Notes on the 1967

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

Reference No. 67-54

Contents of the Volumes

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Editor's Preface

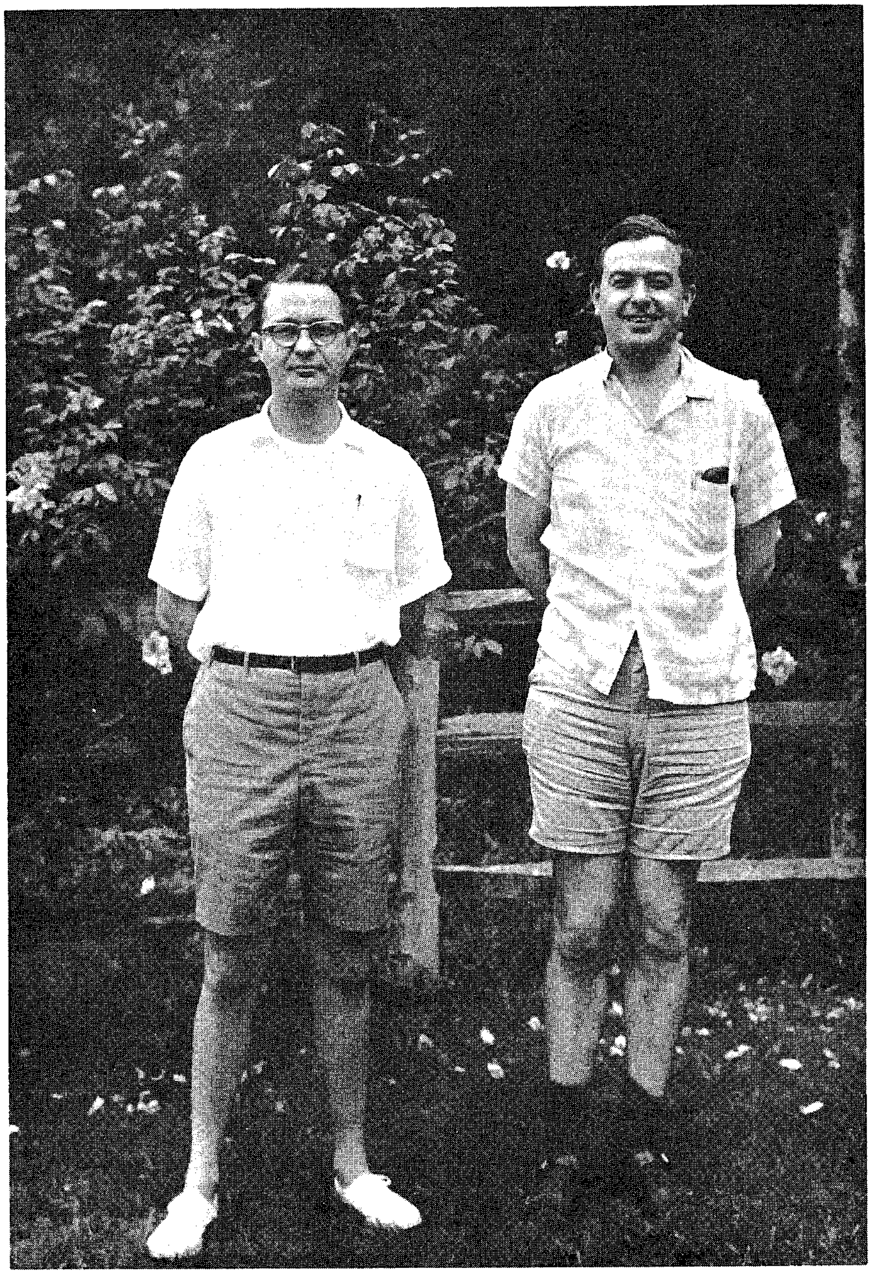
In former years some of the research activities and seminars of the WHOI Geophysical Fluid Dynamics program was concerned with determining the interior structure and motions of stars and galaxies. This year we have focused our attention downward rather than upward and have attempted to learn something about the earth's interior. Freeman Gilbert's lectures on the inverse problem in siesmology discuss one aspect of the geophysicist's attempts to infer something about the earth's interior from the evidence which is available at the surface. Paul Roberts presented a survey of the different attempts to attribute the earth's magnetic field to dynamo action. Willem Malkus, Raymond Hide and Stephen Childress supplemented Roberts' lectures with seminars. As students of our physical environment all of us were entertained and stimulated by this introduction to the netherworld.

The students' notes of Roberts' lectures have been reworked by Roberts to the point where they record much more closely what he said rather than the students' interpretation of his lectures. In that sense this volume departs from the records of the programs of previous years.

Mary C. Thayer has assembled and typed all of the lectures and the participants' reports. Her unique ability to keep the program functioning smoothly continues to amaze and delight all of us who take part in it.

We are grateful to the National Science Foundation for its continued support and encouragement and to Paul M. Fye for making available to us the facilities of the Woods Hole Oceanographic Institution.

George Veronis



Invited Guest Lecturers on the Netherworld.

Freeman Gilbert and Paul Roberts

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Course Lectures

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Dr. Freeman Gilbert, lecturer
University of California at San Diego

Dr. Paul H. Roberts, lecturer
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Dr. Stephen Childress, lecturer

Courant Institute of Mathematical Sciences, New York

Contents of Volume I (continued)

1. A Survey of Continuum Mechanics

Dr. Freeman Gilbert
University of California at San Diego

References: Sedov, 1965: (two translations: Permagon, Academic Press)
Backus, 1961-1965: Course notes in continuum mechanics at University of California at San Diego (Scripps Institute of Oceanography)

a. Conservation Equations

Kinematics of Continua:

Given a point mass P, it is completely specified if we know its position at all times

$$\underline{r}(t) = \hat{x}_i x_i(t).$$

For N point masses, clearly

$$\underline{r}^{(n)}(t) = \hat{x}_i x^{(n)}(t) \qquad n = 1, 2, \ldots, N$$

For the motion of a material body, we approximate the totality of point masses by a continuum. We specify for each infinitesimal volume of the continuum a position vector $\underline{r}$:

$$\underline{r} = \underline{r}(x, t)$$

and $\underline{r}$ is assumed to be a continuous function of position and time.

We define $\underline{r}(\underline{x}, 0) = \underline{x}$

Velocity is given by

$$\underline{u}(\underline{x}, t) = \frac{d}{dt}\underline{r}(\underline{x}, t) = \frac{\partial}{\partial t}\underline{r}(\underline{x}, t).$$

As $\underline{r}$ is a continuous function of $\underline{x}$ and t

$\underline{x}$ is a continuous function of $\underline{r}$ and t

$$J = \det\left(\frac{\partial r_i}{\partial x_j}\right) \neq 0$$

Clearly $J = 1$ at $t = 0$, thus $J > 0$ always.

When we focus our attention on a moving volume and specify it by $\underline{r} = \underline{r}(\underline{x}, t)$, we have the Lagrangian description. Such measurements are often difficult to perform geophysically, and almost always determine quantities of interest at a particular fixed point rather than in a moving volume.

i.e. $\underline{x} = \underline{x}(\underline{r}, t)$ Eulerian description

(However in recent years we have been making Lagrangian measurements in oceanography: neutrally buoyant floats measure temperature, pressure, etc. moving with ocean currents.)

Suppose we have some function ϕ of position and time

$$\phi^L = \phi(\underline{x}, t)$$
$$\phi^E = \phi(\underline{r}, t)$$

These functions are of course numerically equal at a particular point.

Notation: $\dfrac{D\phi}{Dt} \equiv \dfrac{\partial}{\partial t}\phi_L(\underline{x}, t)$ "Material Derivative"

Differentiate

$$\phi_L(x_i, t) = \phi_E(r_i, t) \quad \text{with respect to time}$$

$$\frac{\partial}{\partial t}\phi_L(x_i, t) = \frac{\partial \phi_E}{\partial r_i}\frac{\partial r_i}{\partial t} + \frac{\partial \phi_E}{\partial t} \equiv \frac{D\phi}{Dt}.$$

Note that first term on the right-hand side of the above equation is non-linear (it is often called the "advective" term).

In operator notation write

$$\left.\frac{\partial}{\partial t}\right|_L = \frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)_E \qquad (1.1)$$

"Vanishing Integral Theorem"

Given a function of position $f(\underline{r})$ in V,

if $\int_{V'} dV f(\underline{r}) = 0 \quad V' \text{ in } V$

then $f(\underline{r}) = 0$.

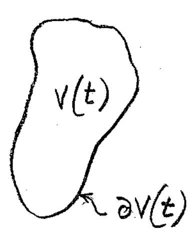
Also $\int_{V'} dV \alpha(\underline{r}) = \int_{V'} dV \beta(\underline{r})$ implies $\alpha(\underline{r}) = \beta(\underline{r})$.

Theorem

Consider a given scalar function of position and time $f(\underline{r},t)$ in a volume $V(t)$; $Q(\underline{r},t)$ is velocity normal to the boundary $\partial V(t)$ of $V(t)$.

Then

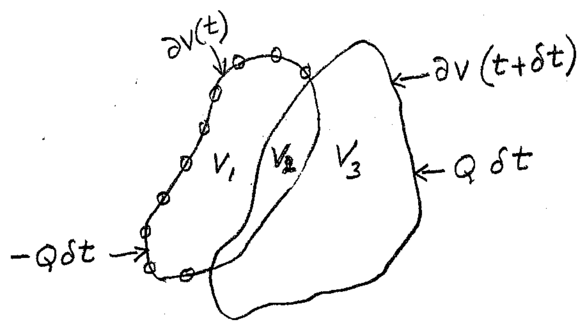
$$\frac{d}{dt}\int_{V(t)} dV f(\underline{r},t) = \underbrace{\int_{V(t)} \frac{\partial}{\partial t} f(\underline{r},t)}_{\substack{\text{Rate of} \\ \text{change in } V(t)}} + \underbrace{\int_{\partial V(t)} dA\, Q(\underline{r},t) f(\underline{r},t)}_{\text{Flux through } \partial V(t)} \tag{1.2}$$



Proof.

Consider

$$\lim_{\partial t\to 0}\left[\frac{1}{\partial t}\left\{\int_{V(t+\delta t)} dV\rho(\underline{r},t+\delta t)-\int_{V(t)} dV\rho(\underline{r},t)\right\}\right]$$



$$\text{Now}\int_{V(t+\delta t)} dV\rho(\underline{r},t+\delta t)=\int_{V_2} dV\rho+\int_{V_3} dV\rho$$

$$\text{and}\int_{V(t)} dV\rho(\underline{r},t)=\int_{V_1} dV\rho+\int_{V_2} dV\rho$$

Then

$$\frac{1}{\delta t}\left\{\int_{V(t+\delta t)} dV\rho(\underline{r},t+\delta t)-\int_{V(t)} dV\rho(\underline{r},t)\right. =$$

$$=\frac{1}{\delta t}\left\{\int_{V_2} dV\rho(\underline{r},t+\delta t)-\int_{V_2} dV\rho(\underline{r},t)\right\}+\frac{1}{\delta t}\left\{\int_{V_3} dV\rho(\underline{r},t+\delta t)-\int_{V_1} dV\rho(\underline{r},t)\right\}.$$

Taking the limit as $\delta t\to 0$, result follows.

Dynamics of Continua

Consider a volume $V(t)$ which moves with the material. $V(0)$ is the initial volume. Points in the volume are denoted by either $\underline{r}(\underline{x},t)$, Lagrangian, or $\underline{x}(\underline{r},t)$, Eulerian, coordinates.

Let $\hat{n}$ be the unit outward normal on $\partial V(t)$.

Then $q(\underline{r},t)=\hat{n}\cdot\underline{u}(\underline{r},t)$ where $\underline{r}$ is on $\partial V(t)$ is the velocity of the surface taken with respect to the moving volume $V(t)$.

Define $\Phi = \int_{V(t)} \phi(\underline{r},t)\,dV$.

Then, using equation (1.2)

$$\frac{d\Phi}{dt} = \int_{V(t)} dV\,\frac{\partial\phi}{\partial t} + \int_{\partial V(t)} dA\,\phi(\underline{r},t)\,\hat{n}\cdot\underline{u}$$

$$= \int_{V(t)} dV\left[\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\,\underline{u})\right] \qquad \text{by Gauss' theorem}$$

i.e. $$\frac{d}{dt}\int_{V(t)} dV\,\phi(\underline{r},t) = \int_{V(t)} dV\left[\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\,\underline{u})\right]. \tag{1.3}$$

<u>Conservation of Mass</u>

Take $\phi=\rho(\underline{r},t)$, density, in equation (1.3).

Then $$\frac{dM}{dt} = \int_{V(t)} dV\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\underline{u})\right] = 0 \qquad \text{if mass conserved.}$$

By the vanishing integral theorem, thus

$$\frac{\partial}{\partial t}\rho(\underline{r},t) + \nabla\cdot(\rho\,\underline{u}) = 0 \tag{1.4}$$

Alternatively

$$\left.\begin{aligned}
\frac{\partial\rho}{\partial t} + \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho &= 0\\
\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} &= 0\\
\frac{D}{Dt}\log\rho + \nabla\cdot\underline{u} &= 0\\
\frac{D}{Dt}\tau - \tau\nabla\cdot\underline{u} &= 0\\
\frac{D}{Dt}\log\tau - \nabla\cdot\underline{u} &= 0
\end{aligned}\right\} \tag{1.5}$$

where $\tau \equiv \frac{1}{\rho}$, specific volume.

<u>Conservation of Momentum</u>

Let $\underline{F}(t)$ be the total force exerted on V(t). Then Newton's second law gives

$$\frac{d}{dt}\int_{V(t)} \rho \underline{u}\, dV = \underline{F}\,.$$

Or, using equation (1.3) and employing indicial notation

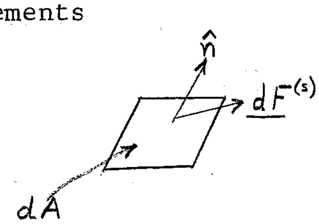
$$F_i = \int_{V(t)} dV \left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial r_j}(\rho u_i u_j)\right]$$

$\underline{F}$ is made up of body and surface forces.

<u>Body forces</u> act on all the matter in V(t)

$$\underline{F}^{(B)} \int_{V(t)} f(\underline{r})\, dV, \quad f(\underline{r}) \quad = \quad \text{body force density.}$$

Surface forces are proportional to the area of infinitesimal surface elements



i.e. $\quad d\underline{F}^{(s)} = \underline{\mathcal{F}}\, dA$

$\underline{\mathcal{F}} = \mathcal{F}(\underline{r}, t, \hat{n})$ is called the <u>stress</u> acting across the surface,

and $\mathcal{F}(\underline{r}, t, \hat{n}) = -\mathcal{F}(\underline{r}, t, -\hat{n})$

Then

$$\underline{F}^{(s)} = \int_{\partial V(t)} dA\; \underline{\mathcal{F}}(\underline{r}, t, \hat{n})$$

and so

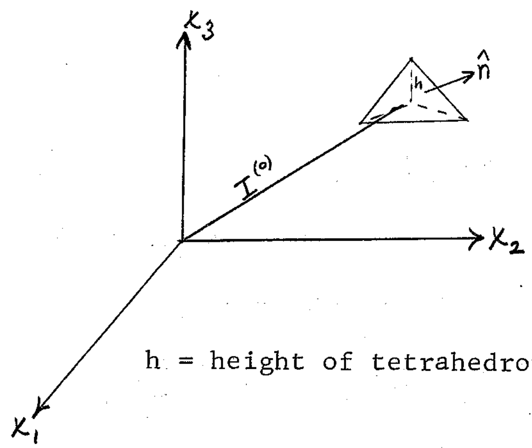
$$F_i = \int_{V(t)} dV f_i + \int_{\partial V(t)} dA\, \mathcal{F}_i(\underline{r}, t, \hat{n})\,.$$

We wish to convert the surface integral to one over V(t).

First we must show that

$$\mathcal{F}_i(\underline{r}, t, \hat{n}) = n_j \mathcal{F}_i(\underline{r}, t, \hat{x}_j) \qquad (1.6)$$

Proof: Consider the equilibrium of an elementary tetrahedron.



h = height of tetrahedron

Let $\mathcal{S}$ = area of slant face.

Three other faces are parallel to cartesian coordinate planes.

Areas of these faces are

$$\mathcal{S}_1 = n_i \mathcal{S}$$

where $\hat{n} = n_i \hat{x}_i$

For equilibrium of body and surface forces on the tetrahedron

$$c\mathcal{S}h\left\{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial r_j}(\rho u_i u_j) - f_i\right\} = \mathcal{S}\left\{\mathcal{F}_i(\underline{r}^{(0)}, t, \hat{n}) - n_j \mathcal{F}_i(\underline{r}^{(0)}, t, \hat{x}_j)\right\} .$$

Taking the limit as h $\longrightarrow$ 0 we obtain the required result (1.6).

$\mathcal{F}_i(\underline{r}, t, x_j) \equiv T_{ij}(\underline{r}, t)$ is called the stress tensor.

