)$$

After manipulation of the entropy term (see Emmons, Fundamentals of Gas Dynamics, p.29) this equation may be integrated to a form similar to the other Bernoulli equations (2) and (3),

$$\frac{1}{2} q^2 + h - \chi = h_s \quad (5)$$

where  $h_s$  is the stagnation enthalpy. As before,  $h_s$  corresponding to  $k$  in (2) and (3), is constant on streamlines. For alternate forms of (4) and (5) the reader is referred to Emmons, section A7.

## 2. Helmholtz Vorticity Theorem

Euler's momentum equation for the case of a barotropic fluid and conservative force field, may be written (lecture #2),

$$\frac{\partial \vec{u}}{\partial t} + (\nabla \times \vec{u}) \times \vec{u} + \nabla \left[ \frac{1}{2} q^2 + \left( \frac{dp}{\rho} - \chi \right) \right] = 0 \quad (6)$$

Vorticity  $\vec{\omega}$  is defined by the equation

$$\vec{\omega} = \nabla \times \vec{u} \quad (7)$$

If we operate on (6) by taking the curl, the resulting equation for the vorticity is a simpler expression,

$$\frac{D\vec{\omega}}{Dt} + (\nabla \cdot \vec{u}) \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u} \quad (8)$$

The relation (8) may be further simplified by using the continuity equation

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{u}) = 0 \quad (9)$$

Let us substitute for  $(\nabla \cdot \vec{u})$  from (9) into (8), and divide by  $\rho$  to obtain the Helmholtz Vorticity Theorem,

$$\frac{D}{Dt} \left( \frac{\vec{\kappa}}{\rho} \right) = \frac{\vec{\kappa}}{\rho} \cdot \nabla u \quad (10)$$

This equation is linear in  $\vec{\kappa}$ , with variable coefficients. By the uniqueness theorem of linear differential equations, if  $\vec{\kappa}$  is given at some time  $t = t_0$  (initial condition), the solution of (10) is uniquely determined. In particular if the motion is irrotational at any time it remains so for all time.

Further simplification occurs in two dimensions. Suppose we let  $\vec{\kappa} = \zeta \vec{k}$ . For this special case the right side of (10) vanishes, leaving

$$\frac{D}{Dt} \left( \frac{\zeta}{\rho} \right) = 0 \quad (11)$$

which is related to a theorem sometimes called the conservation of "potential vorticity".

### 3. Kelvin Circulation Theorem

**Theorem:** When the external forces are conservative, the circulation in any circuit  $c$  which moves with the fluid is constant.

**Proof:** We wish to show

$$\frac{D}{Dt} \int_c u_i dx_i = 0 \quad (12)$$

where  $c$  is described above. Let us change to Lagrangian variables

$X_i$ . Then

$$\begin{aligned} \frac{D}{Dt} \int_c u_i dx_i &= \frac{D}{Dt} \int_c u_i \frac{\partial x_i}{\partial X_j} dX_j \\ &= \int_c \frac{Du_i}{Dt} \frac{\partial x_i}{\partial X_j} dX_j + \int_c u_i \frac{\partial u_i}{\partial X_j} dX_j \\ &= \int_c \frac{Du_i}{Dt} dx_i + \int_c d\left(\frac{1}{2} q^2\right) \end{aligned} \quad (13)$$

The second term on the right side of (13) is an exact derivative, hence the integral over the curve (closed)  $c$  vanishes. Further,

$$\frac{Du_i}{Dt} = \frac{\partial}{\partial x_i} \left( \chi - \int \frac{dp}{\rho} \right) \quad (14)$$

hence the first term vanishes for the same reason, which proves the theorem (12).

#### 4. V. Bjerknes' Theorem

This theorem is analogous to the previous one (12) for a non-barotropic, non-viscous, fluid. The theorem states, for this fluid,

$$\frac{D}{Dt} \int u_i dx_i = \int_S \left[ \nabla p \times \nabla \left( \frac{1}{\rho} \right) \right] \cdot \vec{n} dS$$

The proof is similar to the preceding one, and may be found in detail in Emmons, p.32. If  $\nabla p \times \nabla \left( \frac{1}{\rho} \right) \neq 0$ , corresponding to each point in the fluid flow is a  $(p, \rho)$  point, at least locally. This theorem relates the rate of change in circulation to an integral over the  $(p, \rho)$  surface which can be interpreted as the "area" in the  $(p, \frac{1}{\rho})$  plane.

#### 5. Taylor-Proudman Theorem

This theorem is concerned with a non-viscous, barotropic fluid, which is in a rotating system. The full momentum equation governing such a fluid is (see lecture #2),

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + 2 \vec{\omega} \times \vec{u} + \frac{1}{\rho} \nabla p = \nabla \chi \quad (16)$$

where  $p$  is a modified pressure including centrifugal force, and

$\vec{\omega}$  is a vector representing the axis, magnitude, and sense of rotation.

If we assume the fluid flow deviates only slightly from a rigid body rotation, then we may be justified in neglecting the quantity  $\vec{u} \nabla \vec{u}$  in (16). If we neglect  $\vec{u} \nabla \vec{u}$  and take the curl through (16), there results the Taylor-Proudman theorem,

$$\frac{\partial \vec{\omega}}{\partial t} - 2\vec{\omega} \cdot \nabla \vec{u} = 0 \quad (17)$$

Let us now consider steady flow, i.e.  $\frac{\partial \vec{\omega}}{\partial t} = 0$ , and assume the rotation is such that we may write  $\vec{\omega} = \omega \vec{k}$ .

With these assumptions, the Taylor-Proudman theorem simplifies to

$$\frac{\partial \vec{u}}{\partial z} = 0 \quad (18)$$

Let us now suppose  $w = 0$  on some solid bottom. Then (18) implies the fluid motion is two dimensional, a rather surprising result.

#### 6. General Vorticity Theorems.

Let us consider two dimensional motion of an unbounded viscous incompressible fluid. If  $\vec{u}$  approaches zero at infinity, and if  $\zeta$  approaches zero sufficiently rapidly (faster than  $|\vec{u}|$ ) at infinity, the following results hold which are called the Theorems of Poincaré:

$$\text{I} \quad \frac{\partial}{\partial t} \iint \zeta \, dA = 0$$

$$\text{II} \quad \frac{\partial}{\partial t} \iint \vec{x} \zeta \, dA = 0$$

$$\text{III} \quad \frac{\partial}{\partial t} \iint r^2 \zeta \, dA = 4 \nu \iint \zeta \, dA \quad \text{where } r = |\vec{x}|$$

Dr. L. Howard (MIT) has shown no further results regarding moments may be found by the method employed for I, II, and III.

## References:

1. Truesdell - "Kinematics of Vorticity".
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7. Energy Theorems

I. Among all flows of a non-viscous fluid with prescribed values of the normal velocity at the boundary, the irrotational flow is the one having least kinetic energy.

Proof: Milne-Thomson, Hydrodynamics, p.89.

Remark:

$$\int_S \vec{u} \cdot \vec{n} \, ds = 0 \text{ is always assumed.}$$

II. Among all incompressible flows in a bounded region  $R$  with given values of  $u$  on  $\partial R$ , Stokes flow has the least viscous dissipation .

Def. Stokes flow,  $\nabla \cdot \vec{u} = 0$ ,  $\nu \nabla^2 u = \nabla p$ .

We shall sketch the proof to this theorem. Consider the bilinear form:

$$D[\vec{u}, \vec{v}] = \int_R (u_{i,j} + u_{j,i})(v_{i,j} + v_{j,i}) \, dV \quad (19)$$

Suppose  $\vec{v}$  corresponds to the Stokes flow. Let us denote by  $\vec{u}$  some other flow field.

$$\begin{aligned} D[\vec{u}, \vec{u}] &= D[(\vec{u} - \vec{v}) + \vec{v}, (\vec{u} - \vec{v}) + \vec{v}] \\ D[\vec{u}, \vec{u}] &= D[\vec{u} - \vec{v}, \vec{u} - \vec{v}] + D[\vec{v}, \vec{v}] \\ &\quad + 2D[\vec{u} - \vec{v}, \vec{v}] \end{aligned} \quad (20)$$

Expanding  $D[\vec{u} - \vec{v}, \vec{v}]$ , and applying the divergence theorem with boundary conditions, we find

$$D[\vec{u} - \vec{v}, \vec{v}] = 0$$

Since  $D[\vec{f}, \vec{f}]$  is a positive definite form, the theorem is proved.

#### Appendix

It is possible to prove several other theorems related to I. For these the reader is referred to Milne-Thomson.

(submitted by R. L. Duty)

GRAVITY WAVES: Suppose we have an incompressible irrotational flow in a gravitational field (gravity in the negative z-direction). Suppose further that, in equilibrium, the fluid has a free-surface which is a plane at  $z = 0$ , and a constant depth  $z = -H$ . We then have  $\vec{u} = \vec{\nabla} \phi$  and  $\nabla^2 \phi = 0$ . Let us denote the (top) surface, in general, by  $z = \zeta(x, y, t)$ . We need boundary conditions on the fluid so as to produce a well-defined potential problem. The boundary condition on the bottom,  $z = -H$ , is given by the vanishing of the vertical velocity component - viz.,  $\frac{\partial \phi}{\partial z} = 0$  on  $z = -H$ . We use the Bernoulli equation,  $\phi_t + \frac{1}{2} q^2 + \frac{p}{\rho} + gz = 0$  to obtain the boundary condition at the surface. Supposing constant pressure on the surface  $z = \zeta(x, y, t)$ , we can clearly choose  $p$  such that  $p = 0$  on the surface  $z = \zeta$ . Then since  $z - \zeta$  is a locus of particles we have  $\frac{D}{Dt} (z - \zeta) = 0$ . This equation and Bernoulli's equation on the surface become,

$$\phi_t + \frac{1}{2} q^2 + g\zeta = 0, \text{ and } w - \left\{ \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \right\} = 0;$$

where  $(u, v, w) = \vec{u}$ . If the motion is "small", we may neglect the non-linear terms in the above equations; i.e.,

$$\phi_t + g\zeta = 0, \text{ and } w - \zeta_t = 0 \text{ on } z = \zeta(x, y, t),$$

or to the same order of approximation, on  $z = 0$ .

Eliminating  $\zeta$  between these equations yields the b.c. at the surface; viz.,

$$\phi_{tt} + g\phi_z = 0 \quad (\text{on } z = 0).$$

Writing  $\phi(x, y, z, t) = e^{i\sigma t} \phi(x, y, z)$ , we have the well-defined potential problem:

$$\begin{aligned} \nabla^2 \phi &= 0 && \text{(in interior of fluid)} \\ \phi_z &= 0 && \text{(b.c. on bottom of fluid)} \\ -\sigma^2 \phi + g \phi_z &= 0 && \text{(b.c. on surface of fluid)} \end{aligned} \quad ; \quad (1)$$

with, of course, suitable b.c. in  $x$  and  $y$  (e.g., vertical rigid walls,  $\frac{\partial \phi}{\partial n} = 0$ ).

(Note: In the work below  $\phi$  always refers to  $\phi(x, y, z)$ .)

We now apply the method of "separation of variables" to (1);

viz, we look for a solution  $\phi = \sum_{n=0}^{\infty} a_n(x, y) \phi_n(z)$  where  $\phi_n$  are the ( $z$ -component) eigenfunctions. Substituting this expression in (1) in the customary manner yields the separation of  $\phi_n$ ,

$$\begin{aligned} \phi_n''(z) &= \lambda_n \phi_n(z) ; \\ \phi_n'(-H) &= 0 , \\ \phi_n'(0) &= \frac{\sigma^2}{g} \phi_n(0) \end{aligned} \quad (2)$$

where the prime denotes  $\frac{d}{dz}$ . Note that in the b.c. at the surface we have neglected  $\zeta$  as being small. We now wish to determine the eigenvalues  $\lambda_n$  and the eigenfunctions  $\phi_n$ .

(a) Suppose  $\lambda_n > 0$ : Then let  $\lambda_n = k^2$ . The solution to (2) is then  $\phi(z) = A \cosh k(z + H)$ , the b.c. at  $z = 0$  giving the equation for  $\lambda_n$ , which is  $k \sinh kH = \frac{\sigma^2}{g} \cosh kH$ , or  $\coth(kH) = (kH) \frac{g}{\sigma^2 H}$ .

Graphical plots of  $\coth(kH)$  and  $(kH)\frac{g}{\sigma^2 H}$  versus  $kH$  show that there is just one eigenvalue (for this case;  $\lambda_n \geq 0$ ), which is given by the solution to the equation  $\sigma^2 = gk \tanh(kH)$ .

(b) It is clear that  $\lambda_n = 0$  is not an eigenvalue (for if  $\lambda_n = 0$ , we have  $\phi_n'(z) = \text{constant} = \phi_n'(-H) = 0 \Rightarrow \phi_n(z) \equiv 0$ ).

(c) Suppose  $\lambda_n < 0$ : Then let  $\lambda_n = -K^2$ . The solution to (2) is then  $\phi(z) = A \cos K(z + H)$ , the b.c. at  $z = 0$  yielding the equation for  $\lambda_n$ , which is  $-K \sin KH = \frac{\sigma^2}{g} \cos KH$ , or  $\cot(KH) = -KH \left( \frac{g}{\sigma^2 H} \right)$ .

Graphical solution of this equation show that there is a denumerable infinity of distinct eigenvalues (for this case;  $\lambda_n < 0$ ) such that when arranged in decreasing order,  $\lambda_1 > \lambda_2 > \dots > \lambda_n > \dots$ ,  $\lim_{n \rightarrow \infty} \lambda_n = -\infty$ .

It is easy to show, furthermore, that the eigenfunctions  $\phi_n$  are

orthogonal (viz., from eqs (2), one gets

$$0 = (\phi_m \phi_n' - \phi_n \phi_m') \Big|_{-H}^0 = (\lambda_n - \lambda_m) \int_{-H}^0 \phi_n \phi_m dz, \text{ and by suitable}$$

normalization, the  $\phi_n$  can be made orthonormal.

Then for the solution to (1), we have

$$\phi = \sum_{n=1}^{\infty} a_n(x,y) \phi_n(z) + a_0 \phi_0$$

where  $\phi_n''(z) = -K_n^2 \phi_n(z)$ ,  $n = 1, 2, 3, \dots$  (3)

$$\phi_0''(z) = +k^2 \phi_0.$$

For the determination of  $a_n(x,y)$ , we have the equations

$$\left. \begin{aligned} \Delta_2 a_0 + k^2 a_0 &= 0 ; \\ \Delta_2 a_n - K_n^2 a_n &= 0, \quad n = 1, 2, 3, \dots \end{aligned} \right\} \quad (4)$$

where  $\Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ .

We can again apply "separation of variables", this time to (4). When the b.c. for  $x, y$  variables are given, the solution of (4) substituted into (3) gives the solution to the problem.

If, for example, the motion were given as uniform in the  $y$  direction, then (4) would become

$$\frac{d^2}{dx^2} X(x) + k^2 X(x) = 0 ;$$

$$\frac{d^2}{dx^2} X_n(x) - K_n^2 X_n(x) = 0, \quad n = 1, 2, 3, \dots ;$$

where we have used  $a_0(x,y) = X(x)$ ,  $a_n(x,y) = X_n(x)$ . If in addition we have for b.c. on  $x$  that the solutions are bounded as  $x \rightarrow \pm \infty$ , then we get  $X_n \equiv 0$  ( $n = 1, 2, 3, \dots$ ) and  $X = e^{\pm ikx}$ . The solution to this potential problem (eq. (1)) is then

$$\phi(x,y,z,t) = Ae^{i(\sigma t \pm kx)} \cosh k(z+H), \quad (5)$$

where  $k$  is determined by the equation  $\sigma^2 = gk \tanh kH$ .

From (5) we distinguish two types of waves: (1) Shallow water waves for the case  $kH \ll 1$ ; (2) deep water waves for the case  $kH \gg 1$ . Since for  $kH \ll 1$ ,  $\tanh kH \approx kH$ ; and for  $kH \gg 1$ ,  $\tanh kH \approx 1$ ; we have

$$\left. \begin{array}{l} \sigma^2 \cong gk^2 H \quad \text{for shallow water waves} \\ \text{and } \sigma^2 \cong kg \quad \text{for deep water waves.} \end{array} \right\} \quad (6)$$

Hence, shallow water waves propagate with constant velocity  $c = \sqrt{gH}$ ; but for deep water waves the velocity of propagation,  $c = \sqrt{\frac{g}{k}}$ , depends on the wave number  $k$ . Therefore, deep water waves are dispersive.

exercise 1: Solve the linear wave-maker problem (using the above results). i.e. the end  $x = 0$  of a semi-infinite channel ( $x \geq 0$ ,  $-H \leq z \leq 0$ ,  $0 \leq y \leq b$ ) moves so that the horizontal velocity  $\phi_x$  at  $x = 0$  is a given function  $e^{i\sigma t} f(z)$ . Find the resulting periodic motion assuming that as  $x \rightarrow +\infty$  only an outgoing wave ( $e^{i(\sigma t - kx)}$ ) is present.

exercise 2: Discuss the free standing waves (normal modes) in a closed circular lake of depth  $H$  and radius  $R$ .

exercise 3: A speedboat belonging to A.A. (noted ocean-going physicist at W.H.O.I.) passing through Woods Hole going roughly parallel to shore produces a group of waves which reach the shore ten minutes later. They are observed to have a period of two seconds. Estimate the distance off-shore of the boat (assume infinite depth up to the shore).

Shallow Water Waves (Non-rotating system)

Shallow water waves are defined to be those in which the wavelength is long compared with depth. Having thus defined them, we assume the substantial derivative ( $Dw/Dt$ ) may be neglected in Euler's momentum equation relative to the other terms in the z-component equation, ("Hydrostatic approximation"). We further assume the fluid velocity to be small enough that we may neglect products of the fluid velocity.

With these assumptions we wish to derive the wave equation for the case when the depth of the water's floor varies with position. The case  $H = \text{constant}$  has been previously discussed, and we retain the notation used there. The momentum equations for this case are written

$$p = -\rho g(z - \zeta) \quad (7)$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (8)$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \quad (9)$$

By hypothesis,

$$\zeta = \zeta(x, y, t) \quad (10)$$

Differentiating (7), also using (10), and substituting in (8) and (9) we have the result that

$$\left. \begin{aligned} u &= u(x, y, t) \\ v &= v(x, y, t) \end{aligned} \right\} \quad (11)$$

Let us write the equation for the fluid bottom

$$H = H(x, y) \quad (12)$$

Then the equation of continuity implies the following relation between fluid velocities and other physical variables

$$(u_x + v_y) [H(x,y) + \zeta] + w(\zeta) - w(-H) = 0 \quad (13)$$

The boundary condition that the fluid surface and fluid bottom should always consist of the same particles is expressed

$$\frac{D}{Dt} (z - \zeta) = 0 \quad (14)$$

$$\frac{D}{Dt} (z + H) = 0 \quad (15)$$

If we expand (14) and (15) and substitute into (13) we have the result

$$\zeta_t + (uH)_x + (vH)_y = 0 \quad (16)$$

Eliminating  $u$  and  $v$  from (16) by using (8) and (9), we have the governing equation written in terms of the surface variable  $\zeta$ ,

$$\zeta_{tt} - [(gH\zeta_x)_x + (gH\zeta_y)_y] = 0 \quad (17)$$

If we seek wave solutions of (17) of the usual form

$$\zeta(x,y,t) = e^{i\sigma t} \zeta(x,y) \quad (18)$$

there results the expression for  $\sigma$ ,

$$(H\zeta_x)_x + (H\zeta_y)_y + \frac{\sigma^2}{g} \zeta = 0 \quad (19)$$

Equation (19) may be successfully attacked by variational principles (with appropriate boundary conditions). If we require the normal derivative of  $\phi$  to vanish at a vertical wall we may express  $\sigma$  in the variational ratio

$$\frac{\sigma^2}{g} = \frac{\int_R H(\zeta_x^2 + \zeta_y^2) dA}{\int_R \zeta^2 dA} \quad (20)$$

(Specifically, the minimum of this quotient over all continuously differentiable functions  $\zeta$  on  $R$  subject to  $\int_R \zeta dA = 0$  is the lowest eigenvalue. This minimum is attained for a function  $\zeta$  which satisfies  $\frac{\partial \zeta}{\partial n} = 0$  on  $\partial R$ .)

### Shallow Water Waves in a Rotating System

We wish to consider the previous problem in a rotating system. Rotation produces Coriolis forces and a centrifugal force in addition to those previously found. This results in a system of equations similar to (7), (8), and (9). These may be written

$$p = -f g(z - \zeta) \quad (21)$$

$$\frac{\partial u}{\partial t} - fv + g \zeta_x = 0 \quad (22)$$

$$\frac{\partial v}{\partial t} + fu + g \zeta_y = 0 \quad (23)$$

and continuity

$$\zeta_t + (Hu)_x + (Hv)_y = 0 \quad (24)$$

where  $f$  is the Coriolis frequency ( $= 2\omega$ ).

If we again seek wave solutions of the form (18), the equations of momentum (22) and (23) and continuity (24), become (taking  $g \equiv 1$ )

$$\left. \begin{aligned} i\sigma u - fv + \zeta_x &= 0 \\ i\sigma v + fu + \zeta_y &= 0 \\ i\sigma \zeta + (Hu)_x + (Hv)_y &= 0 \end{aligned} \right\} \quad (25)$$

It is possible to solve the first two equations of (25) for  $u$  and  $v$  in terms of  $\zeta$ , provided the determinant

$$\Delta = \begin{vmatrix} i\sigma - f & \\ f & i\sigma \end{vmatrix} = f^2 - \sigma^2 \quad (26)$$

does not vanish. If  $\Delta \neq 0$ , then the equation corresponding to (17) is

$$\begin{aligned} \left[ H(-\sigma\zeta_x + if\zeta_y) \right]_x + \left[ H(-\sigma\zeta_y - if\zeta_x) \right]_y & \quad (27) \\ & = (\sigma^2 - f^2)\sigma\zeta \end{aligned}$$

If the rotating fluid is supported by a slightly concave bottom, over small regions in the  $(x,y)$  plane  $H$  is approximately constant. The case  $H$  constant is an interesting mathematical situation for which the equation (27) simplifies to

$$-H\sigma \nabla^2 \zeta = \sigma(\sigma^2 - f^2)\zeta \quad (28)$$

If we exclude wave frequency  $\sigma = 0$ , equation (28) may be written

$$\nabla^2 \zeta + k^2 \zeta = 0 \quad (29)$$

where  $k^2 = \frac{\sigma^2 - f^2}{H}$

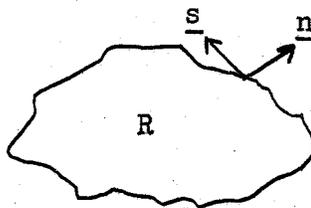
or

$$\sigma^2 = Hk^2 + f^2 \quad (30)$$

The condition for the normal components of  $u$  and  $v$  to vanish on a vertical wall is expressible in the form

$$i\sigma \frac{\partial \zeta}{\partial n} + f \frac{\partial \zeta}{\partial s} = 0 \quad (31)$$

where  $\underline{n}$  and  $\underline{s}$  are oriented as follows:



It is interesting to notice that for a given value of  $k$ , the  $\sigma$  produced by (30) is higher than that for the non-rotating system, unless imaginary  $k$  occurs.

Mathematically, (31) is interesting since if  $f \neq 0$ , the eigenvalue  $\sigma$  appears in the boundary condition which is not the standard Neumann problem which we would have for the case  $f = 0$ .

submitted by R. Duty  
K. Gross

Waves in a Rotating System (con't)Shallow Water Approximation.

The relevant equations for wave motion in a rotating frame, restricted to shallow water were derived in the previous lecture. Assuming a periodic solution in time of the form  $e^{i\sigma t}$  the equations become:

$$i\zeta_x - ifv = \sigma u \quad 1(a)$$

$$i\zeta_y + ifu = \sigma v \quad 1(b)$$

$$i(Hu)_x + i(Hv)_y = \sigma\zeta \quad 1(c)$$

if  $\sigma^2 \neq f^2$  it is possible to solve for  $u$  and  $v$  in terms of  $\zeta$ , giving:

$$u = \frac{1}{\sigma^2 - f^2} \left\{ i\sigma\zeta_x + f\zeta_y \right\} \quad 2(a)$$

$$v = \frac{1}{\sigma^2 - f^2} \left\{ i\sigma\zeta_y - f\zeta_x \right\} \quad 2(b)$$

Now if  $H$ , the water depth, is a constant, the insertion of equations 2(a) and 2(b) into 1(c) gives

$$-\sigma H \Delta_2 \zeta_2 = \sigma(\sigma^2 - f^2)\zeta \quad 2(a)$$

where  $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  the 2-d laplacian.

The associated boundary conditions were presented in the previous lecture as:

$$i\sigma \frac{\partial \zeta}{\partial n} + f \frac{\partial \zeta}{\partial s} = 0, \text{ on the boundary} \quad 3(b)$$

It is interesting to note that the eigenvalue  $\sigma$  appears in the boundary conditions as well as in the differential equation. This means in turn that one can't apply the usual theorems for the

eigenvalues and eigenfunctions of a self adjoint operation.

Veltkamp<sup>1</sup> showed that with the original set of equations that the usual theorems can be applied in that case, with a suitable Hilbert Space formulation.

Geostrophic Approximation.

We investigate the steady solutions of equations (a,b,c) i.e. those solutions with  $\sigma = 0$ . It is necessary to return in this situation to the original set since equation (3a) becomes the trivial equation  $0 = 0$  for  $\sigma = 0$ .

Equations (1, a, b, c) become:

$$v = \frac{\zeta_x}{f} \quad (4a)$$

$$u = -\frac{\zeta_y}{f} \quad (4b)$$

$$(Hu)_x + (Hv)_y = 0 \quad (4c)$$

Substitution of (4a) and (4b) into (4c) yields:

$$(H\zeta_x)_y - (H\zeta_y)_x = 0$$

We note that this equation is satisfied by any  $\zeta$  if H is constant.

For the case where H is not a constant we find that:

$$H_x \zeta_y - H_y \zeta_x = 0 \quad (5)$$

This may be recognized as simply  $\frac{\partial(\zeta, H)}{\partial(x, y)} = 0$ .

Equation (5) then tells us that lines of constant  $\zeta$  will follow

<sup>1</sup>Veltkamp, G.W. Spectral Properties of Hilbert Space Operators Associated with Tidal Motions, April 1960. Doctoral Thesis Royal Univ. of Utrecht, Utrecht, Holland.

the contours of constant  $H$ , and any such  $\zeta$  gives a solution.

Waves on an Infinite Plane

Consider the case where  $\sigma \neq 0$ ,  $\sigma \neq f$ , and  $H = \text{constant}$ . Then the governing equation is (3a).

$$\Delta_2 \zeta + \frac{\sigma^2 - f^2}{H} \zeta = 0 \quad (6)$$

Assuming  $\zeta = e^{-ikx} e^{i\sigma t}$  we find that the requirement demanded by (6) is that:

$$\sigma^2 = f^2 + k^2 H \quad (7)$$

This shows that  $\sigma^2 > f^2$  is required for a plane wave solution. The velocity component perpendicular to the wave normal (here we have arbitrarily picked the wave normal along  $x$ ) is from (2b):

$$v = \frac{1}{\sigma^2 - f^2} ikf e^{-ikx} e^{i\sigma t} \quad (8)$$

Case where  $\sigma = \pm f$

In the situation where  $\sigma = \pm f$  we can no longer solve for  $u$  and  $v$  in terms of  $\zeta$ . The original set of equations becomes:

$$\pm f \{u \pm iv\} = i \zeta_x \quad (9a)$$

$$-if \{u \pm iv\} = i \zeta_y \quad (9b)$$

These imply that

$$\frac{i \zeta_x}{i \zeta_y} = \frac{\pm}{-i} = \pm i$$

or

$$i(\zeta_x \mp i \zeta_y) = 0 \quad (10)$$

Now if we write  $\zeta = a + ib$

Then using the above equation

$$a_x + ib_x \mp i(a_y + ib_y) = 0$$

Setting the real and imaginary parts separately equal to zero gives two equations for a and b:

$$a_x \mp b_y = 0 \quad (11a)$$

$$b_x \mp a_y = 0 \quad (11b)$$

For the lower sign:

$$a_x = b_y \quad (12a)$$

$$b_x = -a_y \quad (12b)$$

We recognize these as the Cauchy-Riemann equations which shows that a and b are harmonic functions and that  $\zeta$  is a complex analytic function of  $x \mp iy$ .

If we define

$$z = x \mp iy$$

$$\bar{z} = x \pm iy$$

then we can show that the condition (10) is equivalent to the statement that

$$\frac{\partial \zeta}{\partial \bar{z}} = 0 \quad (13)$$

and so  $\zeta = \zeta(z)$

#### Case of Variable Depth

The governing equation for wave propagation in shallow water of variable depth, derived in the previous lecture (eq.(27)) is:

$$\begin{aligned}
 & H \left[ -\sigma \Delta_2 \zeta \right] + H_x \left[ -\sigma \zeta_x + i f \zeta_y \right] \\
 & + H_y \left[ -\sigma \zeta_y - i f \zeta_x \right] = \sigma (\sigma^2 - f^2) \zeta
 \end{aligned} \tag{14}$$

If we suppose that  $H$  varies in a linear fashion as:

$$H = H_0 + \beta y$$

and if  $\beta$  is small we will neglect the variation in all situations except where derivatives of  $H$  are involved. Equation (14) becomes:

$$H_0 \left[ -\sigma \Delta \zeta \right] + \beta \left[ -\sigma \zeta_y - i f \zeta_x \right] = \sigma (\sigma^2 - f^2) \zeta \tag{15}$$

If we look at solutions of the form

$$\zeta = e^{-i(kx - \sigma t)}$$

then equation (15) requires that

$$H_0 \left[ +\sigma k^2 \right] - \beta k f = \sigma (\sigma^2 - f^2) \tag{16}$$

or:

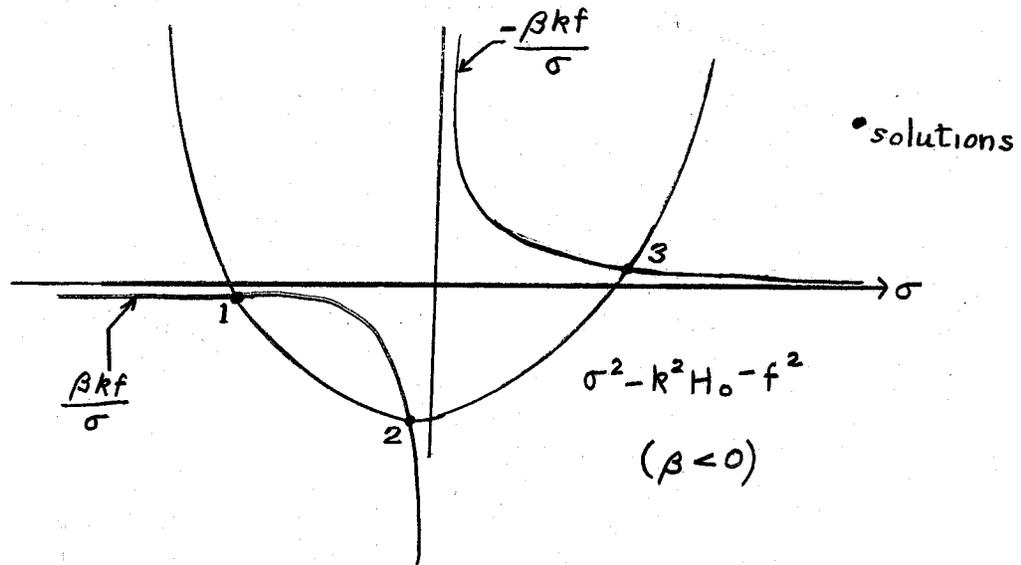
$$\sigma^2 = f^2 + k^2 H_0 - \frac{\beta k f}{\sigma}$$

This is now a cubic equation for  $\sigma$ . When  $\beta = 0$  we return to our quadratic form, and the solution for constant depth shallow water waves.

We may solve equation (16) by rewriting (16) in the form:

$$\sigma^2 - f^2 - k^2 H_0 = -\frac{\beta k f}{\sigma}$$

Plotting each side of the equation separately and finding the solutions at the intersections.



Solutions 1 and 3 correspond to the waves associated with near constant depth motion while solution 2 represents very low frequency waves and are called Rossby waves. Note that for the Rossby wave we always have  $\frac{\beta k f}{\sigma} > 0$ , i.e. waves travel (phase velocity) in only one direction.

Infinite half-plane with periodically oscillating boundary:

Consider the infinite half-plane  $x \geq 0$  with the boundary  $x = 0$  oscillating periodically, i.e.,

$$u|_{x=0} = U e^{i\sigma t} \quad (17)$$

Case I:  $\sigma^2 > f^2$

The equations to be satisfied are, as before,

$$u = \frac{1}{(\sigma^2 - f^2)} (i\sigma \zeta_x + f \zeta_y) \quad (18)$$

$$\Delta \zeta + \frac{(\sigma^2 - f^2)}{H} \zeta = 0 \quad (19)$$

The solution for  $\zeta$  is

$$\zeta = A e^{-ikx} \quad (20)$$

and substitution into (19) shows

$$k^2 = \frac{\sigma^2 - f^2}{H} \quad (21)$$

Substituting  $\zeta$  into (18) and remembering that the time dependence of  $\zeta$  is  $e^{i\sigma t}$ , one sees that at  $x = 0$

$$U e^{i\sigma t} = \frac{1}{\sigma^2 - f^2} (i\sigma)(-ikA) e^{i\sigma t}$$

$$\begin{aligned} \text{or} \quad A &= \frac{U(\sigma^2 - f^2)}{k\sigma} \\ &= \frac{U\sqrt{H}}{\sigma} \sqrt{\sigma^2 - f^2} \end{aligned} \quad (22)$$

Thus the equation for the surface is  $\zeta = A e^{-ikx}$ , where  $k$  and  $A$  are given by (21) and (22) respectively.

Case II:  $\sigma^2 < f^2$

The solution of the above equations for  $\zeta$  (i.e. equations (19)) is

$$\zeta = A e^{-kx}, \quad (23)$$

where

$$k^2 = \frac{f^2 - \sigma^2}{H} \quad (24)$$

The solution of the form  $\zeta = B e^{kx}$  must have  $B \equiv 0$  since the height of the surface must remain finite as  $x \rightarrow \infty$ .

A is found in the same manner as before to be

$$A = -\frac{iU}{\sigma} \sqrt{H} \sqrt{f^2 - \sigma^2} \quad (25)$$

From equation 2(b)

$$\begin{aligned} v &= \frac{1}{\sigma^2 - f^2} \left[ -fA(-k) \right] e^{-kx} \\ &= \frac{ifU}{\sigma} e^{-\sqrt{\frac{f^2 - \sigma^2}{H}} x}, \end{aligned} \quad (26)$$

using the above expressions for A and k. Thus there is a component of velocity in the y direction. (There is also, of course, when  $\sigma^2 > f^2$ ). Note the restriction to the neighborhood of the coastline.

Infinite half-plane with fixed boundary:

The equation for  $\zeta$  is as before

$$\Delta \zeta + \frac{(\sigma^2 - f^2)}{H} \zeta = 0 \quad (27)$$

and the boundary condition is again

$$i\sigma \zeta_x + f\zeta_y = 0 \quad \text{at } x = 0 \quad (28)$$

We try a solution of the form  $\zeta = e^{-iky} \phi(x)$ . Substitution into equation (27) shows

$$\phi''' - \phi k^2 + \frac{(\sigma^2 - f^2)}{H} \phi = 0$$

or

$$\phi'' + \frac{(\sigma^2 - f^2 - k^2 H)}{H} \phi = 0 \quad (29)$$

The boundary condition, equation (28), becomes

$$i\sigma \phi'_{(0)} - ikf \phi_{(0)} = 0 \quad (30)$$

and  $\phi(x)$  must remain finite as  $x \rightarrow \infty$ .

Thus  $\phi$  is given by

$$\phi = e^{-\sqrt{\frac{f^2 + k^2 H - \sigma^2}{H}} x}, \quad (31)$$

$$\text{if } \sigma^2 < f^2 + k^2 H$$

and equation (28) becomes

$$-\sigma \sqrt{\frac{f^2 + k^2 H - \sigma^2}{H}} - kf = 0 \quad (32)$$

Equation (32) implies that  $\sigma$  and  $k$  must have opposite signs. Since the phase velocity of the wave in the  $y$  direction is given by  $\frac{\sigma}{k}$ , the wave must be traveling in the negative direction.

Rewriting equation (32)

$$\frac{f^2 + k^2 H - \sigma^2}{H} = \frac{k^2 f^2}{\sigma^2}$$

or

$$\sigma^2(\sigma^2 - f^2 - k^2 H) + Hk^2 f^2 = 0$$

or

$$(\sigma^2 - f^2)(\sigma^2 - k^2 H) = 0 \quad (33)$$

Since the above analysis is based on  $\sigma^2 \neq f^2$ .

Equation (33) implies

$$\sigma^2 = k^2 H \quad (34)$$

If  $\sigma^2 > f^2 + k^2 H$  we get complex exponentials in  $x$ .

To produce a standing wave one must have an incoming and an outgoing wave. In that case, in view of equation (31) and the corresponding equation with the sign changed in the exponential, the equation for  $\phi$  would be

$$\phi = \cos \sqrt{\frac{\sigma^2 - f^2 - k^2 H}{H}} (x + \alpha), \quad (35)$$

where  $\alpha$  can be determined by equation (28).

The wave number in the  $x$  direction is given by

$$\ell = \sqrt{\frac{\sigma^2 - f^2 - k^2 H}{H}}$$

or

$$\sigma^2 = f^2 + (\ell^2 + k^2)H \quad (36)$$

Veltkamp has investigated in a general way the solutions for an arbitrary domain, also including a different boundary condition appropriate to a portion of the boundary open to a deep ocean.

Some of his results are as follows:

1) For a box of constant depth and solid walls the spectrum of eigenvalues consists of isolated points including zero. For every eigenvalue except zero there is a finite number of eigenfunctions.

2) For a box of variable depth and solid walls those eigenvalues not contained in  $[-f, f]$  have the same character as those

in the above case. The eigenvalues contained in  $[-f, f]$  consist of either an infinite number of point eigenvalues (probably with zero as a limit point, although this has not been proved) or some continuous spectra of eigenvalues.

3) If the boundary is partly "coast" and partly open sea, then the spectrum of eigenvalues contains the whole interval  $[-f, f]$ .

Submitted by J. Pedlosky and R. Ellis

Circular lake with uniform depth

Consider a circular lake whose radius has been normalized to unity and a time scale such that the acceleration due to gravity is also unity. The differential equation satisfied by  $\zeta$ , the height of the surface, was derived in lecture five to be

$$\Delta \zeta + \frac{\sigma^2 - f^2}{H} \zeta = 0 \quad (1)$$

The boundary conditions that there be zero radial velocity at the edge of the lake ( $r = 1$ ) is, in polar coordinates ( $r, \theta$ ),

$$i\sigma \zeta_r + f \zeta_\theta = 0 \quad \text{at } r = 1 \quad (2)$$

We look for a solution of the form

$$\zeta = e^{im\theta} \phi(r) \quad (3)$$

The boundary condition then becomes

$$\phi'(1) + \frac{fm}{\sigma} \phi(1) = 0 \quad (4)$$

and the differential equation becomes

$$\phi'' + \frac{1}{r} \phi' - \frac{m^2}{r^2} \phi + \frac{\sigma^2 - f^2}{H} \phi = 0 \quad (5)$$

$$0 \leq r < 1$$

The singularity at the origin is characteristic of polar coordinates.

Case I:  $m = 0$

If  $\sigma^2 < f^2$  the solution to (5) is

$$\phi = I_0 \left( \sqrt{\frac{f^2 - \sigma^2}{H}} r \right) \quad (6)$$

where  $I_0(r)$  is the zero-th order modified Bessel function which equals  $J_0(ir)$ .

But the boundary condition is now

$$\phi'(1) = 0 \quad (7)$$

Since  $\phi'(r) = 0$  only for  $r = 0$ , there are no eigenfunctions for  $\sigma^2 < f^2$ .

The case  $\sigma^2 = f^2$  would have to be investigated separately since equation (1) is based on  $\sigma^2 \neq f^2$ . (In fact there are no eigenfunctions here either). If  $\sigma^2 > f^2$  the solution to equation (5) is

$$\phi = J_0 \left( \sqrt{\frac{\sigma^2 - f^2}{H}} r \right) \quad (8)$$

where  $J_0$  is the zero-th order Bessel function.

By equation (7)

$$J_0' \left( \sqrt{\frac{\sigma^2 - f^2}{H}} \right) = 0 \quad (9)$$

Thus, the eigenvalues for this case, which we shall denote by  $\sigma_m^{(0)}$  to indicate that they correspond to  $m = 0$ , are given by

$$\sigma_m^{(0)} = \pm \sqrt{f^2 + K_n^{(0)2} H}, \quad (10)$$

where  $J_0'(K_n^{(0)}) = -J_1(K_n^{(0)}) = 0$

It can be shown that in this case there is a  $\Theta$ -component of velocity but that there is axial symmetry.

Case II:  $m > 0$

Let  $\alpha$ ,  $\lambda$  and  $K$  be defined as follows:

$$\frac{fm}{\sigma} \equiv \alpha \quad (11)$$

$$\sigma^2 - f^2 \equiv \lambda \equiv K^2 H \quad (12)$$

Then equations (4) and (5) become

$$\phi'' + \frac{1}{r} \phi' - \frac{m^2}{r^2} \phi + \frac{\lambda}{H} \phi = 0 \quad (13)$$

$$\phi'(1) + \alpha \phi(1) = 0 \quad (14)$$

We will now consider  $\lambda$  to be the eigenvalues and  $\alpha$  to be a parameter and find the possible values of  $\lambda$ .

For  $\lambda > 0$  the solution to equation (13) is given in terms of Bessel functions, viz:

$$\phi = A J_m(Kr), \quad (15)$$

where  $K$  is given by equation (12). The boundary condition, equation (14), is now

$$K J_m'(K) = -\alpha J_m(K), \quad (16)$$

where the prime denotes differentiation with respect to  $Kr$ .

To find the possible values of  $K$  according to equation (16) (and hence the possible values of  $\lambda$  in view of equation (12)) we graph  $K J_m'(K)$  and  $-\alpha J_m(K)$  and find their points of intersection.

For  $\alpha > 0$  the graph is as follows:

$$\textcircled{1} \quad y = -\alpha J_m(K)$$

$$\textcircled{2} \quad y = K J_m'(K)$$

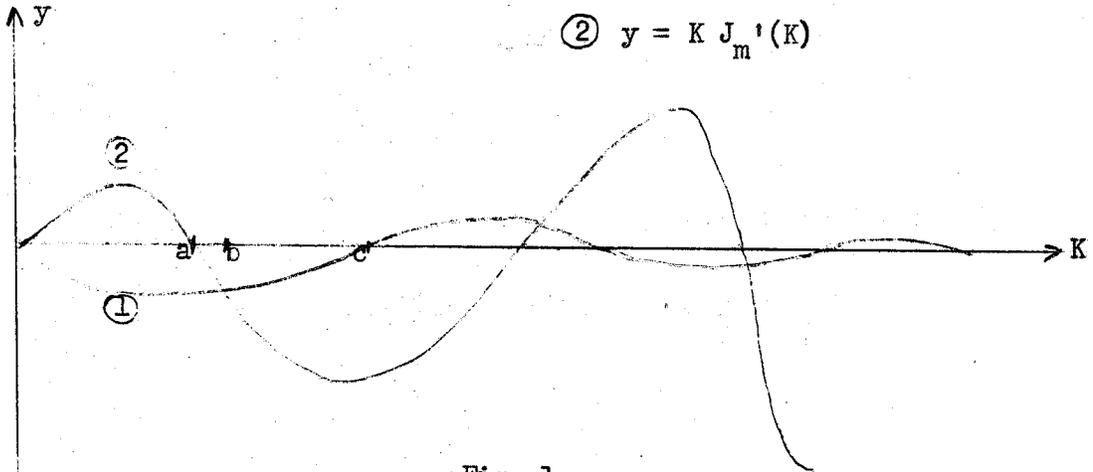


Fig. 1

Let  $b$  denote the value of  $K$  for which the graphs intersect (see Fig. 1). As  $\alpha$  gets small  $b$  approaches point  $a$  on the graph, i.e.  $b$  approaches a root of  $K J_m'(K)$ . As  $\alpha$  gets large  $b$  approaches point  $c$ , i.e. a root of  $-\alpha J_m(K)$ .

For  $\alpha < 0$  the graph  $K J_m'(K)$  is unaltered but the graph of  $-\alpha J_m(K)$  is reflected through the  $K$ -axis. Thus for small negative  $\alpha$  the solutions to equation (16) are still the roots of  $K J_m'(K)$  as before but for large negative  $\alpha$  the solutions approach the roots of  $-\alpha J_m(K)$  which are the roots immediately preceding the roots of  $-\alpha J_m(K)$  approached by the solutions for large positive  $\alpha$ . Thus if  $b_{10}$  approached the tenth root of  $-\alpha J_m(K)$  for large positive  $\alpha$ , it would approach the ninth root for large negative  $\alpha$ .

These results are most easily seen on a graph of the eigenvalues (which we will now denote by  $\lambda_n$  to distinguish them) plotted against  $\alpha$  as in figure two.

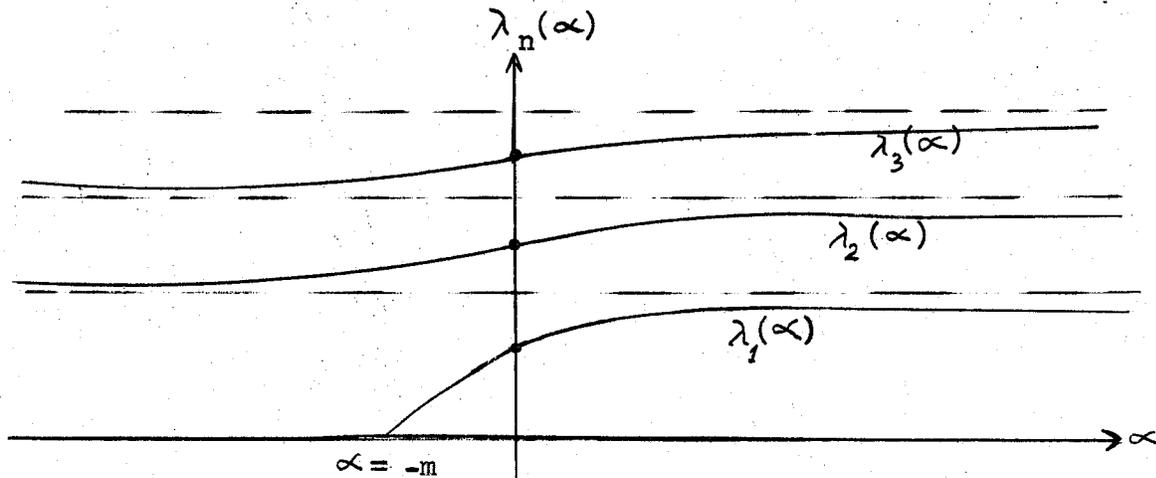


Fig. 2

The  $\lambda$  intercepts on this graph are equal to  $K_n^2 H$  where  $K_n$  is a root of  $K^2 J_m(K)$ . The asymptotes (dashed lines) are the lines  $\lambda_n(\alpha) = K_n^2 H$  where  $K_n$  is a root of  $-\alpha J_m$ . That the first eigenvalue intercepts the line  $\lambda_n(\alpha) = 0$  at  $\alpha = -m$  can be seen by considering the asymptotic behavior of  $J_m(K)$  for small  $K$  (i.e. small  $\lambda$ ). For small  $K$

$$J_m(K) \sim cK^m, \quad (17)$$

where  $c$  is a constant. Therefore,

$$K J_m' \sim m c K^m \quad (18)$$

and equation (16) becomes

$$m c K^m = -\alpha c K^m \quad (19)$$

or

$$m = -\alpha \quad (20)$$

To complete the graph in figure two we must consider the case  $\lambda < 0$ . Then the solution to equation (13) is

$$\phi = A I_m(Kr), \quad (21)$$

where  $I_m$  is the modified Bessel function of order  $m$ .

Equation (14) then becomes

$$K I_m'(K) = -\alpha I_m(K) \quad (22)$$

Since  $I_m$  and  $I_m'$  have the same sign, there are no eigenvalues for  $\alpha > 0$ . For  $\alpha < 0$  we graph  $K I_m'(K)$  and  $-\alpha I_m(K)$  as before in figure three.

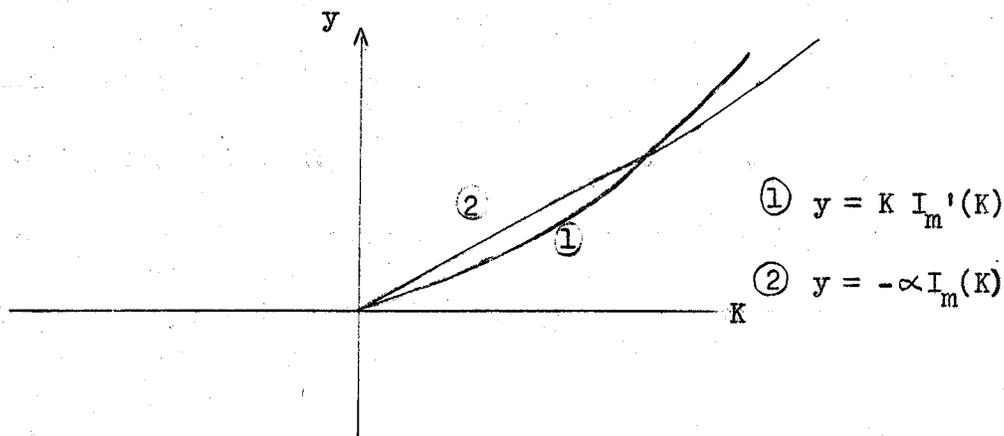


Fig. 3

Thus there is a single point of intersection and therefore only one eigenvalue less than zero. As  $\alpha$  increases negatively the eigenvalue increases negatively as shown by figure three. Since  $I_m(x) = i^m J_m(ix)$ ,  $J_m$  and  $I_m$  have the same sort of asymptotic behavior; therefore, the graphs of  $\lambda_1(\alpha)$  for  $\lambda$  negative and for  $\lambda$  positive join together at  $\alpha = -m$ .

The complete set of eigenvalue curves is shown in figure four. Also shown in figure four is a typical curve which shows the relationship between  $\lambda$  and  $\alpha$  according to equations (11) and (12). Substituting the expression for  $\sigma$  in equation (11)

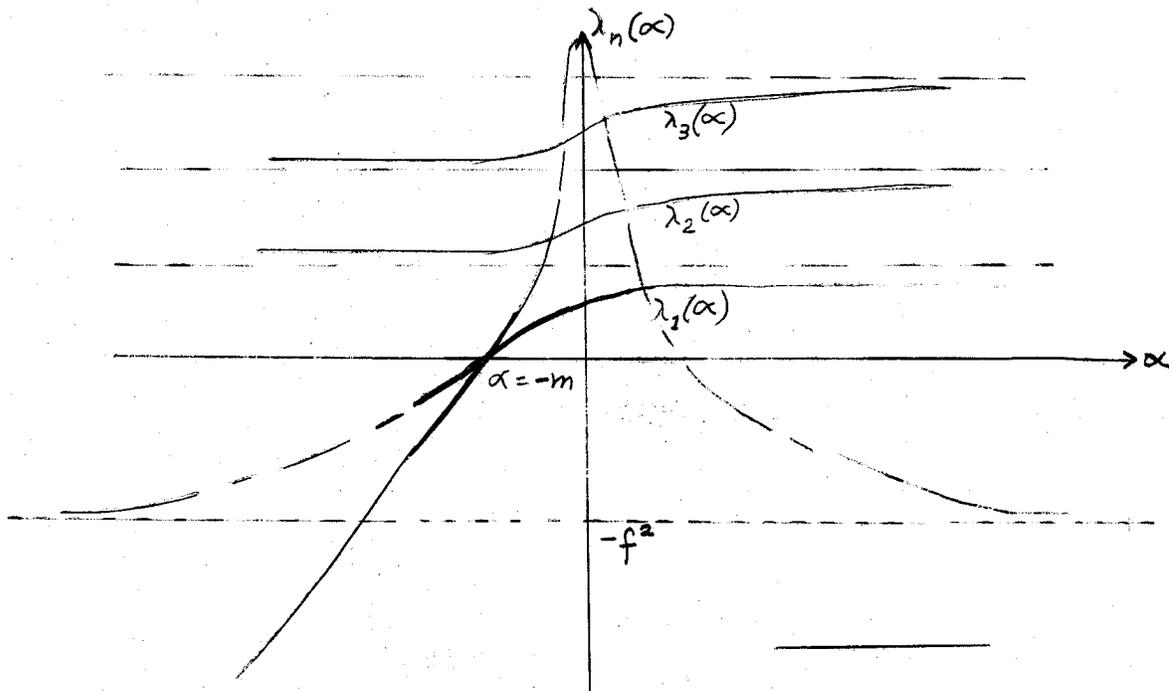


Fig. 4

into equation (12), one obtains

$$\lambda = f^2 \left( \frac{m^2}{\alpha^2} - 1 \right) \quad (23)$$

The intersections of the two sets of curves in figure four determine the eigenvalues for the problem at hand. We are now thinking of  $f$  as a parameter and noting the variation in  $\sigma$  with  $f$  as determined by these intersections. For small values of  $f$ ,  $\sigma^2 \approx \lambda$  and from equation (23) and figure four one can see that the possible

values of  $\lambda$  are approximately equal to the vertical intercepts of the  $\lambda_n(\infty)$  curves (i.e. by equation (12)),

$$\sigma_n \approx K_n^2 H, \quad (24)$$

where  $K_n$  are the roots of  $K J_m'(K)$ . Rewriting equation (12),

one obtains

$$\sigma^2 = f^2 + \lambda \quad (25)$$

Therefore, for all positive values of  $\lambda$ ,  $|\sigma| > |f|$ .

From figure four all  $\lambda_n$ 's are positive for all values of  $f$  ( $f = \frac{\sigma \infty}{m}$ )

except  $\lambda_1$  and are bounded by two successive values of  $K_n^2 H$  where

$K_n$  are the roots of  $-\infty J_m(K)$ . We now have the result that the

$\sigma_n$  curves ( $n > 1$ ) lie outside the wedge  $\sigma = \pm f$  and between two

hyperbolas (see equation (25)) determined by two successive roots

of  $-\infty J_m(K)$ .

The  $\sigma_1$  curve also starts at  $K_1^2 H$  for  $f = 0$ , where  $K_1$  is the first root of  $K J_m'(K)$ , and is bounded from below by the hyperbola corresponding to the first root of  $-\infty J_m(K)$ , but it crosses the wedge.

When  $\lambda_1 = 0$ ,  $\sigma = \pm f$  by equation (25). It can be shown that the

$\sigma_1$  curve is as appears in figure five, where all the above results are shown.

There is a diagram similar to figure five for each value of  $m$ . There will be exactly one  $\sigma$  curve which penetrates the wedge for each value of  $m$ . Each line  $f = \text{const.}$  will intersect some of these curves

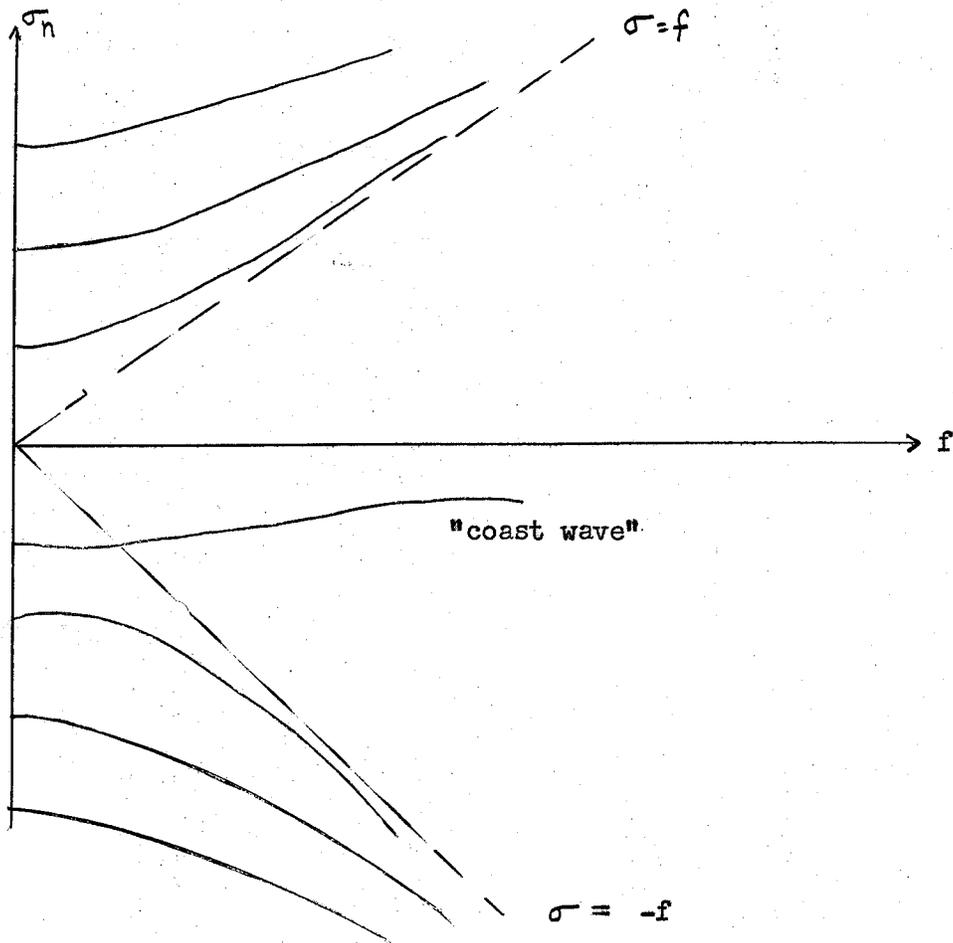


Fig. 5

inside the wedge and some outside the wedge; therefore, as  $f$  is increased the number of "coast waves" is increased.

Variable Depth: The case of a circular lake with a paraboloidal bottom was investigated by Lamb and the case of a depth given by  $H = H_0 + \beta r^2$  was investigated by Veltkamp. Some results for a circular lake with a depth function  $H = H(r)$ , where  $H$  is arbitrary but sufficiently "well-behaved", are as follows:

I. Surface variation independent of  $\theta$ :

For this case the differential equation for the surface variation is

$$\nabla \cdot (H \nabla \zeta) + (\sigma^2 - f^2) \zeta = 0 \quad (26)$$

If  $\sigma_n^{(0)}$  are the eigenvalues for no rotation, then the eigenvalues for the rotational case are given by

$$\sigma_n = \pm \sqrt{[\sigma_n^{(0)}]^2 + f^2} \quad (27)$$

II. Surface variations given by  $\zeta = e^{im\theta} \phi(r)$ :

In this case the differential equation for  $\phi(r)$  can be written

$$(rH\phi')' + \left[ -\frac{m^2}{r} H + \frac{mf}{\sigma} H' + r(\sigma^2 - f^2) \right] \phi = 0 \quad (28)$$

We also require that  $\phi$  be regular at the origin and satisfy the following boundary condition (zero radial velocity at the edge of the lake):

$$\phi'(1) + \frac{mf}{\sigma} \phi(1) = 0 \quad (29)$$

Again we have normalized the radius ( $r = 1$ ) and chosen a system of time units so that  $g = 1$ .

We proceed as before by making the following definitions:

$$\alpha = \frac{mf}{\sigma} \quad (30)$$

$$\lambda = \sigma^2 - f^2 \quad (31)$$

Then equations (28) and (29) become

$$(rH\phi')' + \left[ -\frac{m^2}{r} H + \alpha H' + r\lambda \right] \phi = 0 \quad (32)$$

$$\phi'(1) + \alpha \phi(1) = 0$$

The discussion of this problem will be continued in supplementary notes.

submitted by R. Ellis

The eigenvalues  $\lambda$  are given by the extreme values of the quotient of two integrals, as usual in such Sturm-Liouville problems:

$$\lambda = \frac{\int_0^1 \left\{ r H \phi'^2 + \left( \frac{m^2}{r} H - \alpha H' \right) \phi^2 \right\} dr + \alpha H(1) \phi(1)^2}{\int_0^1 r \phi^2 dr}$$

From this we see that if  $H' < 0$ , (i.e. the lake is deeper in the center), then  $\lambda > 0$  for  $\alpha > 0$  so if  $mf/\sigma > 0$ ,  $\sigma^2 - f^2 > 0$ ; if we take  $m > 0$  this means that there are no waves with  $0 < \sigma < f$ , though the possibility  $-f < \sigma < 0$  is not excluded. Again if  $H' > 0$  (e.g. for a rotating horizontal tank with a flat bottom) we find  $\lambda > 0$  if  $\alpha < 0$  i.e. for  $m > 0$  there are no waves in  $-f < \sigma < 0$  but possibly are in  $0 < \sigma < f$ .