Hence using equation (1.6) and Gauss' theorem we have

$$F_i = \int_{V(t)} dV \left[f_i + \frac{\partial}{\partial r_j} T_{ij}(\underline{r}, t)\right] .$$

Putting this back into Newton's second law and using the vanishing integral theorem, we have

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial r_j}(\rho u_i u_j) - f_i - \frac{\partial}{\partial r_j} T_{ij}(\underline{r}, t) = 0$$

or $$\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r_j}(\rho u_j)\right] + \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial r_j} u_i = f_i + \frac{\partial}{\partial r_j} T_{ij} .$$

If mass is conserved, the first term vanishes, and we obtain

$$\rho \frac{Du_i}{Dt} = \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial r_j} u_i = f_i + \frac{\partial}{\partial r_j} T_{ij}(\underline{r}, t) \qquad (1.7)$$

Conservation of Angular Momentum

Allowing for intrinsic angular momentum $\underline{\ell}$, the total angular momentum in dV is

$$\left[\underline{r} \times (\rho \underline{u}) + \rho \underline{\ell}\right] dV .$$

Equate the time rate of change of angular momentum to $\underline{M}$, the sum of the torques of the forces for the moving volume V(t):

$$\int_{V(t)} dV \left[\frac{\partial}{\partial t}\left\{\rho(\epsilon_{ijk} r_j u_k + \ell_i)\right\} + \frac{\partial}{\partial r_q}\left\{\rho(\epsilon_{ijk} r_j u_k + \ell_i) u_q\right\}\right] = M_i .$$

Allowing for body torques $\underline{m}$ and surface torques $\underline{M}(\underline{r}, t, \hat{n})$ we have

$$M_i = \int_{V(t)} dV \left\{m_i + \epsilon_{ijk} r_j f_k\right\} + \int_{\partial V(t)} dA \left\{M_i(\underline{r}, t \hat{n}) + \epsilon_{ijk} r_j T_{kj} n_q\right\} .$$

In a manner analogous to the proof of equation (1.6), it may be shown

$$M_i(\underline{r}, t, \hat{n}) = n_j M_i(\underline{r}, t, \hat{x}_j)$$

and

$$M_i(\underline{r}, t, \hat{x}_j) = M_{ij}(\underline{r}, t)$$ the <u>surface</u> <u>torque</u> <u>tensor</u>.

Then using Gauss's theorem on the surface integral, have

$$M_i = \int_{V(t)} dV \left\{m_i + \epsilon_{ijk} r_j f_k + \frac{\partial}{\partial r_q}\left(M_{iq} + \epsilon_{ijk} r_j T_{kq}\right)\right\} .$$

Hence, using the above in the angular momentum equation and the vanishing integral theorem, have

$$\frac{\partial}{\partial t}\left\{\rho(\epsilon_{ijk} r_j u_k + \ell_i)\right\} + \frac{\partial}{\partial r_q}\left\{\rho(\epsilon_{ijk} r_j u_k + \ell_i) u_q\right\} =$$

$$= m_i + \epsilon_{ijk} r_j f_k + \frac{\partial}{\partial r_q}\left(M_{iq} + \epsilon_{ijk} r_j T_{kq}\right) .$$

Finally

$$\rho \frac{D}{Dt} l_i = m_i + \frac{\partial}{\partial r_j} M_{ij} + \epsilon_{ijk} T_{kj} \tag{1.8}$$

In the case where internal angular momentum, torque stresses and body torques vanish, we obtain the usual result

$$\epsilon_{ijk} T_{kj} = 0$$

$$\text{i.e.} \qquad T_{kj} = T_{jk} \tag{1.9}$$

These notes submitted by

Wayne R. Thatcher

2. A Survey of Continuum Mechanics (continued)

Dr. Freeman Gilbert
University of California at San Diego

Conservation of Energy

Unlike mass, linear momentum or angular momentum, energy is always conserved in the universe. But energy can exist under different forms, and so the equation which expresses conservation of energy may also take several special forms. Here we shall be concerned only with the procedure of deriving such an energy equation.

We define the work performed in unit time by a force field $\underline{f}(\underline{r}, t)$ in a volume V(t) by the quantity

$$W_V = \int_{V(t)} dV \, \underline{u} \cdot \underline{f} \ .$$

In an analogous manner we define the work done in unit time by surface stresses $\underline{\mathcal{F}}(\underline{r},t,\hat{n})$ on the boundary $\partial V(t)$ of the volume (V(t) as

$$W_{\partial V} = \int_{\partial V(t)} dA\, \underline{u}\cdot\underline{\mathcal{F}} .$$

As

$$\underline{\mathcal{F}}(\underline{r},t,\hat{n}) = \underline{\underline{T}}\cdot\hat{n} ,$$

the divergence theorem allows us to transform the surface integral $W_{\partial V}$ into a volume integral, getting

$$W_{\partial V} = \int_{\partial V(t)} dA\, \underline{u}\cdot\underline{\underline{T}}\cdot\hat{n} = \int_{V(t)} dV\, \nabla\cdot(\underline{u}\cdot\underline{\underline{T}}) .$$

Therefore, the total work performed in unit time in the volume V(t) is

$$W = W_V + W_{\partial V} = \int_{V(t)} dV\, (\underline{u}\cdot\underline{f} + \nabla\cdot(\underline{u}\cdot\underline{\underline{T}})) .$$

Considering now the kinetic energy

$$K = \int_{V(t)} dV\, \tfrac{1}{2}\rho\, \underline{u}\cdot\underline{u} ,$$

we see at once that

$$\frac{dK}{dt} \neq W .$$

To be somewhat general, we must also consider radiative transfer as well as heat conduction by molecular processes. To include these effects in our problem, we define the change in unit time in the radiation energy content of the volume V(t) as

$$\mathcal{K} = \int_{V(t)} dV\, h(\underline{r},t) ,$$

and the heat flow through the surface $\partial V(t)$ as

$$\mathcal{H} = \int_{\partial V(t)} dA\ H(\underline{r},t,\hat{n}) .$$

Introducing the heat flow vector $\underline{H}(\underline{r},t)$, the latter expression becomes

$$\mathcal{H} = \int_{\partial V(t)} dA\, \hat{n}\cdot\underline{H} = \int_{V(t)} dV\, \nabla\cdot\underline{H} .$$

Generally, we still have

$$\frac{dK}{dt} \neq W + \mathcal{N} - \mathcal{H}.$$

We are thus obliged to define a quantity $\mathcal{U}$ called "internal energy" that gives us

$$\frac{d}{dt}(K + \mathcal{U}) = W + \mathcal{N} - \mathcal{H}.$$

The quantity $K + \mathcal{U}$ being the total energy per unit volume.

Let
$$E = \frac{1}{2}u^2 + U$$

be the total energy per unit mass $\left(U = \frac{\mathcal{U}}{\rho}\right)$. Then

$$\frac{d}{dt}\int_{V(t)} dV \rho E = \int_{V(t)} dV\left(\underline{u}\cdot\underline{f} + \nabla\cdot(\underline{u}\cdot\underline{\underline{T}}) + h - \nabla\cdot\underline{H}\right).$$

V(t) being any volume, this relation must hold locally, i.e.

$$\boxed{\frac{\partial}{\partial t}(\rho E) + \nabla\cdot\left(\rho E\underline{u} + \underline{H} - \underline{u}\cdot\underline{\underline{T}}\right) = \underline{u}\cdot\underline{f} + h} \qquad (1.10)$$

(Total Energy Equation)

<u>General Form of Conservation Equations</u>

Let ϕ and k be two scalars, $\underline{K}$ a vector, assuming

$$\begin{aligned} \phi &= \phi(\underline{r}, t) \\ k &= k(\underline{r}, t) \\ \underline{K} &= \underline{K}(\underline{r}, t). \end{aligned}$$

All conservation equations take the general form

$$\boxed{\frac{\partial \phi}{\partial t} + \nabla\cdot\underline{K} = k} \qquad (1.11)$$

Consider a volume V fixed in space and integrate equation (1.11) over this volume, obtaining
$$\int_V dV\left(\frac{\partial\phi}{\partial t} + \nabla\cdot\underline{K} - k\right) = 0.$$

The fact that V is fixed in space implies that the operators $\int_V$ and $\frac{\partial}{\partial t}$ commute. Applying this and the divergence theorem, we may finally write equation (1.11) in the following integral form:

$$\frac{d}{dt}\int_V dV\phi = \int_V dVk - \int_{\partial V} dA\,\hat{n}\cdot\underline{K} \tag{1.12}$$

The latter integral equation clearly shows that the rate of change of the "phi-stuff" $\int_V dV\phi$ is equal to the rate of production of "phi-stuff" in the volume V minus the rate at which "phi-stuff" escapes from V through the surface ∂V . Thus ϕ is the physical quantity being conserved, k is the local production rate often called "source function", and $\underline{K}$ is the local transport vector.

The following table gives a summary of the principal conservation equations of continuum mechanics. The $\underline{K}$-parts of the form $\phi\underline{u}$ are called "advective" terms.

"phi-stuff"	ϕ	K_j	k
mass	ρ	ρu_j	0
linear momentum	ρu_i	$\rho u_i u_j - T_{ij}$	f_i
angular momentum	ρl_i	$\rho l_i u_j - M_{ij}$	$m_i + \epsilon_{ijk} T_{kj}$
total energy	ρE	$\rho E u_j + H_j - u_i T_{ij}$	$u_i f_i + h$
internal energy	ρU	$\rho U u_j + H_j$	$T_{ij}\frac{\partial u_i}{\partial r_j} + h$

b. Boundary Conditions

Introduction.

Until now we always considered a single unspecified continuum of infinite extent. But owing to specific internal structure or finite extent, any real body always possesses some boundary, i.e. a very thin region where we pass from a given continuum into another given continuum. In all the problems of interest to us, we shall only be concerned with simple interfaces between two different continua. At such a boundary both continua are supposed to be always and everywhere in contact (no cavitations) and completely immiscible (the boundary is a surface with no mass and no individual properties). It is clear that generally some physical quantities cannot experience any change when passing through such an interface; they must actually fulfill some definite conditions, which we call "boundary conditions".

Kinematic Boundary Conditions

Let the boundary surface between two continua 1 and 2 be described by the equation

$$\psi(\underline{r}, t) = 0$$

As this boundary moves, all the particles which initially are on the boundary, will always remain on the boundary; so we have

$$\frac{D}{Dt}\psi = 0.$$

The particles are assumed to move in the first medium with velocity $\underline{u}_1$, in the second medium with velocity $\underline{u}_2$. On the boundary, supposed to be a simple interface, the latter equation allows us then to write

$$\frac{\partial \psi}{\partial t} + \underline{u}_1 \cdot \nabla \psi = \frac{\partial \psi}{\partial t} + \underline{u}_2 \cdot \nabla \psi = 0$$

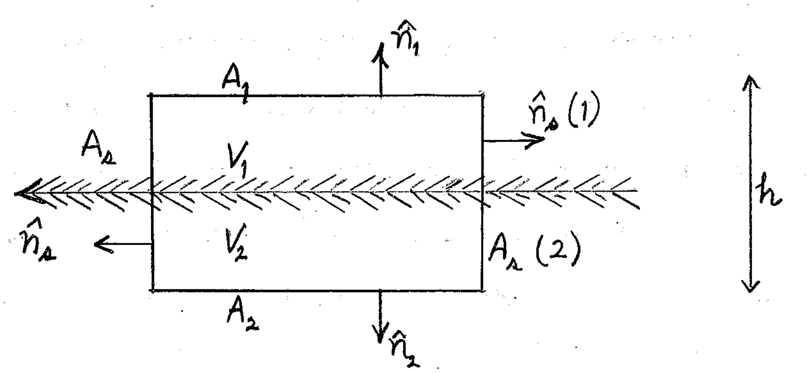
From this we easily deduce the kinematic boundary conditions:

for a simple surface $\hat{n} \cdot \underline{u}_1 = \hat{n} \cdot \underline{u}_2$ slip may occur (1.13)

for a "welded" surface $\underline{u}_1 = \underline{u}_2$ no slip is possible (1.14)

Dynamic Boundary Conditions

Consider a small pillbox across two media as illustrated by the following schematic figure:



From the linear momentum and the continuity equations applied to the whole volume of this pillbox it follows that

$$\frac{d}{dt}\int_{V_1(t)+V_2(t)} dV \rho \underline{u} = \int_{A_1+A_2+A_\Delta} dA\ \underline{\underline{T}} \cdot \hat{n} + \int_{V_1(t)+V_2(t)} dV\ \underline{f} \ .$$

Letting now the height h of the pillbox go to zero, $V_1(t)$, $V_2(t)$ as well as A_Δ go to zero; A_1 and A_2 tend to a part A of the boundary surface. Assuming now that all the body forces $\underline{f}$ are continuous and that no change of linear momentum is occurring, we are left with

$$\int_{A_1+A_2} dA\ \underline{\underline{T}} \cdot \hat{n} = 0$$

or
$$\int_A dA\left(\underline{\underline{T}}^{(1)} - \underline{\underline{T}}^{(2)}\right)\cdot\hat{n} = 0 .$$

The surface A being any part of the boundary, we must have

$$\underline{\underline{T}}^{(1)}\cdot\hat{n} = \underline{\underline{T}}^{(2)}\cdot\hat{n} \qquad (1.14)$$

or, in components

$$T_{ij}^{(1)} n_j = T_{ij}^{(2)} n_j \qquad (i,j = 1,2,3) \qquad (1.14a)$$

The dynamical boundary conditions thus simply state that the normal stress vector is continuous at a boundary surface.

c. Constitutive Relations

There is some simple relation between stress and strain, which we shall establish now.

Consider a point mass $\underline{x} = x_i \hat{X}_i$, the position of which at any time t will be characterized by the position vector $\underline{r}(\underline{x}, t)$. With this notation, we have

at time $t = 0 : \underline{r}(\underline{x}, 0) = \underline{x}$

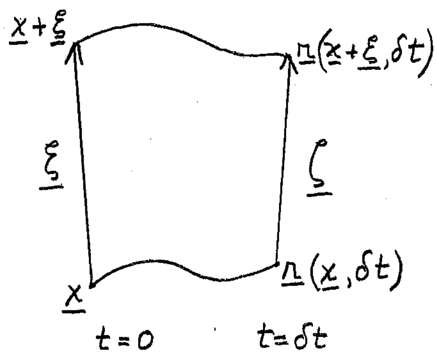
at time $t = \delta t : \underline{r}(\underline{x}, \delta t) = \underline{x} + \underline{\xi}$.

This means that during the time δt the particle labelled $\underline{x}$ has been moved over a distance ξ in the $\hat{\xi}$ direction. During the same time δt , the point mass initially at

$$\underline{r}(\underline{x} + \underline{\xi}, 0) = \underline{x} + \underline{\xi}$$

has moved to a position

$$\underline{r}(\underline{x} + \underline{\xi}, \delta t).$$



We shall study the evolution during time of the relative position of particles labelled $\underline{x}$ and $\underline{x}+\underline{\xi}$. For this purpose it is most convenient to introduce the displacement vector

$$\underline{\zeta}(t)=\underline{r}(\underline{x}+\underline{\xi},t)-\underline{r}(\underline{x},t).$$

We have of course

$$\underline{\zeta}(0)=\underline{x}+\underline{\xi}-\underline{x}=\underline{\xi}$$

If we assume that $\delta t \ll 1$, we may expand the function $\underline{r}(\underline{x}+\underline{\xi},\delta t)$ to the first order in time and get

$$\underline{r}(\underline{x}+\underline{\xi},\delta t)=\underline{r}(\underline{x}+\underline{\xi},0)+\delta t\frac{D}{Dt}\underline{r}(\underline{x}+\underline{\xi},0)+O(\delta t^2)$$
$$\approx \underline{x}+\underline{\xi}+\delta t\,\underline{u}(\underline{x}+\underline{\xi},0)$$

Expanding the displacement vector $\underline{\zeta}\equiv\underline{\zeta}(\delta t)$ also to the first order in time, we have

$$\begin{aligned}\underline{\zeta}=\underline{\zeta}(\delta t)&\approx\underline{\zeta}(0)+\delta t\frac{D}{Dt}\underline{\zeta}(0)\\&=\underline{\xi}+\delta t\frac{D}{Dt}\left[\underline{r}(\underline{x}+\underline{\xi},0)-\underline{r}(\underline{x},0)\right]\\&=\underline{\xi}+\delta t\left[\underline{u}(\underline{x}+\underline{\xi},0)-\underline{u}(\underline{x},0)\right]\end{aligned}\qquad\text{(a)}$$

Expanding then $\underline{u}(\underline{x}+\underline{\xi},0)$ to the first order in $\underline{\xi}$ ($\underline{\xi}$ assumed to be very small):

$$\underline{u}(\underline{x}+\underline{\xi},0)=\underline{u}(\underline{x},0)+\frac{\partial\underline{u}(\underline{x},0)}{\partial\underline{x}}\cdot\underline{\xi}\qquad\text{(b)}$$

and putting the expansion (b) into (a), we get finally

$$\underline{\zeta} = \underline{\xi} + \delta t \, \frac{\partial \underline{u}(\underline{x},0)}{\partial \underline{x}} \cdot \underline{\xi} \quad .$$

In component notation

$$\zeta_i = \xi_i + \frac{\partial u_i(\underline{x},0)}{\partial x_j} \xi_j \delta t \quad . \tag{1.15}$$

$\dfrac{\partial \underline{u}(\underline{x},0)}{\partial \underline{x}}$ represents a second order tensor that belongs to particle $\underline{x}$ at time $t=0$. It is called distortion rate tensor.

A second order tensor has in general $3^2 = 9$ independent entries, and may be decomposed into two irreducible tensors, one symmetric (at most 6 distinct entries), the other antisymmetric (at most 3 distinct entries). We shall put

$$\frac{\partial u_i}{\partial x_j} = \epsilon_{ij} + \omega_{ij} \tag{1.16}$$

with

$$\left.\begin{aligned} \epsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ \omega_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \end{aligned}\right\} \tag{1.17}$$

Introducing the Kronecker symbol δ_{ij} , equation (1.15) may be written now

$$\zeta_i = (\delta_{ij} + \delta t \, \omega_{ij} + \delta t \, \epsilon_{ij}) \xi_j . \tag{1.15a}$$

The term $\delta_{ij} \xi_j$ clearly corresponds to a translation of the initial displacement vector $\underline{\xi}$, the term $\delta t \, \omega_{ij} \xi_j$ corresponds to a rotation; so the term $\delta t \, \epsilon_{ij} \xi_j$ must be associated with a symmetric deformation.