We now investigate the behaviour of  $\lambda$  as a function of  $\alpha$ . If we vary  $\alpha$ , the eigenfunction  $\phi$  changes also, but from the extremal property of the eigenfunctions this part of the variation of the quotient does not affect  $\lambda$  to first order. We thus have:

$$\frac{d\lambda}{d\alpha} = \frac{-\int_0^1 H' \phi^2 dr + H(1) \phi(1)^2}{\int_0^1 r \phi^2 dr}$$

and this holds for each eigenvalue,  $\phi$  being the appropriate eigenfunction. Let us consider from now on the case of a lake which gets shallower toward the edge, i.e.  $H' < 0$ . Then we see that  $\frac{d\lambda}{d\alpha} > 0$ , so that each of the eigenvalues is a monotone increasing function of  $\alpha$ . Using the original formula for  $\lambda$ , the above expression for the derivative can be written:

$$\frac{d\lambda}{d\alpha} = \frac{1}{\alpha} \left\{ \lambda - \frac{\int_0^1 \left[ r H \phi'^2 + \frac{m^2}{r} H \phi^2 \right] dr}{\int_0^1 r \phi^2 dr} \right\}$$

or

$$\frac{d}{d\alpha} \left( \frac{\lambda}{\alpha} \right) = - \frac{1}{\alpha^2} \frac{\int_0^1 [r H \phi'^2 + \frac{m^2}{r} H \phi^2] dr}{\int_0^1 r \phi^2 dr}$$

It is clear from the original formula that all eigenvalues are positive at  $\alpha = 0$ , and since  $\frac{d\lambda}{d\alpha} > 0$  there, this is certainly true also for small positive  $\alpha$ . Let  $\alpha_1$  be a small positive  $\alpha$  for which  $\lambda = \lambda_1 > 0$ . Then since  $\frac{d}{d\alpha} \left( \frac{\lambda}{\alpha} \right) < 0$  we have, for  $\alpha > \alpha_1$ :  $\frac{\lambda(\alpha)}{\alpha} - \frac{\lambda_1}{\alpha_1} < 0$ , or  $\lambda(\alpha) < \frac{\lambda_1}{\alpha_1} \alpha$ . Thus for positive  $\alpha$ ,  $\lambda(\alpha)$  increases, but no faster than linearly. (Actually, usually much more slowly than this). As  $\alpha$  decreases from 0, the various eigenvalues  $\lambda(\alpha)$  also decrease. We can in fact show that they all approach  $-\infty$  as  $\alpha \rightarrow -\infty$  under the assumption  $-H' > K r$ , for some positive number K. For then from the original formula for  $\lambda$  we have ( $\alpha < 0$ )

$$\begin{aligned} \lambda &< \frac{\int_0^1 (r H \phi'^2 + \frac{m^2}{r} H \phi^2) dr}{\int_0^1 r \phi^2 dr} + \alpha K \\ &< \frac{H_0 \int_0^1 [r \phi'^2 + \frac{m^2}{r} \phi^2] dr}{\int_0^1 r \phi^2 dr} + \alpha K \end{aligned}$$

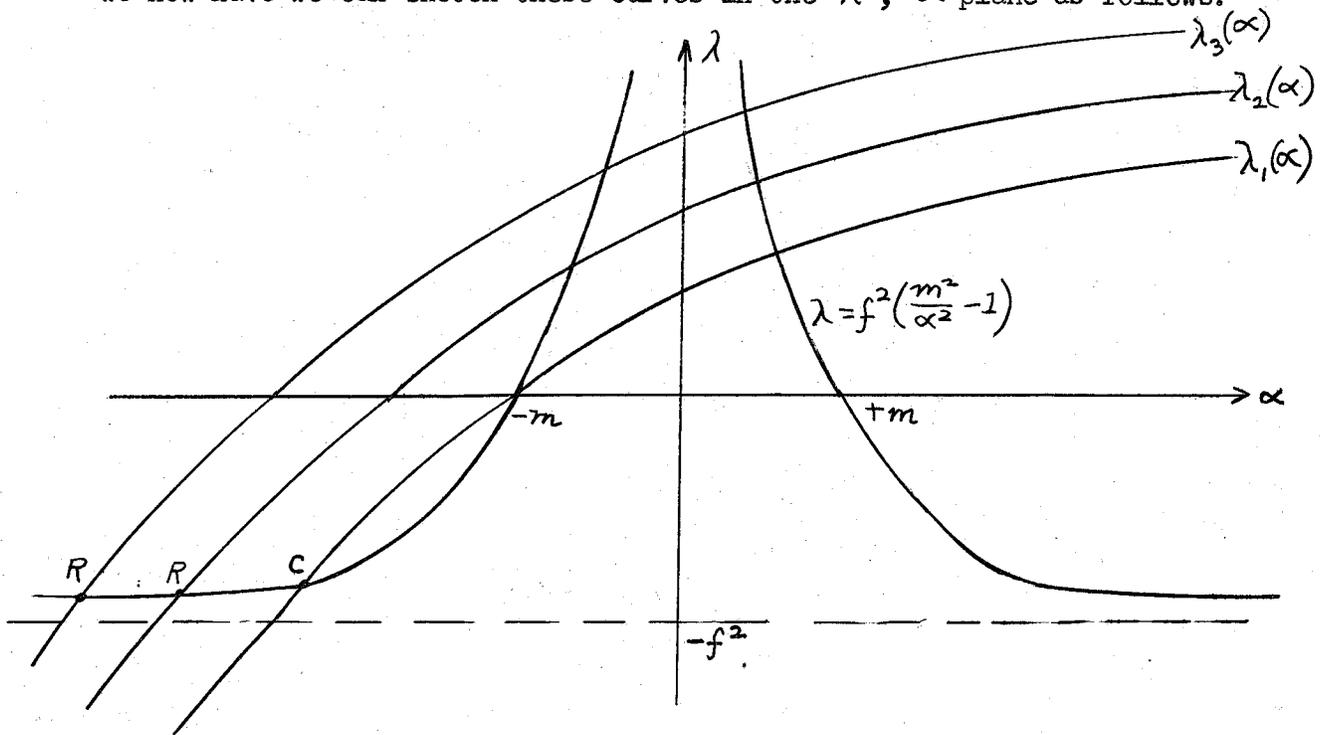
where  $H_0$  is the maximum depth.

Now Courant's minimax principle characterizes the  $n^{\text{th}}$  eigenvalue of the original problem as follows: let  $v_1, \dots, v_{n-1}$  be  $n-1$  fixed functions. Calculate  $\lambda$  from the original quotient of integrals, using a  $\phi$  which is orthogonal to  $v_1, \dots, v_{n-1}$  in the sense  $\int_0^1 r \phi v_k dr = 0$ . Among all such  $\phi$ 's, find the minimum value of  $\lambda$ . Now vary the  $v$ 's, and find the maximum of these minima. This will be the  $n^{\text{th}}$  eigenvalue. Now, in the above inequality for  $\lambda$ , we observe that for the fixed set  $v_1, \dots, v_{n-1}$ , the

left-hand side is always less than the right-hand side, and thus the minimum of the left-hand side is less than the minimum of the right-hand side, for each fixed set  $\nu_1, \dots, \nu_{n-1}$ . But then this also holds for the maxima of these minima as we vary the  $\nu$ 's. But the maximum of the minima on the right-hand side is  $\propto K +$  the  $n^{\text{th}}$  eigenvalue for the problem with constant depth  $H_0$  and no rotation, a certain fixed number  $\lambda_n^{(0)}$ , so we have

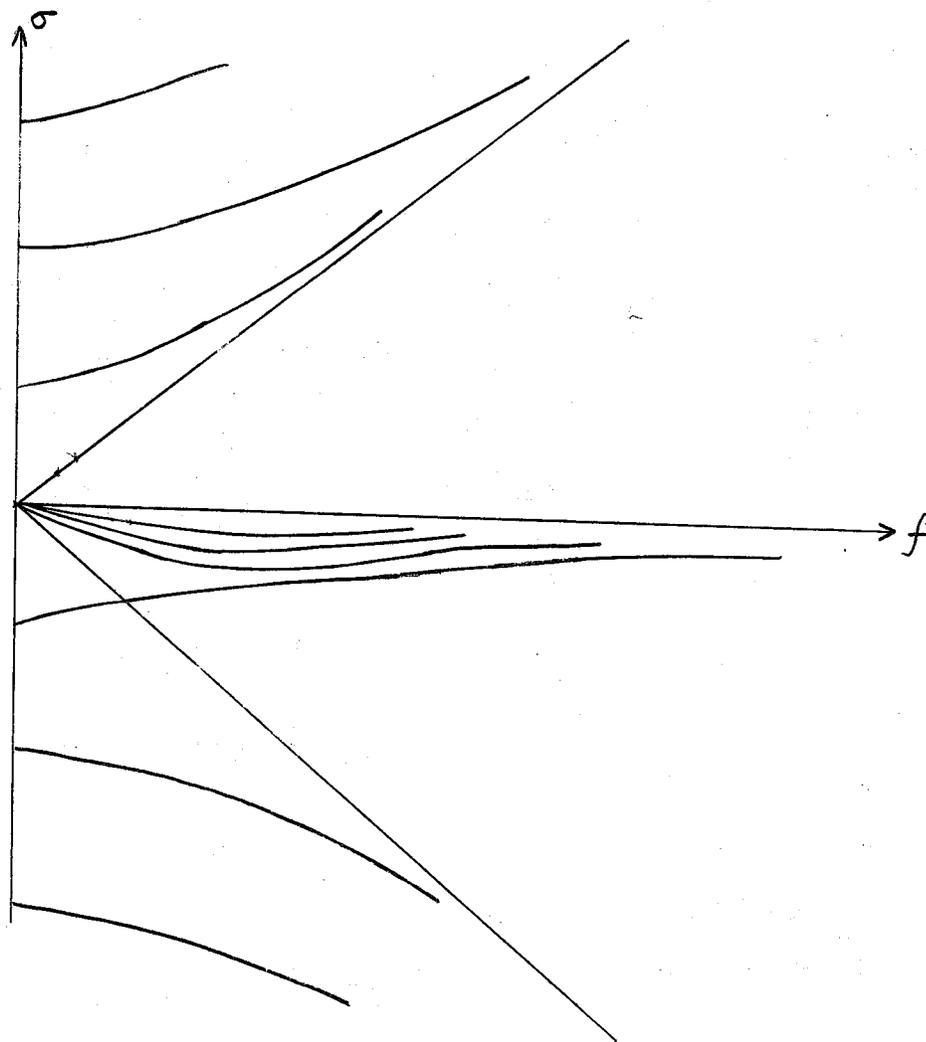
$$\lambda_n(\alpha) < \lambda_n^{(0)} + \alpha K$$

which shows that  $\lambda_n(\alpha) \rightarrow -\infty$  as  $\alpha \rightarrow -\infty$ . To find the  $\sigma$ - $f$  diagram for the problem we must find the intersections of the curves  $\lambda = \lambda(\alpha)$  with  $\lambda = f^2 \left( \frac{m^2}{\alpha^2} - 1 \right)$ . With the information we now have we can sketch these curves in the  $\lambda, \alpha$  plane as follows:



The intersections give the eigenvalues  $\lambda$  and hence the frequencies of the waves. The intersection at  $\lambda = 0, \alpha = -m$ , corresponding to  $\sigma^2 = f^2$

may be extraneous. For investigation of the condition under which it really gives a wave see problem 11. The intersections marked R correspond to "Rossby waves", C is the "coast wave", and the other waves are relatively small variations of the waves present with no rotation. The  $\sigma - f$  diagram is easily obtained qualitatively from the above and is given below.



For an explicit example, see problem 12.

Reference on Courant's minimax principle: Courant-Hilbert,  
"Methods of Mathematical Physics", vol. 1, chap. 6, § 1, 4;  
p.405 in the English edition.

In the last few lectures the motions of incompressible fluids of constant density in rotating systems were considered. Some examples of incompressible fluids with variable density in non-rotating systems will now be considered.

Two Immiscible Layers:

Consider a fluid composed of two immiscible layers, each with constant density but different from that of the other layer, as shown in figure one.

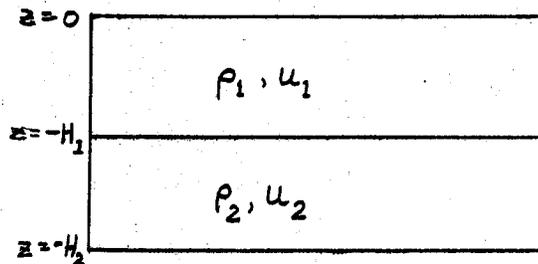


Fig. 1

The equation of the disturbed free surface will be given by

$$z = \zeta_1, \quad (1)$$

and that of the disturbed interface

$$z = -H_1 + \zeta_2 \quad (2)$$

In this case Euler's equations are applicable (see Lecture 2). Thus,

$$\vec{u}_{it} + \nabla \left( \frac{p_i}{\rho_i} + gz \right) = 0, \quad i = 1, 2 \quad (3)$$

$$\nabla \cdot \vec{u}_i = 0 \quad (4)$$

We look for waves in the x-direction with no y variation, i.e., we assume

$$\vec{u}_i = \hat{u}_i(z) e^{i(\sigma t - kx)} \quad (5)$$

The following notations will be used:

$$p_i = p_i^{(0)} + \hat{p}_i e^{i(\sigma t - kx)}, \quad (6)$$

where

$$p_1^{(0)} = -\rho_1 g z \quad (7)$$

$$p_2^{(0)} = \rho_1 g H_1 - \rho_2 g (z + H_1) \quad (8)$$

As usual we write  $\vec{u}_i = u_i \vec{i} + w_i \vec{k} + v_i \vec{j}$

and equation three becomes

$$i \sigma \hat{u}_1 - \frac{ik \hat{p}_1}{\rho_1} = 0 \quad (9)$$

$$i \sigma \hat{u}_2 - \frac{ik \hat{p}_2}{\rho_2} = 0 \quad (10)$$

$$i \sigma \hat{w}_1 + \frac{\hat{p}_1'}{\rho_1} = 0 \quad (11)$$

$$i \sigma \hat{w}_2 + \frac{\hat{p}_2'}{\rho_2} = 0 \quad (12)$$

The equation of continuity (see Lecture 1) states

$$-ik \hat{u}_1 + \hat{w}_1' = 0 \quad (13)$$

$$-ik \hat{u}_2 + \hat{w}_2' = 0 \quad (14)$$

At the surface  $z = \zeta_1$  we must have  $p_1 = 0$  and that the vertical component of the surface velocity equals the vertical component of the particle velocity, i.e.,

$$-\rho_1 g \hat{\zeta}_1 + \hat{p}_1^{(0)} = 0 \quad (15)$$

and 
$$i \sigma \hat{\zeta}_1 = \hat{w}_1^{(0)} \quad (16)$$

In equations (15) and (16)  $\hat{w}_1(\zeta_1)$  and  $\hat{p}_1(\zeta_1)$  have been replaced by  $\hat{w}_1(0)$  and  $\hat{p}_1(0)$ . i.e., we are assuming small disturbances.

At the interface the vertical component of the surface velocity must equal the vertical component of the particle velocity and it must be true that

$$\lim_{z \rightarrow \zeta_2 - H_1} \hat{w}_1 = \lim_{z \rightarrow \zeta_2 - H_1} \hat{w}_2 \quad (17)$$

$$\lim_{z \rightarrow \zeta_2 - H_1} \hat{p}_1 = \lim_{z \rightarrow \zeta_2 - H_1} \hat{p}_2, \quad (18)$$

i.e., the pressure and the velocity are continuous at the interface.

Thus we obtain the following equations:

$$\begin{aligned} & -\rho_1 g(-H_1 + \zeta_2) + \hat{p}_1(-H_1) e^{i(\sigma t - kx)} \\ &= \rho_1 g H_1 - \rho_2 g \zeta_2 + \hat{p}_2(-H_1) e^{i(\sigma t - kx)} \end{aligned} \quad (19)$$

$$\hat{w}_2(-H_1) = i\sigma \zeta_2 = \hat{w}_1(-H_1) \quad (20)$$

Again we have assumed small disturbance at the interface in writing these equations so that, e.g.,  $w_2(-H_1 + \zeta_2)$  is replaced by  $\hat{w}_2(-H)$ .

Equation (19) may be rewritten as

$$(\rho_2 - \rho_1) g \zeta_2 = \hat{p}_2(-H_1) - \hat{p}_1(-H_1) \quad (21)$$

We now have six variables ( $u_i, w_i, p_i$ ), six differential equations and six boundary conditions which can be solved by ordinary

techniques to yield

$$\hat{w}_i'' - K^2 \hat{w}_i' = 0, \quad (22)$$

$$\hat{w}_1'(0) = \frac{gK^2}{\sigma^2} \hat{w}_1(0) \quad (23)$$

$$\frac{\rho_2 \hat{w}_2'(-H_1) - \rho_1 \hat{w}_1'(-H_1)}{\rho_2 - \rho_1} = \frac{gK^2}{\sigma^2} \hat{w}_1(-H_1) = \frac{gK^2}{\sigma^2} \hat{w}_2(-H_1), \quad (24)$$

and 
$$\hat{w}_2(-H_2) = 0 \quad (25)$$

Thus  $\hat{w}_1$  and  $\hat{w}_2$  are given by

$$\hat{w}_1 = A_1 \left[ \frac{gK}{\sigma^2} \sinh kz + \cosh kz \right] \quad (26)$$

$$\hat{w}_2 = A_2 \sinh k(z + H_2) \quad (27)$$

If these are substituted into equations (24), two linear homogeneous equations result. In order for there to be a non-trivial solution the determinant of the system must vanish. This yields

$$\begin{aligned} & (\rho_2 + \rho_1 \tanh KH_1 \tanh KD) \left( \frac{\sigma^2}{gK} \right)^2 \\ & - \rho_2 (\tanh KH_1 + \tanh KD) \frac{\sigma^2}{gK} \end{aligned} \quad (28)$$

$$+ (\rho_2 - \rho_1) \tanh KH_1 \tanh KD = 0, \text{ where } D \equiv H_2 - H_1,$$

which is a quadratic in  $\left(\frac{\sigma^2}{gK}\right)$ . The approximate solutions of this

equation for a few cases will be considered.

Case I: Wavelengths short (or K large) with respect to  $H_1$  and D,

i.e.,  $KH_1 \gg 1$ ,  $KD \gg 1$ .

In this case  $\tanh KH_1 \approx \tanh KD \approx 1$  and equation (28) becomes

$$(\rho_2 + \rho_1) \left( \frac{\sigma^2}{gK} \right)^2 - 2\rho_2 \left( \frac{\sigma^2}{gK} \right) + \rho_2 - \rho_1 = 0, \quad (29)$$

which can be factored into

$$\left[ \frac{\sigma^2}{gK} - 1 \right] \left[ (\rho_2 + \rho_1) \frac{\sigma^2}{gK} - (\rho_2 - \rho_1) \right] = 0 \quad (30)$$

The two solutions of this equation are  $\frac{\sigma^2}{gK} = 1$  and  $\frac{\sigma^2}{gK} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$ .

From equations (16) and (20)

$$\frac{\hat{\zeta}_2}{\hat{\zeta}_1} = \frac{\hat{w}_1(-H_1)}{\hat{w}_1(0)} \quad (31)$$

For  $\frac{\sigma^2}{gK} = 1$  equation (31) becomes

$$\frac{\hat{\zeta}_2}{\hat{\zeta}_1} = e^{-KH_1} \ll 1 \quad (32)$$

Therefore, for  $\sigma^2 = gK$ , the amplitude of the surface wave is much greater than that of the wave at the interface, the so-called internal wave. The two waves are also in phase. For  $\frac{\sigma^2}{gK} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$

$$\begin{aligned} \frac{\zeta_2}{\zeta_1} &= \left[ \cosh KH_1 - \left( \frac{\rho_2 + \rho_1}{\rho_2 - \rho_1} \right) \sinh KH_1 \right] \\ &= \frac{1}{\rho_2 - \rho_1} \left[ \rho_2 e^{-KH_1} - \rho_2 e^{KH_1} \right], \end{aligned} \quad (33)$$

which is  $\gg 1$  since  $KH_1 \gg 1$  and since  $\rho_2 \approx \rho_1$  in most cases. One way to interpret the internal wave is to think of the effective value of the acceleration due to gravity as being  $g \left( \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right)$ . The surface

and interface motions are also out of phase (since  $\rho_2 > \rho_1$ ) for the internal mode.

Case II:  $KD \equiv K(H_2 - H_1) \gg 1$ ,  $KH_1$  not necessarily large.

In this case  $\tanh KD \approx 1$  and equation (28) becomes

$$\begin{aligned} & (\rho_2 + \rho_1 \tanh KH_1) \left( \frac{\sigma^2}{gK} \right)^2 - \rho_2 (1 + \tanh KH_1) \frac{\sigma^2}{gK} \\ & + (\rho_2 - \rho_1) \tanh KH_1 = 0 \quad , \end{aligned} \quad (34)$$

which can be factored into

$$\left( \frac{\sigma^2}{gK} - 1 \right) \left[ \frac{\sigma^2}{gK} - \frac{(\rho_2 - \rho_1) \tanh KH_1}{\rho_2 + \rho_1 \tanh KH_1} \right] = 0 \quad (35)$$

Note that if  $KH_1 \gg 1$ , this reduces to equation (30) as expected.

Also if  $\rho_2 = \infty$  which would correspond to a hard bottom, this reduces to the formula for a lake of finite depth and constant density.

For  $\frac{\sigma^2}{gK} = 1$ , equation (31) becomes

$$\frac{\hat{\xi}_2}{\hat{\xi}_1} = e^{-KH_1} \quad (36)$$

and the two waves are seen to be in phase, though  $\hat{\xi}_2$  is not necessarily small compared to  $\hat{\xi}_1$ .

For  $\frac{\sigma^2}{gK} = \frac{(\rho_2 - \rho_1) \tanh KH_1}{\rho_2 + \rho_1 \tanh KH_1}$ , equation (31) becomes

$$\frac{\hat{\xi}_2}{\hat{\xi}_1} = \frac{-\rho_1}{(\rho_2 - \rho_1)} e^{KH_1} \quad , \quad (37)$$

and the waves are seen to be out of phase. If  $\rho_2$  differs only a little from  $\rho_1$ ,  $\xi_2$  will be large compared to  $\xi_1$ , even if  $KH_1$  is not large.

Case III:  $KH_1 \ll 1$  and  $KD \ll 1$ :

In this case the hyperbolic tangents may be replaced by their arguments and  $\rho_1 K^2 H_1 D$  may be neglected compared to  $\rho_2$ .

Then equation (28) becomes

$$\rho_2 \left( \frac{\sigma^2}{gK} \right)^2 - \rho_2 (KH_2) \frac{\sigma^2}{gK} + (\rho_2 - \rho_1)(KH_1)(KD) = 0, \quad (38)$$

The solution to this equation is given by

$$\frac{\sigma^2}{gK} = \frac{KH_2 \pm \sqrt{(KH_2)^2 - 4(1 - \frac{\rho_1}{\rho_2})(KH_1)(KD)}}{2} \quad (39)$$

If  $\rho_1 \approx \rho_2$ , then  $1 - \frac{\rho_1}{\rho_2} \ll 1$ , and the roots of equation (39)

are nearly

$$\frac{\sigma^2}{gK} = KH_2 \quad (40)$$

$$\frac{\sigma^2}{gK} = \left(1 - \frac{\rho_1}{\rho_2}\right) K \frac{DH_1}{H_2} \quad (41)$$

where these roots are obtained by a binomial expansion and by neglecting second order and higher terms.

Since the phase velocity is given by  $\frac{\sigma}{K}$ , it is seen that there is no dispersion in this case.

For  $\frac{\sigma^2}{gK} = KH_2$  equation (31) becomes

$$\frac{\hat{\xi}_2}{\hat{\xi}_1} \approx \frac{D}{H_2} \quad (42)$$

which implies that the two waves are in phase.

For the other root

$$\frac{\hat{\xi}_2}{\hat{\xi}_1} \approx \frac{-H_2}{D(1 - \frac{\rho_1}{\rho_2})} \quad (43)$$

which implies that the two waves are out of phase, and usually

$$\hat{\xi}_2 \gg \hat{\xi}_1.$$

#### Continuous Variation of Density:

The hydrostatic pressure  $p_0$  for a fluid with continuously varying density  $\rho_0$  is given by

$$p_0(z) = \int_z^0 \rho_0(z) g dz \quad (44)$$

This implies

$$\nabla p_0(z) = -g \rho_0(z) \vec{k} \quad (45)$$

The linearized equations for this case are as follows:

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p + g \rho \vec{k} = 0 \quad (46)$$

$$\nabla \cdot \vec{u} = 0 \quad (47)$$

$$\frac{\partial}{\partial t} (\rho - \rho_0) + \vec{u} \cdot \nabla \rho_0 = 0 \quad (48)$$

Subtracting equation (45) from equation (46),

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla(p - p_0) + g(\rho - \rho_0) \vec{k} = 0 \quad (49)$$

We make the following notational definitions:

$$p - p_0 \equiv \hat{p} e^{i(\sigma t - Kx)} \quad (50)$$

$$\rho - \rho_0 \equiv \hat{\rho} e^{i(\sigma t - Kx)} \quad (51)$$

and look for waves in the x-direction, i.e.

$$u = \hat{u}(z) e^{i(\sigma t - Kx)}, \text{ etc.} \quad (52)$$

The above equations imply

$$(\rho_0 \hat{w}')' - \left[ g \frac{K^2}{\sigma^2} \rho_0' + K^2 \rho_0 \right] \hat{w} = 0 \quad (53)$$

and the boundary conditions are

$$w'(0) = g \frac{K^2}{\sigma^2} \hat{w}(0) \quad (54)$$

$$\hat{w}(-H) = 0 \quad (55)$$

It can be shown that if  $H = \infty$ , then  $\hat{w} = e^{Kz}$  and

$\frac{\sigma^2}{gK} = 1$  is a solution. This solution is the surface wave. If

$H \neq \infty$  there is also a "surface mode", but we cannot give it explicitly until  $\rho_0$  is specified.

If  $\alpha \equiv \frac{gK^2}{\sigma^2}$ , then the variational formulation of the

Sturm-Liouville problem states that

$$-K^2 = \frac{\int_{-H}^0 (\rho_0 \hat{w}'^2 + \alpha \rho_0' \hat{w}^2) dz - \alpha \rho_0(0) \hat{w}^2(0)}{\int_{-H}^0 \rho_0 \hat{w}^2 dz} \quad (56)$$

If the two terms involving  $\alpha$  were absent, then the right-hand side of equation (56) would be positive and no (negative) eigenvalues would exist. The term  $\alpha \rho_o(0) \hat{w}^2(0)$  may be regarded as responsible for the surface wave, the other term as responsible for the internal waves.

We now determine how  $K$  varies as  $\alpha$  varies. By the extremal properties of  $-K^2$  there is no change in  $-K^2$  as  $\hat{w}$  varies; therefore, the total derivative of  $-K^2$  with respect to  $\alpha$  is equal to the partial derivative of  $-K^2$  with respect to  $\alpha$ , i.e.,

$$-\frac{dK^2}{d\alpha} = \frac{\int_{-H}^0 \rho_o' w^2 dz - \rho_o(0) \hat{w}^2(0)}{\int_{-H}^0 \rho_o \hat{w}^2 dz} \quad (57)$$

If  $\rho_o > 0$  and  $\rho_o' < 0$ , then  $\frac{dK^2}{d\alpha} \geq 0$  which implies that the phase speed is an increasing function of wavelength.

Groen showed that the period is an increasing function of wavelength. We get this result by a different method as follows:

$$-\frac{dK^2}{d\alpha} = \frac{1}{\alpha} \left( -K^2 - \frac{\int_{-H}^0 \rho_o \hat{w}'^2 dz}{\int_{-H}^0 \rho_o \hat{w}^2 dz} \right) \quad (58)$$

This follows directly from equations (56) and (57) and can be rewritten as

$$\frac{\alpha}{K^2} \frac{dK^2}{d\alpha} = 1 + \frac{\int_{-H}^0 \rho_o \hat{w}'^2 dz}{K^2 \int_{-H}^0 \rho_o \hat{w}^2 dz} \quad (59)$$

$$\text{But } \frac{\alpha}{K^2} \frac{dK^2}{d\alpha} = \frac{d \log K^2}{d \log \alpha}, \quad (60)$$

and since  $\alpha = \frac{gK^2}{\sigma^2}$  we can write

$$\log \alpha = \log g + \log K^2 - \log \sigma^2 \quad (61)$$

$$\text{which implies } \frac{d \log \alpha}{d \log K^2} = 1 - \frac{d \log \sigma^2}{d \log K^2} = 1 - \frac{d \log \sigma}{d \log K} \quad (62)$$

These equations imply

$$\frac{d \log \sigma^2}{d \log K^2} = \frac{\int_{-H}^0 \rho_0 \hat{w}'^2 dz}{K^2 \int_{-H}^0 \rho_0 \hat{w}^2 dz + \int_{-H}^0 \rho_0 \hat{w}'^2 dz} \quad (63)$$

which clearly shows

$$0 \leq \frac{d \log \sigma}{d \log K} \leq 1 \quad (64)$$

This implies that  $\frac{d\sigma}{dK} \geq 0$  or that the period is an increasing function of wavelength.

$$\frac{d \log \sigma}{d \log K} \leq 1 \Rightarrow \log \frac{\sigma}{\sigma_0} \leq \log \frac{K}{K_0} \Rightarrow \sigma \leq \frac{\sigma_0}{K_0} K$$

so  $\sigma$  cannot grow faster than linearly with  $K$ . Actually it grows much more slowly.

Groen also showed that there exists an upper limit to  $\sigma$  for internal waves. If we set  $\hat{w}(0) = 0$ , then the surface wave will be suppressed and we have only internal waves. Then

$$K^2 = \frac{\int_{-H}^0 -\rho_0' \hat{w}^2 dz - \int_{-H}^0 \rho_0 \hat{w}^2 dz}{\int_{-H}^0 \rho_0 \hat{w}^2 dz} \quad (65)$$

$$\leq \frac{\int_{-H}^0 -\rho_0' \hat{w}^2 dz}{\int_{-H}^0 \rho_0 \hat{w}^2 dz}$$

since  $\rho_0 > 0$ . This can be written

$$K^2 \leq \frac{\int_{-H}^0 -\frac{\rho_0'}{\rho_0} \rho_0 \hat{w}^2 dz}{\int_{-H}^0 \rho_0 \hat{w}^2 dz}$$

$$\leq \alpha \max \left( -\frac{\rho_0'}{\rho_0} \right) = \frac{gK^2}{\sigma^2} \max \left( -\frac{\rho_0'}{\rho_0} \right)$$

This implies  $\sigma^2 \leq g \max \left( -\frac{\rho_0'}{\rho_0} \right)$  Q.E.D. (66)

For a surface wave  $\sigma^2 = gK$ . These results are shown in figure two

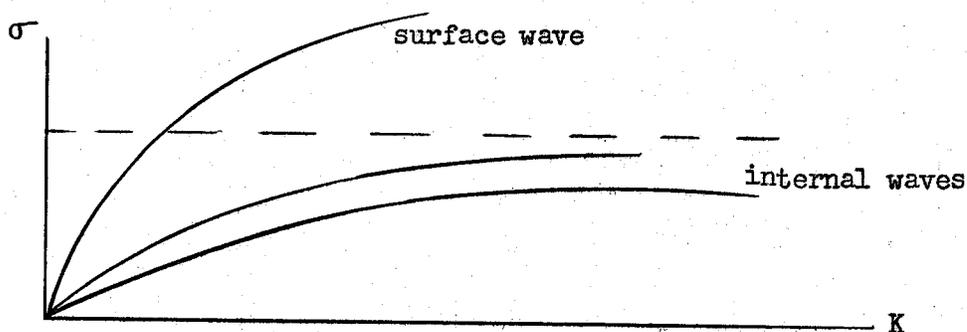


Fig. 2

Hydrodynamical Stability:

Let us consider the flow of a non-viscous fluid of uniform density between two plates with no external forces acting. Let the basic velocity  $\vec{u}_0$  be given by

$$\vec{u}_0 = \bar{u}(y) \vec{i} \quad (67)$$

We want to consider what happens to the fluid if the motion is perturbed. If we let  $\vec{u}$  be the perturbed velocity, then the linearized equation of motion is

$$\frac{\partial \vec{u}}{\partial t} + \bar{u}(y) \frac{\partial \vec{u}}{\partial x} + v \bar{u}'(y) \vec{i} + \nabla \frac{p}{\rho} = 0, \quad (68)$$

where  $v \equiv \vec{u} \cdot \vec{j}$ , the component of  $\vec{u}$  in the positive y-direction.

The equation of continuity reduces to

$$\nabla \cdot \vec{u} = 0 \quad (69)$$

and the boundary conditions are

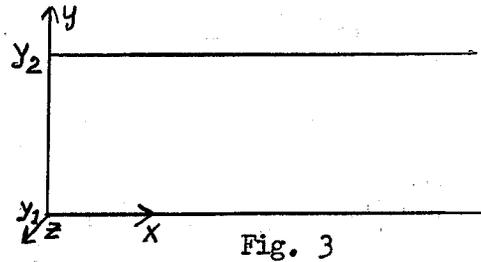
$$v(y_1) = v(y_2) = 0, \quad (70)$$

where  $y_1$  and  $y_2$  are the ordinates of the two plates.

The Rayleigh approach is to look for a solution with an exponential time dependence, i.e.,

$$u = e^{i(Kx + lz - \sigma t)} \quad (71)$$

If  $\sigma$  has a positive imaginary part, then the perturbation will grow exponentially.



If equation (71) is substituted into (68) and (69), the following equations result:

$$u_t + \bar{u} iku + v\bar{u}' + ik \frac{p}{\rho} = 0 \quad (70)$$

$$v_t + \bar{u} ikv + \frac{p_y}{\rho} = 0 \quad (71)$$

$$w_t + \bar{u} ikw + i\ell \frac{p}{\rho} = 0 \quad (72)$$

$$iku + i\ell w + v_y = 0 \quad (73)$$

Combining equations (70) and (72) yields

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right) \left(\frac{ku + \ell w}{\sqrt{k^2 + \ell^2}}\right) + \frac{k}{\sqrt{k^2 + \ell^2}} v\bar{u}' + i\sqrt{k^2 + \ell^2} \frac{p}{\rho} = 0 \quad (74)$$

Rewriting equations (71) and (73) yields

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right) v + \frac{p_y}{\rho} = 0 \quad (75)$$

$$i(ku + \ell w) + v_y = 0 \quad (76)$$

If we make the following definitions:

$$U \equiv \frac{ku + \ell w}{\sqrt{k^2 + \ell^2}} \quad (77)$$

$$\alpha^2 = k^2 + \ell^2 \quad (78)$$

$$p^* = \frac{\alpha}{k} p \quad (79)$$

$$t^* = \frac{k}{\alpha} t \quad (80)$$

then equations (74), (75) and (76) can be written

$$\left( \frac{\partial}{\partial t^*} + i\alpha \bar{u} \right) U + \bar{v} U' + i\alpha \frac{p^*}{\rho} = 0 \quad (81)$$

$$\left( \frac{\partial}{\partial t^*} + i\alpha \bar{u} \right) v + \frac{p y^*}{\rho} = 0 \quad (82)$$

$$i\alpha U + v_y = 0 \quad (83)$$

These equations are seen to be the same as equations (70) through (73) with  $w = 0$ ,  $u = U$ ,  $t = t^*$ ,  $p = p^*$ ,  $k = \alpha$  ( $\alpha$  is called the total wave number). Thus the three dimensional problem has been reduced to a two dimensional problem (since  $w = 0$ ). This is "Squire's Theorem" which also holds when viscosity is included. We see that if the starred equations are unstable, then so are the previous equations although a longer time is required in  $t$  than in  $t^*$ , i.e. the most unstable wave is two-dimensional.

If  $v = \hat{v}(y) e^{i\alpha(x-ct)}$ , then the Rayleigh Equation results:

$$(\bar{u} - c)(\hat{v}'' - \alpha^2 \hat{v}) - \bar{u}'' \hat{v} = 0 \quad (84)$$

Here we are considering  $\alpha$  as being fixed and  $c$  as being the eigenvalue although this is not an ordinary eigenvalue problem.

Since  $w = 0$ , it does not appear in the equation and we are free to assign it a new meaning. We make the following definitions, to simplify writing and follow a fairly standard terminology:

$$w(y) \equiv \bar{u}(y) \quad (85)$$

$$\hat{v} \equiv -i\alpha \phi \quad (86)$$

Equation (84) may then be written as

$$(w - c)(\phi'' - \alpha^2 \phi) - w'' \phi = 0$$

References:

1. Veronis & Stommel (1956), Journal of Marine Research, 15, No.1, 43-75.
  - \*2. Groen (1948), Physica 14, 294-300.
  - \*3. Groen & Heyna (1958), Physica 24, 383-389.
  4. Rayleigh, Theory of Sound, Chapter 21.
  5. Squire (1933), Proc.Roy.Soc. A, 142, 621-628.
- \*See also Kon.Ned.Met.Inst.Medeled Verh. E11 (1948).

submitted by R. Ellis

Rayleigh Equation

The equation governing the instability of parallel inviscid flow was found in the last lecture to be:

$$(\omega - c) \{ \phi'' - \alpha^2 \phi \} - \omega'' \phi = 0 \quad (1)$$

with associated boundary conditions for rigid walls at  $y_1$  and  $y_2$ :

$$\phi(y_1) = \phi(y_2) = 0 \quad (2)$$

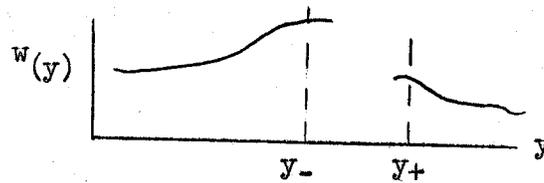
$\phi$  is the stream function and the velocity amplitude in  $y$  is related to  $\phi$  by the relation

$$\hat{u} = -i \alpha \phi$$

and  $\omega$  is the steady flow whose stability we are investigating.

In equation (1) we regard  $c$  as the eigenvalue and think of  $\alpha$  as a known parameter. It is the object of the stability analysis to determine the function  $c = c(\alpha)$ .

In certain circumstances it is beneficial to approximate  $\omega(y)$  in equation (1) as a profile of piecewise linear segments to simplify the analysis. The difficulty encountered in this procedure is associated with the singularities of  $\omega''(y)$  at the corners of the approximate profile. To escape this difficulty we would like to restrict the range of the equation to each linear interval separately and then join solutions at the corners. It is therefore necessary to investigate the jump conditions on the solutions across the discontinuity in the profile. Consider the profile shown below, with a discontinuity at some  $y$ . Let  $y_-$  be to the left of the discontinuity and  $y_+$  be to the right.



Equation (1) may be rewritten as

$$\left[ (w-c)\phi' - w'\phi \right]' - \alpha^2(w-c)\phi = 0 \quad (3)$$

Integrating this equation across the discontinuity yields

$$(w-c)\phi' - w'\phi \Big|_{y-}^{y+} = \alpha^2 \int_{y-}^{y+} (w-c)\phi \, dy \quad (4)$$

If we regard  $\phi$  as bounded in the interval the right hand side of (4) goes to zero as we let  $y_+$  approach  $y_-$ . The jump condition becomes:

$$(w-c)\phi' - w'\phi \text{ is continuous.} \quad (5)$$

This equation is identical with that obtained by integrating the momentum equation for  $\hat{u}$  in the  $y$  direction and demanding continuity of pressure across the discontinuity in  $w(y)$ .

Therefore

$$(w-c)\phi' - w'\phi = \chi; \quad \chi \text{ is a continuous function of } y. \quad (6)$$

Rewriting equation (6):

$$\left( \frac{\phi}{w-c} \right)' = \frac{\chi}{(w-c)^2} \quad (7)$$

Again integrating across the discontinuity:

$$\frac{\phi}{w-c} \Big|_{y-}^{y+} = \int_{y-}^{y+} \frac{\chi}{(w-c)^2} \, dy \quad (8)$$

The right hand side is equal to zero if  $\text{Im } c \neq 0$  when  $y_+ \rightarrow y_-$ .

The second jump condition becomes:

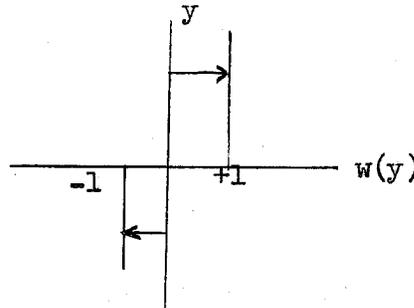
$$\frac{\phi}{w-c} \text{ is continuous} \quad (9)$$

Thus condition is essentially the conservation of streamline tangency across the discontinuity of  $w$ .

### Helmholtz - Instability

As an example of the above development we will investigate the flow

$$w(y) = \text{sgn } y = \frac{y}{|y|}$$



For  $y < 0$  equation (1) becomes

$$(-1-c)(\phi'' - \alpha^2 \phi) = 0 \quad (10)$$

if  $c \neq -1$  (we are looking for unstable solutions with  $\text{Im } c \neq 0$ ) we obtain as a solution to (10)

$$\phi = Ae^{\alpha y} \quad (11)$$

The solution  $e^{-\alpha y}$  is disregarded because it increases without bound as  $y$  goes to  $-\infty$ .

For  $y > 0$

$$(1-c) \{ \phi'' - \alpha^2 \phi \} = 0$$

With the solution

$$\phi = Be^{-\alpha y} \quad (12)$$

$$c \neq 1$$

Using the continuity of pressure and streamline slope { equations

(5) and (9) } we obtain

$$\frac{A}{-1-c} = \frac{B}{1-c} \quad \text{from (9)} \quad (13)$$

$$(-1-c)(\alpha A) = (1-c)(-\alpha B) \quad \text{from (5)} \quad (14)$$

Dividing (14) into (13) yields:

$$(1+c)^2 = -(1-c)^2 \quad (15)$$

Therefore

$$c = \pm i$$

The instability mode thus grows with no propagation as might be guessed by the original symmetry of  $w(y)$ . The growth of  $\phi$  goes as  $e^{\alpha t}$  since

$$\phi = \phi(y) e^{i\alpha(x-ct)}$$

#### Plane Couette Flow

As another example we investigate the stability of the flow

$$w(y) = y \quad -1 \leq y \leq 1$$

Equation (1) becomes:

$$(y-c) \{ \phi''' - \alpha^2 \phi \} = 0 \quad (16)$$

If  $c$  is not in  $[-1, 1]$

$$\phi'' - \alpha^2 \phi = 0 \quad (17)$$

There are no solutions of (17) which satisfy the boundary conditions  $\phi(1) = \phi(-1) = 0$ . The only other alternative is that  $y = c$  at some point in  $[-1, 1]$ . In either case there exists no solution to (16) which is unstable.

To find the eigenfunctions of (16) when  $c$  is in  $[-1, 1]$  we consider the equivalent equation

$$\phi'' - \alpha^2 \phi = \delta(y-c) \quad (18)$$

The solution of this equation is the associated Green's function:

$$\begin{aligned} \phi &= K \sinh \alpha(y+1) \sinh \alpha(c-1) & y < c \\ &= K \sinh \alpha(y-1) \sinh \alpha(c+1) & y > c \end{aligned}$$

where 
$$K = \frac{1}{\alpha \sinh \alpha}$$

We now investigate some general properties of equation (1).

I. Imaginary part of  $c \neq 0$ .

Equation (1) is

$$(w-c) \{ \phi'' - \alpha^2 \phi \} - w'' \phi = 0 \quad (19)$$

The complex conjugate equation is:

$$(w-\bar{c}) \{ \bar{\phi}'' - \alpha^2 \bar{\phi} \} - w'' \bar{\phi} = 0 \quad (20)$$

Where the bar denotes complex conjugation. If we associate  $\bar{\phi}$  with some function  $\psi$  then  $\psi$  will be a solution of (1) with eigenvalue  $\bar{c}$ . In other words if there exists a solution which decays it is accompanied by a solution increasing exponentially in time.

If we multiply (19) by  $\phi$  and integrate over the interval:

$$\int_{y_1}^{y_2} \bar{\phi} \phi'' - \alpha^2 \bar{\phi} \phi - \frac{w''}{w-c} \phi \bar{\phi} = 0 \quad (21)$$

If we integrate the first term by parts we obtain:

$$\int_{y_1}^{y_2} \bar{\phi} \phi'' = \bar{\phi} \phi' \Big|_{y_1}^{y_2} - \int_{y_1}^{y_2} \bar{\phi}' \phi' = - \int_{y_1}^{y_2} \bar{\phi}' \phi'$$

$$\text{since } \bar{\phi}(y_1) = \bar{\phi}(y_2) = 0$$

Then

$$\int_{y_1}^{y_2} \left\{ |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{w''(w-\bar{c})}{|w-c|^2} |\phi|^2 \right\} dy = 0 \quad (22)$$

Writing  $c = c_r + ic_i$  and equating real and imaginary parts of (22) to zero separately we obtain

$$\int_{y_1}^{y_2} \left\{ |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{w''(w-c_r)}{|w-c|^2} |\phi|^2 \right\} = 0 \quad (23)$$

$$c_i \int_{y_1}^{y_2} \frac{w''}{|w-c|^2} |\phi|^2 dy = 0 \quad (24)$$

If (24) is to vanish then clearly  $w''$  must change sign somewhere in the interval so that for  $c_i \neq 0$   $w''$  must be zero somewhere in the interval which leads to Rayleigh's Theorem, which states that:

The existence of an inflection point of  $w(y)$  in  $[y_1, y_2]$

is a necessary condition for instability.

Tollmien has shown that if the profile is even in  $y$  or is a Boundary Layer type profile then this condition is also sufficient. (See Lin "Theory of Hydrodynamic Stability" Paragraph 82).

Equation one may be rewritten since

$$\left\{ (w-c)^2 \left( \frac{\phi}{w-c} \right)' \right\}' = \left\{ (w-c) \phi' - w' \phi \right\}' = (w-c) \phi'' - w'' \phi$$

Equation (1) is then

$$\left\{ (w-c)^2 \left( \frac{\phi}{w-c} \right)' \right\}' = \alpha^2 (w-c)^2 \frac{\phi}{w-c} \quad (25)$$

multiplying by  $\frac{\bar{\phi}}{w-\bar{c}}$  and integrating over the interval:

$$\int_{y_1}^{y_2} \frac{\bar{\phi}}{w-\bar{c}} \left\{ (w-c)^2 \left( \frac{\phi}{w-c} \right)' \right\}' dy = \int_{y_1}^{y_2} \alpha^2 (w-c)^2 \left| \frac{\phi}{w-c} \right|^2 dy \quad (26)$$

Integrating the left hand side by parts and using the vanishing of  $\bar{\phi}$  at  $y_1$  and  $y_2$ , equation (26) becomes

$$-\int_{y_1}^{y_2} \left| \left( \frac{\phi}{w-c} \right)' \right|^2 (w-c)^2 dy = \alpha^2 \int_{y_1}^{y_2} (w-c)^2 \left| \frac{\phi}{w-c} \right|^2 dy \quad (27)$$

$$\text{writing } (w-c)^2 = (w-c_r)^2 - c_i^2 - 2ic_i (w-c_r)$$

we equate the real and imaginary parts separately and equation (27) yields the two equations

$$\int_{y_1}^{y_2} dy \left\{ \left| \left( \frac{\phi}{w-c} \right)' \right|^2 + \alpha^2 \left| \frac{\phi}{w-c} \right|^2 \right\} \left\{ (w-c_r)^2 - c_i^2 \right\} = 0 \quad (28)$$

$$c_i \int_{y_1}^{y_2} dy \left\{ \left| \left( \frac{\phi}{w-c} \right)' \right|^2 + \alpha^2 \left| \frac{\phi}{w-c} \right|^2 \right\} \left\{ w - c_r \right\} = 0 \quad (29)$$

From equation (29) we see that  $c_r$  must lie in the range of  $w$  and we see that equation (28) places a restriction on the magnitude of  $c_i$ , since  $(w-c_r)^2 - c_i^2$  must take on both signs in  $(y_1, y_2)$ .

For an odd profile in a symmetric interval we can indicate that  $c$  will probably be pure imaginary. One argument is by symmetry; i.e. that there is no preferred direction for wave propagation. Or we may say that

$$\text{Since } w(-y) = -w(y)$$

if  $\phi$  satisfies:

$$\phi'' - \alpha^2 \phi - \frac{w''}{w-c} \phi = 0 \quad (30)$$

then  $\chi = \phi(-y)$  is a solution if we change the sign of  $c$ . Now the complex conjugate of  $\phi$  satisfies (30) for another  $c$  namely  $\bar{c}$ . Arguing from the uniqueness of  $c(\alpha)$  we may infer that  $\psi = \bar{\phi}$  so that  $-c = \bar{c}$  which means  $c_r = 0$ .

If  $w$  is odd and  $-a \leq y \leq a$  and  $c$  is pure imaginary and  $w''(0) = 0$  is the only inflection point in this interval we can infer the following: For  $c = ic_i$  equation (23) may be written as:

$$\int_{y_1}^{y_2} \left\{ |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{w''w}{w^2 + |c|^2} |\phi|^2 \right\} dy = 0 \quad (31)$$

This clearly cannot be satisfied if  $w''w \geq 0$  in the complete interval.

We conclude that  $w''w \leq 0$  to be unstable.

To find a condition on the magnitude of  $w''w$  to admit instability for odd flows we will use the relation

$$\int_{-a}^a dy |\phi'|^2 \geq \frac{\pi^2}{4a^2} \int_{-a}^a dy |\phi|^2 \quad (32)$$

which holds for any  $\phi$  of class  $C^1$  which vanishes at  $\pm a$ .

Then using (31) and (32)

$$0 = \int_{-a}^a dy \left\{ |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{w''w}{w^2 + |c|^2} |\phi|^2 \right\} \geq \int_{-a}^a \left( \frac{\pi^2}{4a^2} + \frac{w''w}{w^2} \right) |\phi|^2 dy \quad (33)$$

or

$$\geq \left\{ \frac{\pi^2}{4a^2} + \text{Min} \frac{w''}{w} \right\} \int |\phi|^2 dy$$

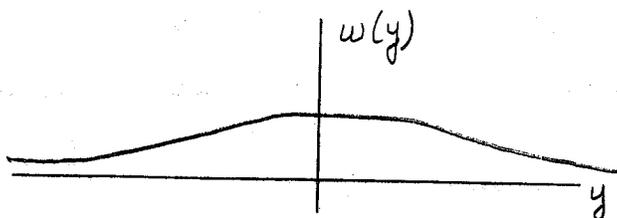
$$0 \geq \frac{\pi^2}{4a^2} + \text{Min} \frac{w''}{w}$$

Thus for instability not only must  $\frac{w''}{w}$  be negative, but  $\max(-\frac{w''}{w})$  must exceed  $\frac{\pi^2}{4a^2}$ .

### Bickley Jet

Consider the flow

$$w(y) = \text{sech}^2 y$$



For this flow

$$\begin{aligned} w' &= -2 \operatorname{sech}^2 y \tanh y \\ w'' &= 4 \operatorname{sech}^2 y \tanh^2 y - 2 \operatorname{sech}^4 y \\ &= 4 \operatorname{sech}^2 y - 6 \operatorname{sech}^2 y \end{aligned}$$

The inflexion point  $w''(y) = 0$  occurs at  $\operatorname{sech}^2 y = 2/3$  i.e. where  $w = 2/3$ .

The result of Tollmien and Lin about the existence of instability when there is an inflection point in certain types of profile gives in fact the existence of a neutrally stable solution with

$$c = w(y_s); \quad w''(y_s) = 0$$

We illustrate this with the present example, setting

$$c = w(y_s) = 2/3.$$

Equation (1) becomes:

$$\left\{ \operatorname{sech}^2 y - 2/3 \right\} \left\{ \phi'' - \alpha^2 \phi \right\} - \left\{ -6 \operatorname{sech}^2 y \right\} \left\{ \operatorname{sech}^2 y - 2/3 \right\} \phi = 0 \quad (35)$$

Rewriting (35)

$$\phi'' - \alpha^2 \phi + 6 \operatorname{sech}^2 y \phi = 0$$

or

$$\phi'' = -6w\phi + \alpha^2 \phi \quad (36)$$

A solution to (36) is  $\phi = \operatorname{sech}^2 y$ ;  $\alpha = 2$ . Note that  $\phi = w$  here. In general this is not the case, as we see by trying  $w$  in eqn. (1):

$$(w-c) \left\{ w'' - \alpha^2 w \right\} - w''w = 0$$

$$-cw'' - \alpha^2(w-c)w = 0$$

which just happens to be satisfied for the  $\text{sech}^2 y$  profile with

$\alpha = 2$ ,  $c = 2/3$ . Another solution is

$\alpha = c = 0$ ,  $\phi = w$ . This holds in general. In a more general case the trivial solution  $\alpha = 0$  is admissible if  $\phi$  is taken equal to  $w-c$  and  $(w-c)_{y_1} = (w-c)_{y_2} = 0$ . Now it is always possible to find the

second solution from the first trivial solution for the second order equation (1). It therefore seems worthwhile to expand solutions around  $\alpha = 0$  to facilitate solving (1). The significance of the  $\alpha = 0$  solution  $\phi = w-c$  is that for  $\alpha = 0$ :

$$u = w(y) + A \phi'(y) = w(y) + Aw'(y) \quad (37)$$

$$v = 0 + 0$$

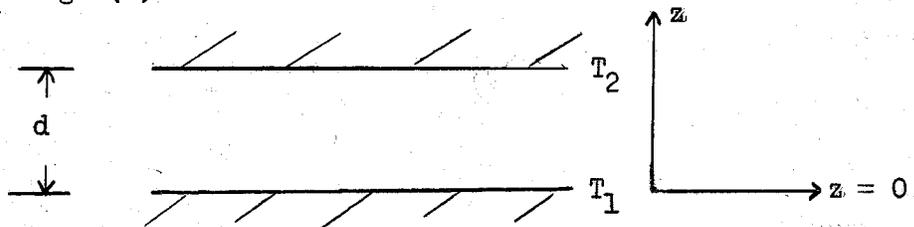
utilizing the fact that  $\phi$  is a stream function of the perturbation and that  $\phi = w-c$  for  $\alpha = 0$ . Then it is clear that this perturbation solution is just a shifting of the basic flow in  $y$  from  $y$  to  $y + A$  as shown from (37) since

$$u = w(y) + Aw'(y) = w(y + A) \text{ to first order.}$$

submitted by J. Pedlosky  
and Bob Ellis

Thermal Convective Instability

We wish to investigate the stability of a layer of fluid subjected to a linear temperature gradient. Consider the situation depicted in Fig. (1).



A layer of fluid of depth  $d$  is kept at temperatures  $T_1$  and  $T_2$  at  $z = 0$  and  $d$  respectively. We wish to consider the stability of the static solution of the hydrodynamical equation with respect to a perturbation in the temperature field.

We will utilize the Boussinesq equations derived in lecture #3. They are, after linearization to investigate stability to infinitesimal disturbances ( $\vec{u}$ ,  $p$ ,  $\theta$ ):

$$\frac{\partial \vec{u}}{\partial t} + \frac{\nabla p}{\rho_0} - \alpha g \theta \vec{k} = \nu \Delta \vec{u} \quad (1)$$

$$\nabla \cdot \vec{u} = 0 \quad (2)$$

$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla T_0 = K \Delta \theta \quad (3)$$

where

$$T_0(z) = T_1 - \beta z = \text{initial, steady temperature}$$

$$\alpha = \text{expansion coefficient of the medium}$$

$$\theta = \text{temperature perturbation}$$

The boundary conditions fall into two classes. For the case where the

planes  $z = 0$ ,  $z = d$  are rigid walls, the boundary conditions become:

$$\text{Rigid walls} \quad \theta(0) = \theta(d) = 0 \quad (4)$$

$$\vec{u}(0) = \vec{u}(d) = 0$$

For the case where the planes  $z = 0$ ,  $z = d$  support no tangential stress (so-called "Free" boundaries) the boundary conditions are:

(The free boundaries are impermeable)

$$\theta(0) = \theta(d) = 0 \quad (5)$$

$$\text{Free boundary} \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad \text{on } z = 0, d \quad (6)$$

$$w(0) = w(d) = 0$$

To reduce the number of variables under consideration we form the curl of equation (1), to obtain

$$\frac{\partial}{\partial t} \text{curl } \vec{u} - \alpha g \text{grad } \theta \times \vec{k} = \nu \Delta \text{curl } \vec{u} \quad (7)$$

We note that the vertical component of the vorticity satisfies a pure diffusion equation. To be able to reduce the number of variables to a more tractable scale we take the curl now of equation (7) obtaining:

$$\frac{\partial}{\partial t} (-\Delta \vec{u}) - \alpha g \left\{ \nabla \theta_z - \Delta \theta \vec{k} \right\} = \nu \Delta \left\{ -\Delta \vec{u} \right\} \quad (8)$$

If we consider the vertical component only:

$$\frac{\partial}{\partial t} \Delta w - \alpha g \left\{ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \theta \right\} = \nu \Delta \Delta w \quad (9)$$

The energy equation (3) becomes

$$\frac{\partial \theta}{\partial t} - \beta w = \kappa \Delta \theta \quad (10)$$

The boundary conditions become: for rigid walls:

$$w(0) = w(d) = 0 \quad (11)$$

and since  $u$  and  $v$  vanish at every point in the planes  $z = 0$  and  $d$  then:

$$\text{at } z = 0, d \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{and therefore from the continuity}$$

equation

$$-\frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{on } z = 0, d \quad (12)$$

For free boundaries equation (11) is also true but equation (12) is replaced by the condition of absence of shear on the boundaries; from continuity:

$$\frac{\partial^2 w}{\partial z^2} = -\frac{\partial}{\partial x} \frac{\partial u}{\partial z} - \frac{\partial}{\partial y} \frac{\partial v}{\partial z} = 0 \quad (13)$$

$z = 0, d$

To solve equations 9 and 10 we expand in a set of eigenfunctions of the equation:

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} U_n + k^2 U_n = 0$$

This equation should have the proper boundary conditions to allow a convenient expansion in our problem of the form

$$\begin{cases} w(x, y, z, t) \\ \theta(x, y, z, t) \end{cases} = \begin{cases} \sum_n w(t, z) U_n(x, y) \\ \sum_n \theta(t, z) U_n(x, y) \end{cases} \quad (14)$$

if we write  $k^2 = \frac{a^2}{d^2}$  and then choose  $d = 1$ .

We may replace the laplacian in equations 9 and 10 as:

$$\Delta = \frac{\partial^2}{\partial z^2} - a^2 \equiv D^2 - a^2 \quad (15)$$

equations 9 and 10 then become:

$$\frac{\partial}{\partial t} \left\{ D^2 - a^2 \right\} w + \alpha g a^2 \theta = \nu (D^2 - a^2)^2 w \quad (16)$$

$$\frac{\partial \theta}{\partial t} - \beta w = K(D^2 - a^2)\theta \quad (17)$$

In every equation here, and to follow, the variables  $w$  and  $\theta$  are the current transformed variables. For example in equations 16 and 17  $w = w(z, k, t)$ . We consider the initial value problem of a temperature disturbance at  $t = 0$  of

$$\left. \begin{aligned} \theta &= \theta_0(z, \vec{k}) \\ w &= 0 \end{aligned} \right\} \text{ at } t = 0 \quad (18)$$

The presence of initial conditions suggests the use of a Laplace transform. Denoting the transform variable by  $p$ , equations 16 and 17 become under transformation

$$\left\{ p - \nu (D^2 - a^2) \right\} (D^2 - a^2)w + \alpha g a^2 \theta = 0 \quad (19)$$

$$\left\{ p - K(D^2 - a^2) \right\} \theta - pw = \theta_0 \quad (20)$$

where now  $\begin{Bmatrix} \theta \\ w \end{Bmatrix} = \begin{Bmatrix} \theta(z, k, p) \\ w(z, k, p) \end{Bmatrix}$

We will investigate the simpler example of free boundaries in more detail. The boundary conditions in this case:

$$w = w'' = 0 \quad \text{on } z = 0, d$$

suggest an expansion of the form

$$\theta = \sum_{n=1}^{\infty} \theta_n \sin n\pi z \quad (21)$$

$$w = \sum_{n=1}^{\infty} w_n \sin n\pi z \quad (22)$$

$$\theta_0 = \sum_{n=1}^{\infty} B_n \sin n\pi z \quad (23)$$

In the rigid wall problem, while  $w$  and  $\theta$  can be expanded in sine series, the boundary conditions are such that it is usually not legitimate to differentiate these series term by term, and the problem is essentially more difficult.

Substituting equations (21), (22) and (23) into (19) and (20) we obtain the following algebraic equations:

$$- \left[ p + \nu \left\{ n^2 \pi^2 + a^2 \right\} \right] \left\{ n^2 \pi^2 + a^2 \right\} w_n + \alpha g a^2 \theta_n = 0 \quad (24)$$

$$\left[ p + K(n^2 \pi^2 + a^2) \right] \theta_n - \beta w_n = B_n \quad (25)$$

Solving these equations for  $w_n$  and  $\theta_n$  we obtain:

$$w_n = \frac{B_n g \alpha a^2}{[\text{Denominator}]} \quad (26)$$

$$\theta_n = \frac{B_n \left[ p + \nu \left\{ n^2 \pi^2 + a^2 \right\} \left( n^2 \pi^2 + a^2 \right) \right]}{[\text{Denominator}]} \quad (27)$$

$$[\text{Denominator}] = \left\{ n^2 \pi^2 + a^2 \right\} \left\{ \left[ p + \nu \left( n^2 \pi^2 + a^2 \right) \right] \left[ p + K \left( n^2 \pi^2 + a^2 \right) \right] - \frac{g \alpha \beta a^2}{n^2 \pi^2 + a^2} \right\} \quad (28)$$

which may be written as:

$$[\text{Denominator}] = \left( n^2 \pi^2 + a^2 \right) \left\{ p - p_+ \right\} \left\{ p - p_- \right\}$$

$$p_{\pm} = \frac{-(K+\nu)(n^2\pi^2+a^2) \pm \sqrt{(K-\nu)^2(n^2\pi^2+a^2) + \frac{4g\alpha\beta a^2}{n^2\pi^2+a^2}}}{2} \quad (29)$$

$\omega_n$  may then be written as:

$$\omega_n = \left\{ \frac{B_n g \alpha a^2}{n^2 \pi^2 + a^2} \right\} \left\{ \frac{1}{p_+ - p_-} \right\} \left[ \frac{1}{p - p_+} - \frac{1}{p - p_-} \right] \quad (30)$$

The solution after taking inverse Laplace transforms will be:

$$\omega_n = \left\{ \frac{B_n g \alpha a^2}{n^2 \pi^2 + a^2} \right\} \left\{ \frac{1}{p_+ - p_-} \right\} \left[ e^{p_+ t} - e^{p_- t} \right] \quad (31)$$

Now the sum of the roots  $p_+ + p_-$  is always negative. The product of the roots is

$$p_+ p_- = \nu K (n^2 \pi^2 + a^2)^2 - \frac{g \alpha \beta a^2}{n^2 \pi^2 + a^2} \quad (32)$$

Since the sum of the roots is always negative at least one of the roots is negative if they are real. It follows therefore that if  $p_+$  and  $p_-$  are real the requirement that at least one root is positive real (instability) is that

$$p_+ p_- < 0 \quad (33)$$

The marginal case  $p = 0$  occurs when

$$\nu K (n^2 \pi^2 + a^2)^2 - \frac{g \alpha \beta a^2}{n^2 \pi^2 + a^2} = 0 \quad (34)$$

If the roots are complex (they will be conjugates) equation (29) shows that they will be damped oscillations not contributing to instability. Equation (32) shows that the most unstable mode for a fixed horizontal wave number,  $a$ , is when  $n = 1$ , i.e. when (32) is most negative.