Indeed, the antisymmetric tensor (ω_{ij}) has necessarily the following matrix representation

$$(\omega_{ij}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

to which we can associate the dual vector

$$\underline{\omega} = \omega_i \hat{X}_i = (\omega_1, \omega_2, \omega_3) .$$

Thus

$$\omega_{ij}\, \xi_j = (\underline{\omega} \times \underline{\xi})_i .$$

From this it follows at once that

$$\underline{\omega} = \frac{1}{2} \nabla \times \underline{u}, \tag{1.16a}$$

i.e. the curl vector equals half the vorticity.

The tensor $\underline{\underline{\epsilon}} = (\epsilon_{ij})$ is called the strain rate tensor, and $\underline{\underline{e}} = (\epsilon_{ij})\delta t$ is called the strain tensor.

Define

$$\psi = e_{ij}\, \xi_i\, \xi_j - a^2,$$

then

$$\frac{\partial \psi}{\partial \xi_i} = e_{ij}\, \xi_j \equiv v_i ,$$

which is the i-component of the strained part $\underline{v}$ of the displacement vector $\underline{\xi}$.

In any frame we have

$$\begin{aligned} v_1 &= e_{11}\xi_1 + e_{12}\xi_2 + e_{13}\xi_3 \\ v_2 &= e_{21}\xi_1 + e_{22}\xi_2 + e_{23}\xi_3 \\ v_3 &= e_{31}\xi_1 + e_{32}\xi_2 + e_{33}\xi_3 . \end{aligned}$$

Let us now define three new mutually orthogonal axes $\hat{Y}_i$ parallel to the principal directions of the matrix (e_{ij}). In these new axes, the matrix (e_{ij}) takes then the diagonal form

$$(e'_{ij}) = \begin{pmatrix} e'_{11} & & \\ & e'_{22} & \\ & & e'_{33} \end{pmatrix}$$

The eigenvalues e'_{11}, e'_{22}, e'_{33} of this matrix are called the principal strains. Let y_i be the corresponding eigenfunctions; then we have

$$\psi = e'_{11} y_1^2 + e'_{22} y_2^2 + e'_{33} y_3^2 .$$

We have also

$$\zeta_1 = (1+e'_{11}) y_1$$
$$\zeta_2 = (1+e'_{22}) y_2$$
$$\zeta_3 = (1+e'_{33}) y_3$$

The fractional change of volume of any body becomes then (to the first order)

$$\frac{Vol(\delta t) - Vol(0)}{Vol(\delta t)} = e'_{11} + e'_{22} + e'_{33} = e'_{ii} .$$

But the trace of any matrix is an invariant; so we have

$$e'_{ii} = e_{ii} = \nabla \cdot \underline{u}\, \delta t .$$

<u>Energy Considerations</u>

From the total energy equation

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial r_j}\left(\rho E u_j + H_j - u_i T_{ij}\right) = u_i f_i + h$$

where

$$E = \tfrac{1}{2} u_i u_i + U ,$$

it follows that

$$\rho \frac{D}{Dt} U = \left(T_{ij} \frac{\partial u_i}{\partial r_j} + h - \frac{\partial H_j}{\partial r_j}\right) .$$

Note that the latter equation implies that mass and linear momentum are conserved.

Following the motion the change in interanl energy of a particle during time δt , and per unit volume, is

$$\rho \delta U = \left(T_{ij} \frac{\partial u_i}{\partial x_j} + h - \frac{\partial H_i}{\partial x_i}\right) \delta t.$$

Assuming that

$$T_{ij} = T_{ji} ;$$

the former equation may be written

$$\rho \delta U = \left(\underbrace{T_{ij} \epsilon_{ij}}_{\substack{\text{rate of} \\ \text{change in} \\ \text{elastic energy}}} + h - \frac{\partial H_i}{\partial x_i}\right) \delta t .$$

But the First Law of Thermodynamics states that

$$\delta U = \frac{1}{\rho} T_{ij} \delta e_{ij} + \delta Q ,$$

where

$\frac{1}{\rho} T_{ij} \delta e_{ij}$ is the total work performed in a unit mass during time δt ;

δQ is the change in heat in a unit mass during time δt .

If S denotes the entropy per unit mass, and θ the actual temperature, we have of course

$$\delta Q = \theta \delta S ,$$

so that finally

$$\delta U = \frac{1}{\rho} T_{ij} \delta e_{ij} + \theta \delta S . \tag{1.18}$$

If the superscript $^{\circ}$ denotes a quantity at time $t = 0$, we may write then

$$U = U^{\circ} + a_{ij}(e_{ij} - e^{\circ}_{ij}) + \tfrac{1}{2} b_{ijkl}(e_{ij} - e^{\circ}_{ij})(e_{kl} - e^{\circ}_{kl}) +$$
$$+ c(S - S^{\circ}) + f(S - S^{\circ})^2 +$$
$$+ g_{ij}(S - S^{\circ})(e_{ij} - e^{\circ}_{ij}) + \ldots \qquad (1.19)$$

Equation (1.18) states that

$$\left.\begin{aligned} \left(\frac{\partial U}{\partial e_{ij}}\right)_S &= \frac{1}{\rho} T_{ij} \\ \left(\frac{\partial U}{\partial S}\right)_{e_{ij}} &= 0 \end{aligned}\right\} \qquad (1.20)$$

We get therefore from (1.19)

$$\frac{1}{\rho} T_{ij} = a_{ij} + b_{ijkl}(e_{kl} - e^{\circ}_{kl}) + g_{ij}(S - S^{\circ})$$
$$\theta = c + f(S - S^{\circ}) + g_{ij}(e_{ij} - e^{\circ}_{ij}) .$$

Finally

$$\left.\begin{aligned} T_{ij} &= T^{\circ}_{ij} + c_{ijkl}\,\delta e_{kl} + w_{ij}\,\delta S \\ &= T^{\circ}_{ij} + \tau_{ij} \end{aligned}\right\} \qquad (1.21)$$

The most general tensor of rank 4 would have in the physical space $3^4 = 81$ entries. However, it is easy to show that (c_{ijkl}) must be symmetric with respect to the two groups of indices $(ij), (kl)$, as well as inside each group. From this it follows that (c_{ijkl}) can at most have 21 distinct entries.

Furthermore, if the continuum is completely isotropic and cannot support shear stress, the tensor (c_{ijkl}) reduces to

$$c_{ijkl} = K\,\delta_{ij}\,\delta_{kl} , \qquad (1.22)$$

where K is the incompressibility.

Many elastic processes behave adiabatically, i.e. $h = 0$, $H_i = 0$, $S = \text{const.}$ or $\delta S = 0$. In the latter case, the term

$$W_{ij}\,\delta S = 0$$

and we are left with the usual stress-strain relation

$$\delta T_{ij} = c_{ijkl}\, \delta e_{kl}$$

In fact there must always be small changes in entropy, because elastic waves are damped, and any dissipation process is associated with a change in energy. However, this damping occurs usually only over great distances, and locally the entropy effects are quite negligible with respect to the strain effects, i.e. $|W_{ij}\,\delta S| \ll |c_{ijkl}\,\delta e_{kl}|$.

2. The Earth's Normal Modes

Linearized Equations

First of all, we have to consider some Earth model. The assumptions we shall make throughout these lectures are the following:

The Earth is a perfect sphere of constant density,
in hydrostatic equilibrium,
not rotating, and
perfectly elastic.

All these assumptions do not hold rigorously of course, but the main deviatory effect, which comes from rotation, is about 4×10^{-2}.

From the foregoing assumptions we may conclude at once that all equilibrium quantities (specified by the subscript $_0$) are radially distributed. In particular, we have

$$\left\{\begin{array}{ll} \underline{u}_0 = 0 & \text{no motion} \\ \phi_0 = \phi(r) & \text{gravitational potential} \\ g_0 = \dfrac{\partial}{\partial r}\phi_0(r) = g_0(r) & \text{gravity} \\ -\hat{r}\, g_0 & \text{gravitational field} \\ \underline{f}_0 = -\nabla\cdot\underline{\underline{T}}_0 = \nabla p_0 & \text{hydrostatic equilibrium} \end{array}\right\} \qquad (2.1)$$

or

$$\frac{d}{dr} p_0 = -\rho_0(r)\, g_0(r).$$

To this we must add the Poisson equation, which takes here the following form

$$\left(\frac{d}{dr} + \frac{2}{r}\right) g_0(r) = 4\pi G \rho_0(r) \tag{2.2}$$

Let us now disturb slightly this equilibrium state by displacing locally some material over a small distance defined by the initial displacement vector $\underline{S}(\underline{r}, o)$. A motion of small amplitude sets in, which we can characterize by the instantaneous local velocity

$$\underline{u} = \underline{u}(\underline{r}, t) = \frac{d}{dt} \underline{S}(\underline{r}, t) .$$

The effect of this motion will be a variation of the physical parameters like the density ρ and the pressure p as well in space as in time; we assume that we may write

$$\begin{aligned} \rho(\underline{r}, t) &= \rho_0(r) + \rho_1(\underline{r}, t) \\ p(\underline{r}, t) &= p_0(r) + p_1(\underline{r}, t) \\ \phi(\underline{r}, t) &= \phi_0(r) + \phi_1(\underline{r}, t) , \end{aligned} \tag{2.3}$$

where the disturbed quantities ρ_1, p_1, ϕ_1 are very small compared to ρ_0, p_0, ϕ_0 respectively.

Because we focus our attention on what happens at a fixed point in space, all the disturbed quantities are Eulerian variations, and therefore the Eulerian description is convenient. Limiting ourselves to the first order in these perturbations, the continuity equation takes the form

$$\rho_1 = -\rho_0 \nabla \cdot \underline{S} - \underline{S} \cdot \nabla \rho_0 , \tag{2.4}$$

whereas the linearized momentum equation becomes

$$\rho_0 \frac{\partial^2}{\partial t^2} \underline{S} = -\rho_0 \nabla \phi_1 - \rho_1 \nabla \phi_0 + \nabla \cdot \underline{\underline{\tau}}^E , \tag{2.5}$$

where τ^E is defined by

$$\underline{\underline{T}}^E = -\underline{\underline{1}}\, p_0 + \underline{\underline{\tau}}^E .$$

Now, we know only the Lagrangian stress tensor $\underline{\underline{\tau}}^L$ defined by the stress-strain relation

$$\left(\tau^L_{ij}\right) = \left(c_{ijkl}\, e_{kl}\right) ,$$

where the superscript "L" refers to the Lagrangian description. As

$$\underline{\underline{T}}^L = \left(1 + \underline{S} \cdot \nabla\right) \underline{\underline{T}}^E ,$$

we find at once

$$\underline{\underline{\tau}}^E = \underline{\underline{\tau}}^L + \underline{\underline{1}}\, \underline{S} \cdot \nabla p_0 .$$

Introducing this into the linearized momentum equation (2.5), the latter takes finally the form

$$\rho_0 \frac{\partial^2}{\partial t^2} \underline{S} = -\rho_0 \nabla \phi_1 + \rho_0\, \nabla \cdot \underline{S}\, \nabla \phi_0 - \rho_0 \nabla\left(\underline{S} \cdot \nabla \phi_0\right) + \nabla \cdot \underline{\underline{\tau}}^L . \tag{2.6}$$

To this we must add the Poisson equation for the perturbed part ϕ_1 of the potential, i.e.

$$\nabla^2 \phi_1 = -4\pi G \nabla \cdot \left(\rho_0 \underline{S}\right) \tag{2.7}$$

Assuming that all the equilibrium values are known, we have thus a system of 4 linear differential equations for the 4 unknown quantities S_1, S_2, S_3, ϕ_1.

We shall try to solve this linear system in two special cases.

(i) Radial Oscillations of a Homogeneous Sphere

We shall here try to find a solution to the former system (2.6)-(2.7) for a homogeneous, compressible, self-gravitating spherical mass of radius a. We shall further assume that the equilibrium state is only perturbed radially, and that the mass is completely isotropic and does not support any shear. We have seen that under the latter circumstances the stress tensor assumes the simple form

$$\underline{\underline{\tau}}^{L} = K \nabla \cdot \underline{S}\, \underline{\underline{1}} = - p' \underline{\underline{1}}, \tag{2.8}$$

where K is a scalar called incompressibility.

We have also to take into account the boundary conditions which arise at the surface $r = a$ of the sphere. Indeed, let us recall that at the surface the following quantities must be continuous:

$$\hat{r} \cdot \underline{S}\,,\ \hat{r} \cdot \underline{\underline{\tau}}^{L},\ \phi_1,\ \frac{\partial}{\partial r}\phi_1 + (\hat{r} \cdot \underline{S}) 4\pi G \rho_0\ . \tag{2.9}$$

Let us now simplify our fundamental equations, knowing that all motions occur radially, that is, along the unit radius vector $\hat{r}$. Introducing the radial displacement $U(r)$ by

$$\underline{S} = \hat{r}\, U,$$

recalling that for a homogeneous sphere

$$\rho_0 = \text{const.}, \qquad K = \text{const.},$$

the momentum equation (2.6) reduces to

$$\rho_0 \frac{\partial^2}{\partial t^2} U \hat{r} = \hat{r}\left(-\rho_0 \frac{\partial}{\partial r}\phi_1 + \rho_0\, g_0 \nabla \cdot \underline{S}\right) - \rho_0 \nabla\left(\underline{S} \cdot \hat{r}\, g_0\right) + K \nabla \nabla \cdot \underline{S} \tag{2.6a}$$

with $\nabla \cdot \underline{S} = \frac{\partial}{\partial r} U + \frac{2}{r} U$.

The Poisson equation takes the radial form

$$\frac{\partial^2}{\partial r^2}\phi_1 + \frac{2}{r}\frac{\partial}{\partial r}\phi_1 = -4\pi G \rho_0 \nabla \cdot \underline{S} \tag{2.7a}$$

Defining the quantity

$$g_1 \equiv \frac{\partial}{\partial r}\phi_1 + 4\pi G\rho_0 U,$$

we have seen that this quantity must be continuous at the boundary, but outside the sphere, g_1 is necessarily zero, and therefore g_1 must be zero on the boundary. From this we deduce that g_1 also vanishes everywhere inside the sphere, i.e.

$$\frac{\partial}{\partial r}\phi + 4\pi G\rho_0 U \equiv 0.$$

Assuming the radial displacement U is determined everywhere, then the disturbed potential ϕ_1 is also uniquely determined everywhere.

As ρ_0 is constant, we have

$$g_0 = \frac{G\int_{v(r)}\rho\, dr}{r^2} = \frac{4}{3}\pi G\rho_0 r. \qquad (2.10)$$

Introducing the constant

$$\xi = \frac{g_0}{r} = \frac{4}{3}\pi G\rho_0 ,$$

and separating out the time dependence by a factor $e^{-i\omega t}$, we are finally left with the following momentum equation

$$\rho_0(\omega^2 + 4\xi)\underline{S} + K\nabla\nabla\cdot\underline{S} = 0 ,$$

the divergence of which takes the form

$$\rho_0(\omega^2 + 4\xi)\nabla\cdot\underline{S} + K\nabla^2\nabla\cdot\underline{S} = 0 .$$

Putting

$$\left.\begin{aligned} f &= \nabla\cdot\underline{S} \\ k^2 &= (\omega^2 + 4\xi)\frac{\rho}{K} \end{aligned}\right\} \qquad (2.11)$$

we have $\nabla^2 f + k^2 f = 0,$

with $\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr}$

The only solution of this equation which is regular at the center $r=0$ is the spherical Bessel function of zero order

$$j_0(kr) = c' \frac{\sin kr}{r} ;$$

therefore

$$f(r) = c \frac{\sin kr}{r} .$$

At the surface, the displacement $\underline{\xi}$ must vanish, implying

$$f(a) = 0 ,$$

or

$$k^2 a^2 = n^2 \pi^2 . \quad (n \text{ integer}) \tag{2.12}$$

Using (2.11) in (2.12), we conclude then that ω^2 satisfies the equation

$$\omega^2 = \frac{n^2 \pi^2 K}{a^2 \rho_0} - \frac{16}{3} \pi G \rho_0 . \tag{2.13}$$

The form of the time factor $e^{-i\omega t}$ shows that

$\omega^2 > 0$ corresponds to stability;

$\omega^2 = 0$ corresponds to marginal stability;

$\omega^2 < 0$ corresponds to instability.

We may put the instability criterion into the following form: instability occurs if

$$\frac{16 G}{3\pi} \frac{\rho_0^2 a^2}{n^2 K} > 1 . \tag{2.14}$$

From this we may argue that radial oscillations tend to be unstable for low mode values n , for small incompressibility K , for high density and for great dimensions of the sphere (radius a).

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3. A Survey of Continuum Mechanics (continued)

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(ii) More General Solutions in Spherical Coordinates

Vector and Tensor Representation Theorem

Reference: Backus, G.E. (1966) Arch.Rat.Mech.Anal.
" " " (1967) Geophys.J. of R.A.S., 14.

What follows is an attempt to represent a vector in spherical coordinates with a minimum of gruesome algebra. Details and rationale are mostly omitted, and those who are interested are urged to refer to the papers of Backus.

ξ_i are a set of curvilinear coordinates and h_i are the scale factors for the coordinate system.