The condition for marginal stability (34) is

$$\frac{g \propto \beta}{\nu k} = \frac{(\pi^2 + a^2)^3}{a^2} \quad (35)$$

When (35) is made dimensional ( $d \neq 1$ ) the non-dimensional parameter on the left hand side of (35) becomes  $\frac{g \propto \beta d^4}{\nu k}$ . This number is called the Rayleigh number ( $\equiv R$ ).

If  $x$  and  $y$  are infinite in extent we can find that horizontal wave number which minimizes  $R$

$$\frac{dR}{da} = 0 \text{ yields } a^2 = \frac{\pi^2}{2} \text{ or that}$$

the minimum Rayleigh number for marginal stability  $R_{\min} = 656$ .

#### "Principle of Exchange of Stabilities"

Pellew and Southwell<sup>1</sup> were able to show that for the case of marginal stability and instability the exponential time dependence must be non-oscillatory, that is  $p$  is pure real for instability and therefore  $p = 0$  for marginal stability in the case of rigid walls. We were able to show this explicitly for the case of free boundaries by examining the roots in equation 29. This is not feasible for rigid walls because of the difficulty of the expansion of the type shown in equations (21, 22, 23) for the rigid wall. However, for the rigid wall, more general methods can be used.

First multiply equations (16) and (17) by  $\bar{w}$  and  $\bar{\theta}$  respectively. (Barred quantities are complex conjugates.) Then we integrate over the

<sup>1</sup>Pellew and Southwell, "Proceedings of the Royal Society" A 176 1940 pp. 312-343.

region  $z = 0$  to  $z = 1$ .

$$\text{Terms like } \int_0^1 dz \bar{w} D^2 w = \bar{w} Dw \Big|_0^1 - \int_0^1 D\bar{w} Dw dz$$

The integrated term vanishes because of the vanishing of  $w$  on the boundary. Terms like

$$\begin{aligned} \int_0^1 \bar{w} D^4 w &= \bar{w} D^3 w \Big|_0^1 - \int_0^1 D\bar{w} D^3 w dz \\ &= \bar{w} D^3 w \Big|_0^1 - D\bar{w} D^2 w \Big|_0^1 + \int_0^1 D^2 \bar{w} D^2 w dz \\ &= \int_0^1 |D^2 w|^2 dz \quad \text{since } Dw = w = 0 \text{ on } z = 0, 1 \end{aligned}$$

Equations (16) and (17) become when integrated in the aforementioned manner after multiplication by  $\bar{w}$  and  $\bar{\theta}$  respectively:

$$\begin{aligned} p \left[ - \int_0^1 dz \left[ |Dw|^2 + a^2 |w|^2 \right] - \gamma \left[ \int_0^1 \left\{ |D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2 \right\} dz \right] \right] \\ + g \alpha a^2 \int_0^1 \bar{w} \theta dz = 0 \end{aligned} \quad (36)$$

$$p \int_0^1 |\theta|^2 dz - K \left[ \int_0^1 -|D\theta|^2 - a^2 |\theta|^2 dz \right] - \beta \int_0^1 \bar{\theta} w dz = 0 \quad (37)$$

Taking the conjugate of (37) and multiplying (37) by  $g \alpha a^2$ , and (36) by  $\beta$  we add to eliminate the terms in  $\int_0^1 \beta \theta dz$ .

The resulting equation is

$$\begin{aligned}
 & p \left[ -\beta \int_0^1 |Dw|^2 dz + \int_0^1 a^2 |w|^2 dz \right] + \bar{p} \left[ g\alpha a^2 \int_0^1 |\theta|^2 dz \right. \\
 & \left. - \gamma \beta \left[ \int_0^1 dz |D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2 \right] + g\alpha^2 K \left[ \int_0^1 |D\theta|^2 + a^2 |\theta|^2 dz \right] \right] \\
 & = 0 \tag{38}
 \end{aligned}$$

In equation (38) only  $p$  is complex, therefore when we equate real and imaginary parts of (38) to zero separately we obtain the two equations:

$$p_{\text{real}} \left\{ -\beta (+) + g\alpha a^2 (+) \right\} - \beta \{ + \} = 0 \tag{39}$$

$$p_{\text{imag}} \left\{ -\beta (+) - g\alpha a^2 (+) \right\} = 0 \tag{40}$$

(+) stands for quantities that are positive definite and multiply the parameters  $\beta$  and  $g\alpha a^2$ .

Now examine equation (40). If  $p_{\text{imag}} \neq 0$  then  $\beta$  must be  $< 0$  for (40) to hold. If  $\beta < 0$  then in (39) the only way for the left hand side to vanish is for  $p_{\text{real}}$  is  $< 0$ . If  $p_{\text{real}} < 0$  our solution is a decaying one since we have assumed a solution with time dependence  $e^{pt}$ . Therefore all oscillatory modes, as in the free boundary case, are damped and  $p$  must be pure real for instability with marginal stability given by the condition  $p = 0$ .

In the case of marginal stability ( $p = 0$ ) the governing equation can be reduced to one-sixth order equation for  $w(z, k, t)$ .

$$(D^2 - a^2)^3 + a^2 R w = 0 \quad (41)$$

The boundary conditions for the rigid walls are

$$w = w' = (D^2 - a^2)^2 w = 0 \quad \text{on } z = 0, 1. \quad (42)$$

To form a variational statement of (41) and (42) we multiply (41) by  $(D^2 - a^2)^2 w$  and integrate between  $z = 0, 1$ . After integrating by parts one obtains:

$$a^2 R = \frac{\int_0^1 [D(D^2 - a^2)^2 w]^2 + a^2 [(D^2 - a^2)^2 w]^2}{\int_0^1 [(D^2 - a^2)^2 w]^2} \quad (43)$$

The reason we do not multiply equation (41) by  $w$  and integrate is because the boundary conditions are such that the terms obtained by integrating by parts do not vanish. One can also show that the eigenfunctions of (41) make (43) a minimum, which sets the usual variational problem  $\delta \alpha^2 R = 0$ .

To approximate  $\alpha^2 R$  in (43) we do not guess a  $w$  since  $w$  is differentiated five times in the integral which will aggravate any error in  $w$ . Instead we guess a function

$$g(z) = (D^2 - a^2)^2 w \quad \text{and solve for } w \text{ utilizing the boundary conditions}$$

$$w(0) = w'(0) = 0$$

$$w(1) = w'(1) = 0$$

and then substitute in (43). Pellew and Southwell<sup>(1)</sup> obtained in the case of rigid boundaries a minimum value of  $R$  of 1707.8 for marginal stability which occurs for a horizontal wave number,  $a$ , of 3.13. For mixed free and rigid boundary conditions they obtained a value for  $R_{\min}$  of 1100.

submitted by Joe Pedlosky and Bob Ellis.

Boundary Layer Theory

Boundary layer theory was introduced into the structure of fluid mechanics by L. Prandtl in an attempt to investigate those flow situations in which the effect of viscosity is confined to a narrow region. In viscous flows one of the most obvious effects of viscosity is the introduction of the boundary condition of zero slip on all solid boundaries. In certain circumstances it is in this region only that viscosity need be considered and this region is called the boundary layer.

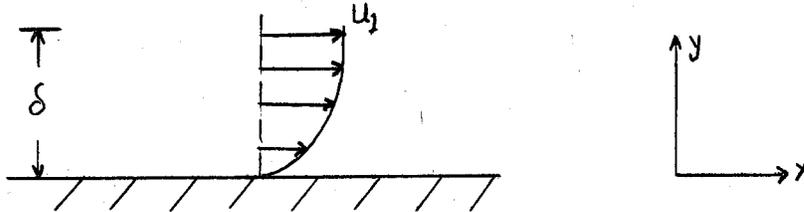
Consider the problem of a flat plate starting to move in its own plane in a fluid initially at rest. If the viscous boundary condition is observed then since the fluid is initially at rest away from the plate and is moving at the velocity of the plate on the plate, there will exist at the plate a sheet of infinite vorticity. The vorticity will diffuse into the fluid and the spread of the vorticity is given by  $\sqrt{4\nu t}$ . In locally steady flows such as the problem of entrance to a channel, where the velocity is initially non zero on the boundaries, the vorticity will be carried downstream as it diffuses from the channel boundary. In such problems the characteristic time  $t$  becomes  $\frac{x}{U}$ , with  $x$  measured from the channel entrance, and the region of the channel normal to the walls, affected by viscosity is therefore on the order of  $\sqrt{\frac{4\nu x}{U}}$ .

In all of these problems the common feature is the possibility of neglecting the action of viscosity external to a region (usually small) near the boundaries, where high shear exists.

Derivation of the Boundary Layer Equations

(Prandtl's approach).

We consider a planar 2-d flow shown below.



$U_1$  is the velocity at the point  $\delta$  where the action of viscosity is no longer important. The distance in  $y$ , the boundary layer thickness  $\delta$ , where this takes place is difficult to define with precision but is qualitatively where the flow reaches a certain value which is agreed upon to be considered a free stream value. We wish to derive the equations of motion within this region  $\delta$ . Our previous discussion leads us to believe that  $\delta = O\left(\sqrt{\frac{\nu x}{U}}\right)$  so that for flows with low viscosity  $\delta$  is to be considered a small quantity.

The Navier-Stokes equations for two dimensional, incompressible, viscous flow are:

$$U_t + UU_x + VU_y = \nu \left\{ U_{xx} + U_{yy} \right\} - \frac{P_x}{\rho} \quad (1)$$

$$V_t + UV_x + VV_y = \nu \left\{ V_{xx} + V_{yy} \right\} - \frac{P_y}{\rho} \quad (2)$$

with the continuity equation

$$U_x + V_y = 0 \quad (3)$$

If the  $U$  velocity is taken to scale with the free stream horizontal velocity which is  $O(1)$  and derivatives in the  $y$  direction

are  $O(1/\delta)$  times the considered function while derivatives in  $x$  are  $O(1)$  times the considered function,

Then

$$\begin{aligned} U &= O(1) \\ U_y &= O\left(\frac{1}{\delta}\right) \\ U_{yy} &= O\left(\frac{1}{\delta^2}\right) \text{ where } \delta \text{ is a small number.} \end{aligned}$$

The continuity equation when examined term by term for order of magnitude i.e.

$$U_x + V_y = 0$$

$$O(1) + \frac{V}{\delta} = 0 \quad \text{shows that } V \text{ must be } = O(\delta).$$

If  $\delta$  is of the order of  $\sqrt{\nu}$  then the same type of examination of the momentum equations leads to:

$$\begin{aligned} U_t + UU_x + VU_y &= \nu \left\{ U_{xx} + U_{yy} \right\} - \frac{1}{\rho} P_x \\ O(1) \cdot O(1) \cdot \delta \frac{1}{\delta} &= \delta^2 \left\{ O(1) \cdot O\left(\frac{1}{\delta}\right) \right\} \end{aligned} \quad (4)$$

$$\begin{aligned} V_t + VV_y + UV_x &= \nu \left\{ V_{xx} + V_{yy} \right\} - \frac{1}{\rho} P_y \\ O(\delta) + O(\delta) \frac{\delta}{\delta} + O(\delta) &= \delta^2 \left\{ O(\delta) + O\left(\frac{1}{\delta}\right) \right\} \end{aligned} \quad (5)$$

Now the equation for  $V$  clearly shows that  $p_y = O(\delta)$  which means that the change in pressure through the narrow boundary layer region is of order  $\delta^2$  and will be neglected.

In equation (4) all terms are of order unity except the first term on the right hand side which we will neglect since it is of higher order in  $\delta$ . The final boundary layer equations become for  $0 \leq y < \delta$ .

$$U_t + UU_x + VU_y + \frac{P_x}{\rho} = \nu U_{yy} \quad (6)$$

$$U_x + U_y = 0 \quad (7)$$

$$P_y = 0 \quad (8)$$

The boundary conditions are:

$$U(x, 0) = V(x, 0) = 0$$

$$U(x, \delta) = U_1(x)$$

The vertical velocity cannot be prescribed at the point  $y = \delta$  reflecting the fact that we have, roughly speaking, substituted a first order equation (8) for the second order equation (2) for  $V$ .

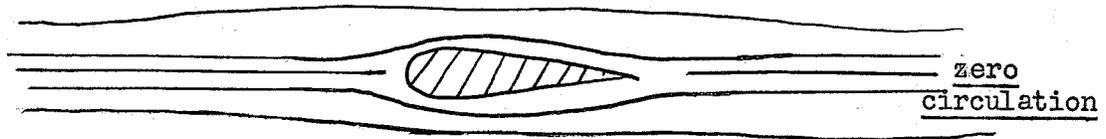
This procedure for solving for the flow within the region  $0 \leq y < \delta$  may be considered the second step of an approximation scheme. The first step would be to find the solution of the Euler equations for the region. These non viscous solutions are used to determine the pressure field within the boundary layer. When equations 6-8 are solved with this information the  $V$  velocity which is introduced by the viscous effects will in general affect the potential flow, acting as a mass source into the potential region. This new potential flow will then be used to calculate a new boundary layer flow. This procedure could be, in principle, continued until the "correct" solution is obtained.

Outside the boundary layer where viscosity is ignored the pressure gradient is determined by the Euler equation

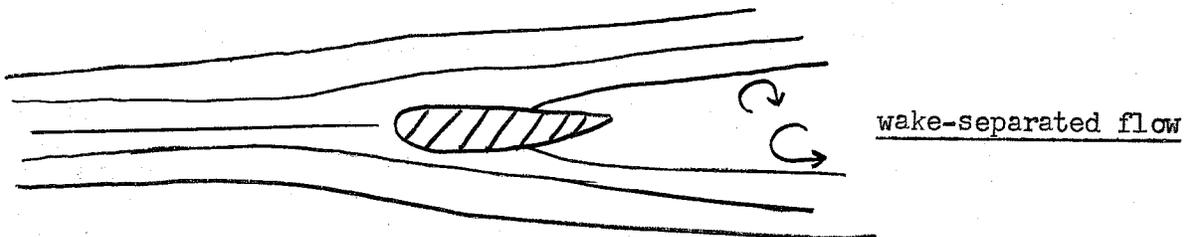
$$U_{1t} + U_1 U_{1x} = -\frac{1}{\rho} p_x \quad (9)$$

if  $p_y = 0$  from equation (8) then within the boundary layer we may use equation (9) for the pressure. This is the real advantage of the boundary layer technique. The problem remains non-linear but the pressure field may be considered a known quantity within the boundary layer.

One of the difficulties associated with this problem is the non-uniqueness of the solutions of the Euler equations. For instance for the geometry shown below two different flows are often seen, the zero circulation flow shown here



and the separated flow shown below.



However once the model of the inviscid flow is decided upon the solutions of the boundary layer equations follow, as outlined above.

The above derivation of the boundary layer equations gives a fairly good physical picture of the basis for the approximation, but is perhaps not so clear from the mathematical standpoint. Another approach is the following. Suppose the physical situation is dimensionally characterized by a length scale  $L$  and a velocity scale  $U_0$ , which might for instance be a measure of the size of the obstacle and the free stream speed at large distances. From these and the kinematic viscosity  $\nu$  we can form the single dimensionless parameter  $R = \frac{U_0 L}{\nu}$  (Reynold's number), and by choosing  $L$  for unit of length,  $L/U_0$  for unit of time, and  $\rho U_0^2$  as unit of pressure the equations take the dimensionless form:

$$U_t + UU_x + VU_y + p_x = \frac{1}{R} (U_{xx} + U_{yy})$$

$$V_t + UV_x + VV_y + p_y = \frac{1}{R} (V_{xx} + V_{yy})$$

$$U_x + V_y = 0$$

We now suppose we have a solution of these equations corresponding to flow past an obstacle with, say, a solid boundary on  $y = 0$ . We consider two limiting processes associated with  $\nu \rightarrow 0$  or  $R \rightarrow \infty$ . In the first, we consider a fixed point in the flow, off the wall, and look at the limits of  $U$ ,  $V$ , etc. as  $R \rightarrow \infty$ . We expect, except for non-uniformities in the convergence, that this limiting flow will be a solution of the limits of the equations as  $R \rightarrow \infty$ , i.e. of the Euler equations. We call this limit  $\lim_1$ , and  $\lim_1 \vec{U} = \vec{U}_e$  the external flow. It can presumably be found by solving the Euler equations, but will in general not satisfy all the viscous boundary conditions. Thus the convergence of  $\vec{U}$  to  $\vec{U}_e$  as  $R \rightarrow \infty$  cannot be uniform, and we may expect the non-uniformity to occur at the wall, the convergence being uniform in closed bounded sets which do not meet the wall. (This is not proved, except in certain vaguely analogous linear problems, but there is good reason to think it is true in many cases of interest.) Boundary layer theory attempts to understand the nature of the non-uniformity by studying a different type of limit of the exact solution,  $\lim_2$ . For this limiting process, we allow the point to move toward the wall as  $R \rightarrow \infty$ , in such a way as to preserve some of the higher derivatives in the equations and so permit the satisfaction of viscous

boundary conditions. We first introduce the new independent variable  $\eta = R^{\frac{1}{2}}y$  and replace  $V$  by  $R^{-\frac{1}{2}}v$ . (The first change of variable almost forces the second if the continuity equation is to be preserved as  $R \rightarrow \infty$ . In general, one would introduce

$\eta = \varepsilon^n y$ ,  $v = \varepsilon^{-n} v$  and then try to determine  $n$  so that the limiting equations will have solutions which can satisfy the boundary conditions at  $y = 0$  and match the external flow away from the boundary.

In many cases, like the present one, this determines  $n$  unambiguously.)

In the new variables the equations become:

$$U_t + UU_x + vU_\eta + P_x = \frac{1}{R} U_{xx} + U\eta\eta$$

$$\frac{1}{R} [v_t + Uv_x + vv_\eta] + P_\eta = \frac{1}{R^2} v_{xx} + \frac{1}{R} v\eta\eta$$

$$U_x + v\eta = 0$$

The formal limit of these equations for  $R \rightarrow \infty$  is

$$U_t + UU_x + vU_\eta + P_x = U\eta\eta$$

$$P_\eta = 0$$

$$U_x + v\eta = 0$$

If these equations are rewritten in the original variables we obtain exactly Prandtl's boundary layer equations. We may expect that, when written in the  $x, \eta$  variables, the exact solution will converge to the boundary layer solution for fixed  $x$  and  $\eta$ , and that this convergence will be uniform for  $(x, \eta)$  in a closed bounded set (in the  $x$ - $\eta$  plane) which meets the wall. This exhibits somewhat more

clearly the senses in which the external flow and the boundary layer flow are approximations to the exact solution: they are both limits of the exact solution for  $R \rightarrow \infty$  but with different variables and different domains of validity. There is, however, a connection between them, at least in those cases where boundary layer theory is useful. To a certain extent, the domains of validity of the two approximations overlap; and it is this which links them together by boundary conditions. The situation appears in fact to be as follows: except for quantities uniformly small to order  $R^{-\frac{1}{2}}$ , the exact solution ( $U$ ) is the sum of a term  $f_0(x,y)$  and a term  $g_0(x, \eta)$  ( $\eta = R^{\frac{1}{2}}y$ ) where  $g_0$  is transcendentally small off  $y = 0$ , i.e.

$$\lim_{R \rightarrow \infty} R^k g_0(x, R^{\frac{1}{2}}y) = 0 \text{ for any fixed } k \text{ and } y \neq 0, \text{ (however } g_0(x, 0) \text{ usually is not } 0), \text{ that is:}$$

$$U(x,y,R) = f_0(x,y) + g_0(x, R^{\frac{1}{2}}y) + O(R^{-\frac{1}{2}})$$

If we apply  $\lim_1$  to this we get: (for  $y \neq 0$ )

$$\lim_1 U(x,y,R) = f_0(x,y)$$

i.e., 
$$f_0(x,y) = U_e(x,y)$$

Applying  $\lim_2$  we get the boundary layer solution:

$$\begin{aligned} U_{BL}(x, \eta) &= \lim \left[ f_0(x, R^{-\frac{1}{2}}\eta) + g_0(x, \eta) + O(R^{-\frac{1}{2}}) \right] \\ &= f_0(x, 0) + g_0(x, \eta) \end{aligned}$$

From this formula we see, since  $g_0 \rightarrow 0$  as  $\eta \rightarrow \infty$ , that

$U_{BL}(x, \infty) = f_0(x, 0) = U_e(x, 0)$ . This gives the boundary condition at  $\infty$  on  $U_{BL}$ . Similar results hold for  $V$ . The two can be combined by use of the stream function  $\psi$ . There seems to be good reason to believe that in the situations where boundary layer techniques are relevant, the complete asymptotic expansion of  $\psi$  is the form:

$$\psi \sim \sum_0^{\infty} R^{-n/2} f_n(x, y) + \sum_0^{\infty} R^{-\frac{n+1}{2}} g_n(x, \eta)$$

where  $\eta = R^{1/2}y$  and the  $g_n$  are transcendentally small for  $\eta \rightarrow \infty$ . The higher boundary layer and external flow approximations are obtained by successive applications of  $\lim_1$  and  $\lim_2$  to  $\psi$ , and involve successively higher terms in the two parts of the above asymptotic expansion.

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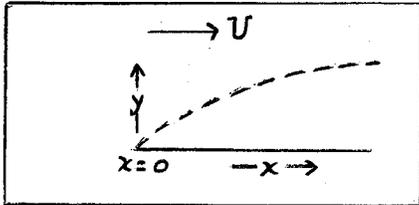
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submitted by J. Pedlosky

APPLICATION OF BOUNDARY LAYER THEORY

I. Blasius Problem. We use the boundary layer technique to study

flow around a semi-infinite plane sheet, as in the diagram.



The free-stream flow  $U$  is in the  $x$ -direction and the sheet is the  $x$ - $z$  plane. We then have the two-dimensional boundary layer problem - the equations being

$$\left. \begin{aligned} uu_x + vu_y &= \nu u_{yy} \\ u_x + v_y &= 0 \end{aligned} \right\}, \quad (1)$$

with suitable boundary conditions on the sheet ( $y = 0$ ) and the matching of  $u$  with  $U$  at the edge of the boundary layer.

Since there is no natural length associated with this problem, let  $L$  be an arbitrary length. Then the stretch coordinate is  $\eta = \frac{y}{L} \sqrt{\frac{UL}{\nu}}$

We can rewrite eqs. (1) as

$$\begin{aligned} u &= \psi_y, \quad v = -\psi_x \\ \psi_y \psi_{xy} - \psi_x \psi_{yy} &= \nu \psi_{yyy} \end{aligned}, \quad (2)$$

where  $\psi$  is the stream function.

Because  $L$  is arbitrary, the solution to this problem must be independent of  $L$  - hence, the solution must be a function of  $\eta \sqrt{\frac{L}{X}}$ .

Thus, we are led to consider the variable  $\theta \equiv y \sqrt{\frac{U}{2\nu x}}$ . Letting

$\psi = \sqrt{2\nu U x} f(\theta)$ , we have only to substitute the functions  $\psi_x$ ,  $\psi_y$ ,  $\psi_{xy}$ ,  $\psi_{yyy}$  into (2) to obtain the Blasius equation,

$$f'''(\theta) + f(\theta)f''(\theta) = 0 \quad (3)$$

$$\left. \begin{array}{l} \text{As b.c., we have } f(0) = f'(0) = 0 \\ \text{and } f'(\infty) = 1 \end{array} \right\} ; \quad (4)$$

which represent the vanishing of  $u$  and  $v$  on the sheet ( $y=0$ ), and the matching of  $u$  with  $U$  at the edge of the boundary layer, respectively.

Herman Weyl<sup>1</sup> worked out an iterative solution to the Blasius problem, which we present below. We first consider the function  $h(\theta)$  which satisfies the problem:

$$\boxed{\begin{array}{l} h''' + hh'' = 0 \\ h(0) = h'(0) = 0 \\ h''(0) = 1 \end{array}} ; \quad (5)$$

which is the Blasius problem with the b.c.  $f'(\infty) = 1$  replaced by the condition  $h''(0) = 1$ .

Then if there exists a solution  $h(\theta)$  to (5), for which  $h'(\theta)$  approaches a positive limit as  $\theta \rightarrow \infty$ , we can obtain a solution to the Blasius problem as follows: Let  $f(\theta) = kh(k\theta)$ . Then we have  $f''' + ff'' = k^4 \{h''' + hh''\} = 0$ , and  $f(0) = f'(0) = 0$ . Now,  $f'(\infty) = k^2 h'(\infty)$ , and assuming we have  $h'(\infty) = M \neq \infty$ ,  $M > 0$ , if we set  $k = 1/\sqrt{M}$ , we get  $f'(\infty) = 1$ .  $\therefore f(\theta)$  is a solution of the Blasius problem. [Note that  $k$  essentially determines  $\frac{\partial u}{\partial y}$  on the sheet  $y = 0$ , viz.,  $\frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = U \frac{\partial}{\partial y} f'(\theta)$

$$= U \sqrt{\frac{U}{2\nu x}} f''(\theta) \Rightarrow \frac{\partial u}{\partial y} \Big|_{y=0} = U \sqrt{\frac{U}{2\nu x}} k^3 h''(0) = U \sqrt{\frac{U}{2\nu x}} k^3. ]$$

<sup>1</sup>Annals of Math. 43, 381 (1942).

We are thus led to try to find a solution to (5) for which  $h'(\infty)$  exists and is positive. Weyl provided such a solution as follows:

From (5), we have  $\frac{h'''}{h''} = -h$ , which when integrated from 0 to  $\theta$  gives

$$\log h''(\theta) = -\int_0^\theta h(t)dt$$

or

$$h''(\theta) = e^{-\int_0^\theta h(t)dt} \quad (6)$$

Using the identity,  $h(t) = \frac{d}{dt}(th(t)) - th'(t)$ ,

we get

$$\begin{aligned} \int_0^\theta h(t)dt &= \theta h(\theta) - \int_0^\theta th'(t)dt \\ &= \theta \int_0^\theta h'(t)dt - \int_0^\theta th'(t)dt \end{aligned}$$

where we have used the b.c.,  $h(0) = 0$ .

$$\therefore \int_0^\theta h(t)dt = \int_0^\theta (\theta - t)h'(t)dt.$$

Applying this process once again, we get

$$\int_0^\theta h(t)dt = \frac{1}{2} \int_0^\theta (\theta - t)^2 h''(t)dt.$$

Substituting this identity into (6), yields an integral equation for  $h''(\theta)$ ,

$$h''(\theta) = e^{-\frac{1}{2} \int_0^\theta (\theta-t)^2 h''(t)dt} \quad (7)$$

We will apply the method of successive approximations on eq.(7) to yield a solution which when integrated twice produces the solution to(5).

Define the operator  $F$  by  $F\{\phi\} = e^{-\frac{1}{2} \int_0^\theta (\theta-t)^2 \phi(t) dt}$

Then a solution to (7) is given by  $\lim_{n \rightarrow \infty} H_n(\theta)$ , where the sequence  $\{H_n\}$

$$\text{is defined by } \begin{cases} H_n = F\{H_{n-1}\}, & n = 1, 2, 3, \dots \\ H_0 = 1 \end{cases}$$

It is clearly a property of the operator  $F$  that: if  $\phi > \psi$ ,  $F(\phi) < F(\psi)$ . Hence, since  $H_0 = 1$ ,  $H_1 = e^{-\frac{1}{6}\theta^3} < 1$ .

$$\therefore F(H_0) = H_1 < F(H_1) = H_2,$$

$$F(H_1) = H_2 > F(H_2) = H_3,$$

$$F(H_2) = H_3 < F(H_3) = H_4,$$

$$\vdots$$

In general,  $H_{2n} > H_{2n+1}$  and  $H_{2n+1} < H_{2n+2}$  ( $n=0, 1, 2, \dots$ ).

$$\text{Also, } H_2 = F(H_1) < F(H_0) = H_0 = 1 \quad (\text{since } H_1 > 0).$$

$$H_3 = F(H_2) > F(H_0) = H_0 = 1,$$

$$H_4 = F(H_3) < F(H_1) = H_1,$$

$$\vdots$$

$$\vdots$$

In general,  $H_{2n+1} > H_{2n-1}$  and  $H_{2n} < H_{2n-2}$  ( $n = 1, 2, 3, \dots$ )

$$0 < H_1 < H_3 < H_2 < H_0 = 1$$

$$0 < H_1 < H_3 < H_4 < H_2 < H_0 = 1$$

$$0 < H_1 < H_3 < H_5 < H_4 < H_2 < H_0 = 1$$

$$\vdots$$

$$0 < \underbrace{H_1 < H_3 < \dots < H_{2n+1}}_{\text{all odd indices}} < \underbrace{\dots < H_{2n} < \dots < H_4 < H_2}_{\text{all even indices}} < H_0 = 1.$$

Weyl shows that  $H_{2n} - H_{2n+1} \rightarrow 0$  and so that the sequence converges.

Returning to the Blasius Problems,

$$\begin{aligned} h'(\infty) &= \int_0^{\infty} h''(\theta) d\theta \quad (\text{since as a b.c., } h'(0) = 0) \\ &= \int_0^{\infty} \left( \lim_{n \rightarrow \infty} H_n' \right) d\theta < \int_0^{\infty} H_2(\theta) d\theta. \end{aligned}$$

Weyl has shown (footnote<sup>1</sup>) that  $\int_0^{\infty} H_2(\theta) d\theta$  is in fact, finite.

Hence,  $h'(\infty)$  is finite, say  $h'(\infty) = M \neq \infty$  (and  $M > 0$ ), and as previously remarked,  $f(\theta) = \frac{1}{\sqrt{M}} h(\theta/\sqrt{M})$  is a solution to the Blasius

problem ( $k = 1/\sqrt{M}$ ).

It may be noted that the convergence of the approximating sequence is quite rapid - e.g., using the first approximation  $H_1 = e^{-\frac{1}{6}\theta^3}$ , one obtains  $h'(\infty) = \sqrt[3]{6} \Gamma(4/3)$  giving  $k \approx .484$  which compares favorably with the "exact value" of  $k$ ,  $k \approx .470$ . That is, even the first approximation is rather good.

One notices in the formulation of the differential equation of the Blasius problem that there is a singularity along the line  $x = 0$ . Saul Kaplun<sup>2</sup> has studied the general problem of the influence of different coordinate systems on boundary layer theory and has shown the existence of certain "optimal" coordinate systems in which the

<sup>2</sup> Kaplun, S. "The role of coordinate systems in Boundary Layer Theory". Zeits. Ang. Math. u. Phys. 1954 or 1955.

boundary layer solution gives an approximation valid not only in the boundary layer but essentially everywhere. For the Blasius problem, parabolic coordinates turn out to be optimal, and using these the singularities disappear everywhere except at  $x = y = 0$ . The singularity here appears to be unavoidable in boundary layer theory.

II. A second example of boundary layer equations is the well-known Falkner-Skan equation,

$$f''' + ff'' + \beta (1 - f'^2) = 0 \quad (8)$$

which arises in solving the problem of a boundary layer on an infinite plane wedge having a total interior wedge angle  $(\pi\beta)$ . For the physical problem, appropriate boundary conditions are

$$\begin{aligned} f(0) = f'(0) &= 0 \\ f'(\infty) &= 1 \end{aligned} \quad (9)$$

As we are particularly interested in studying the use of boundary layer methods for solving differential equations having a small parameter multiplying the highest order derivative, let us formally allow  $\beta$  to become large and not attempt to interpret the results in terms of the wedge problem which gave rise to the original equation(8).

If we divide (8) by  $\beta$  and formally set  $\beta = \infty$  we obtain a differential equation of lower order than the original which we call the reduced equation corresponding to (8). The reduced equation is

$$1 - f'^2 = 0 \quad (10)$$

If we consider  $x = 0$  as a boundary, then it is clear  $f(x)$  as given by (10) will satisfy the boundary condition at infinity, and may be made to satisfy  $f(0) = 0$  by choosing the constant of integration in (10). To satisfy all of the boundary conditions (9) it is necessary to make the usual stretching transformation of independent variable

$$\eta = x\beta^a$$

Further, since the equation (8) is non-linear it may also be necessary to stretch the dependent variable

$$f(x) = g(\eta)\beta^{-b}$$

With these changes of variable the original expression (8) becomes

$$g''' \beta^{-b+3a} + g g'' \beta^{-2b+2a} + \beta(1 - \beta^{2(a-b)} g'^2) = 0 \quad (11)$$

We must now choose  $a$  and  $b$  so that the boundary conditions  $f = f' = 0$  at  $x = 0$  can be satisfied by the boundary layer solution  $g$  of the formal limit of (11) for  $\beta \rightarrow \infty$ , which also must match up with the external solution ( $f' = 1$ ) as  $\eta \rightarrow \infty$ . Now  $f'(x) = g'(\eta)\beta^{a-b}$ . The one non-homogeneous boundary condition on  $g$  is then that  $g'(\eta)\beta^{a-b} \rightarrow 1$  as  $\eta \rightarrow \infty$ . This indicates that we should take  $a = b$ . (11) then becomes

$$g''' \beta^{2a} + g g'' + \beta(1 - g'^2) = 0$$

If  $a < \frac{1}{2}$ , the formal limit is  $1 - g'^2 = 0$ , the solutions of which cannot satisfy the boundary conditions at 0. If  $a > \frac{1}{2}$ , the formal limit is  $g''' = 0$ , and the general solution is a quadratic polynomial, which cannot satisfy both  $g'(0) = 0$  and  $g'(\infty) = 1$ . We are thus

finally led to  $a = b = \frac{1}{2}$  as the only possibilities for a usable boundary layer theory. The final "boundary layer" equation is:

$$g''' + (1 - g'^2) = 0 \quad (12)$$

It is to be solved subject to boundary conditions

$$\begin{aligned} g(0) = g'(0) &= 0 \\ g'(\infty) &= 1 \end{aligned} \quad (13)$$

If we multiply (12) by  $g''$  and integrate we find

$$\frac{1}{2} g''^2 + g' - \frac{1}{3} g'^3 = \text{constant} \quad (14)$$

The condition (13) that  $g'(0) = 0$  implies that the constant in (14) is  $[1/2 g''(0)]^2$ . Further, letting  $\eta \rightarrow \infty$  in (7) and applying the condition  $g'(\infty) = 1$  we see the constant also equals  $2/3$ , which then implies

$$[1/2 g''(0)]^2 = 2/3 \quad (15)$$

In terms of the original variables,

$$f''(0) = \sqrt{\frac{4\beta}{3}} \quad (16)$$

If we now integrate (14) we have  $g(\eta)$  expressed as a simple quadrature. This quadrature may be executed explicitly if we substitute the value  $2/3$  for the constant, and factor the resulting cubic into a perfect square times a linear term.

Problem: Complete the solution of this boundary layer approximation to the Falkner-Skan equation and show that

$$f_{BL}(\chi) = \chi - \frac{\beta^{-\frac{1}{2}} \sqrt{2} \tanh\left(\frac{\chi \beta^{\frac{1}{2}}}{\sqrt{2}}\right)}{1 + \sqrt{\frac{2}{3}} \tanh\left(\frac{\chi \beta^{\frac{1}{2}}}{\sqrt{2}}\right)} \quad (17)$$

To compare the second derivatives of  $f$  as given by (16) and the numerically exact solution we present the table

$\beta$	$\sqrt{\frac{4\beta}{3}}$	$f''(0)$
0	0	0.47
.4	.73	0.86
.8	1.02	1.12
1.2	1.26	1.34
2.0	1.62	1.68

Rather close agreement is seen for even moderate values of  $\beta$ . This boundary layer treatment of the Falkner-Skan equation is due to P.A. Lagerstrom. (High Speed Aero. and Jet Prop. Vol.IV)

### III. Ekman B.L. Equations

We wish to derive the governing equations for viscous flow in a rotating system. If we assume the fluid is of constant depth, i.e.  $H = \text{constant}$ , and the Coriolis frequency  $f$  is constant, we have the momentum equation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + f \vec{k} \times \vec{u} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \vec{u} - g \vec{k} \quad (18)$$

Let us hereafter assume the flow is steady, and neglect non-linear terms. If  $\nu$  is small we expect the "external flow" to be a solution of the equations

$$f \vec{k} \times \vec{u} + \frac{1}{\rho} \nabla p = -g \vec{k} \quad (19)$$

$$\nabla \cdot \vec{u} = 0$$

The solution to the above equations is called the steady geostrophic flow.

If we take  $z = \zeta(x, y)$  as the equation of the fluid surface, and take  $p = 0$  on the surface, we have the system below

$$p = g\rho(\zeta - z) \quad (20)$$

which is the "hydrostatic approximation", and

$$\begin{aligned} -fv + g\zeta_x &= 0 \\ fu + g\zeta_y &= 0 \end{aligned} \quad (21)$$

If we solve the equation of continuity for  $\frac{\partial w}{\partial z}$ , and note from (21) that  $f(u_x + v_y) = 0$ , we see the solutions,  $u_1$ , and  $v_1$ , of the geostrophic equations are independent of  $z$ .

Let us denote the flow velocities in the boundary layer on the fluid bottom by  $u$ ,  $v$ , and  $w$ . If we take  $z = 0$  as the fluid bottom, then boundary conditions to be applied are

$$u = v = w = 0 \quad \text{on } z = 0 \quad (22)$$

As usual, we introduce a stretched coordinate in the boundary layer,

$$\eta = z\sqrt{\frac{b}{\nu}} \quad (23)$$

where  $\nu$  is assumed to be small. In terms of the  $\eta$  variable, we have from (11)

$$\begin{aligned} -fv + \frac{1}{\rho} p_x &= \nu \left[ \frac{b}{\nu} u_{\eta\eta} + u_{xx} + u_{yy} \right] \\ fu + \frac{1}{\rho} p_y &= \nu \left[ \frac{b}{\nu} v_{\eta\eta} + v_{xx} + v_{yy} \right] \\ \frac{1}{\rho} p_z + g &= \nu \left[ \frac{b}{\nu} w_{\eta\eta} + w_{xx} + w_{yy} \right] \end{aligned} \quad (24)$$

and the equation of continuity

$$u_x + v_y + \sqrt{\frac{f}{\nu}} w_z = 0 \quad (25)$$

From (25) we see if  $u$  and  $v$  are to be of order one, it is necessary that  $w_z$  be of order  $\sqrt{\frac{\nu}{f}}$ . If we now retain terms of order one, we have the boundary layer equations

$$\begin{aligned} -f(v-v_i) &= f u_{zz} \\ f(u-u_i) &= f v_{zz} \\ \frac{1}{\rho} p_z + g &= 0 \end{aligned} \quad (26)$$

The first two equations of (26) may be written as

$$\frac{\partial^2}{\partial z^2} [(u-u_i) + i(v-v_i)] = i [(u-u_i) + i(v-v_i)] \quad (27)$$

which is called the Ekman boundary layer equation.

Problem: Find the solution of the Ekman b.l. equation with boundary conditions

$$u = v = 0 \text{ on } \eta = 0 \text{ and } u - u_i = v - v_i = 0 \text{ as } \eta \rightarrow \infty.$$

Problem: Investigate the Ekman layer at the fluid surface if horizontal stresses  $\tau_x$  and  $\tau_y$  are applied there and if the surface is approximately plane. In particular, calculate  $w$  just below the boundary layer.

submitted by K. Gross

R. Duty

Munk's Theory of the Gulf Stream:

Consider fluid in a rectangular region which has been scaled so that the fluid lies between the planes  $y = 0$  and  $y = \pi$  and between the planes  $x = 0$  and  $x = a$ .

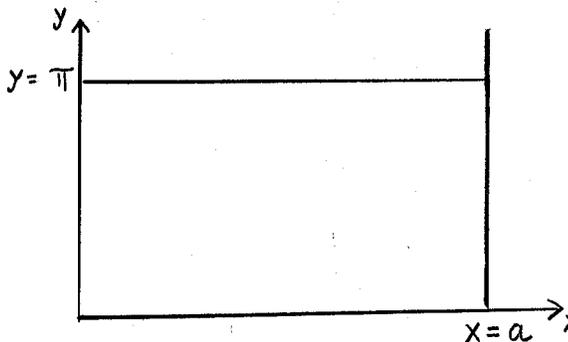


Fig. 1

The region is large enough

that  $f$ , the Coriolis parameter, varies throughout the fluid; in particular we assume

$$f = f_0 + \beta y, \tag{1}$$

where  $\beta$  is small and will be neglected except when derivatives of  $f$  are involved. We also assume that the motion of the fluid is driven by wind stress which varies sinusoidally with  $y$ , being maximum in opposite directions at  $y = \pi$  and  $y = 0$  and vanishing at  $y = \frac{\pi}{2}$ .

The dimensionless, linearized Navier-Stokes equations are

$$-fv + p_x = \nu \Delta u - A \cos y \tag{2}$$

$$fu + p_y = \nu \Delta u, \tag{3}$$

where  $A \cos y$  is the term due to the wind stress (modeled, in this shallow water theory, by a body force) and  $\Delta$  is the Laplacian operator. The equation of continuity is

$$u_x + v_y = 0, \tag{4}$$

i.e., we assume the flow to be independent of  $z$ .

Differentiating equation (2) with respect to  $y$  and equation (3) with respect to  $x$  and subtracting the results, we obtain

$$-f_y v - f v_y + p_{xy} - f u_x - p_{xy} = \nu (u_y - v_x) + A \sin y \quad (5)$$

But by equation (1)  $f_y = \beta$  and by equation (4)  $u dy - v dx$  is a total differential and therefore there exists a scalar function

$\psi(x,y)$  such that

$$u = \frac{A}{\beta} \psi_y \quad (6)$$

$$v = -\frac{A}{\beta} \psi_x \quad (7)$$

Therefore, equation (5) can be written as

$$A \psi_x - \nu \frac{A}{\beta} \Delta (\psi_{xx} + \psi_{yy}) + A \sin y$$

or

$$\psi_x = \frac{\nu}{\beta} \Delta \Delta \psi + \sin y \quad (8)$$

The boundary conditions on  $\psi$  (we take, for simplicity,  $\psi(x,0) = \psi(x,\pi) = \psi_{yy}(x,0) = \psi_{yy}(x,\pi) = 0$ ) permit us to expand  $\psi$  in a Fourier sine series in  $y$ . But equation (8) implies that all the coefficients must vanish except the one for  $\sin y$ , i.e.,

$$\psi = f(x) \sin y \quad (9)$$

The value of  $\frac{\nu}{\beta}$  for the Gulf Stream is approximately  $10^{-4}$ , using Munk's value of  $5 \times 10^7$  for the "effective"  $\nu$  due to horizontal turbulence. Away from the boundaries, where  $\Delta \Delta \psi$  is not large, the first term on the right in equation (8) may be neglected and the

"external flow" is given by

$$\psi_e = (x + k) \sin y \quad (10)$$

Substituting equation (9) into equation (8) and putting  $\epsilon = \frac{\nu}{\beta}$ , one obtains

$$f' = \epsilon (f'''' + 2f'' + f) + 1 \quad (11)$$

We wish to examine the boundary layer at  $x = 0$ . As usual we make the change of variable

$$x = \epsilon^{1/3} \xi \quad (12)$$

Then equation (11) becomes

$$\epsilon^{-1/3} f_\xi = \epsilon \left( \epsilon^{-4/3} f_{\xi\xi\xi\xi} + 2\epsilon^{-2/3} f_{\xi\xi\eta\eta} + f_{\eta\eta\eta\eta} \right) + 1 \quad (13)$$

Since  $\epsilon$  is small we neglect all terms of higher order in  $\epsilon$ , i.e.,

$$f_\xi = f_{\xi\xi\xi\xi} \quad (14)$$

The boundary conditions on  $f$  are  $f(0) = f_\xi(0) = 0$

since  $x = 0$  is a streamline and there is no horizontal velocity at  $x = 0$ .

We now define a new function  $g$  as follows:

$$f = x + k + g \quad (15)$$

Substituting into equation (14)

$$g_\xi + \epsilon^{1/3} = g_{\xi\xi\xi\xi}$$

or

$$g_\xi = g_{\xi\xi\xi\xi} \quad (16)$$

since  $\epsilon$  is small. The boundary conditions on  $g$  corresponding to those on  $f$  are

$$g(0) = -k \quad (17)$$

$$g_{\xi}(0) = 0 \quad (18)$$

and by equation (10)  $g \rightarrow 0$  as  $\xi \rightarrow \infty$

The solution to equation (16) is found by elementary techniques to be

$$g = C_1 + C_2 e^{\xi} + C_3 e^{-\xi/2} \cos \frac{\sqrt{3}}{2} \xi + C_4 e^{-\xi/2} \sin \frac{\sqrt{3}}{2} \xi \quad (19)$$

The boundary conditions imply

$$C_1 = C_2 = 0, \quad C_3 = -k, \quad C_4 = -\frac{k}{\sqrt{3}}$$

Therefore the complete solution is

$$g = -k e^{-\xi/2} \left[ \cos \frac{\sqrt{3}}{2} \xi + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} \xi \right] \quad (20)$$

(cf. Stommel, H., "The Gulf Stream", Chapter 7.)

#### Non-linear Shallow Water Theory:

The equation for flow in shallow water in a rotating system with gravity is given by

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p + f \vec{k} \times \vec{u} + g \vec{k} = 0 \quad (21)$$

The depth of the fluid will be given by  $H(x,y)$  and the height of the surface by  $\zeta(x,y)$ . We will assume  $u$  and  $v$  to be independent of  $z$  and the pressure to be given by the hydrostatic pressure, i.e.,

$$p = \rho g(\zeta - z) \quad (22)$$

Under the conditions the  $x$  and  $y$  components of equation (21) may be written as

$$u_t + uu_x + vv_y + g\zeta_x - fv = 0 \quad (23)$$

$$v_t + uv_x + vv_y + g\zeta_y + fu = 0 \quad (24)$$

If the equation of continuity ( $\nabla \cdot \mathbf{u} = 0$ ) is integrated vertically between  $z = -H(x,y)$  and  $z = \zeta(x,y)$ , there results

$$w(\zeta) - w(-H) = -(u_x + v_y)(\zeta + H) \quad (25)$$

At  $z = -H$

$$w = -\frac{DH}{DZ} = -\frac{\partial H}{\partial t} - uH_x - vH_y = -uH_x - vH_y \quad (26)$$

since the depth is constant in time.

At  $z = \zeta$

$$w = \frac{D\zeta}{Dt} = \zeta_t + u\zeta_x + v\zeta_y \quad (27)$$

Substituting equations (26) and (27) into equation (25), one obtains,

$$uH_x + vH_y + \zeta_t + u\zeta_x + v\zeta_y + (\zeta + H)(u_x + v_y) = 0$$

which can be written as

$$\zeta_t + [u(\zeta + H)]_x + [v(\zeta + H)]_y = 0 \quad (28)$$

Equations (23), (24), and (28) are the three equations in  $u$ ,  $v$  and  $\zeta$  for the non-linear shallow water theory.

#### Steady State Equations:

Let us now consider the above equations in the steady state, i.e., no time dependence. Let the velocity scale be  $U$  and the length scale be  $L$ , and define  $h$  as follows:

$$H = H_0 h, \quad (29)$$

where  $H_0$  is a constant. Equation (23) may be written with  $u$ 's and  $v$ 's as follows (Note: we will use the same notation as before; thus, in what follows  $u$  and  $v$  will be dimensionless):

$$\frac{U^2}{L} (uu_x + vv_y) + \frac{g}{L} \zeta_x - fUv = 0 \quad (30)$$

We now define a dimensionless  $\zeta^*$ .

$$\zeta^* = \frac{g}{LfU} \zeta \quad (31)$$

Using this definition and changing our notation as before so that  $\zeta$  is now dimensionless, we can write equation (30) as

$$\frac{U^2}{L} (uu_x + vv_y) + fU\zeta_x - fUv = 0 \quad (32)$$

Similarly equations (24) and (28) can be written as

$$\frac{U^2}{L} (uv_x + vv_y) + fU\zeta_y + fUu = 0 \quad (33)$$

$$\left[ u \left( \frac{LfU}{g} \zeta + H_0 h \right) \right]_x + \left[ v \left( \frac{LfU}{g} \zeta + H_0 h \right) \right]_y = 0 \quad (34)$$

The dimensionless parameter  $\epsilon$  called the Rossby number is defined by

$$\epsilon \equiv \frac{U}{fL} \quad (35)$$

and another dimensionless parameter  $N$  is defined by

$$\frac{LfU}{gH_0} = \frac{U}{fL} \cdot \frac{(Lf)^2}{gH_0} \equiv \epsilon N \quad (36)$$

For an ocean  $\epsilon$  is of the order of  $10^{-4}$  and  $N$  is of the order of 1.

Using these definitions the dimensionless steady state equations

can be written as

$$\epsilon (uu_x + vu_y) + \zeta_x - v = 0 \quad (37)$$

$$\epsilon (uv_x + vv_y) + \zeta_y + u = 0 \quad (38)$$

$$[u(h + \epsilon N\zeta)]_x + [v(h + \epsilon N\zeta)]_y = 0 \quad (39)$$

These are the equations of an inertial boundary layer.

submitted by R. Ellis

We wish now to study these equations for the case of small  $\epsilon$  and  $N$  of order 1. The equations for the "external flow" are obtained by formally setting  $\epsilon = 0$ :

$$\zeta_x - v = 0 \quad (40)$$

$$\zeta_y + u = 0 \quad (41)$$

$$(uh)_x + (vh)_y = 0 \quad (42)$$

These are the equations of geostrophic flow, of lecture 6, p.2. If  $h$  is not constant they show that the lines of constant  $\zeta$  are parallel to the lines of constant  $h$ , and give also the streamlines. In particular, if  $h$  is, say, a function of  $y$  alone, the external flow must be along lines  $y = \text{constant}$ , and accordingly will not be able to satisfy appropriate boundary conditions on a solid wall  $x = \text{const.}$  We thus expect the development of boundary layers, "inertial boundary layers", on such walls along which depth is not constant, in which

the inertial terms become important. We examine a boundary layer at  $x = 0$  by setting  $x = \varepsilon^a \xi$ ,  $v = \varepsilon^{-b} V$ . We then have:

$$\varepsilon^{1-a} u u_{\xi} + \varepsilon^{1-b} v v_y - \varepsilon^{-b} V + \varepsilon^{-a} \zeta_{\xi} = 0 \quad (43)$$

$$\varepsilon^{1-a-b} u v_{\xi} + \varepsilon^{1-2b} v v_y + u + \zeta_y = 0 \quad (44)$$

$$(\varepsilon^{-a} u_{\xi} + \varepsilon^{-b} v_y) h + \varepsilon^{-b} V h'(y) = 0 \quad (45)$$

We must have  $u = 0$  on  $\xi = 0$ , and, for the boundary layer flow,  $u = u_1(y)$ ,  $V = 0$ , as  $\xi \rightarrow \infty$  where  $u_1(y)$  is the external geostrophic flow. If  $b > a$ , the boundary layer approximation to the first equation will be  $V = 0$ , which is not usable. If  $b < a$  the boundary layer approximation to the third equation will be  $u_{\xi} = 0$ , which is also not usable, since it cannot agree both with  $u(0, y) = 0$  and  $u(\infty, y) = u_1(y)$ . Thus we must take  $a = b$ , and the first and third boundary layer equations are

$$-V + \zeta_{\xi} = 0 \quad (46)$$

$$(u_{\xi} + v_y) h + V h'(y) = 0 \quad (47)$$

If  $a < \frac{1}{2}$  the second boundary layer equation is  $u + \zeta_y = 0$ , i.e. we are back to the geostrophic equations which cannot satisfy all conditions. If  $a > \frac{1}{2}$  the second equation is  $u v_{\xi} + v v_y = 0$ , i.e.  $V$  is constant on streamlines, and since  $V = 0$  at  $\infty$ ,  $V = 0$  everywhere which does not work. We are thus led finally to  $a = b = \frac{1}{2}$ , and the second boundary layer equation becomes:

$$uV_{\xi} + vV_y + u + \zeta_y = 0 \quad (48)$$

If we eliminate  $\zeta$  from these equations we get:

$$\begin{cases} (uV_{\xi} + vV_y)_{\xi} + u_{\xi} + v_y = 0 \\ (uh)_{\xi} + (vh)_y = 0 \end{cases} \quad (49)$$

$$(50)$$

A first integral of these is obtained as follows:

$$uV_{\xi\xi} + vV_{\xi y} + (u_{\xi} + v_y)(1 + v_{\xi}) = 0 \quad (51)$$

$$u h_{\xi} + v h_y + (u_{\xi} + v_y) h = 0 \quad (52)$$

$$\therefore (u \frac{\partial}{\partial \xi} + v \frac{\partial}{\partial y}) \left( \frac{v_{\xi} + 1}{h} \right) = 0 \quad (53)$$

$$\text{Let } uh = \psi_y, \quad vh = -\psi_{\xi}$$

Thus  $\frac{v_{\xi} + 1}{h}$  is constant on streamlines, i.e. is a function of  $\psi$ ,

and our first integral is

$$\frac{1}{h} \left[ - \left( \frac{\psi_{\xi}}{h} \right)_{\xi} + 1 \right] = F(\psi) \quad (54)$$

This expresses the conservation of "potential vorticity",

$\frac{v_{\xi} + 1}{h}$ , along streamlines.

Example: Suppose  $h(y) = \frac{1}{1+\beta y}$  for  $y \geq 0$  and suppose the external

flow field has  $\psi_e(o,y) = -y$ . Letting  $\xi \rightarrow \infty$ ,  $\psi_{\xi} \rightarrow 0$  and so we have

$$\frac{1}{h(y)} [0 + 1] = F(\psi_e(o,y)) = F(-y) \quad (55)$$

This determines  $F$ :  $F(\Psi) = \frac{1}{h(-\Psi)} = 1 - \beta \Psi$ . We then have:

$$\frac{1}{h} \left[ -\frac{1}{h} \Psi \xi \xi + 1 \right] = 1 - \beta \Psi, \text{ or}$$

$$\Psi \xi \xi - \beta h^2 \Psi = h - h^2 \quad (57)$$

$$\therefore \Psi = -\frac{h-h^2}{\beta h^2} + A(y) e^{-\sqrt{\beta} h(y) \xi} + B(y) e^{+\sqrt{\beta} h(y) \xi} \quad (58)$$

Since we are taking  $0 \leq \xi < \infty$  we must have  $B = 0$ , and since  $\frac{h-h^2}{\beta h^2} = \frac{1}{\beta} \left[ \frac{1}{h} - 1 \right] = \frac{1}{\beta} [1 + \beta y - 1] = y$  we have,

making  $\Psi = 0$  on  $\xi = 0$ ,  $A(y) = y$ . Thus

$$\Psi = -y [1 - e^{-\sqrt{\beta} h(y) \xi}] \quad (59)$$

Isotropic Tensors

1. An isotropic tensor is one whose components are the same in all orthogonal Cartesian coordinate systems. If the components are the same in all right-handed systems, and simply change sign for the left-handed systems, the tensor may be called pseudo-isotropic. (It is easy to see that if the components of a pseudo-isotropic tensor in right-handed systems are taken also in left-handed systems, we obtain a system of components defining an isotropic pseudo-tensor.) Here we shall determine all isotropic tensors of order 2 and 4, in  $n$  dimensions ( $n > 1$ ), and will also point out the pseudo-isotropic tensors. We do this by using the invariance of the components for a certain class of coordinate transformations (the "infinitesimal orthogonal transformations") to deduce necessary conditions for isotropy, and verify afterwards that these conditions are sufficient. In fact, since the transformations we use carry right-handed systems into right-handed systems, the necessary conditions will actually be satisfied by pseudo-isotropic as well as isotropic tensors; however, it turns out to be easy afterwards to identify the pseudo-isotropic tensors, and these have indeed an interest of their own.

2. We now consider a family of orthogonal transformations:

$x_i \rightarrow A_{ij}x_j$  depending differentiably on a parameter  $t$ , and such that

$A_{ij}(0) = \delta_{ij}$ . Thus "small  $t$ " means that the transformation is "close"

to the identity transformation, and we have in fact

$A_{ij}(t) = \delta_{ij} + tF_{ij} + o(t)$  where  $F_{ij} = \left. \frac{d}{dt}(A_{ij}) \right|_{t=0}$ . We say that the

"infinitesimal transformation" is determined by the coefficients  $F_{ij}$ ,

which in fact form an antisymmetric matrix:  $F_{ij} = -F_{ji}$ . This follows

from the orthogonality of  $A_{ij}$ , for differentiating  $A_{ij}A_{ik} = \delta_{jk}$  and

setting  $t = 0$  we obtain  $F_{ij} \delta_{ik} + \delta_{ij} F_{ik} = 0$ , or  $F_{kj} + F_{jk} = 0$ . Furthermore, given any antisymmetric matrix  $F = [F_{ij}]$  we can find a family of orthogonal matrices  $A(t) = [A_{ij}(t)]$  for which  $F$  defines the associated infinitesimal transformation. This can be done in many ways - one is to set  $A(t) = \exp(tF)$ , as is easily verified. Since  $A(t)$  is orthogonal, its determinant is  $\pm 1$ , but since it depends continuously on  $t$  and is  $+1$  at  $t = 0$  it must be  $+1$  always. Thus these transformations all carry right-handed systems into right-handed systems.

3. We now consider an isotropic second order tensor with components  $T_{ij}$ . From invariance under the transformations  $A(t)$  we have  $A_{ki} A_{lj} T_{ij} = T_{kl}$ . Differentiate this with respect to  $t$  and set  $t = 0$  to obtain:  $F_{ki} T_{il} + F_{lj} T_{kj} = 0$ , or  $[\delta_{im} T_{nj} + \delta_{jm} T_{in}] F_{mn} = 0$ . For any isotropic tensor  $T_{ij}$  this relation must hold, and it must hold for arbitrary antisymmetric  $F_{mn}$ . Thus the quantity in the bracket must in fact be symmetric in  $m$  and  $n$ , i.e.:

$$\delta_{im} T_{nj} + \delta_{jm} T_{in} = \delta_{in} T_{mj} + \delta_{jn} T_{im} \quad (1)$$

This system of linear equations must be satisfied by the components of any isotropic tensor. Any multiple of  $\delta_{ij}$  is easily seen to be isotropic, so if  $T_{ij}$  is isotropic,  $S_{ij} = T_{ij} - \frac{1}{n} T_{kk} \delta_{ij}$  is also.  $S_{ij}$  has zero trace,  $S_{kk} = 0$ . We shall show in fact that  $S_{ij} = 0$ , so that  $T_{ij}$  will be a multiple of  $\delta_{ij}$ .  $S_{ij}$  must satisfy equation (1). Putting this in and contracting on the indices  $i, m$  we get:

$$n S_{nj} + S_{jn} = S_{nj} + \delta_{jn} S_{mm}, \text{ or since } S_{mm} = 0,$$

$$(n-1) S_{nj} + S_{jn} = 0 \quad (2)$$

Interchanging  $n$  and  $j$  in this we obtain

$$S_{nj} + (n-1) S_{jn} = 0 \quad (3)$$

We may regard (2) and (3) as simultaneous equations for  $S_{nj}$  and  $S_{jn}$ . The coefficient determinant is  $(n-1)^2 - 1 = n(n-2) \neq 0$

unless  $n = 2$ . ( $n > 1$  in any case.) Thus for  $n \neq 2$  we find  $S_{nj} = 0$  and the only possible isotropic tensors of second order are multiples of  $\delta_{ij}$ , which in fact are isotropic. If  $n = 2$ , equation (2) gives  $S_{nj} + S_{jn} = 0$ , the only solutions of which are easily seen to be multiples of  $\epsilon_{ij}$ . We conclude that in this case the only possibilities for isotropic second order tensors are of the form

$$T_{ij} = \lambda \delta_{ij} + \mu \epsilon_{ij}$$

These do have the same components in all right-handed systems, but on going over to a left-handed system one easily checks that

$$\epsilon_{ij} \longrightarrow -\epsilon_{ij}, \text{ hence the only isotropic tensors are multiples}$$

of  $\delta_{ij}$  but there are pseudo-isotropic tensors with components  $\mu \epsilon_{ij}$  in right-handed systems and  $-\mu \epsilon_{ij}$  in left-handed systems, and these are the only pseudo-isotropic second order tensors.

4. We proceed now to isotropic fourth order tensors  $T_{ijkl}$ .

Such tensors have three possibly different scalar traces, namely

$T_{mmnn}$ ,  $T_{mnmn}$ , and  $T_{nmmn}$ . Three obviously isotropic fourth order

tensors are  $\delta_{ij} \delta_{kl}$ ,  $\delta_{ik} \delta_{jl}$ ,  $\delta_{il} \delta_{jk}$ . We wish to show that

every isotropic fourth order tensor is a linear combination of these. To do this, we first write

$$S_{ijkl} = T_{ijkl} - A \delta_{ij} \delta_{kl} - B \delta_{ik} \delta_{jl} - C \delta_{il} \delta_{jk}$$

and determine A, B, and C so that all the scalar traces of  $S_{ijkl}$  are zero. This is possible; in fact we have only to solve the equations:

$$n^2 A + nB + nC = T_{mmnn}$$

$$nA + n^2 B + nC = T_{mnmn}$$

$$nA + nB + n^2 C = T_{mnmn}$$

whose coefficient determinant is  $n^3(n-1)^2(n+2) \neq 0$  for  $n > 1$ .

We now investigate  $S_{ijkl}$ . Proceeding as above, we find that invariance under infinitesimal orthogonal transformations implies  $F_{in} S_{njkl} + F_{jn} S_{inkl} + F_{kn} S_{ijnl} + F_{ln} S_{ijkn} = 0$  for arbitrary anti-symmetric  $F_{mn}$ . This then implies as before that

$$\begin{aligned} & \delta_{im} S_{njkl} + \delta_{jm} S_{inkl} + \delta_{km} S_{ijnl} + \delta_{lm} S_{ijkn} \\ &= \delta_{in} S_{mjkl} + \delta_{jn} S_{imkl} + \delta_{kn} S_{ijml} + \delta_{ln} S_{ijkm} \end{aligned} \quad (4)$$

This is analogous to equation (1).