Define the following vector functions:

$$\underline{P} = \hat{\xi}_1 U(\xi_1, \xi_2, \xi_3)$$
$$\underline{C} = \nabla \times [\hat{\xi}_1 c(\xi_1, \xi_2, \xi_3)]$$
$$\underline{B} = \hat{\xi}_1 \times \nabla \times [\hat{\xi}_1 b(\xi_1, \xi_2, \xi_3)]$$

In spherical coordinates

$$\xi_1 = r \qquad h_1 = 1$$
$$\xi_2 = \theta \qquad h_2 = r$$
$$\xi_3 = \varphi \qquad h_3 = r \sin\theta$$

Then

$$\underline{P} = \hat{r}\, U(r, \theta, \varphi)$$
$$\underline{C} = r^{-1}\left[\frac{\hat{\theta}}{\sin\theta}\frac{\partial c}{\partial\theta} - \hat{\varphi}\frac{\partial c}{\partial\varphi}\right]$$
$$\underline{B} = r^{-1}\left[\hat{\theta}\frac{\partial b}{\partial\theta} + \frac{\varphi}{\sin\theta}\frac{\partial b}{\partial\varphi}\right]$$

Here put $\mathscr{a} = rW$, $\mathscr{b} = rV$

Use the notation $\nabla_1 \equiv \hat{\theta}\,\frac{\partial}{\partial\theta} + \frac{\hat{\varphi}}{\sin\theta}\,\frac{\partial}{\partial\varphi}$

Then in this notation, represent any vector $\underline{v}$ as

$$\underline{v} = \hat{r}\,U(r,\theta,\varphi) + \nabla_1 V(r,\theta,\varphi) - \hat{r}\times\nabla_1 W(r,\theta,\varphi) \tag{2.15}$$

The following useful relations are then immediate:

$$\left.\begin{aligned}
\nabla\cdot\underline{v} &= \frac{\partial}{\partial r}U + \frac{2}{r}U + r^{-1}\nabla_1^2 V \\
\nabla\times\underline{v} &= -\hat{r}\,r^{-1}\nabla^2 W + \nabla_1\left[\frac{\partial}{\partial r}W + \frac{3}{r}W\right] - \hat{r}\times\nabla_1\left[r^{-1}U - r^{-1}V - \frac{\partial}{\partial r}V\right] \\
(\hat{r}\cdot\nabla)\underline{v} + r^{-1}\nabla(\underline{r}\cdot\underline{v}) &= \hat{r}\,2\,\frac{\partial}{\partial r}U + \nabla_1\left[\frac{\partial}{\partial r}V - r^{-1}V - r^{-1}W\right] - \\
&\quad - \hat{r}\times\nabla_1\left[r^{-1}U - r^{-1}V - \frac{\partial}{\partial r}V\right]
\end{aligned}\right\} \tag{2.16}$$

Non-radial Free Oscillations

Consider the linearized momentum equation (2.6) for harmonic motion $\left(\frac{\partial^2}{\partial t^2} = -\omega^2\right)$.

Then

$$\rho\,\frac{\partial^2}{\partial t^2}\underline{s} = -\rho\nabla\phi_1 + \rho\nabla\cdot\underline{s}\,\nabla\phi_0 - \rho\nabla(\underline{s}\cdot\nabla\phi_0) + \nabla(\chi\,\nabla\cdot\underline{s})$$

becomes

$$\rho\omega^2\underline{s} - \rho\nabla\phi_1 + \rho\xi\underline{r}\,\nabla\cdot\underline{s} - \rho\,\xi\,\nabla^2(\underline{r}\cdot\underline{s}) + \chi\,\nabla\nabla\cdot\underline{s}\,.$$

Call $\Theta = \nabla\cdot\underline{s}$

Take, in turn, the divergence and the curl of the momentum equation and obtain

$$\rho\omega^2\Theta - \rho\nabla^2\phi_1 - \rho\xi\nabla\cdot(\underline{r}\,\Theta) - \rho\,\xi\,\nabla^2(\underline{r}\cdot\underline{s}) + \chi\,\nabla^2\Theta = 0$$

$$\rho\,\omega^2\,\nabla\times\underline{s} + \rho\,\xi\,\nabla\times(\underline{r}\,\Theta) = 0$$

Performing the algebraic manipulations we obtain

$$\rho(\omega^2+4\xi)+\rho\frac{\xi^2}{\omega^2}\nabla_1^2\Theta+K\nabla^2\Theta=0$$

Now expand Θ in spherical harmonics

i.e. $$\Theta=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Theta_\ell^m(r)\,Y_\ell^m(\theta,\varphi).$$

Using $\nabla_1^2 Y_\ell^m+\ell(\ell+1)Y_\ell^m=0$ we find

$$\rho(\omega^2+4\xi)\Theta_\ell^m-\ell(\ell+1)\rho\frac{\xi^2}{\omega^2}\Theta_\ell^m+K\nabla_r^2\Theta_\ell^m=0$$

where $\nabla_r^2=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\ell(\ell+1)}{r^2}$

This ordinary differential equation is of the form

$$\nabla_r^2\Theta_\ell^m+k^2\Theta_\ell^m=0$$

with $k^2=\left[\omega^2+4\xi-\ell(\ell+1)\frac{\xi^2}{\omega^2}\right]\frac{\rho}{K}$.

Solutions regular at the origin are $j_\ell(kr)$.

Boundary condition at $r=a$ gives

$$ka=X_{\ell m}$$

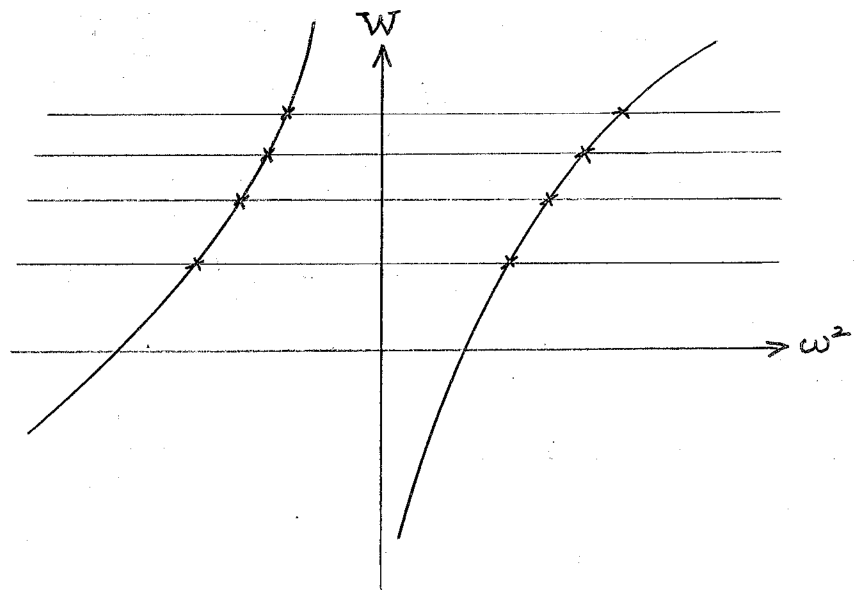
where $X_{\ell m}$ are zeros of the spherical Bessel function j_ℓ.

Then

$$\omega^2+4\xi-\ell(\ell+1)\frac{\xi^2}{\omega^2}=\frac{K}{\rho}\frac{X_{\ell m}^2}{a^2}=W,\ \text{say}\quad . \quad . \quad . \qquad (2.17)$$

and $W>0$, as all zeros are real.

Schematically then, we have the following:



That is, there are an infinite number of positive and negative values of ω^2. Thus there are an infinite number of stable and unstable modes.

These notes submitted by

Wayne R. Thatcher

4. The Momuntum Equation in Spherical Coordinates and the Inverse Eigenvalue Problem

We now return to the linearized momentum equation,

$$\rho^0 \frac{\partial^2 \vec{s}}{\partial t^2} = -\rho^0 \nabla \phi_1 + \rho^0 \nabla \cdot \vec{s}\, g \hat{r} - \rho^0 \nabla (gU) - \nabla P \tag{1}$$

where $\vec{s} = \hat{r}\, U + \nabla_1 V - \hat{r} \times \nabla_1 W,$

and consider its solution for a sphere with variable $K(r), \rho(r)$.

Now,

$$\nabla (g U) = g \nabla U + \hat{r}\, U (4\pi G \rho - 2 r^{-1} g) \tag{2}$$

The equation for ϕ_1 is $\quad \nabla^2 \phi_1 = 4\pi G \rho'$

which, in spherical coordinates becomes

$$\left[\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla_1^2\right]\phi_1 = 4\pi G \rho' \tag{3}$$

From the linearized continuity equation we have for ρ_1 ,

$$\frac{\partial}{\partial t}\rho' + \rho^0 \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho^0 = 0$$

which in terms of the displacement $\vec{s}\left(\vec{u} = \frac{\partial \vec{s}}{\partial t}\right)$ becomes

$$\rho' = -\rho^0 \nabla \cdot \vec{s} - \vec{s} \cdot \nabla \rho^0 .$$

In spherical coordinates we have

$$\rho' = -\rho'\left(\frac{\partial U}{\partial r} + \frac{2}{r}U + \frac{\nabla_1^2 V}{r}\right) - U\frac{\partial \rho^0}{\partial r} \tag{4}$$

Substituting (4) into (3) we have

$$\left[\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla_1^2\right]\phi_1 = -4\pi G\left[\rho^0\left(\frac{\partial U}{\partial r} + \frac{2}{r}U + \frac{1}{r}\nabla_1^2 V\right) + U\frac{\partial \rho^0}{\partial r}\right] \tag{5}$$

Let $g_1 \equiv \frac{\partial \phi_1}{\partial r} + 4\pi G\rho^\circ U$ and then (5) becomes

$$\frac{\partial g_1}{\partial r} = -\frac{2}{r} g_1 - \frac{1}{r^2}\nabla_1^2\phi_1 - \frac{4\pi G}{r}\rho^\circ \nabla_1^2 V \tag{6}$$

The radial component of (1) can be written in terms of g_1, as

$$\frac{\partial P}{\partial r} = \rho_0\left[-\frac{\partial^2 U}{\partial t^2} - g_1 + \frac{4g}{r}U + \frac{g}{r}\nabla_1^2 V\right] \tag{7}$$

and the angular component of (1) is

$$\nabla_1\left[P - \rho^\circ\left(-\frac{\partial^2 V}{\partial t^2} - \phi_1 - gU\right)\right] = 0 \tag{8}$$

and $\rho^\circ \frac{\partial^2}{\partial t^2}\left(-\hat{r}\times\nabla_1 W\right) = 0.$

Now, $P = -K\nabla\cdot\vec{s}$

or $$\frac{\partial U}{\partial r} = -\frac{2}{r}U - \nabla_1^2\frac{V}{r} - K^{-1}P. \tag{9}$$

Now, expand everything in spherical harmonics.

Equation (9) then becomes

$$\frac{\partial U}{\partial r} = -\frac{2U}{r} + \frac{\ell(\ell+1)}{r} - \frac{P}{K} \tag{10}$$

For (7) and (8) we have (writing $\frac{\partial^2}{\partial t} = -\omega^2$)

$$\frac{\partial P}{\partial r} = \rho^\circ\left(\omega^2 + \frac{4g}{r}\right)U - \ell(\ell+1)\frac{\rho^\circ g}{r}V - \rho^\circ g_1 \tag{11}$$

$$\rho^\circ\omega^2 V = \rho^\circ g U + P + \rho^\circ\phi_1 \tag{12}$$

For ϕ_1 we have (from the definition of g_1)

$$\frac{\partial \phi_1}{\partial r} = -4\pi G\rho^\circ U + g_1 \tag{13}$$

and for g_1 we have from (6)

$$\frac{\partial g_1}{\partial r} = 4\pi G\rho^0 \frac{\ell(\ell+1)}{r} V + \frac{\ell(\ell+1)}{r^2}\phi_1 - \frac{2}{r} g_1 \tag{14}$$

Thus we have four ordinary differential equations (10), (11), (13), (14) in the four unknowns U, P, ϕ_1, g_1 (note that V is known in terms of U, P, ϕ_1 from (12)). Now U, P, ϕ_1 and g_1 must be continuous. The boundary conditions at the free surface $r = a$ are $P = 0$ and ϕ and g_1 must equal their external values. Outside the sphere, ϕ_1 must be a solution to Laplace's equation

$$\phi_1^e = c\, r^{-\ell-1}$$

and at $r = a$ ϕ_1^e must equal its internal value. Externally, $\vec{s} = 0$ so

$$g_1 = \frac{\partial \phi_1}{\partial r} = -\frac{(\ell+1)}{r}\phi_1 \tag{15}$$

and at $r = a$

$$g_1 = -\frac{(\ell+1)}{a}\phi_1 \tag{16}$$

If we define

$$\psi = g_1 + \frac{(\ell+1)}{r}\phi_1 \tag{17}$$

we see that at $r = a$, $\psi = 0$. ψ is also a continuous function of r because g_1 and ϕ_1 are, so we may use ψ in place of g_1 as one of the dependent variables in the problem. Thus we have a system of coupled linear ordinary differential equations for U, P, ϕ_1, ψ which we can write as:

$$\frac{d}{dr}\begin{pmatrix} U \\ P \\ \phi_1 \\ \psi \end{pmatrix} = \begin{pmatrix} -2r^{-1}+\frac{\ell(\ell+1)g}{\omega^2 r} & -\frac{1}{K}+\frac{\ell(\ell+1)}{\rho\omega^2 r} & \frac{\ell(\ell+1)}{\omega^2 r^2} & 0 \\ \rho\omega^2+4\rho g r^{-1}-\frac{\ell(\ell+1)\rho g^2}{\omega^2} & -\frac{\ell(\ell+1)g}{\omega^2 r} & \frac{(\ell+1)\rho}{r}-\frac{\ell(\ell+1)\rho g}{\omega^2 r} & -\rho \\ -4\pi G\rho & 0 & -\frac{(\ell+1)}{r} & 1 \\ 4\pi G\rho & \frac{\ell(\ell+1)4\pi G}{\omega^2 r^2} & \frac{\ell(\ell+1)4\pi G\rho}{\omega^2 r^2} & \frac{\ell-1}{r} \end{pmatrix}\begin{pmatrix} U \\ P \\ \phi_1 \\ \psi \end{pmatrix} \tag{18}$$

There are four linearly independent solutions to this system of equations, of which only two are regular at $r=0$; the two boundary conditions needed to uniquely determine the solution to within a multiplicative constant (provided ω^2 is an eigenvalue) are $\psi_2=0$ and $P=0$ at the surface.

The method for solution of the direct problem would be to assume a value of ω^2 , and integrate up to the surface by some numerical scheme, such as the Runge-Kutta method. If the solution satisfies both boundary conditions there, then we have found an eigenvalue; otherwise a new value of ω^2 must be tried. Due to the singularities in the coefficient matrix at $r=0$, the homogeneous earth solution is used to integrate a short distance away from $r=0$, whereafter the numerical technique is initiated.

The matrix formulation is useful because it allows one to vary the step size in the computer program; this is done by determining the eigenvalues of the 4x4 matrix for a given location (r) ; these eigenvalues are the growth rates of the solution, which are needed to determine computation errors as a function of step size. Naturally, the step size is selected to keep the error within some pre-determined tolerance.

However, we want to solve the inverse problem - given the eigenvalues what can we say about the distribution of ρ, λ and μ ? Certainly, there are an infinite number of possible distributions corresponding to any finite set of eigenvalues. This may be seen simply by counting the degrees of freedom allowed in determining a function. Therefore we must start with some model of the earth and allow small deviations from this model to yield the correct eigenvalues. The following is the method used to determine corrections to the assumed model.

A variational principle is used. Consider the linearized momentum equation

$$\rho\,\omega^2 s_i = \rho\,\frac{\partial\phi_1}{\partial r_i} - \rho\,\frac{\partial}{\partial r_i} s_j \frac{\partial}{\partial r_j}\phi_0 - \frac{\partial}{\partial r_j}\tau_{ij} \tag{19}$$

and Poisson's equation

$$\frac{\partial^2}{\partial r_i^2}\phi_1 = -4\pi G\,\frac{\partial}{\partial r_j}(\rho s_j) \tag{20}$$

Multiply the momentum equation by s_i and Poisson's equation by ϕ_1 , subtract them, integrate, and simplify by integrating by parts to obtain

$$\omega^2\int_V \rho s^2 dV = \int_V dV\left\{\tau_{ij}\frac{\partial s_i}{\partial r_j} + \rho\left[s_i s_j \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\phi_0 + \frac{\partial}{\partial r_j}\phi_0\left(s_i\frac{\partial s_j}{\partial r_i} - s_j\frac{\partial s_i}{\partial r_i}\right) + 2 s_i\frac{\partial\phi}{\partial r_i}\right]\right\} +$$

$$+ \int_{\substack{\text{ALL}\\ \text{SPACE}}} dV(4\pi G)^{-1}|\nabla\phi_1|^2 . \tag{21}$$

Consider changing the density by $\delta\rho$ and elastic parameters by $\delta\lambda$ and $\delta\mu$. The solution of the equations will change, as will the eigenvalue ω^2. By taking the variation of the above integral relation, we obtain a relation among the changes in the functions and parameters. The integral relation was constructed in such a way as to force the coefficients of δs and $\delta\phi_1$ to vanish by using the momentum equation and Poisson's equation after integration by parts.

$$\frac{\delta(\omega^2)}{\omega^2}\int \omega^2 \rho s^2 dV = \int_V dV\left\{\delta\lambda\,\mathcal{L} + \delta\mu\,\mathcal{M} + \delta\rho\,\mathcal{R}\right\} \tag{22}$$

is the form of the equation. By a suitable normalization and letting a subscript "j" correspond to each eigenvalue,

$$\epsilon_j \equiv \frac{\delta(\omega_j^2)}{\omega_j^2} = \int_V dV\left\{\delta\lambda\,\mathcal{L}_j + \delta\mu\,\mathcal{M}_j + \delta\rho\,\mathcal{R}_j\right\} \tag{23}$$

First one assumes a "model" earth, that is, assumed initial guesses for the functions ρ, λ and μ. Then, one computes the eigenvalues ω_j^2 corresponding to this model of the earth, and the corresponding values ϵ_j, which are measures of (ω^2 computed - ω^2 observed) where ω^2 observed are data for the real earth. There are, of course, many combinations of functions $\delta\rho$, $\delta\mu$ and $\delta\lambda$ that give the correct values of ϵ_j' when substituted into the above equation. Hopefully, the original model is not too different from the "real" earth, so we would like to choose $\delta\rho$, $\delta\mu$, and $\delta\lambda$ to be as "small as possible" in some sense, provided of course they satisfy the above criterion. A mathematically convenient way of expressing this idea is to require

$$\frac{1}{2}\int_0^a r^2 dr\left[(\delta\lambda)^2 + (\delta\mu)^2 + (\delta\rho)^2\right] \tag{24}$$

to be a minimum relative to all choices of $\delta\lambda, \delta\mu$ and $\delta\rho$ satisfying the set of constraints in equation (23). Notice that the functions are volume weighted. If there are N eigenvalues to be fitted, and thus N values of $\mathcal{E}_j$, we must minimize

$$\frac{1}{2}\int_0^a r^2 dr\left[(\delta\lambda)^2+(\delta\mu)^2+(\delta\rho)^2\right]+\sum_{j=1}^{N}\nu_j\left[\mathcal{E}_j-\int_0^a r^2 dr(\delta\lambda)\mathcal{L}_j+(\delta\mu)\mathcal{M}_j+(\delta\rho)\mathcal{R}_j)\right] \tag{25}$$

This expression is to be minimized for independent choices of $\delta\lambda$, $\delta\mu$, $\delta\rho$. Taking the variance of this expression, we obtain the Euler Lagrange equations

$$\left.\begin{aligned} \delta\lambda-\sum\nu_j\mathcal{L}_j &= 0\\ \delta\rho-\sum\nu_j\mathcal{R}_j &= 0\\ \delta\mu-\sum\nu_j\mathcal{M}_j &= 0\end{aligned}\right\} \tag{26}$$

and consequently we would have the required functions $\delta\rho$, $\delta\mu$ and $\delta\lambda$ if the N Lagrange multipliers ν_j were known. To determine these ν_j's, substitute equations (26) into (23), immediately giving

$$\mathcal{E}_j=\sum_{i=1}^{N}\int_0^a r^2 dr\left[\mathcal{L}_i\mathcal{L}_j+\mathcal{M}_i\mathcal{M}_j+\mathcal{R}_i\mathcal{R}_j\right]\nu_i\ . \tag{27}$$

This is a system of N inhomogeneous equations ($\mathcal{E}_j$ are given) in N unknowns, ν_i. Thus the ν_i's are known, and $\delta\rho$, $\delta\mu$ and $\delta\lambda$ may be calculated from (26) (except in the rare circumstance when the determinant of the relevant matrix vanishes).

The process is then repeated, this time with a new model designated by the corrected values of ρ, μ, λ; the eigenvalue problem is solved giving new values of ω^2; and $\mathcal{E}_j$, and of course, new functions $\mathcal{L}_i$,

$\mathcal{M}_i$ and $\mathcal{R}_i$. Hopefully, this procedure will converge on some earth model that possesses the correct eigenvalues. It is important to note that this method gives the model of the earth that is closest to the initial guess <u>and</u> compatible with the imposed constraints, i.e., it must possess the correct eigenfrequencies.

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5. Some Numerical Results of Earth Model Studies

Dr. Freeman Gilbert

References

1966 "Numerical Applications of a Formalism for Geophysical Inverse Problems". Geophysical Journal (Royal Astronomical Society) (to appear).

1967 "Approximate Solutions to the Inverse Normal Mode Problem" Bull.Seismological Society of America (submitted)

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Results

The first model used is called the Gutenberg model wherein the quantities ρ, V_p (velocity of primary wave), and V_s (velocity of secondary wave) have the assumed values shown in Figure (1). Note that ρ, V_p and V_s suffice to furnish us with ρ, μ and λ. Sixteen normal modes were used in the solving of the inverse problem, with periods ranging from 84 seconds to 20 minutes. Letting the first number be the number of modes in the radial part of the (separable) wave function, the last number being the value of "ℓ" for the spherical harmonic, and the letter denoting the type of wave (S: spheriodal; and T: toroidal), the following normal modes were used:

${}_0S_0$	${}_0S_{64}$
${}_1S_0$	${}_0S_9$
${}_3S_0$	${}_0T_4$
${}_0S_4$	${}_0T_8$
${}_0S_8$	${}_0T_{16}$
${}_1S_8$	${}_0T_{32}$
${}_0S_{16}$	${}_0T_{64}$
${}_0S_{32}$	${}_0T_{105}$

The root-mean-square error of the Gutenberg model using the frequencies associated with the sixteen normal modes being considered is about 1 percent. Three iterations of the technique described in the last lecture resulted in the r.m.s. error being reduced to .024 percent (see Figure 2). Although the technique could have reduced the error even further, the accuracy of the data is not good enough to make any further reduction meaningful.

The technique used relies very strongly on the choice of the initial model, since the technique, in effect, converges on the "closest" model (to the initial model) that possesses the eigenvalues. In order to see how, in a quantitative manner, the choice of initial model affects the final model, a so-called quadratic-Gutenberg model was tried next; it is shown in Figure (3). It has the same core, but a different mantle which

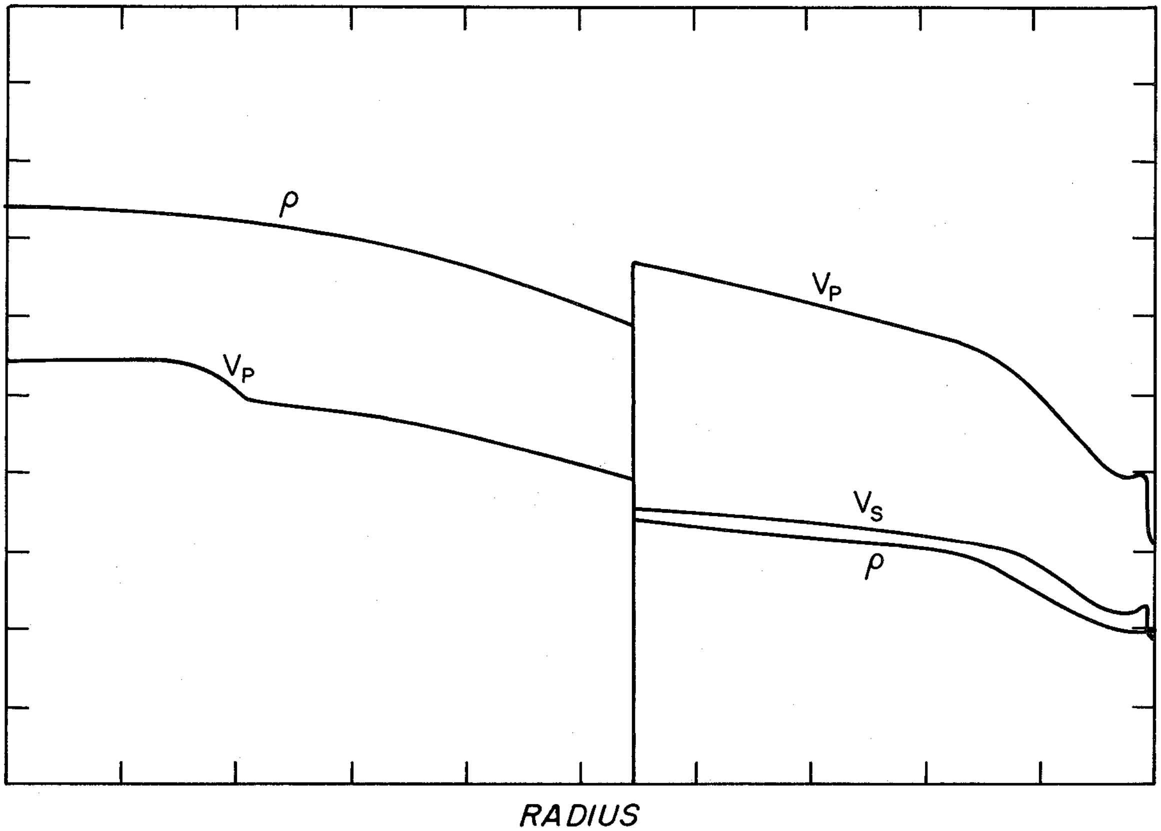

Figure 1.

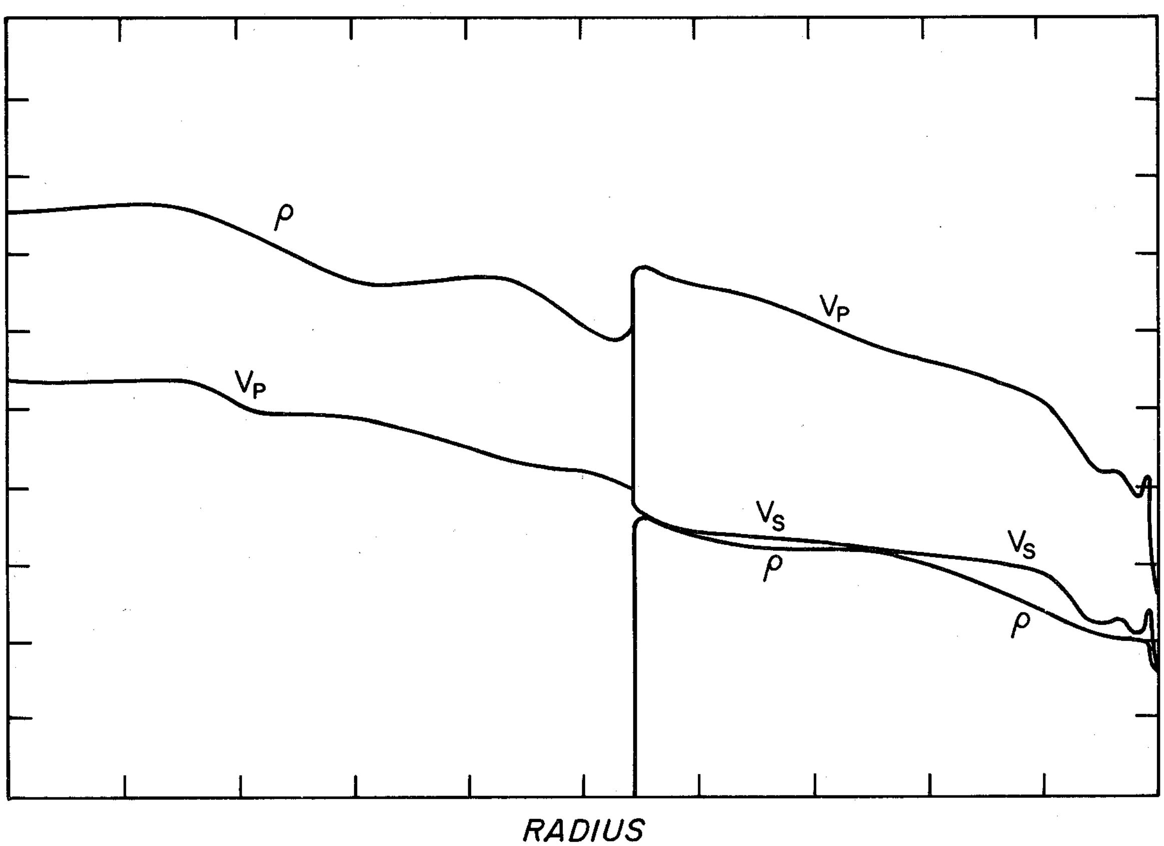

Figure 2.

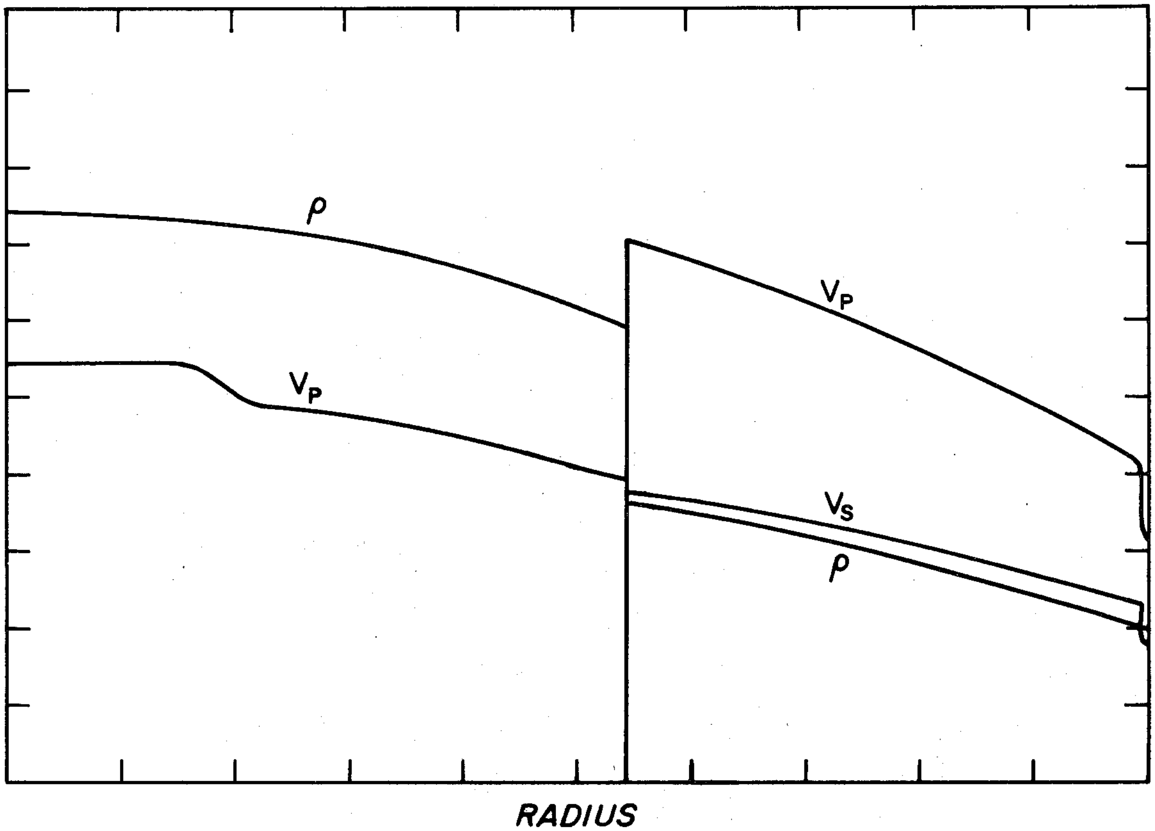

Figure 3.

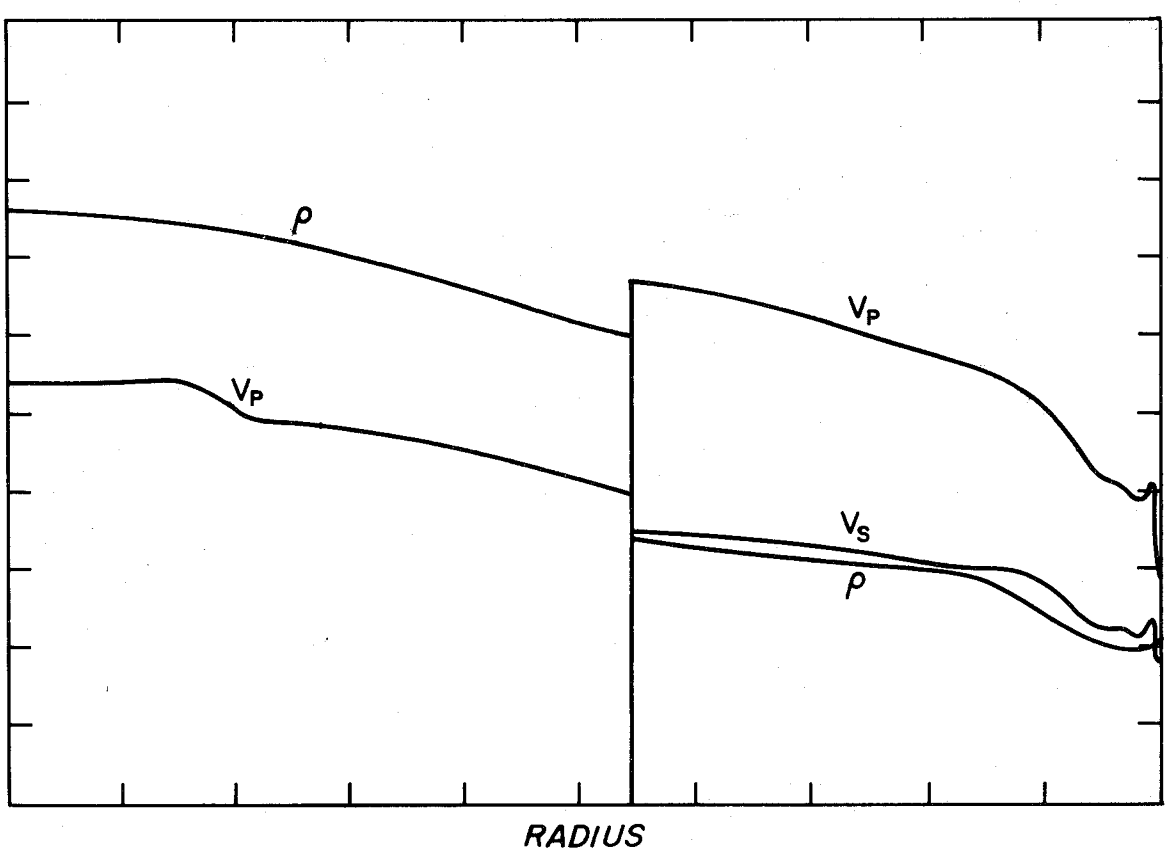

Figure 4.

is smoother than the Gutenberg model. It was chosen to see how much structure the numerical technique puts into the model. The initial model has an r.m.s. error of 3 percent, which was reduced to .038 percent by the third iteration (see Figure 4). It is not too surprising that the final model obtained by starting with the quadratic Gutenberg model differed substantially from the final model evolving from the Gutenberg model. Naturally, the 16 eigenvalues that were used as data agreed for the two corrected models, but it turned out that all 200 eigenfrequencies calculated matched quite well, with the maximum difference between them being about .2 percent. This result is significant in the light of the dissimilarities in the models, except in the outermost region of the mantle.