Now we have arranged that all the scalar traces of  $S_{ijkl}$  are zero, but this implies that also all the partial traces, like  $S_{mmkl}$ , are zero also. For  $S_{mmkl}$  is an isotropic second order tensor, hence must actually have the form  $S_{mmkl} = \frac{1}{n} S_{mmnn} \delta_{kl}$  and so be zero. A similar argument applies to the other partial traces. Note that if we also allow pseudo-isotropic tensors, there are other possibilities

(134)

if  $n = 2$ , but we shall not discuss these. Now contract (4) on the indices  $i, m$ , using the fact that all the partial traces vanish, to get:

$$(n-1)S_{njkl} + S_{jnkl} + S_{kijnl} + S_{ljkn} = 0 \quad (5)$$

This is analogous to equation (2). By permuting the indices and manipulating the resulting equations we can now show that  $S_{ijkl} = 0$ , if  $n \neq 4$ . Because of the greater complexity here, however, it is best to use a more efficient notation. We rewrite equation (5) as

$$\left[ (n-1)I + (12) + (13) + (14) \right] S_{njkl} = 0 \quad (6)$$

where we think of the permutation symbols as operators on the (linear) space of isotropic 4th order tensors with zero traces.

As such operators, (6) says simply

$$(n-1)I + (12) + (13) + (14) = 0 \quad (7)$$

(Here 0 means the zero operator.) We shall deduce from (7) that  $I = 0$  (if  $n \neq 4$ ) and so that the "space" of the  $S_{ijkl}$  consists only of the zero tensor.

Multiplying (7) on the left and right by (12), (13), and (14) successively we get:

$$(n-1)I + (12) + (23) + (24) = 0$$

$$(n-1)I + (23) + (13) + (34) = 0$$

$$(n-1)I + (24) + (34) + (14) = 0$$

Subtracting (7) from the sum of these three equations and dividing by 2 gives:

$$(n-1)I + (23) + (24) + (34) = 0 \quad (8)$$

Now multiply this on the left by (234) to get:

$$(n-1)(234) + (24) + (34) + (23) = 0$$

Using this with (8) we have  $I = (234)$ , and similarly (or by squaring this last result) we see also that  $I = (243)$ . Now multiply (8) on the left by (23), getting  $(n-1)(23) + I + (243) + (234) = 0$  and so

$$3I + (n-1)(23) = 0$$

Multiplying this by (23) gives

$$(n-1)I + 3(23) = 0$$

and so  $I = (23) = 0$  unless  $3^2 - (n-1)^2 = -(n-4)(n+2) = 0$ , which can occur only when  $n = 4$ , since  $n > 1$ . Thus, with this exception, we have  $I = 0$ , which implies that the only isotropic 4th order tensors are of the form

$$T_{ijkl} = A \delta_{ij} \delta_{kl} + B \delta_{ik} \delta_{jl} + C \delta_{il} \delta_{jk}$$

By investigating the exceptional case of  $n = 4$  it can be shown that the only possible form for  $S_{ijkl}$  is a multiple of  $\epsilon_{ijkl}$  and that this gives a pseudo-isotropic tensor, not an isotropic one. Thus the result holds in general.

It is sometimes more convenient to write the general isotropic fourth order tensor in the equivalent form

$$T_{ijkl} = A \delta_{ij} \delta_{kl} + B' (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + C' (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

where the coefficient of  $B'$  is symmetric in  $ij$  (and  $kl$ ) and the coefficient of  $C'$  is antisymmetric in these indices.

## 5. Exercises for the Enterprising Student.

a) Use the above techniques to show that there are no isotropic third order tensors (except 0), but for  $n = 3$  there are isotropic pseudo-tensors of the form  $\lambda \epsilon_{ijk}$ .

b) Investigate the exceptional cases  $n = 2$  and  $4$  for  $n$ th order tensors.

c) In 3 dimensions, show with a minimum of computation, by using the above facts about isotropic tensors, that

$$\epsilon_{ijn} \epsilon_{kln} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$$

1. Show that  $\Phi \geq 0$  for all velocity fields is equivalent to  $\mu \geq 0$  and  $3\lambda + 2\mu \geq 0$ .

$$\Phi = \lambda (\nabla \cdot \vec{u})^2 + \frac{\mu}{2} \sum_{i,j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2$$

Regard  $\Phi$  as a quadratic form in the six variables

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = d_{ij} \text{ , say, } 1 \leq i \leq j \leq 3 :$$

$$\Phi = \frac{\lambda}{4} [d_{11} + d_{22} + d_{33}]^2 + \frac{\mu}{2} [(d_{11}^2 + d_{22}^2 + d_{33}^2) + 2(d_{12}^2 + d_{13}^2 + d_{23}^2)]$$

$$= \left\{ \frac{\lambda}{4} [d_{11} + d_{22} + d_{33}]^2 + \frac{\mu}{2} [d_{11}^2 + d_{22}^2 + d_{33}^2] \right\} + \mu \{ d_{12}^2 + d_{13}^2 + d_{23}^2 \}$$

This is the sum of two three-variable quadratic forms, in  $d_{11}, d_{22}, d_{33}$  and  $d_{12}, d_{13}, d_{23}$ , and so is positive semidefinite if and only if they both are. For the second, this is the case if and only if  $\mu \geq 0$ . For the first, it will be positive semidefinite if and only if all eigenvalues of its matrix are non-negative. The matrix is:

$$\begin{bmatrix} \frac{\lambda}{4} + \frac{\mu}{2} & \frac{\lambda}{4} & \frac{\lambda}{4} \\ \frac{\lambda}{4} & \frac{\lambda}{4} + \frac{\mu}{2} & \frac{\lambda}{4} \\ \frac{\lambda}{4} & \frac{\lambda}{4} & \frac{\lambda}{4} + \frac{\mu}{2} \end{bmatrix} = \frac{\lambda}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{\mu}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let  $a_1, a_2, a_3$  be the eigenvalues of  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ . Then clearly the

eigenvalues we are seeking are just  $\frac{\lambda}{4} a_i + \mu/2$ . But the  $a_i$  are easily computed to be 0, 0, and 3, hence the eigenvalues which must be  $\geq 0$  are  $\mu/2$ ,  $\mu/2$  and  $\frac{1}{4}(3\lambda + 2\mu)$ . Thus  $\Phi \geq 0$  for all  $d_{ij}$  if and only if both  $\mu \geq 0$  and  $3\lambda + 2\mu \geq 0$ .

2. Show that relative to a system rotating with constant angular velocity  $\vec{\omega}$ , the acceleration term  $\frac{D\vec{u}}{Dt}$  in the momentum equation should be replaced by  $\frac{D\vec{u}}{Dt} + 2\vec{\omega} \times \vec{u}$ , and the body force term  $\rho \vec{f}$  increased by  $\rho \nabla \frac{1}{2} \omega^2 R^2$ , where  $R$  is the distance from the axis of rotation.

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Let  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$  be the position vector. As a vector, it is the same in both systems, but its representation in terms of components is different, say  $(xyz)$  in the fixed system,  $(x'y'z')$  in the rotating system. (We assume the origins are coincident and on the axis of rotation.) The unit vectors  $\vec{i}'$ ,  $\vec{j}'$ ,  $\vec{k}'$  of course change with time (as seen from the fixed system), and in such a way that the position vector of a point fixed in the rotating system ( $x'y'z'$  const.) rotates with angular velocity  $\vec{\omega}$ . Thus for such a fixed point we have:

$$\vec{\omega} \times \vec{r} = \frac{d\vec{r}}{dt} = x' \frac{d\vec{i}'}{dt} + y' \frac{d\vec{j}'}{dt} + z' \frac{d\vec{k}'}{dt}$$

Consider now the particle velocity  $\vec{u} = \frac{D\vec{r}}{Dt}$  in a fluid motion. We have

$$\begin{aligned}\vec{u} &= \frac{D\vec{h}}{Dt} = \frac{Dx'}{Dt}\vec{e}' + \dots + x' \frac{D\vec{e}'}{Dt} + \dots \\ &= \vec{u}' + \vec{\omega} + \vec{h}\end{aligned}$$

Since  $\vec{u}'$ , the apparent velocity in the rotating system, is the vector with components  $\frac{Dx'}{Dt}$  . . . and  $\vec{e}'$  etc. are functions of  $t$  alone so that all time derivatives of them are the same.

In the same way we calculate the acceleration:

$$\begin{aligned}\frac{D\vec{u}}{Dt} &= \frac{D}{Dt} (\vec{u}'\vec{e}' + \dots) + \vec{\omega} \times \frac{D\vec{h}}{Dt} \\ &= \frac{Du'}{Dt}\vec{e}' + \dots + \vec{\omega} \times \vec{u}' + \omega \times \vec{u} \\ &= \left(\frac{D\vec{u}'}{Dt'}\right) + 2\vec{\omega} \times \vec{u}' + \vec{\omega} \times (\vec{\omega} \times \vec{h})\end{aligned}$$

where  $\frac{D\vec{u}'}{Dt'}$  is the apparent acceleration in the rotating system.

$$\text{Now } \vec{\omega} \times (\vec{\omega} \times \vec{h}) = \vec{\omega} \cdot \vec{h} \omega - \omega^2 \vec{h} = -\omega^2 \left(\vec{h} - \frac{\vec{\omega} \vec{\omega} \cdot \vec{h}}{\omega^2}\right) = -\omega^2 \vec{R}$$

where  $\vec{R}$  is the part of  $\vec{h}$  perpendicular to  $\vec{\omega}$ . Evidently

$$R = \frac{1}{2} \nabla R^2 \text{ where } R = |\vec{R}|. \text{ We can now substitute into the equations}$$

of motion in the fixed system and find the equations in terms of

$\vec{u}'$  etc. in the rotating system. This gives

$$\rho \left( \frac{D\vec{u}'}{Dt'} + 2\vec{\omega} \times \vec{u}' \right) - \rho \nabla \left( \frac{\omega^2 R^2}{2} \right) + \nabla p = \nabla \cdot \vec{\tau} + \rho \vec{f}$$

where  $\vec{\tau}$  is the stress tensor. This equation then is similar to the equation in the non-rotating system with the addition of the Coriolis acceleration  $2\vec{\omega} \times \vec{u}'$  and the centrifugal force  $\frac{1}{2}\rho \nabla (\omega^2 R^2)$  (which can be added into the  $\vec{f}$  term). We should also check that the stress tensor is given by the same formula in the rotating as in the fixed system. For the Navier-Stokes stress tensor it is sufficient to show this for the deformation tensor  $d$  whose components are

$$d_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

This is clearly the same tensor in both systems (though its components will be different) from its geometrical interpretation in terms of the rate of deformation which is not affected by a rigid rotation; however, this can be explicitly verified as follows: Let  $\hat{u}_i$  be the components of  $\vec{u}'$  in the fixed system, i.e.  $\vec{u}' = u'_i \vec{i}' + \dots = \hat{u}_i \vec{i} + \dots$ . Then  $u_i = \hat{u}_i + \epsilon_{ilk} \omega_l x_k$  and so

$$\begin{aligned} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} &= \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} + \epsilon_{ilk} \omega_l \delta_{jk} + \epsilon_{jlk} \omega_l \delta_{ik} \\ &= \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} + (\epsilon_{ilj} + \epsilon_{jli}) \omega_l \\ &= \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}, \text{ since } \epsilon_{ilk} = -\epsilon_{kli} \end{aligned}$$

Thus  $d$  is the same as the deformation tensor of the relative velocity, at least for the components in the fixed system. But then this holds also for the components in the rotating system, since no time derivatives are involved here and for a fixed time the basis vectors  $\vec{i}'$  etc. for the rotating system are obtained from  $\vec{i}$  etc. by an orthogonal transformation, which of course leaves equality of tensor components unchanged.

3. A deep cylindrical tank contains water rotating with more or less uniform angular velocity. A small plug is pulled out of the bottom, and a strong vortex is observed to form, which, at least for sufficiently strong initial circulation and a large enough hole, has a hollow core extending down to the bottom and out the

drain. Discuss this from the point of view of the general hydrodynamical theorems, e.g. The Bernoulli, Kelvin, Helmholtz, and Taylor-Proudman theorems.

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Remarks: This problem does not of course have a simple unique answer. Some observations which contribute to an understanding of the flow are given here.

a) The most striking feature of the flow is the very long thin column of air down the center. (Remark: the experiment can easily be set up by the home handyman and is well worth performing). An intuitive and not wholly incorrect explanation of this is given by the Taylor-Proudman theorem. We suppose the motion referred to a coordinate system rotating with the original motion of the fluid. If the flow were steady after the drain is opened (which it isn't, but never mind for now) the simple form of the Taylor-Proudman theorem, equ. (18) of lecture 4, gives  $\frac{\partial \omega}{\partial z} = 0$ , i.e.  $\omega$  is constant along vertical lines, and since  $\omega$  is zero on the solid bottom and vertically downward on the drain hole, we conclude that  $\omega$  is zero everywhere except directly above the drain, i.e. the cylinder above the drain falls out the bottom and the remaining water is held in position by Taylor and Proudman. This is of course not really a legitimate application of the theorem, for the initial motion is certainly not steady and the motion which develops after a few seconds, while roughly steady, is certainly not a small perturbation

of the original uniform rotation, the angular speed near the center being greatly increased. A legitimate application of the theorem can be made, however, if the time dependent form (lecture 4, eq. (17)) is used and attention is restricted to short times after pulling the plug, before the deviation from steady rotation has become large. The vertical component of the equation is:

$$\frac{\partial \Omega_3}{\partial t} = 2 \omega \frac{\partial w}{\partial z}$$

If  $\frac{\partial w}{\partial z}$  were zero, the central core would just fall out the bottom as described above. It is clear, however, that the tendency for this to happen will be to some extent offset by horizontal motions carrying water toward the center. This means that there will be a negative horizontal divergence  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$  near the center and consequently, from the continuity equation, a positive value of  $\frac{\partial w}{\partial z}$  there. Thus if  $\omega$  is positive,  $\frac{\partial \Omega_3}{\partial t} > 0$  and we see that we must expect the development of vertical vorticity near the center, with the same sense of rotation as the original steady rotation. This is just what happens. The problem of studying the initial development of the motion, using linearized equations, deserves more attention but will not be discussed further since we are here attempting only a general description of the motion.

b) After the development of the vortex, the motion appears to be essentially steady, except for the slow downward displacement of the surface. Neglecting the latter, or supposing it eliminated by a slow seepage of water in through the solid bottom, we consider a circle of particles lying on the surface. From symmetry this circle of particles remains a circle as the particles spiral in toward the

center and down the vortex. Returning to a fixed coordinate system, we describe the free surface in cylindrical coordinates by  $z = f(r)$  and let  $v$  be the  $\theta$ -component of velocity. From the Kelvin circulation theorem, the circulation around the above circle of particles, namely  $2\pi r v(r, f(r))$  is a constant, and so this function must in fact be constant over the free surface since the surface is traversed by the particles. We may estimate this constant by taking  $v$  at the outer edge of the cylinder,  $r = R$ , say, to be the same as it was before the plug was pulled, namely  $R\omega$ . Thus we have

$$v(r, f(r)) \cong \frac{R^2\omega}{r}$$

Thus the angular velocity increases rapidly as one approaches the center. If the experiment is performed and a bit of wood is dropped on the surface, it spirals in toward the center, and is seen to descend rapidly down the vortex, but usually in a rather tight spiral, indicating that, although the downward speed is considerable near the center, the increase in  $v$  is so great that the velocity vector is still nearly horizontal.

c) Bernoulli's theorem tells us that for steady flow  $\frac{1}{2} q^2 + p + gz$  is constant along streamlines. The free surface is covered by spiral streamlines, and by symmetry the above constant will be the same for all streamlines on the surface and hence constant over the surface. Neglecting effects of surface tension,  $p$  is zero on the surface and so  $\frac{1}{2} q^2 + gz$  must be constant on the surface. If we use the experimental observation that the spiral

streamlines on the surface are rather tight we may approximate the speed  $q$  by the  $v$  component, and this gives an equation for the surface:

$$q \cong v \cong \frac{R^2 \omega}{r}, \quad \text{hence } \frac{1}{2} \left( \frac{R^2 \omega}{r} \right)^2 + g f(r) \cong \text{const.}$$

$$\text{or } z = f(r) \cong C - \frac{R^4 \omega^2}{g r^2}$$

This form of the free surface is certainly in good qualitative agreement with the one that actually occurs.

4. Solve the linear wave-maker problem, i.e. the end  $x = 0$  of a semi-infinite channel ( $x \geq 0, -H \leq z \leq 0, 0 \leq y \leq b$ ) moves so that the horizontal velocity  $\phi_x$  at  $x = 0$  is a given function  $e^{i\sigma t} f(z)$ . Find the resulting periodic motion, assuming that as  $x \rightarrow +\infty$  only an outgoing wave ( $e^{i(\sigma t - kx)}$ ) is present.

---

(cf. Lecture 5, pp 1-4).

The motion is to be independent of  $y$ , and the vertical eigenfunctions are (not normalized)

$$\phi_0 = \cosh k(z+H)$$

$$\phi_n = \cos K_n(z+H) \quad n = 1, 2, \dots$$

where  $\sigma^2 = gk \tanh kH = -g K_n \tan K_n H$

$$\text{Thus } \phi = A_0 e^{-ikx} \cosh k(z+H) + \sum_1^{\infty} A_n e^{-k_n x} \cos K_n(z+H)$$

Taking account of the fact that when the  $e^{i\sigma t}$  factor is restored we must have  $\phi \sim \text{const } e^{i(\sigma t - kx)} \cosh k(z+H)$  from the out-going wave

condition at  $+\infty$ . This satisfies Laplace's equation, the free surface condition, and the boundary condition at  $\infty$ . At  $x = 0$  we have  $\phi_x = f(z)$ , or

$$f(z) = -ikA_0 \cosh k(z+H) - \sum_{n=1}^{\infty} K_n A_n \cos K_n (z+H)$$

From the orthogonality of the eigenfunctions, we get

$$-ikA_0 \int_{-H}^0 \cosh^2 k(z+H) dz = \int_{-H}^0 f(z) \cosh k(z+H) dz$$

$$-K_n A_n \int_{-H}^0 \cos^2 K_n (z+H) dz = \int_{-H}^0 f(z) \cos K_n (z+H) dz$$

Further details are left to the reader. Note that if  $f(z)$  is real, i.e. all parts of the wave maker are in phase, then all the damped waves have real amplitudes  $A_n$  and are standing waves in phase with the wave maker. Note also that it is entirely possible for the wave maker to produce no wave at large distances. This occurs if  $f(z)$  is orthogonal to  $\phi_0(z)$ , for example if  $f(z)$  is one of the  $\phi_n$ ,  $n \geq 1$ .

5. Discuss the free standing waves (normal modes) in a closed circular lake of depth  $H$  and radius  $R$ .

---

Here we have vertical eigenfunctions

$$\phi_0 = \cosh k(z+H), \quad \phi_n = \cos K_n (z+H) \quad \text{with}$$

$$\sigma^2 = gk \tanh kH = -g K_n \tan K_n H$$

as before, but now the values of  $k$  and  $K_n$  are to be determined from the homogeneous boundary condition of zero radial velocity, and this

determines the various natural frequencies  $\sigma$ . If we write  
 $\phi = a_0(x, y)\phi_0 + \sum a_n \phi_n$  we have  $\frac{\partial \phi}{\partial r} = 0$  on  $r = R$ ,  
 hence  $\frac{\partial a_n}{\partial r} = 0$  on  $r = R$ . Also  $\Delta a_0 + k^2 a_0 = 0, \Delta a_n - k_n^2 a_n = 0$ .

In general, these imply  $a_0 = a_n = 0$ , and this is always  
 the case for  $a_n$  since all eigenvalues of  $\Delta$  are negative, but  
 for special values of  $k$ ,  $a_0$  may differ from zero. If we write  
 $a_0 = e^{im\theta} f(r)$  we get

$$f'' + \frac{1}{r} f' + \left(k^2 - \frac{m^2}{r^2}\right) f = 0$$

So  $f = C J_m(kr)$  and the possible values of  $k$  are deter-  
 mined from

$$J_m'(kR) = 0$$

If the  $l^{\text{th}}$  positive root of  $J_m'(x)$  is denoted by  $\alpha_l^{(m)}$

we thus have for possible values of  $k$ :

$$k_l^{(m)} = \frac{1}{R} \alpha_l^{(m)}$$

which gives for the natural frequencies

$$[\sigma_l^{(m)}]^2 = \frac{g}{R} \alpha_l^{(m)} \tanh(\alpha_l^{(m)} H)$$

$$m = 0, \pm 1, \pm 2, \dots$$

$$l = 1, 2, \dots$$

The associated eigenfunctions are:

$$\phi = e^{i(\sigma_l^{(m)} t + m\theta)} J_m(k_l^{(m)} r)$$

6. A speedboat going roughly parallel to shore produces a group of waves which reach the shore 10 min. later and have a period of 2 sec. Estimate the distance off shore of the boat.

---

A really careful treatment of this problem is quite difficult, but for a rough estimate we compute the group velocity of 2 sec. waves. This is

$$c_g = \frac{d\sigma}{dk} = \frac{d}{dk} (\sqrt{gk}) = \frac{1}{2} \sqrt{\frac{g}{k}}$$

$$= \frac{1}{2} \frac{g}{\sigma}$$

For  $\sigma = 2\pi/2 \text{ sec}^{-1}$  this gives  $c_g = \frac{1}{2} \frac{g}{\pi}$

In ten minutes, a group of waves with this speed goes a distance

$$D \cong \frac{1}{2\pi} 10 \frac{\text{m}}{\text{sec}} \cdot 10 \cdot 60 \cong 1 \text{ km.}$$

7. Investigate the natural frequencies of a shallow paraboloidal lake.  
(cf. Lamb, § 193).

---

Take  $H = H_0 (1 - (\frac{r}{R})^2)$ . Using cylindrical coordinates, we have, setting  $\zeta = f(r) \cos n\theta$

$$\nabla \cdot (H \nabla \zeta) = H \nabla^2 \zeta + H'(r) \frac{\partial \zeta}{\partial r} = \left\{ H \left[ f'' + \frac{1}{r} f' - \frac{n^2}{r^2} f \right] - 2 \frac{H_0}{R^2} r f' \right\} \cos n\theta = -\frac{\sigma^2}{g} f \cos n\theta$$

setting  $x = r/R$ ,  $f(r) = g(x)$  this gives

$$(1-x^2) \left[ g'' + \frac{1}{x} g' - \frac{n^2}{x^2} g \right] - 2xg' + \lambda g = 0$$

where 
$$\lambda = \frac{\sigma^2 R^2}{g H_0}$$

Boundary conditions are regularity of  $g(x)$  at  $x = 0$  and boundedness at  $x = 1$ .

Let  $g(x) = x^n \sum_0^{\infty} a_k x^k$  and substitute this in the differential equation to get:

$$\sum_0^{\infty} a_k \left[ (n+k)^2 - n^2 \right] x^{n+k-2} + \sum_0^{\infty} a_k \left[ n^2 - (n+k)^2 - 2(n+k) + \lambda \right] x^{n+k} = 0$$

Thus we must have  $n = n$ ,  $a_1 = 0$  and then

$$a_{k+2} = - \frac{n^2 - (n+k)(n+k+2) + \lambda}{(k+2)(2n+k+2)} a_k, \quad k \geq 0$$

The solution will be a polynomial and so certainly bounded at  $x = 1$  if  $\lambda$  is one of the values

$$\begin{aligned} \lambda_{nm} &= (n+m)(m+m+2) - n^2 \\ &= n(2m+2) + m(m+2) \end{aligned}$$

for  $n = 0, 1, 2, \dots$   $m = 0, 2, 4, 6, \dots$

with  $n = m = 0$  excluded on account of the condition  $\iint \zeta dx dy = 0$ .

(More detailed investigation of the differential equation shows that these are the only eigenvalues, i.e. the solution is not bounded at  $x = 1$  unless it is a polynomial)

The lowest eigenvalue is  $\lambda_{10} = 2$ ,  $g(x) = x$ ,

$$\zeta = A \cos \theta e^{i\sigma t}, \quad \sigma^2 = 2 \frac{g H_0}{R^2}$$

The lowest symmetrical mode has  $n = 0$ ,  $m = 2$

$$\lambda_{0,2} = 8, \quad \zeta = \left(1 - 2 \frac{\kappa^2}{R^2}\right) e^{i\sigma t}$$

$$\sigma^2 = 8 \frac{gH_0}{R^2}$$

Note that this has just twice the frequency of the lowest mode. Both of these modes are easily excited and have frequencies convenient for measurement with a watch in an ordinary soup-bowl.

These frequencies can also be computed approximately by using trial eigenfunctions in the variational formula eq. (20) of lecture 5. With the obvious polynomial trial functions, the exact results would be obtained, since the exact solutions happen to be polynomials.

8. Investigate the generation of waves by a radially oscillating circular wall, in a rotating system. (Uniform shallow depth  $H$ , outside a circle of radius  $R$ .)

The equations are: ( $g = 1$ )

$$i \nabla \zeta + if \vec{k} \times \vec{u} = \sigma \vec{u}$$

$$iH \nabla \cdot \vec{u} = \sigma \zeta$$

Using cylindrical coordinates  $\vec{u} = u \vec{e}_r + v \vec{e}_\theta$ , we have

$u = U$  on  $r = R$  and assuming that  $u$ ,  $v$ , and  $\zeta$  are functions of  $r$  alone we have:

$$i \zeta_r - ifv = \sigma u$$

$$ifu = \sigma v$$

$$iH \left[ u_r + \frac{u}{r} \right] = \sigma \zeta$$

Eliminating  $\zeta$  and  $v$  we get

$$u_{xx} + \frac{u_x}{H} - \frac{u}{H^2} + \frac{\sigma^2 - f^2}{H} u = 0$$

If  $\sigma^2 > f^2$  we take outgoing waves at  $\infty$  and so

$$u = A H_1^{(2)} \left( \sqrt{\frac{\sigma^2 - f^2}{H}} \kappa \right) e^{i\sigma t}$$

If  $\sigma^2 < f^2$  we have  $u \rightarrow 0$  at  $\infty$  and

$$u = A K_1 \left( \sqrt{\frac{f^2 - \sigma^2}{H}} \kappa \right)$$

In either case A is determined from  $u = U$  on  $\kappa = R$ .

If  $\sigma = f$  we have  $u = \frac{UR}{\kappa}$ .

9. Investigate free waves in a rotating doubly infinite channel.

The equations are ( $g = 1$ )

$$i \zeta_x - i f v = \sigma u$$

$$i \zeta_y + i f u = \sigma v$$

$$i H (u_x + v_y) = \sigma \zeta$$

with  $v = 0$  on  $y = 0, L$ . Assuming the x-dependence is of the form  $e^{ikx}$  and eliminating  $u$  and  $\zeta$  we obtain

$$v_{yy} + \left[ \frac{\sigma^2 - f^2}{H} - k^2 \right] v = 0$$

if we set  $\frac{\sigma^2 - f^2}{H} - k^2 = l^2$  we obtain solutions with  $v = A \sin ly$

if  $lL = n\pi$ . Thus for each  $k$  we get waves with

$$\sigma^2 = f^2 + H \left( k^2 + \frac{n^2 \pi^2}{L^2} \right) \quad n = 1, 2, \dots$$

We must also examine the possibility that  $v \equiv 0$ . We then have

$$\begin{aligned} -k\zeta &= \sigma u \\ i\zeta_y + ifu &= 0 \\ -kHu &= \sigma\zeta \end{aligned}$$

The first and third imply  $u = \zeta = 0$  unless  $\sigma^2 = k^2 H$ .  
In this case we also get waves, ("coast waves") with  $\zeta = A e^{f(k/\sigma)y}$ .

$$u = -\frac{k}{\sigma} A e^{f(k/\sigma)y}, \quad v = 0.$$

For large positive  $f k/\sigma$  these waves are mainly near the coast  $y = L$  and are travelling in the negative  $x$ -direction. For large negative  $f k/\sigma$  they travel in the positive  $x$ -direction and are mainly near the coast at  $x = 0$ . Thus for each  $k$  we have a denumerable set of waves with frequencies all greater than  $f$ , and two "coast waves" with frequencies  $\sigma = \pm k\sqrt{H}$ .

10. Investigate the possibility of waves in a rotating circular basin of uniform (shallow) depth with  $\sigma^2 = f^2$ .

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We take the radius to be 1 and  $g = 1$ . Let  $\sigma = \pm f$ . Then the equations are: (cf. Lecture 6, eqs. 9a, b).

$$\begin{aligned} \pm f(u \pm iv) &= i\zeta_x \\ -if(u \pm iv) &= i\zeta_y \end{aligned}$$

and

$$iH(u_x + v_y) = \pm f\zeta$$

From the first two of these it easily follows that  $\zeta$  must be a complex analytic function of  $z = x + iy$ , and so  $u \pm iv$  must be too. Let  $\zeta = F'(z)$ . Then

$$u \pm iv = -\frac{1}{f} \zeta_y = \frac{\pm i}{f} F''(\bar{z}).$$

Using this, the third equation can be written:

$$\frac{\partial}{\partial \bar{z}} (u \mp iv) = \mp \frac{if}{H} F'(\bar{z})$$

from which we get

$$u \mp iv = \mp \frac{if}{H} F(\bar{z}) + C(\bar{z})$$

We thus have the general representation in terms of two complex analytic functions  $F(z)$  and  $C(\bar{z})$ :

$$\begin{cases} \zeta = F'(z) \\ u = \frac{1}{2} \left[ \pm \frac{i}{f} F'' \mp \frac{if}{H} F + C(\bar{z}) \right] \\ v = \frac{1}{2} \left[ \frac{1}{f} F'' + \frac{f}{H} F \pm i C(\bar{z}) \right] \end{cases}$$

This holds for any basin of uniform depth. For the circular basin, we naturally consider waves with  $\zeta = \phi(r) e^{im\theta}$ . For this  $F'(z) = A z^m$  and we must take the lower sign,  $\sigma = -f$ , if  $m > 0$ .

If  $m < 0$  we must take the upper sign. We include both cases by supposing instead an angle dependence  $e^{\mp im\theta}$  and so taking

$F'(z) = A z^m$ ,  $m = 1, 2, 3, \dots$  (It is easily seen that for  $m = 0$  we must always have  $\sigma^2 > f^2$  unless  $\sigma = 0$ , so this case is not of interest here). We take  $F(z) = \frac{A}{m+1} z^{m+1}$ , and compute the radial velocity component,

$$\begin{aligned} U &= u \cos \theta + v \sin \theta = \frac{1}{2} \left[ \pm \frac{i}{f} F'' e^{\mp i\theta} \mp \frac{if}{H} F e^{\pm i\theta} + C(\bar{z}) e^{\pm i\theta} \right] \\ &= \frac{1}{2} \left[ \pm \frac{i}{f} A^m \frac{z^m}{r} \mp \frac{if}{H} \frac{A}{m+1} r z^m + C(\bar{z}) e^{\pm i\theta} \right] \end{aligned}$$

Since  $C(\bar{z})$  must be regular at the origin it cannot be  $K \bar{z}^{-(m+1)}$  which is necessary to produce the  $e^{\mp im\theta}$  angle dependence, unless

$K = 0$ . Thus  $C = 0$  and we have

$$U = \pm \frac{iA_m}{2f} \left[ r^{m-1} - \frac{f^2}{H_m(m+1)} r^{m+1} \right] e^{\mp im\theta}$$

Since  $U$  is to vanish on  $r = 1$ , this wave is possible if and only if  $f^2 = H_m(m+1)$ . The tangential velocity component

$V = -u \sin \theta + v \cos \theta$  is readily computed and the final result is that waves with  $\sigma = \pm f$  exist provided  $f^2 = H_m(m+1)$  and are given by:

$$\zeta = A r^m e^{\mp im\theta}$$

$$U = \pm i \frac{A_m}{2f} (r^{m-1} - r^{m+1}) e^{\mp im\theta}$$

$$V = \frac{A_m}{2f} (r^{m-1} + r^{m+1}) e^{\mp im\theta}$$

11. Show in general for a circular lake with depth function

$H(r) (0 \leq r \leq 1, g=1)$  that there is just one eigenfunction for  $\sigma = \pm f$  with angle dependence  $e^{\mp im\theta} (m \geq 1)$  and this occurs for

$$f^2 = 2m^2(m+1) \int_0^1 r^{2m-1} H(r) dr$$

In polar coordinates with radial velocity  $u$  and tangential velocity  $v$  the basic equations are:

$$i\zeta_r - ifv = \pm f u$$

$$\frac{i}{r}\zeta_\theta + ifu = \pm f v$$

$$i \left[ (Hu)_r + rHu + \frac{1}{r} (Hv)_\theta \right] = \pm f \zeta$$

As in the previous problem we find from the first two equations that  $\zeta$  must be an analytic function of  $x + iy = r e^{\mp i\theta}$ .