Next, a second set of computer runs was made, this time the data comprised eigenfrequencies with periods 1.5 to 6 minutes, with values of ℓ between 20 and 100. Since the corresponding eigenfunctions are very small except in the mantle, the calculated eigenfrequencies would be very insensitive to the actual model of the core. For this reason, only a mantle model was used. The Gutenberg mantle model was used first; beginning with a 1 percent r.m.s. error, the calculated corrected model had an r.m.s. error in the eigenfrequencies of .2 percent, which is small relative to the expected data error of .5 percent.

Another mantle model was tried, being composed of linear functions for the density and two velocities, quite different from the Gutenberg model. Although the calculated models for this second set differed by as much as 20 percent from the corrected Gutenburg model, the eigenfrequencies were, for all practical purposes, identical, differing by only .04 percent. In addition, all four final models had the exact same structure near the surface, each showing marked low velocity zones near the surface.

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6. Effects of Rotation and Dissipation

Dr. Freeman Gilbert

Thus far, all the analysis has assumed that the earth does not rotate. This assumption is justified to lowest order whenever the periods of the normal modes are a small fraction of a day, but the effects of rotation should be considered not only to find the first order correction for each eigenfrequency, but also to resolve the $(2\ell+1)$-fold degeneracy of the eigenfrequencies. This $(2\ell+1)$-fold degeneracy arises from the fact that the index m in the spherical harmonic expansion $\left[P_\ell^m(\cos\theta)e^{im\phi}\right]$ does not appear in the resulting equations.

For the long period (low frequency) modes, the effect of rotation will be the most significant perturbation since the perturbation due to ellipticity is of the order of 1 part in 300, and that due to rotation is about ten times greater.

The analysis is done with a system of N particles, rather than with a continuum, in order to use the results of matrix theory at the end instead of the theory of differential operators. Note that the Einstein summation convection is not used.

Let the N point masses have positions $\vec{r}_\alpha$ and masses m_α. The potential energy depends on the positions of the N particles: $U=U(\vec{r}_1,\ldots,\vec{r}_N)$. In a rotating coordinate system one must add "pseudo-forces" which are the Coriolis and centrifugal forces if the rotation is uniform. For each point mass then,

$$m_\alpha\left[\ddot{\vec{r}}_\alpha+2\vec{\Omega}\times\dot{\vec{r}}_\alpha+\vec{\Omega}\times(\vec{\Omega}\times\vec{r}_\alpha)\right]+\vec{\nabla}_\alpha U=0, \text{ for } \alpha=1,2,\ldots,N. \tag{1}$$

As usual, the equations will be linearized about a state of rest by letting

$$\vec{r}_\alpha(t)\equiv\vec{R}_\alpha+\vec{S}_\alpha(t) \text{ for } \alpha=1,2,\ldots,N. \tag{2}$$

Since $\vec{R}_\alpha$ is independent of time and large compared to $\vec{S}_\alpha(t)$ the linearized set of equations is:

$$m_\alpha\left[\ddot{\vec{S}}_\alpha+2\vec{\Omega}\times\dot{\vec{S}}_\alpha+\vec{\Omega}\times(\vec{\Omega}\times\vec{S}_\alpha)\right]+\vec{\nabla}_\beta\vec{\nabla}_\alpha U\cdot\vec{S}_\beta=0. \tag{3}$$

The last term arises from the expansion of U in the following Taylor series:

$$\vec{\nabla}_\alpha U(\vec{r}_\beta)=\vec{\nabla}_\alpha U(\vec{R}_\beta+\vec{S}_\beta)=\vec{\nabla}_\alpha U(\vec{R}_\beta)+\sum_\beta\vec{\nabla}_\alpha\vec{\nabla}_\beta U(\vec{R}_\beta)\cdot\vec{S}_\beta+\ldots \tag{4}$$

The equations look neater if we adopt the following definitions:

$$\vec{q}_\alpha\equiv m_\alpha^{1/2}\vec{S}_\alpha \tag{5}$$

$$\vec{\vec{U}}_{\alpha\beta}=m_\alpha^{-1/2}m_\beta^{-1/2}\vec{\nabla}_\alpha\vec{\nabla}_\beta U \tag{6}$$

Then, looking for a normal mode solution

$$-\omega^2\vec{q}_\alpha + 2i\omega\vec{\Omega}\times\vec{q}_\alpha + \vec{\Omega}\times(\vec{\Omega}\times\vec{q}_\alpha) + \vec{\vec{U}}_{\alpha\beta}\vec{q}_\beta = 0 \qquad (7)$$

Now, assume that rotation is a small (regular) perturbation, and expand the frequency and solution in the following asymptotic series

$$\omega = \omega_0\left(1 + \frac{\Omega}{\omega_0}\sigma_1 + \left(\frac{\Omega}{\omega_0}\right)^2\sigma_2 + \ldots\right) \qquad (8)$$

$$\vec{q}_\alpha = \vec{q}_\alpha^{(0)} + \frac{\Omega}{\omega_0}\vec{q}_\alpha^{(1)} + \left(\frac{\Omega}{\omega_0}\right)^2\vec{q}_\alpha^{(2)} + \ldots \qquad (9)$$

As usual, the zeroth-order solution is the solution obtained when rotation is neglected.

$$\sum_\beta\left(\vec{\vec{I}}\delta_{\alpha\beta} - \omega_0^{-2}\vec{\vec{U}}_{\alpha\beta}\right)\cdot\vec{q}_\beta^{(0)} = \vec{0}\,. \qquad (10)$$

The solution of this problem yields the eigenvalues, one of which is ω_0, and eigenfunctions, one of which is

$$\vec{q}_\alpha^{(0)}\ (\alpha = 1, 2, \ldots, N)\,. \qquad (11)$$

The first-order equation, obtained by equating the coefficient of $\left(\frac{\Omega}{\omega_0}\right)$ in the expansion of equation (9), has the same homogeneous operator as the zeroth order problem, i.e., equation (10).

$$\sum_\beta\left(\vec{\vec{I}}\delta_{\alpha\beta} - \omega_0^{-2}\vec{\vec{U}}_{\alpha\beta}\right)\cdot\vec{q}_\beta^{(1)} = -2\sigma_1\vec{q}_\alpha^{(0)} + 2i\hat{z}\times q_\alpha^{(0)} \qquad (12)$$

The necessary and sufficient condition for there to be a solution to this problem is that the inhomogeneous terms (the right-hand side of the equation) must be orthogonal to all solutions of the homogeneous problem (equation (10)). Assume that the solution to equation (10) for a particular ω_0 is n'th-fold degenerate, that is, there are n eigenfunctions corresponding to the eigenfrequency ω_0, denoted by $\vec{q}_\alpha^{(0)1}, \vec{q}_\alpha^{(0)2}, \vec{q}_\alpha^{(0)3}, \ldots, \vec{q}_\alpha^{(0)n}$. It should be noted that an eigenfunction here consists of a column vector each element of which is a three-dimensional vector, so the degenerate eigenfunctions are:

$$\begin{pmatrix} \vec{q}_1^{(0)1} \\ \vec{q}_2^{(0)1} \\ \cdot \\ \cdot \\ \cdot \\ \vec{q}_N^{(0)1} \end{pmatrix}, \begin{pmatrix} \vec{q}_1^{(0)2} \\ \vec{q}_2^{(0)2} \\ \cdot \\ \cdot \\ \cdot \\ \vec{q}_N^{(0)2} \end{pmatrix}, \ldots \begin{pmatrix} \vec{q}_1^{(0)n} \\ \vec{q}_2^{(0)n} \\ \cdot \\ \cdot \\ \cdot \\ \vec{q}_N^{(0)n} \end{pmatrix} \qquad (13)$$

In mathematical notation, the condition of orthogonality is orthogonality of <u>solutions</u>, not orthogonality of three-dimensional vectors.

$$\sum_\alpha \vec{q}_\alpha^{(0)j} \cdot \left\{ -2\sigma_1 \vec{q}_\alpha^{(0)} + 2i\hat{z} \times \vec{q}_\alpha^{(0)} \right\} = 0, \text{ for } j = 1, 2, \ldots, N \qquad (14)$$

Here, $\vec{q}_\alpha^{(0)}$ may be any linear combination of the unperturbed eigenfunctions.

Expanding the solution $\vec{q}_\alpha^{(0)}$ the eigenfunctions (which are orthonormal)

$$\vec{q}_\alpha^{(0)} = \sum_{k=1}^{n} a_j \vec{q}_\alpha^{(0)k} \qquad (15)$$

then substituting this form into equation (14), and rewriting we have

$$\sigma_1 a_k = \sum_{\alpha, j} \left[-i\hat{z} \cdot \vec{q}_\alpha^{(0)*k} \times \vec{q}_\alpha^{(0)j} \right] a_j \,.$$

The term in brackets is a matrix for the subscripts j and k; calling this matrix B_{jk},

$$\sum_j \left(\sigma_1 \delta_{jk} - B_{jk} \right) a_j = 0 \tag{16}$$

which has solutions only if σ_1 is an eigenvalue of B_{jk}. For the present case, if one considers the continuous model, by choosing the unperturbed normal modes to be $P_\ell^m(\cos\theta)e^{im\varphi}$, the matrix B_{jk} would be diagonal. This means that these modes, and not some linear combination of them, correspond to each of the perturbed frequencies. In other words, each line in the spectrum has a particular value of m attached to it; this fact was certainly not obvious from the outset since it was conceivable that a particular line corresponded to the eigenfunction, say, $P_2^1(\cos\theta)e^{i\varphi} + 2P_2^{-1}(\cos\theta)e^{-i\varphi}$.

Because B_{jk} is diagonal, its eigenvalues are its diagonal elements. The possible choices of σ_1 are therefore B_{jj} (<u>no</u> summation implied), for $j = 1, 2, \ldots, n$. Written explicitly,

$$\sigma_1 = \frac{-\sum_\alpha -i\hat{z} \cdot \vec{q}_\alpha^{(0)*j} \times \vec{q}_\alpha^{(0)j}}{\sum_\alpha \vec{q}_\alpha^{(0)*j} \cdot \vec{q}_\alpha^{(0)j}} \qquad j = 1, 2, \ldots, n. \tag{17}$$

where the denominator has already been chosen to be unity by the orthonormality condition.

The result of the continuous model may be derived from the discrete point-mass model by replacing the sum over mass points by an integral over the appropriate volume, so

$$\beta \equiv \frac{\sigma_1}{m} = \frac{\int_{volume} \rho \hat{z} \cdot (\vec{s}_\ell^m \times \vec{s}_\ell^{m*}) dV}{\int_{volume} \rho |s_\ell^m|^2 dV} \tag{18}$$

This parameter β which is independent of m, is called the splitting parameter since the eigenfrequencies may be written in the form $\omega = \omega_0 + \beta m \Omega$. It may be calculated by performing the integration above. For the toroidal modes $\beta = \frac{1}{\ell(\ell+1)}$, independent of the model used; if the Gutenberg model is used in the calculation of β for the spheriodal modes, the following are the results for the n = 0 modes for various ℓ : $\beta_{\ell=2}$ = .3964, $\beta_{\ell=3}$ = .1839, $\beta_{\ell=4}$ = .09985, $\beta_{\ell=5}$ = .05868, and $\beta_{\ell=6}$ = .03423. Notice that β is diminishing in size as ℓ increases; this means that the spacing between the lines caused by the rotating becomes finer, eventually so fine that other perturbations must be considered, such as ellipticity.

Naturally, one does not see lines in the spectrum, since there is some dissipation in the earth; instead peaks are observed, the width of which would allow the amount of dissipation to be computed if it were not for the splitting of these peaks due to rotation. If the width of the individual lines is small compared to the splitting, then the Q for each line may be determined. If the width is considerably larger than the spacing between the lines, then the Q for the whole set of lines may be found. However, for the case where the width and the spacing due to splitting are of the same order of magnitude, it is not obvious that a meaningful value of Q may be calculated. This problem arises for the values of ℓ between about 4 and 15. Below $\ell \leq 4$ the splitting is sufficiently great to be able to determine a value of Q for each value of m, and for $\ell \geq 15$, the splitting is sufficiently small to allow all the lines to be treated as one.

The rotational splitting not only affects the eigenfrequencies, but perturbs the eigenfunctions as well. An order ℓ toroidal mode has a perturbation which contains orders $\ell+1$ spheroidal modes, and an order ℓ spheroidal mode has perturbations which are orders $\ell \pm 1$ toroidal modes. This means that a gravimeter which theoretically only measures spheroidal modes will have toroidal modes coupled in by rotation.

For some source the motion in the absence of rotation, for a given ℓ, can be written as a linear vector operation

$$\vec{s} = \underline{\mathcal{L}} P_\ell \left[\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos(\phi-\phi_s)\right]\sin\omega_o t \tag{19}$$

Using the addition theorem for spherical harmonics we can write (19) as

$$\vec{s} = -\underline{\mathcal{L}}\, I\, m \sum_{m=-\ell}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta)\, P_\ell^m(\cos\theta_s)\, e^{im(\theta-\theta_s)-i\omega_o t} \tag{20}$$

When there is rotation we replace ω_o by $\omega_o - m\beta\Omega$ and (20) is

$$\vec{s} = -\underline{\mathcal{L}}\, I\, m \sum_{m=-\ell}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta)\, P_\ell^m(\cos\theta_s)\, e^{im[(\theta-\theta_s)+\beta\Omega t]-i\omega_o t} \tag{21}$$

which reduces to

$$\vec{s} = \underline{\mathcal{L}} P_\ell \left[\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos(\theta-\theta_s+\beta_\ell\Omega t)\right]\sin\omega_o t \tag{22}$$

Thus, the geographical amplitude pattern, for a given ℓ , on a non-rotating earth, is stationary. But, on a rotating earth the amplitude pattern drifts to the west at the rate $\beta\Omega$ radians/sec.

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The Dynamo Problem

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1. Foundations

1.1 Statement of the problem

The dynamo problem concerns itself with the question of how large cosmic bodies, such as the Earth, can maintain a magnetic field within themselves over geological times. For example, the science of paleomagnetism has shown that the Earth's geomagnetic field is of the order of 10^9 years old. Ideas of permanent magnetism or of intrinsic magnetism due to rotation can be ruled out, or made to look extremely unattractive (cf. e.g. Bullard, 1948, 1949). It is virtually certain that the Earth's field is maintained by currents in the core and, perhaps, the lower regions of the mantle. The question is, then, how these currents are maintained. It is known that in the absence of electromotive forces, they would decay in a time of the order of 10^4 to 10^5 years (cf. equation (1.2.3) below). Several possibilities have been suggested but the most plausible seems to be the dynamo mechanism. Here the motion, $\underline{u}$, of the core of the Earth (assumed to be electrically conducting) across the geomagnetic lines of force, $\underline{B}$, creates an electromotive force $\underline{u} \times \underline{B}$. The currents, $\underline{j}$, which result, drive the very field by which the process started. It should be pointed out that this is exactly how the conventional laboratory, or engineering, self-exciting dynamo functions. The laboratory dynamo, however, is multiply connected and highly asymmetric whereas the Earth's core is simply connected and approximately spherical. It is not clear that these differences are not disastrous. Obviously, if the laboratory dynamo were to be made homogeneous in structure by immersing it in a conducting fluid, it would cease to function. The real question is whether the motions within a homogeneous body (such as the Earth) can, by their asymmetry in form, atone for the evident lack of asymmetry in the container. This is the problem to be examined.

The dynamo problem exists at several levels of difficulty. The simplest (and this is difficult enough!) is the kinematic dynamo problem. Here we specify the velocity $\underline{u}$ in some reasonable way (e.g. we may suppose it is continuously differentiable and contains no sources or sinks of mass), and ask whether it can support a field $\underline{B}$ which does not decay to zero as the time t tends to infinity. We would generally suppose that $\underline{u}$ is itself time independent (the Backus-dynamo is an exception). The search has generally been restricted to steady solutions, $\partial \underline{B}/\partial t = 0$, although really this is a fairly severe limitation mathematically. If we suppose $\underline{B} \propto e^{st}$ the dynamo problem poses an eigenvalue problem for s. And, on the face of it, we are more likely to be able to find a flow for which $\mathcal{I}(s)$ is zero than one for which both $\mathcal{I}(s)$ and $\mathcal{R}(s)$ vanish; i.e. we are more likely to be able to construct an oscillatory dynamo field than a steady one.

If we find a solution to the dynamo equations we would expect by increasing $\underline{u}$ everywhere by some constant factor, to obtain an eigenvalue s which is positive or has a positive real part. The field $\underline{B}$ would then grow without bound, a physically unacceptable conclusion. This highlights the importance of <u>the hydromagnetic dynamo problem</u>. In this, we specify $\underline{F}$, the body force per unit mass on the fluid, in some reasonable way, and ask whether we can simultaneously solve the induction equation and the Navier-Stokes equation to obtain a flow $\underline{u}$ and a field $\underline{B}$ which does not die away to zero as $t \to \infty$. If we start from a state of rest, we would expect that as $\underline{u}$ increased under the action of $\underline{F}$, a stray field $\underline{B}$ might grow, but now, even if it is inclined to grow to infinity on the kinematic dynamo picture, the Lorentz force $\underline{j} \times \underline{B}$ will grow quadratically. This will modify and reduce $\underline{u}$ (Lenz's law) and will therefore bring the growth of $\underline{B}$ to a halt. Generally we would suppose that $\underline{F}$ is time-independent.

The hydromagnetic dynamo problem is far more realistic and far more difficult than the kinematic dynamo problem. The hope is that by studying the kinematic dynamo problem (which is linear in $\underline{B}$, for given $\underline{u}$) we can separate some of the difficulty. Ideally, we might like it to be a "black box" which will produce a $\underline{B}$ (if it exists) when a $\underline{u}$ is fed into it. We will confine our attention to the kinematic dynamo problem until further notice.

It should be noted that a proof, that a steady fluid motion is capable of increasing initially the energy of a stray magnetic field, does not establish dynamo action. It is possible to construct examples in which the magnetic energy reaches an enormous peak through the motions, but for which, nevertheless, the field decays ultimately to zero as $t \to \infty$.

The basic equations governing the kinematic dynamo are the pre-Maxwell equations (in m.k.s. units, assuming constant μ and ϵ, the permeability and dielectric constant)

$$\text{curl } \underline{B} = \mu \underline{j}, \tag{1.1.1}$$

$$\text{div } \underline{B} = 0, \tag{1.1.2}$$

$$\text{curl } \underline{E} = -\frac{\partial \underline{B}}{\partial t}, \tag{1.1.3}$$

$$\text{div } E = \frac{\vartheta}{\epsilon}. \tag{1.1.4}$$

Here $\underline{E}$ is the electric field. Equation (1.1.4) merely defines the volume charge ϑ from $\underline{E}$. Ohm's law for a moving conductor (non-relativistic approximation) is given by

$$\underline{j} = \sigma\left(\underline{E} + \underline{u} \times \underline{B}\right), \tag{1.1.5}$$

where σ is the electrical conductivity. Generally we will suppose the fluid is incompressible,

$$\text{div } \underline{u} = 0. \tag{1.1.6}$$

At any surface of discontinuity S, we must have (assuming here and henceforth that $\langle\mu\rangle = 0$)

$$\langle \underline{B} \rangle = 0 \text{ and } \langle \underline{n} \times \underline{E} \rangle = 0, \tag{1.1.7}$$

where $<\;>$ denotes a jump of a quantity across S. The usual situation we will consider is one in which S is fixed in time and separates the conducting fluid volume, V, from a surrounding insulator, $\hat{V}$. In the insulator $\hat{V}$, we will have

$$\text{curl}\ \hat{\underline{B}} = 0, \tag{1.1.1}$$

$$\text{div}\ \hat{\underline{B}} = 0, \tag{1.1.2}$$

$$\text{curl}\ \hat{\underline{E}} = -\frac{\partial \hat{\underline{B}}}{\partial t} \tag{1.1.3}$$

$$\text{div}\ \hat{\underline{E}} = 0\,. \tag{1.1.4}$$

The fields $\hat{\underline{B}}$ and $\hat{\underline{E}}$ must satisfy the requirement that the field is self-excited, i.e., these fields can contain no sources "at infinity". The formal mathematical statement of this is

$$\hat{\underline{B}} = O(r^{-3}),\quad r \to \infty, \tag{1.1.8}$$

where r denotes distance from some origin in V.

Equations (1.1.1), (1.1.3) and (1.1.4) imply that

$$\frac{\partial \underline{B}}{\partial t} = \text{curl}\,(\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B}, \tag{1.1.9}$$

where

$$\eta = \frac{1}{\mu\sigma}\,. \tag{1.1.10}$$

Equation (1.1.9) is often called "the equation of electromagnetic induction" and η is termed "the magnetic diffusivity". Also from equations (1.1.5) and (1.1.1) we have

$$\underline{E} = -\underline{u} \times \underline{B} + \eta \operatorname{curl} \underline{B} . \tag{1.1.11}$$

Let $\mathcal{L}$ be a length characteristic of $\underline{B}$, and let U be a typical fluid velocity. The dimensionless quantity

$$R = \frac{U\mathcal{L}}{\eta} = \mu\sigma U\mathcal{L} \tag{1.1.12}$$

is termed "the magnetic Reynolds number". The behavior of $\underline{B}$ depends very much on the relative magnitude of the three terms occurring in (1.1.9) and (1.1.11). If R is large it may be a good approximation, particularly far from the boundary S , to neglect the η terms, giving

$$\frac{\partial \underline{B}}{\partial t} = \operatorname{curl}(\underline{u} \times \underline{B}), \text{ and } \underline{E} = -\underline{u} \times \underline{B}, \text{ for } R \longrightarrow \infty, \tag{1.1.13}$$

the case of the perfect conductor. At the other extreme, we might take R to be small, then to a first approximation

$$\frac{\partial \underline{B}}{\partial t} = \eta \nabla^2 \underline{B}, \text{ and } \underline{E} = \eta \operatorname{curl} \underline{B}, \text{ for } R \longrightarrow 0, \tag{1.1.14}$$

the case of no motion. Alternatively, the left-hand side of equation (1.1.9) and (1.1.11) might be neglected if

$$R^* = \frac{\mathcal{L}^2 \mathcal{T}}{\eta} = \mu\sigma\mathcal{L}^2\mathcal{T} \tag{1.1.15}$$

is small, where $\mathcal{T}$ is a typical time scale of $\underline{B}$. In this case, for $R^* \longrightarrow 0$, we have, to a first approximation

$$0 = \text{curl}(\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B}, \text{ and } 0 = -\underline{u} \times \underline{B} + \eta \,\text{curl}\, \underline{B},$$

the steady case. We will not consider this possibility here, and will not usually distinguish between $\mathcal{R}$ and $\mathcal{R}^*$. The steady dynamo is, however, really a case of $\mathcal{R}^* = 0$ and $\mathcal{R} = O(1)$.

Values of parameters for the Earth

It is difficult to be sure of the value of η on the Earth. Estimates of σ vary generally from about 10^5 to 6×10^5 mho/m. We will take $\sigma = 3.10^5$ mho/m giving $\eta = 3$ m^2/sec.

Estimates of U are more uncertain varying from 3×10^{-5} m/sec to 10^{-3} m/sec. It may be seen that

$$U = 3.10^{-5} \text{ m/sec implies } \mathcal{R} = \frac{3.10^{-5} \times 3 \times 10^6}{3} = 30,$$

and

$$U = 10^{-3} \text{ m/sec implies } \mathcal{R} = \frac{10^{-3} \times 3 \times 10^6}{3} = 1000.$$

In both of these estimates we have taken for $\mathcal{L}$ the radius of the core (3.5×10^6m).

For reasons which may become clear later, I am more inclined to believe that $\mathcal{R}$ is nearer 1000 than 30, though whether this is because σ is smaller than 3×10^5 mho/m, or whether U is larger than 3×10^{-5} m/sec is not obvious, nor is this question properly in the province of kinematic dynamo theory. At all events, it is fairly likely that $\mathcal{R} \gg 1$.

1.2 The stationary conductor, the perfect conductor and intermediate cases.

Two features of the electromagnetic fields in solid conductors are appreciated simply by writing (1.1.14) in dimensionless form

$$\frac{B}{\tau} = \frac{\eta B}{\mathcal{L}^2}, \tag{1.2.1}$$

which we can rewrite either as

$$\tau = \frac{\mathcal{L}^2}{\eta} \quad \text{or as} \quad \mathcal{L} = (\eta\tau)^{1/2}. \tag{1.2.2}$$

The first of these defines "the electromagnetic diffusion time"

$$\tau_\eta = \frac{\mathcal{L}^2}{\eta}. \tag{1.2.3}$$

This is the characteristic time a field of scale $\mathcal{L}$ will persist in the face of ohmic decay and in the absence of applied electromotive forces. For the Earth's core this would be

$$\tau_\eta = \frac{(3.5\times 10^6)}{3}\ \text{secs} = 3\times 10^{12}\ \text{secs} = 10^5\ \text{years}.$$

(Actually this is a slight overestimate even for the largest e-folding time for fields in this case. A precise calculation shows that the decay modes of a sphere of radius $\mathcal{L}$ which last longest are poloidal $n = 1$ modes, for which the e - folding time is $\tau_\eta/\pi^2 \approx 10^4$ years for the core.)

The second of (1.2.2) defines "the electromagnetic penetration depth" of a solid conductor,

$$d_\eta = (\eta\tau)^{1/2}. \tag{1.2.4}$$

This is the characteristic distance beneath the surface of a solid conductor to which an applied oscillating field, of period τ, will penetrate. This concept has little direct relevance to a fluid conductor for which disturbances initiated at the surface can be radiated inwards by Alfvén waves. It is, however appropriate to the mantle. It is observed that periodicities in the secular variation (i.e. the changes in the geomagnetic field of internal origin) of time scale less than about 3 years are never observed. If we supposed, as a crude model of the conductivity variations in the mantle, that its lowest 10^3 km was of conductivity 10^2 mho/m, we would obtain for τ =3 years = 10^8 sec, a value of d_η of 10^6 m; i.e. periods smaller than this will not "get through" the 10^3 km layer.

One feature of the stationary state may be noted. According to (1.1.1), (1.1.4) and (1.1.5) we have, in general,

$$\vartheta = \epsilon \operatorname{div} \underline{E} = -\epsilon \operatorname{div}(\underline{u} \times \underline{B}), \tag{1.2.5}$$

i.e. the volume charge density does not vanish. In a stationary conductor, however, we always have $\vartheta = 0$. In both cases, however, free charges may be present at a discontinuity in σ, e.g. at the surface of V.

In a perfect fluid conductor, it is particularly helpful to think about flux tubes. This is appreciated when it is realized that, by (1.1.13),

$$\frac{\partial \underline{B}}{\partial t} = \operatorname{curl}(\underline{u} \times \underline{B}). \tag{1.2.6}$$

Comparing this with the analogous (Helmholtz) equation for the vorticity in a uniform fluid moving under conservative forces, we see that magnetic tubes of flux move with a fluid. Actually the relationship is not one-sided: The Lorentz forces created by $\underline{B}$ may be pictured in terms of (Maxwell) stresses on the fluid. These comprise a tension $B^2/2\mu$ per unit area acting down each flux tube, and a repulsion $B^2/2\mu$ per unit area between a flux tube and the surrounding field. The generation of magnetic energy by a fluid motion can be pictured as a stretching and crowding together of flux tubes by the flow against this tension and repulsion. Also, Alfvén waves can be conveniently pictured as transverse waves travelling down the flux tubes like waves down an elastic string, the above stresses supplying the restoring force. It may be noticed, however, that in the perfectly conducting case, material particles are rigidly attached to flux tubes. There is, therefore, no possibility of changing the topology of the field lines, or of creating flux tubes where none previously existed. The net flux (of one sign) of field out of a fluid body cannot be altered, the fluid motions can merely rearrange it.

We have now examined the limits $R \longrightarrow 0$ and $R \longrightarrow \infty$. In intermediate cases there is some diffusion of flux relative to the material particles and some advection of flux by the flow. These processes will not necessarily proceed at the same rate throughout the system. Where field gradients are high or fluid velocities are low, the diffusion will be relatively more important. This is well illustrated by two significant processes: the severing and coalescence of flux tubes. These are illustrated by the schematic sequence shown in Fig. 1.1. The relatively high

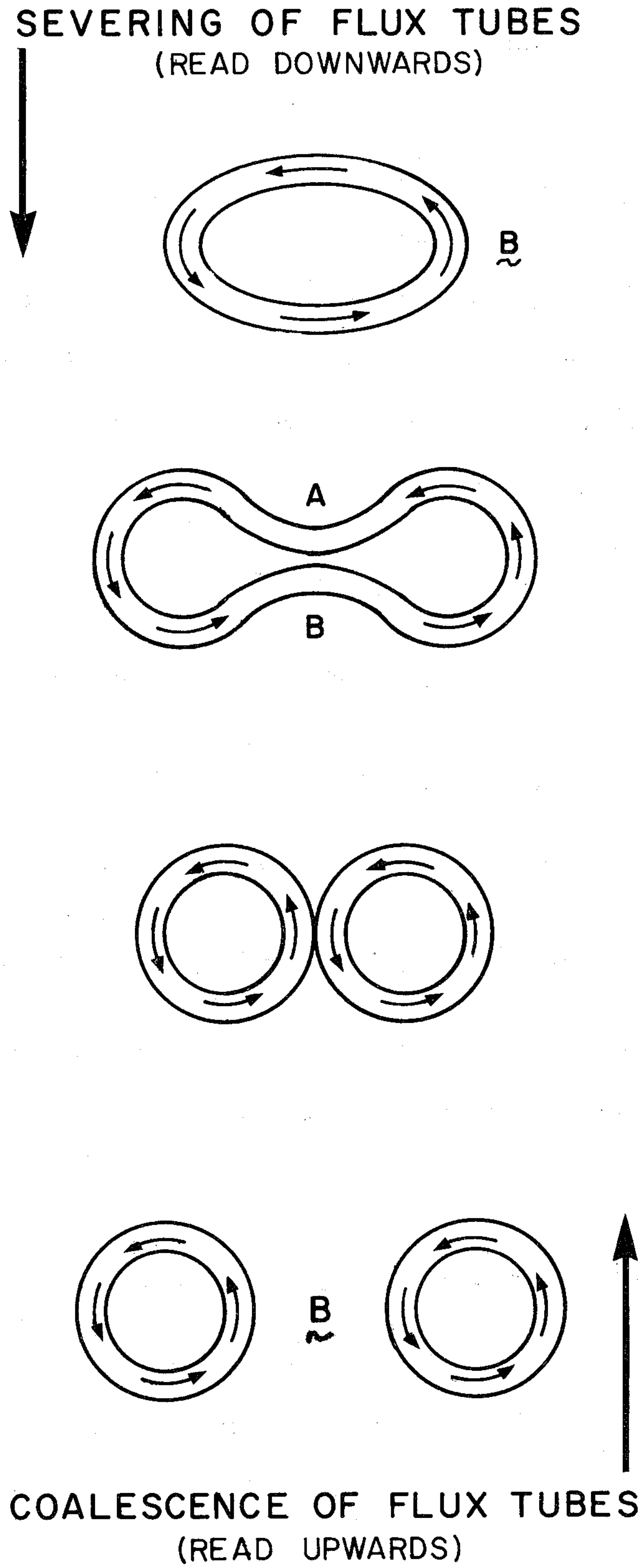
SEVERING OF FLUX TUBES
(READ DOWNWARDS)
B
A
B
B
COALESCENCE OF FLUX TUBES
(READ UPWARDS)

FIG. 1.1

field gradients created at AB by the fluid motion enhance the processes of diffusion and cause the redistribution of flux indicated. The processes illustrate how the topology of field lines can be altered within a finitely conducting fluid.

The advantages of picturing the changes of the magnetic field in terms of moving flux tubes are evidently considerable, and it is natural to seek, even for an imperfect conductor, a fictitious "velocity of magnetic flux", $\underline{U}$ satisfying

$$\frac{\partial \underline{B}}{\partial t} = \operatorname{curl}\left(\underline{U} \times \underline{B}\right). \tag{1.2.7}$$

By (1.1.3) a single-valued Φ exists such that

$$\underline{E} = -\operatorname{grad}\Phi - \underline{U} \times \underline{B}. \tag{1.2.8}$$

Within the fluid, where (1.1.5) applies, we have, by (1.2.8)

$$\frac{\underline{j}}{\sigma} = -\operatorname{grad}\Phi - \left(\underline{U} - \underline{u}\right) \times \underline{B}. \tag{1.2.9}$$

Consider, for example, an axisymmetric case in which $\underline{B}$ lies in meridian planes and $\underline{j}$ is azimuthal. It is easily seen from (1.2.9) that $\Phi = 0$ everywhere, giving

$$\underline{U}_d = \underline{U} - \underline{u} = \frac{\underline{j} \times \underline{B}}{\sigma B^2} + \psi \underline{B}, \tag{1.2.10}$$

where ψ is arbitrary and reflects the fact that we are attempting to identify flux tubes at two different times but not individual points on them. We may call $\underline{U}_d$ the drift velocity between fluid and flux tubes.

By dimensional analysis $U_d \sim \eta B^2/\mathcal{L} B^2 = \eta/\mathcal{L}$. The condition $R >> 1$ can be written as

$$\frac{\eta}{\mathcal{L}} << u, \quad \text{i.e. } U_d << u,$$

i.e. convection of field is rapid compared with its drift relative to the material particles. Note that on a symmetric curve ℓ on which $\underline{B}$ is zero, $\underline{U}$ is infinite and directed inwards on ℓ (see Fig. 1.2).

If $\underline{E} \cdot \underline{B} \neq 0$, then $\operatorname{grad} \Phi \neq 0$ but similar arguments apply with an added difficulty. By (1.2.8), we have

$$\Phi(\underline{x}, t) = -\int_{\underline{x}_s}^{\underline{x}} (\underline{E} \cdot \underline{B}) \frac{d\ell}{B} + \Phi(\underline{x}_s, t), \tag{1.2.11}$$

where the integral is taken along the line of force through $\underline{x}$, from the point $\underline{x}_s$, where it meets some conveniently defined reference surface, $\mathcal{S}$. If a closed line of force exists in the system, we must have

$$\oint (\underline{E} \cdot \underline{B}) \frac{d\ell}{B} = 0. \tag{1.2.12}$$

Unless this is the case, the whole idea of such a velocity $\underline{U}$ fails. Even when the lines of force do not close, a similar difficulty about non-single-valuedness of Φ occurs: Following a line of force from a specified value at a point P_1 on $\mathcal{S}$, then (1.2.11) determines Φ at P_2, the point where the line of force next meets $\mathcal{S}$. Continuing on that line of force, we obtain values of Φ at (perhaps) an infinite set of points P_1, $P_2 \ldots$ on $\mathcal{S}$. And it is not clear that a continuous $\Phi(\underline{x}_s, t)$ can result. Indeed it may be differentiable nowhere!