For  $e^{\mp im\theta}$  angle dependence we must thus have  $\zeta = A r^m e^{\mp im\theta}$

and then from either of the first two equations,

$$(u \pm iv) = \pm \frac{imA}{f} r^{m-1} e^{Fim\theta}$$

Now setting  $u = U(r) e^{Fim\theta}$ ,  $v = V(r) e^{Fim\theta}$ , the third equation becomes:

$$i \left[ (HU)' + \frac{1}{r} HU \mp \frac{im}{r} HV \right] = \pm f A r^m$$

But from the above expression for  $u \pm iv$  we have

$$\mp iV = U \mp \frac{im}{f} A r^{m-1}, \quad \text{and so}$$

$$i \left[ (HU)' + \frac{1}{r} (HU) + \frac{m}{r} HU \mp \frac{im^2}{f} A r^{m-2} H \right] = \pm f A r^m$$

$$\therefore i \left[ \frac{d}{dr} (r^{m+1} HU) \mp \frac{im^2}{f} A r^{2m-1} H \right] = \pm f A r^{2m+1}$$

From this equation  $U$  can be determined, provided the boundary condition  $HU = 0$  on  $r = 1$  can be satisfied, and integrating the equation from 0 to 1 this condition gives:

$$\frac{m^2}{f} A \int_0^1 r^{2m-1} H dr = \frac{fA}{2(m+1)}$$

or

$$f^2 = 2m^2(m+1) \int_0^1 r^{2m-1} H(r) dr$$

If this condition is satisfied, a wave with  $\sigma = \pm f$  exists, and it is clear from the above equations that  $U$  and  $V$  determined uniquely.

12. Determine approximately the  $\sigma$ - $f$  diagram for waves in a rotating shallow paraboloidal lake.

We take  $g = 1$ , radius of lake = 1,  $H(r) = H_0 (1 - r^2)$  and set

$$\zeta = \phi(r) e^{im\theta}. \quad \text{The equation for } \zeta \text{ is:}$$

$$-\sigma \nabla \cdot (H \nabla \zeta) + if (H_x \zeta_y - H_y \zeta_x) = \sigma (\sigma^2 - f^2) \zeta$$

which gives for  $\phi$  :

$$(\kappa H \phi')' + \left[ -\frac{m^2}{\kappa} H + \frac{mf}{\sigma} H' + \kappa (\sigma^2 - f^2) \right] \phi = 0$$

with  $\phi$  regular at  $\kappa = 0$  and  $\phi' + \frac{mf}{\sigma} \phi$  bounded as  $\kappa \rightarrow 1$ .

The case  $f = 0$  was investigated in problem 7, and the eigenfrequencies are:

$$\sigma_{mk}^{(0)2} = H_0 [2m(2k+1) + 4k(k+1)]$$

$k = 0, 1, 2, \dots$ ,  $k = 0$  omitted if  $m = 0$ .

In the present case, the results are the same if  $m = 0$  except that  $f^2$  must be added to  $\sigma^2$ . We consider from now on the case of a fixed positive  $m$ . With the given  $H(\kappa)$  we have:

$$(\kappa(1-\kappa^2)\phi')' + \left[ -\frac{m^2}{\kappa}(1-\kappa^2) - \frac{2mf\kappa}{\sigma} + \kappa \frac{(\sigma^2 - f^2)}{H_0} \right] \phi = 0$$

This happens to be the same equation as in the non-rotating case, if we set  $\lambda = \frac{\sigma^2 - f^2}{H_0} - \frac{2mf}{\sigma}$  for the eigenvalue instead of the previous  $\sigma^2/H_0$ . For  $\phi$  bounded as  $\kappa \rightarrow 1$  we get the same eigenvalues as before and so the  $\sigma_{mk}$  are to be determined as the solutions of

$$\sigma^2 = f^2 + H_0 \left[ \frac{2mf}{\sigma} \right] + \sigma_{mk}^{(0)2}$$

By replacing  $\sigma$  by  $\sigma \sqrt{H_0}$  and  $f$  by  $f \sqrt{H_0}$  we get an equation of the same form as the original with  $H_0 = 1$ , so we consider only the case  $H_0 = 1$ .

The cubic is then

$$\sigma^3 - (f^2 + 2m(2k+1) + 4k(k+1))\sigma - 2mf = 0$$

For  $k = 0$  this factors into  $(\sigma + f)(\sigma^2 - f\sigma - 2m) = 0$

The root  $\sigma = -f$  is usually extraneous; in fact the formula of problem 11 shows that it is relevant only when  $f^2 = m$ , which is when it coincides with a root of the other factor. Thus we consider only the quadratic factor, from which we find

$$\sigma = \frac{1}{2} \left[ f \pm \sqrt{f^2 + 8m} \right]$$

The root with the lower sign is the "coast-wave"; it crosses into  $\sigma^2 < f^2$  at  $f = \sqrt{m}$  and for large  $f$  is approximately  $\sigma \cong -\frac{2m}{f}$ . For  $k \geq 1$  the cubic always has 3 real and unequal roots. For small  $f$  two of them are near  $\pm \sigma^{(0)}$ , in fact we have for them

$$\sigma_{\pm} \cong \pm \sqrt{2m(2k+1) + 4k(k+1)} \left( 1 + \frac{mf}{[2m(2k+1) + 4k(k+1)]^{3/2}} + O(f^2) \right)$$

The third root is small, and is given approximately by:

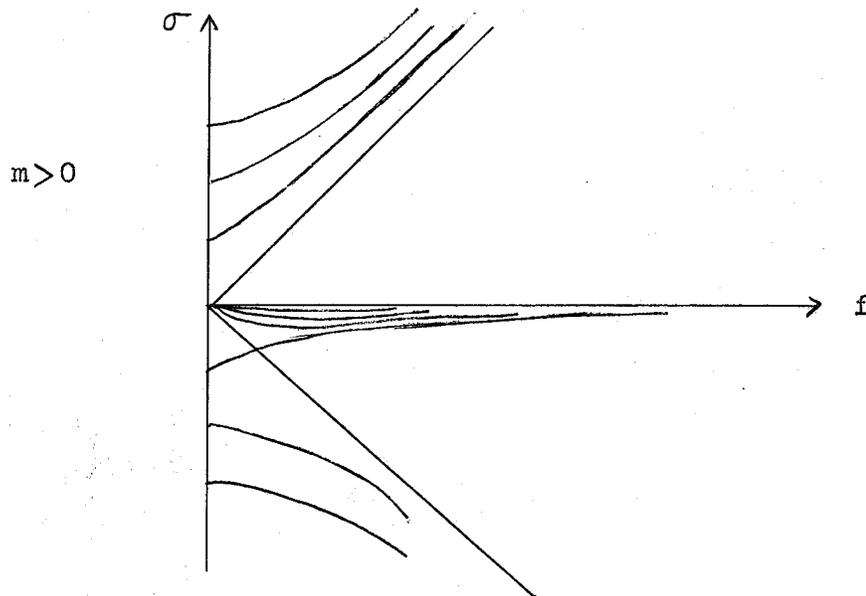
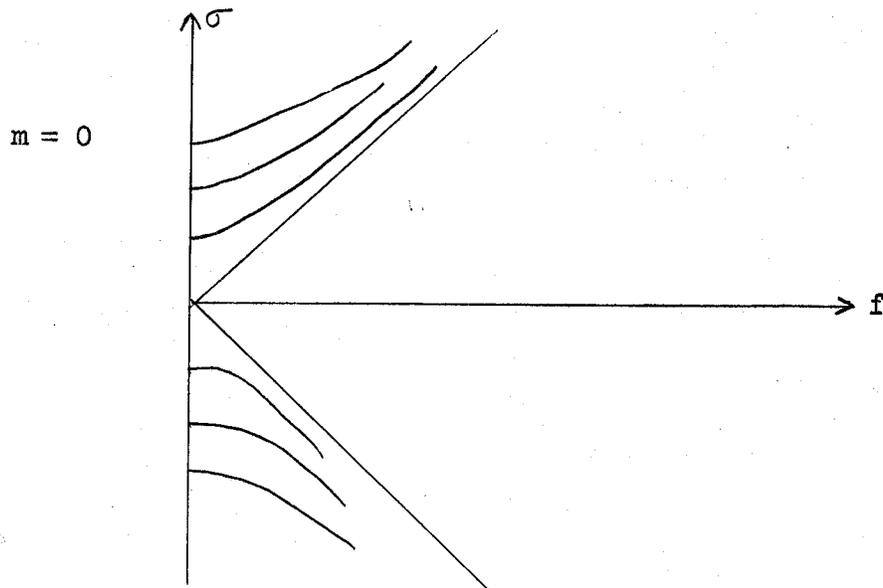
$$\sigma_0 \cong -\frac{2mf}{2m(2k+1) + 4k(k+1)} + O(f^3)$$

For large  $f$  the roots are given approximately by

$$\sigma_{\pm} \cong \pm f \left[ 1 + \frac{m(2k+1) + 2k(k+1) \pm m}{f^2} + O(f^{-4}) \right]$$

$$\sigma_0 \cong -\frac{2m}{f} \left[ 1 - \frac{2m(2k+1) + 4k(k+1)}{f^2} + O(f^{-4}) \right]$$

These results enable us to sketch roughly the  $\sigma-f$  diagrams for this problem, as follows:



Note that for  $m = 0$  we have a "coast wave" and an infinite number of "Rossby Waves". (see also Lamb § 212)

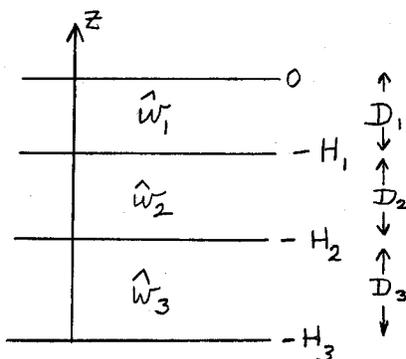
13. Investigate internal waves in a three layer system, in the case of waves short compared to all three thicknesses.

This problem can be solved just as the 2-layer problem of Lecture 8 (pp. 1-8) was. After elimination of all variables but the vertical velocities we have

$$\hat{w}_n'' = k^2 \hat{w}_n \quad (n = 1, 2, 3)$$

with:

$$\hat{w}_1 = \frac{\sigma^2}{gk^2} \hat{w}_1' \quad \text{on } z = 0$$



$$\hat{w}_2 = \hat{w}_1 = \frac{\sigma^2}{gk^2} \frac{\rho_2 \hat{w}_2' - \rho_1 \hat{w}_1'}{\rho_2 - \rho_1} \quad \text{on } z = -H_1$$

$$\hat{w}_3 = \hat{w}_2 = \frac{\sigma^2}{gk^2} \frac{\rho_3 \hat{w}_3' - \rho_2 \hat{w}_2'}{\rho_3 - \rho_2} \quad \text{on } z = -H_2$$

$$\hat{w}_3 = 0 \quad \text{on } z = -H_3$$

From the differential equations and the conditions

$$\hat{w}_3(-H_3) = 0, \quad \hat{w}_3(-H_2) = \hat{w}_2(-H_2),$$

$$\hat{w}_2(-H_1) = \hat{w}_1(-H_1) \quad \text{one can obtain at once that}$$

$$\hat{w}_3 = A_3 \sinh k(z + H_3)$$

$$\hat{w}_2 = A_2 \sinh k(z + H_2) + A_3 \sinh k D_3 \cosh k(z + H_2)$$

$$\hat{w}_1 = A_1 \sinh k(z + H_1) + [A_2 \sinh k D_2 + A_3 \sinh k D_3 \cosh k D_2] \cosh k(z + H_1)$$

Inserting these into the other three boundary conditions gives three homogeneous linear equations for the  $A_n$ . The vanishing of the deter-

minant of this system gives the relation between  $\sigma$  and  $k$ . For waves short compared with all layer thicknesses this equation is easily handled and one obtains a surface wave with  $\sigma^2 = gk$ , an internal wave near  $z = -H$ , with  $\frac{\sigma^2}{gk} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$  and an internal wave near  $z = -H_2$  with

$$\frac{\sigma^2}{gk} = \frac{\rho_3 - \rho_2}{\rho_3 + \rho_2}$$

14. Investigate internal waves in a fluid whose density varies exponentially over a finite range. Consider first the case of a rigid top so that there is no surface wave.

---

Take  $H = -1$ ,  $g = 1$ ,  $\rho_0(z) = e^{-az}$ ,  $\alpha = gk^2/\sigma^2$ .  
 Then (cf. Lecture 8, eq. (53))  $(e^{-az}\hat{w}')' + \alpha e^{-az}\hat{w} - k^2 e^{-az}\hat{w} = 0$   
 with  $\hat{w}(-1) = 0$ ,  $\hat{w}(0) = 0$  (rigid top)  
 or  $\hat{w}'(0) = \alpha \hat{w}(0)$  (free surface)

The differential equation can be written

$$\hat{w}'' - a\hat{w}' + (\alpha a - k^2)\hat{w} = 0$$

The solution of this which vanishes at  $z = -1$  is:

$$\hat{w} = e^{\frac{a}{2}(z+1)} \sinh \frac{1}{2} \sqrt{a^2 + 4k^2 - 4a\alpha} (z+1)$$

with  $\hat{w}(0) = 0$  we have

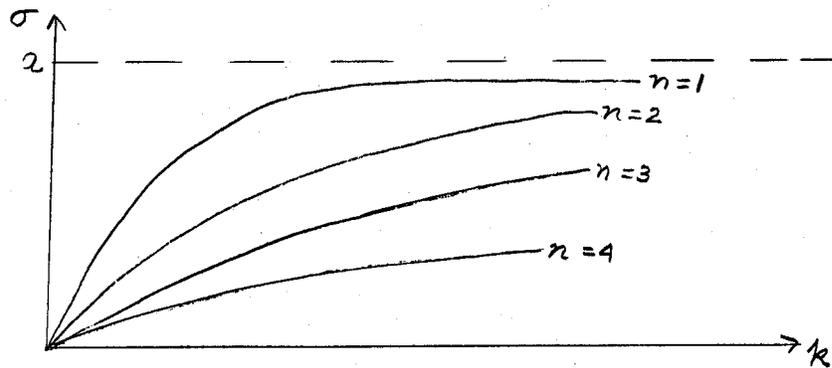
$$\frac{1}{2} \sqrt{a^2 + 4k^2 - 4a\alpha} = n\pi i, \quad n = 1, 2, \dots$$

or

$$k^2 = a\alpha - \frac{a^2}{4} - n^2\pi^2$$

since  $\alpha = k^2/\sigma^2$  this gives

$$\sigma^2 = \frac{ak^2}{k^2 + a^2/4 + n^2\pi^2}$$



Note that  $-\frac{\rho_0'}{\rho_0} = a$  verifying Groen's limit. The case with the free surface boundary condition cannot be handled quite so explicitly, but qualitatively similar results are obtained, with the addition of the surface wave.

15. Study the stability of the Helmholtz flow when the densities in the two layers are different.

---

(See Lamb § 232 for the solution to this problem)

16. Investigate the stability of the following flows:

a)  $w = 1 - |y| \quad -1 \leq y \leq 1$

b)  $w = -1 \quad -\infty < y < -1$   
 $= y \quad -1 < y < 1$   
 $= +1 \quad 1 < y < \infty$

c)  $w = 1 \quad |y| < 1$   
 $= 0 \quad |y| > 1$

In particular find  $c(\alpha)$ .

---

Part (a)

$$\varphi'' - \alpha^2 \varphi = 0 \quad \text{in } (-1, 0) \text{ and } (0, 1)$$

$$\text{with } \varphi(-1) = \varphi(+1) = 0$$

$$\frac{\varphi}{w-c} \text{ and } (w-c)\varphi' - w'\varphi \text{ continuous at } y=0$$

$$\text{For } -1 < y < 0, \quad \varphi = A \sinh \alpha (y+1)$$

$$\text{For } 0 < y < 1 \quad \varphi = B \sinh \alpha (y-1)$$

Since  $w$  is continuous,  $\varphi$  must be continuous at  $y=0$ , i.e.  $B = -A$ .

The other condition gives:

$$\begin{aligned} (1-c)\alpha A \cosh \alpha - A \sinh \alpha \\ = (1-c)\alpha B \cosh(-\alpha) - (-1) B \sinh(-\alpha) \end{aligned}$$

$$\text{or } (1-c)2\alpha \cosh \alpha = 2 \sinh \alpha$$

$$c = 1 - \frac{\tanh \alpha}{\alpha}$$

(stable)

Part (b)

$$-\infty < y < -1; \quad \varphi = A e^{\alpha(y+1)}$$

$$-1 < y < 1; \quad \varphi = B e^{\alpha(y+1)} + C e^{-\alpha(y-1)}$$

$$1 < y < \infty \quad \varphi = D e^{-\alpha(y-1)}$$

Since  $w$  is continuous,  $\varphi$  must be also, so

$$A = B + C e^{2\alpha}$$

$$D = B e^{2\alpha} + C$$

Using these and the other jump conditions,

$$(-1-c)\alpha(B + C e^{2\alpha}) = (-1-c)[\alpha B - \alpha C e^{2\alpha}] - [B + C e^{2\alpha}]$$

$$(1-c)[\alpha B e^{2\alpha} - \alpha C] - [B e^{2\alpha} + C] = (1-c)(-\alpha)(B e^{2\alpha} + C)$$

$$\text{or } B + e^{2\alpha} C [1 - 2\alpha(1+c)] = 0$$

$$e^{2\alpha} B [1 - 2\alpha(1-c)] + C = 0$$

$$\therefore 1 - e^{4\alpha} [1 - 4\alpha + 4\alpha^2(1-c^2)] = 0$$

$$\text{or } 1 - c^2 = \frac{1 - e^{4\alpha}(1-4\alpha)}{4\alpha^2 e^{4\alpha}}$$

$$c^2 = \frac{-e^{-4\alpha} + (1-2\alpha)^2}{4\alpha^2}$$

For small  $\alpha$  we have

$$c^2 = \frac{-1 + 4\alpha - 8\alpha^2 + \frac{32}{3}\alpha^3, \dots + 1 - 4\alpha + 4\alpha^2}{4\alpha^2}$$

$$= -1 + \frac{8}{3}\alpha \dots$$

$$\therefore c = \pm i \left(1 - \frac{4}{3}\alpha + \dots\right)$$

and the flow is unstable to long waves. On the other hand, it is clear that  $c^2$  is positive for sufficiently large  $\alpha$ , the change in sign occurring where  $e^{-4\alpha} = (1-2\alpha)^2$ , or  $e^{-2\alpha} = 2\alpha - 1$ .

Note that for those values of  $\alpha$  for which instability occurs the instability is a standing wave, i.e.  $c^2$  is always real. This is the usual situation with odd profiles.

Part (c). In this case, since  $w$  is an even function, we can consider even and odd eigenfunctions separately.

Even eigenfunction:

$$\begin{aligned} -\infty < y < -1 & \quad \varphi = A e^{\alpha(y+1)} \\ -1 < y < 1 & \quad \varphi = B \cosh \alpha y \\ 1 < y < \infty & \quad \varphi = A e^{-\alpha(y-1)} \end{aligned}$$

$$\frac{A}{-c} = \frac{B \cosh \alpha}{1-c}$$

$$-c\alpha A = (1-c)(\alpha B \sinh(-\alpha))$$

$$\therefore \left( \frac{c}{1-c} \right)^2 = -\tanh \alpha$$

Thus this flow is unstable for all  $\alpha$ .

The case of the odd eigenfunction is handled similarly, with the result

$$\left( \frac{1-c}{c} \right)^2 = -\tanh \alpha$$

again showing instability. In the case of  $\varphi$  even, the disturbance is a sinuous deviation of the jet which for long waves is nearly a growing standing wave. In the case of  $\varphi$  odd the disturbance is the development of varicosities in the jet, moving (for small  $\alpha$ ) with approximately the speed of the jet.

17. Use the Pellew-Southwell variational formula,

$$a^2 R = \frac{\int_0^1 \left\{ [D(D^2 - a^2)w]^2 + a^2 [(D^2 - a^2)w]^2 \right\} dz}{\int_0^1 [(D^2 - a^2)w]^2 dz}$$

to compute approximately the minimum critical Rayleigh number

(1708) for rigid boundaries, taking for a trial:

$$(D^2 - a^2)w = \sin \pi z$$

and solving this with  $w = w' = 0$  on  $z = 0, 1$  to find  $(D^2 - a^2)w$ .

The solution for  $w$  is easily obtained and is

$$w = \frac{\sin \pi z}{(\pi^2 + a^2)^2} + A \cosh a(z - \frac{1}{2}) + B(z - \frac{1}{2}) \sinh a(z - \frac{1}{2})$$

$$\text{where } B = \frac{\pi / (\pi^2 + a^2)^2}{\sinh \frac{a}{2} + \frac{a}{2} \operatorname{sech} \frac{a}{2}}$$

$$\text{and } A = -\frac{1}{2} \tanh \frac{a}{2} B$$

$$\text{From this, } (D^2 - a^2)w = -\frac{\sin \pi z}{\pi^2 + a^2} + 2aB \cosh a(z - \frac{1}{2})$$

$$\text{and then } \int_0^1 [(D^2 - a^2)w]^2 dz = \frac{\frac{1}{2}}{(a^2 + \pi^2)^2} - \frac{8a\pi^2 \cosh a/2}{(a^2 + \pi^2)^4 (\sinh \frac{a}{2} + \frac{a}{2} \operatorname{sech} \frac{a}{2})} \\ + \frac{2a^2\pi^2}{(a^2 + \pi^2)^4} \frac{(\frac{\sinh a}{a} + 1)}{(\sinh \frac{a}{2} + \frac{a}{2} \operatorname{sech} \frac{a}{2})^2}$$

Finally we get

$$R = \frac{(a^2 + \pi^2)^3}{a^2} \left\{ 1 - \frac{4a\pi^2}{(a^2 + \pi^2)^2} \left[ \frac{4 \cosh a/2}{\sinh a/2 + \frac{a}{2} \operatorname{sech} a/2} - \frac{\sinh a + a}{(\sinh a/2 + \frac{a}{2} \operatorname{sech} a/2)^2} \right] \right\}^{-1}$$

Calculations with a slide rule from this give:

$a$	$R$	$R$ exact (Pellew and Southwell)
2	2180	2177.4
3	1711	1711.3
4	1889	1879.3

This calculation is at any rate good enough to show that  $R \cong 1700$  and  $a$  is a little larger than 3. Pellew and Southwell, using a trial function containing one adjustable parameter, obtain by this method almost exactly the same results as they obtained by direct solution of the differential equation, namely  $R = 1707.8$ ,  $a = 3.13$ .

18. Show that in the case of free boundaries  $R$  is given by:

$$a^2 R = \frac{\int_0^1 \{ [D(D^2 - a^2)w]^2 + a^2 [(D^2 - a^2)w]^2 \} dz}{\int_0^1 w^2 dz}$$

and that this is stationary.

The free boundary conditions are  $w = w' = w'' = 0$ .

If  $w$  satisfies these and  $(D^2 - a^2)^3 w + a^2 R w = 0$ , the above follows almost at once from

$$\int_0^1 w (D^2 - a^2)^3 w dz + a^2 R \int_0^1 w^2 dz = 0$$

by three integrations by parts. Furthermore if  $w$  satisfies the equation and boundary conditions,

$$\delta(a^2 R) = \frac{2 \int_0^1 D(D^2 - a^2)w D(D^2 - a^2)\delta w + a^2 (D^2 - a^2)w (D^2 - a^2)\delta w}{\int_0^1 w^2}$$

$$-a^2 R = \frac{2 \int_0^1 w \delta w}{\int_0^1 w^2} = \frac{2}{\int_0^1 w^2} \int_0^1 \left\{ -D^2 (D^2 - a^2)^2 w \delta w + a^2 (D^2 - a^2)^2 w \delta w - a^2 R w \delta w \right\}$$

$$= \frac{-2}{\int_0^1 w^2} \int_0^1 \left\{ (D^2 - a^2)^3 w + a^2 R w \right\} \delta w = 0$$

19. Study the problem

$$\begin{aligned} \nu y'' + y' - y &= 0 \\ y(0) &= 0 \quad y(1) = 1 \end{aligned}$$

for small  $\nu$  by examining the exact solution and by boundary layer techniques.

The exact solution is easily found and is:

$$y = e^{\frac{1-x}{2\nu}} \frac{\sinh\left(\frac{\sqrt{1+4\nu}}{2\nu} x\right)}{\sinh\left(\frac{\sqrt{1+4\nu}}{2\nu}\right)}$$

For fixed  $x$ , compute  $\lim_{\nu \rightarrow 0^+} y$

$$y = e^{\frac{1-x}{2\nu}} \frac{e^{\frac{\sqrt{1+4\nu}}{2\nu}(x-1)} - e^{-\frac{\sqrt{1+4\nu}}{2\nu}(x+1)}}{1 - e^{-\frac{\sqrt{1+4\nu}}{\nu}}}$$

$$= \left[ e^{\frac{\sqrt{1+4\nu}-1}{2\nu}(x-1)} - e^{\frac{1-\sqrt{1+4\nu}}{2\nu} - x \frac{1+\sqrt{1+4\nu}}{2\nu}} \right] \left[ 1 - e^{-\frac{\sqrt{1+4\nu}}{\nu}} \right]^{-1}$$

Now  $\frac{\sqrt{1+4\nu}-1}{2\nu} = \frac{1+2\nu-2\nu\sqrt{1+4\nu}-1}{2\nu} = 1-\nu+\dots$

Thus if  $x > 0$  we have

$$\lim_{\nu \rightarrow 0^+} y(x, \nu) = e^{x-1}, \text{ but if } x=0, \lim_{\nu \rightarrow 0^+} y = 0$$

Thus there is a non-uniformity at  $x=0$ . Note also that for  $x > 0$  the limit is the solution of the equation with  $\nu = 0$  which satisfies the boundary condition at  $x=1$ , not the boundary condition at  $x=0$ . For small  $\nu$  we have in any case,

$$y = e^{x-1} - e^{-\frac{x}{\nu}} e^{-(x+1)} + o(\nu)$$

If we set

$$x = \nu \xi, \lim_{\nu \rightarrow 0} y(\nu \xi, \nu) = e^{-1}(1 - e^{-\xi}).$$

Without use of the exact solution, the boundary layer approach is as follows: The "external solution" is  $y = A e^x$ , and we expect either  $A = 0$  if the boundary layer is at  $x=1$  or  $A = e^{-1}$  if it is at  $x=0$ . To examine the possibility of a boundary layer at  $x=1$ , set  $1-x = \nu^a \xi$ , and rewrite the equation as

$$\nu^{1-2a} \frac{d^2 y}{d\xi^2} - \nu^{-a} \frac{dy}{d\xi} - y = 0$$

To preserve the second derivative we must have  $a \geq 1$ . If  $a > 1$ , the B.L. equation would be  $\frac{d^2 y}{d\xi^2} = 0$  which does not have a

solution with  $y(0) = 1$  and  $y \rightarrow 0$  (the external solution at  $x = 1$ , if the boundary layer is at  $x = 1$ ) as  $\xi \rightarrow \infty$ . If  $a = 1$  the

B.L. equation is  $\frac{d^2 y}{d\xi^2} - \frac{dy}{d\xi} = 0$  whose general solution

$A e^\xi + B$  again cannot satisfy the required conditions. We con-

clude that the boundary layer must be at  $x = 0$ , and start over,

setting  $x = \nu^a \xi$  and taking  $y_e = e^{x-1}$ . The B.L. equation is

$\frac{d^2 y}{d\xi^2} = 0$  if  $a > 1$ , which does not work, and setting  $a = 1$  we have

$$\frac{d^2 y}{d\xi^2} + \frac{dy}{d\xi} = 0$$

This has a solution with the required properties, namely,

$$y_{BL} = e^{-1} (1 - e^{-\xi}) = e^{-1} (1 - e^{-x/\nu})$$

This is evidently in agreement with the results from the exact solution.

20. Study the Ekman boundary layer at the free surface of a deep ocean if horizontal stresses  $\tau_x$  and  $\tau_y$  are applied there. In particular find  $w$  at the bottom of the boundary layer.

---

With origin at the free surface and  $\eta = z \sqrt{\frac{f}{\nu}}$  the Ekman boundary layer equations are:

$$\frac{d^2}{d\eta^2} ((u-u_1) + i(v-v_1)) = i((u-u_1) + i(v-v_1))$$

where  $u_1, v_1$  gives the geostrophic flow outside the boundary layer. The solution of this equation which goes to zero as

$\eta \rightarrow -\infty$  is

$$(u-u_1) + i(v-v_1) = A e^{\frac{1+i}{\sqrt{2}} z}$$

The boundary conditions are, on  $z = 0$ :

$$\rho v \frac{\partial u}{\partial z} = \tau_x; \quad \rho v \frac{\partial v}{\partial z} = \tau_y$$

or 
$$\frac{\partial}{\partial \eta} (u + iv) = \frac{1}{\rho f v} (\tau_x + i \tau_y)$$

Since  $u_1$  and  $v_1$  are independent of  $\eta$  this gives

$$A \left( \frac{1+i}{\sqrt{2}} \right) = \frac{1}{\rho f v} (\tau_x + i \tau_y)$$

from which we find:

$$u = u_1 + \frac{e^{z/a}}{\rho \sqrt{2} f v} [(\tau_x + \tau_y) \cos z/a + (\tau_x - \tau_y) \sin z/a]$$

$$v = v_1 + \frac{e^{z/a}}{\rho \sqrt{2} f v} [(\tau_x + \tau_y) \sin z/a + (\tau_y - \tau_x) \cos z/a]$$

where  $a = \sqrt{\frac{2v}{f}}$  is the Ekman depth. From these we can find

by using the continuity equation:

$$\begin{aligned} -w_z = u_x + v_y &= \frac{e^{z/a}}{\rho \sqrt{2} f v} \left[ \cos z/a \left( \frac{\partial}{\partial x} (\tau_x + \tau_y) + \frac{\partial}{\partial y} (\tau_y - \tau_x) \right) \right. \\ &\quad \left. + \sin z/a \left( \frac{\partial}{\partial x} (\tau_x - \tau_y) + \frac{\partial}{\partial y} (\tau_x + \tau_y) \right) \right] \end{aligned}$$

From this we find  $w$  at the bottom of the boundary layer:

$$w(-\infty) = \int_{-\infty}^0 -w_z dz = \frac{1}{\rho f} \left[ \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right]$$