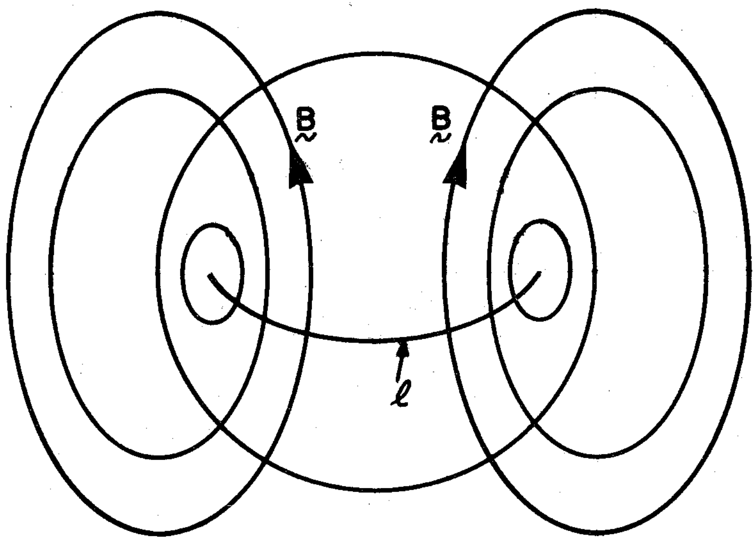

Fig. 1.2

Before leaving these comparisons between the $R \to 0$, $R \to \infty$ and R finite cases, we should remark that, when we deal with a phenomenon whose time-scale is short compared with τ_η , a good first approximation may be obtained by supposing that the fluid is perfectly conducting $(\eta = 0, R = \infty)$. For over such times there cannot be much relative motion between flux tubes and fluid. In the dynamo question, however, we are by definition, concerned with maintaining a field over time scales of the order of τ_η ; thus, the notion of the perfectly conducting fluid, while perhaps a useful aid to thought, is foreign to the mathematical proof. In the case of the Earth, the idea of a perfectly conducting core may be applied profitably (Roberts, 1966) to the secular variation, which has a time-scale of a decade. It cannot be used to explain the existence of the main geomagnetic field.

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2. Some bounds. Cowling's theorem.

2.1 Two bounds for dynamo action.

The induction equation

$$\frac{\partial \underline{B}}{\partial t} = \operatorname{curl}\,(\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B}\,, \tag{2.1.1}$$

gives, when integrated over any fixed volume V,

$$\begin{aligned}
\frac{\partial}{\partial t}\int \frac{1}{2\mu} B^2 dV &= \int \frac{1}{\mu} B_i \frac{\partial B_i}{\partial t} dV \\
&= \int \frac{dV}{\mu} B_i \left[B_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial B_i}{\partial x_j} - B_i \frac{\partial u_j}{\partial x_j} - \eta \operatorname{curl}_i^2 \underline{B} \right] \\
&= \int \frac{dV}{\mu} B_i B_j \frac{\partial u_i}{\partial x_j} - \int \frac{\partial}{\partial x_j}\left(\frac{1}{2\mu} B^2 u_j\right) dV - \int \frac{B^2}{2\mu} \frac{\partial u_j}{\partial x_j} dV - \\
&\quad - \frac{\eta}{\mu}\int (\operatorname{curl} \underline{B})^2 dV + \frac{\eta}{\mu}\int d\underline{S} \cdot (\underline{B} \times \operatorname{curl} \underline{B}) \\
&= \int \left(\frac{1}{\mu} B_i B_j - \frac{1}{2\mu} B^2 \delta_{ij}\right) \frac{\partial u_i}{\partial x_j} dV - \frac{1}{\sigma}\int j^2 dV - \\
&\quad - \int \frac{1}{2\mu} B^2 \underline{u} \cdot d\underline{S} + \frac{1}{\mu}\int \left[B^2 \underline{u} - (\underline{B}\cdot\underline{u})\underline{B}\right] \cdot d\underline{S} - \frac{1}{\mu}\int (\underline{E} \times \underline{B}) \cdot d\underline{S}\,,
\end{aligned} \tag{2.1.2}$$

giving finally

$$\frac{\partial}{\partial t}\int \frac{B^2}{2\mu} dV = \int m_{ij} \frac{\partial u_i}{\partial x_j} dV - \int \frac{j^2}{\sigma} dV - \int u_i m_{ij} dS_j - \frac{1}{\mu}\int \left((\underline{E} \times \underline{B}) \cdot d\underline{S}\right),$$

where m_{ij} is the magnetic stress tensor:

$$m_{ij} = \frac{1}{\mu} B_i B_j - \frac{1}{2\mu} B^2 \delta_{ij}\,.$$

For an incompressible fluid obeying the no-slip condition,

$$\underline{u} = 0 \qquad \text{on } S\,, \tag{2.1.3}$$

equation (2.1.2) reduces to

$$\frac{\partial}{\partial t}\int \frac{B^2}{2\mu}dV = \frac{1}{\mu}\int B_i B_j \frac{\partial u_i}{\partial x_j}dV - \int \frac{j^2}{\sigma}dV - \frac{1}{\mu}\int (\underline{E}\times\underline{B})\cdot d\underline{S}. \qquad (2.1.4)$$

For the exterior region $\hat{V}$ (if it is an insulator), we have similarly,

$$\frac{\partial}{\partial t}\int \frac{\hat{B}^2}{2\mu}dV = -\frac{1}{\mu}\int (\hat{\underline{E}}\times\hat{\underline{B}})\cdot d\hat{\underline{S}}. \qquad (2.1.5)$$

Since $\underline{B}$ and $\underline{E}\times\underline{n}$ are continuous and

$$d\underline{S} = -d\hat{\underline{S}},$$

we obtain, adding (2.1.4) and (2.1.5),

$$\frac{\partial}{\partial t}\left[\int \frac{B^2}{2\mu}dV + \int \frac{\hat{B}^2}{2\mu}d\hat{V}\right] = \frac{1}{\mu}\int B_i B_j e_{ij}dV - \int \frac{j^2}{\sigma}dV, \qquad (2.1.6)$$

where we have used

$$B_i B_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2}B_i B_j\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = B_i B_j e_{ij}.$$

Here

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (2.1.7)$$

is the rate of strain tensor.

Consider the class of all solenoidal fields $(\underline{B}, \hat{\underline{B}})$ continuous across S, twice continuously differentiable everywhere except possibly on S, and vanishing at infinity (cf. 1.1.8). Let $\underline{j}$ be defined from $\underline{B}$ by Ampère's law (1.1.1). We ask two questions: (i) how small can

$$\epsilon_\eta = \int \frac{j^2}{\sigma}dV \qquad (2.1.8)$$

be, (ii) how large can $\int B_i B_j e_{ij}\, dV$ become, when in each case the total magnetic energy

$$\mathcal{M} = \frac{1}{2\mu}\int B^2 dV + \frac{1}{2\mu}\int \hat{B}^2 d\hat{V}, \tag{2.1.9}$$

is given. To answer the first of these questions, suppose the current distribution $\underline{j}$ for the solution is known. Let $\underline{j} + \delta\underline{j}$ be a neighbouring distribution satisfying, like $\underline{j}$, the conditions

$$\operatorname{div} \delta\underline{j} = 0 \tag{2.1.10}$$

$$\underline{n} \cdot \delta\underline{j} = 0, \text{ on } S. \tag{2.1.11}$$

Let $\delta\underline{B}$ and $\delta\underline{\hat{B}}$ be the fields which arise by Amperé's law (1.1.1), from $\delta\underline{j}$. Then

$$\begin{aligned}
\delta\mathcal{M} &= \frac{1}{\mu}\int \underline{B}\cdot\delta\underline{B}\, dV + \frac{1}{\mu}\int \underline{\hat{B}}\cdot\delta\underline{\hat{B}}\, d\hat{V} \\
&= \frac{1}{\mu}\int \operatorname{curl}\delta\underline{A}\cdot\underline{B}\, dV + \frac{1}{\mu}\int \operatorname{curl}\delta\underline{\hat{A}}\cdot\underline{\hat{B}}\, dV \\
&= \frac{1}{\mu}\int \underline{A}\cdot\operatorname{curl}\delta\underline{B}\, dV + \frac{1}{\mu}\int \underline{\hat{A}}\cdot\operatorname{curl}\delta\underline{\hat{B}}\, dV + \\
&\quad + \frac{1}{\mu}\int(\delta\underline{B}\times\underline{A})\cdot d\underline{S} + \frac{1}{\mu}\int(\delta\underline{\hat{B}}\times\underline{\hat{A}})\cdot d\underline{\hat{S}} \\
&= \int \underline{A}\cdot\delta\underline{j}\, dV + \frac{1}{\mu}\int \delta\underline{B}\cdot\left[(\underline{A}-\underline{\hat{A}})\times d\underline{S}\right],
\end{aligned}$$

where $\underline{B} = \operatorname{curl}\underline{A}$. Since $\langle\underline{n}\cdot\underline{B}\rangle = 0$ implies that $\langle\underline{n}\times\underline{A}\rangle = 0$, we have

$$\delta\mathcal{M} = \int \delta\underline{j}\cdot\underline{A}\, dV. \tag{2.1.12}$$

Also, clearly,

$$\delta \epsilon_\eta = \frac{2}{\sigma} \int \delta \underline{j} \cdot \underline{j} \, dV. \tag{2.1.13}$$

By the usual processes of the calculus of variations, the solution for which $\delta \epsilon_\eta = 0$, for all $\delta \underline{j}$ obeying (2.1.10) and the condition $\delta M = 0$ is

$$\underline{j} = -s\sigma \underline{A} + \text{grad}\, \Phi, \tag{2.1.14}$$

where $-s\sigma$ is a constant multiplier and Φ is a function multiplier. Taking the curl of (2.1.14), we obtain

$$\eta \nabla^2 \underline{B} = s \underline{B}. \tag{2.1.15}$$

Thus the solution we seek is a normal mode $\underline{B} \propto e^{st}$ of the induction equation for a stationary volume V. Since this problem is self-adjoint, it can readily be proved that s is real; in fact since by (2.1.15) $-s = \epsilon_\eta / 2M$ is the ratio of two positive definite integrals, s is negative. The sequence of all possible s has a limit point, $s = -\infty$. The least negative $s (= -c$, say) provides the smallest value of ϵ_η, i.e.

$$\epsilon_\eta \geqq 2cM, \tag{2.1.16}$$

with equality if, and only if, $\underline{B}$ is the decay mode with the largest "growth" rate (i.e. smallest decay rate).

Now let us consider the second question posed above. Let $\lambda(\underline{x})$ be the largest real eigenvalue of the symmetric matrix e_{ij} at the point $\underline{x}$. Since $e_{ii} = \text{div}\, \underline{u} = 0$, this eigenvalue is in fact positive. Let Λ be the largest value of λ in V or on S. Then

$$\int B_i B_j e_{ij}\, dV \leqq \int \lambda B^2 dV \leqq \Lambda \int B^2 dV \leqq \Lambda \left[\int B^2 dV + \int \hat{B}^2 dV \right] = 2\mu \Lambda M . \tag{2.1.17}$$

Returning to (2.1.6), we now see that (2.1.15) and (2.1.16) imply that

$$\frac{dM}{dt} \leqq \frac{2\mu \Lambda M}{\mu} - 2cM = 2(\Lambda - c)M .$$

It follows that

$$M(t) \leqq M(0)e^{2(\Lambda - c)t} . \tag{2.1.18}$$

Thus for dynamo action, it is necessary that

$$\Lambda \geqq c , \tag{2.1.19}$$

(Backus, 1958). This condition may be written alternatively. It is easily shown from (2.1.15) by dimensional argument that

$$c = \frac{k\eta}{\mathcal{L}^2} ,$$

where $k = k(V)$ is dimensionless. (For a sphere of radius $\mathcal{L}$, for example, $k = \pi^2$.) Then (2.1.19) may also be expressed as a bound for a magnetic Reynolds Number:

$$\widetilde{R} \equiv \frac{\Lambda \mathcal{L}^2}{\eta} \geq k . \tag{2.1.19a}$$

A second bound (Herzenberg and Lowes, 1957) can be obtained from the integral formulation of the dynamo problem. It is necessary in this case, however, to suppose that $\hat{V}$ is a stationary medium of the same conductivity as V. For in this case $\underline{j}$ is continuous everywhere, and so $\underline{B}$ can be written as

$$\begin{aligned} \underline{B}(\underline{r}) &= \frac{\operatorname{curl}}{4\pi} \int \frac{\operatorname{curl}' \underline{B}(\underline{r}')}{|\underline{r} - \underline{r}'|} dV' = \frac{1}{4\pi} \int \operatorname{curl}' \underline{B}(\underline{r}') \times \operatorname{grad}' \frac{1}{|\underline{r} - \underline{r}'|} dV' \\ &= \frac{1}{4\pi\eta} \int \left[-\operatorname{grad}' \Phi' + \underline{u}' \times \underline{B}' \right] \times \operatorname{grad}' \frac{1}{|\underline{r} - \underline{r}'|} dV' \\ &= \frac{1}{4\pi\eta} \int \frac{\operatorname{grad}' \Phi' \times d\underline{S}'}{|\underline{r} - \underline{r}'|} + \frac{1}{4\pi\eta} \int (\underline{u}' \times \underline{B}') \times \operatorname{grad}' \frac{1}{|\underline{r} - \underline{r}'|} dV' ; \end{aligned}$$

i.e. since the integral over the sphere at infinity vanishes,

$$\underline{B}(\underline{r}) = \frac{1}{4\pi\eta}\int \frac{(\underline{u}'\times \underline{B}')\times(\underline{r}-\underline{r}')\,dV'}{|\underline{r}-\underline{r}'|^3} . \tag{2.1.20}$$

Let B_{max} be the largest value of B in V or on S, and suppose it occurs at $\underline{r} = \underline{r}''$. Applying (2.1.20) at this point, we obtain

$$B_{max} \leq \frac{1}{4\pi\eta}\, u_{max}\, B_{max}\left[\int \frac{dV}{|\underline{r}''-\underline{r}'|^2}\right]_{max} , \tag{2.1.21}$$

where u_{max} is the maximum value of u anywhere in V or on S.

If $$\ell = \left[\int \frac{dV'}{|\underline{r}-\underline{r}'|^2}\right]_{max} ,$$

we then have $B_{max} = 0$ or

$$\hat{R} = \frac{u_{max}\,\ell}{\eta} \geqq 4\pi . \tag{2.1.22}$$

This condition, like (2.1.19a) is necessary for dynamo action, but is not sufficient and, indeed, part of the mathematical charm of dynamo theory lies here. Hand-waving arguments based on the qualitative motions of § 1.1.2 of (partially) frozen-in fields might suggest that, if R is large enough, regeneration of field must occur. As we will see in the next sections, however, this is far from the case; much more sophisticated requirements arise.

2.2 Cowling's theorem

We temporarily reverse the dynamo question. Instead of specifying $\underline{u}$ and seeking $\underline{B}$, we can specify a $\underline{B}$ and ask if we can find a $\underline{u}$ to create it. In other words, we try to construct a drift velocity

$\underline{U}_d = -\underline{u}$ which has reasonable physical behavior, e.g. no infinities, no sources or sinks of mass, etc. Consider successively all the possible types of axisymmetric fields. First suppose an axisymmetric field with both meridional and azimuthal components. Then in the steady state we must have,

$$\oint \underline{B} \cdot \mathrm{curl}\, \underline{B} \, \frac{ds}{B} = 0 , \tag{2.2.1}$$

for every closed line of force in the fluid (cf. (1.1.22)). If we examine the meridional projection of the field, we can easily locate such a closed line (see Fig. 1.2). Since this lies in the fluid and since by axisymmetry, $\underline{B} \cdot \mathrm{curl}\, \underline{B} \, (\neq 0)$ has the same sign everywhere on ℓ, (2.2.1) cannot hold. The same result holds true even if the field has no azimuthal component, although the argument is different. There is still a curve ℓ which neighbouring lines of force loop around, but (2.2.1) does not apply, since ℓ is not itself a line of force. Everywhere in V,

$$\underline{j} \cdot \underline{B} = 0 ,$$

whence (cf. (1.2.10))

$$\underline{u} = -\frac{\underline{j} \times \underline{B}}{B^2} + \psi \underline{B} , \quad \text{where } \psi \text{ is arbitrary.} \tag{2.2.2}$$

This velocity clearly tends to infinity as the critical curve ℓ on which $\underline{B}$ is zero is approached. This result is known as Cowling's theorem, which can be proved more formally and also holds good (as does the earlier one) even in non-axisymmetric situations in which a closed curve, ℓ, exists, about which all the neighbouring lines of force twist in the same sense.

The axisymmetric case in which the field is purely azimuthal also fails, although the argument is not so straightforward. Here Φ does not vanish, and the Cowling type line is the axis of symmetry and does not close on itself. There exists a surface (shown dotted in Fig. 2.1), which may be S itself, on which B is zero, and within which B is of one sign. Consider a path lying on this surface which is completed by the axis of symmetry. Since $B = 0$ everywhere on this curve, we must have

$$\oint \operatorname{grad} \Phi \cdot d\underline{r} = \frac{1}{\sigma} \oint \underline{j} \cdot d\underline{r} \,,$$

and, since $\underline{j}$ is in the same sense everywhere on it, the integral on the right is non-zero. Since Φ is single valued, this is impossible.

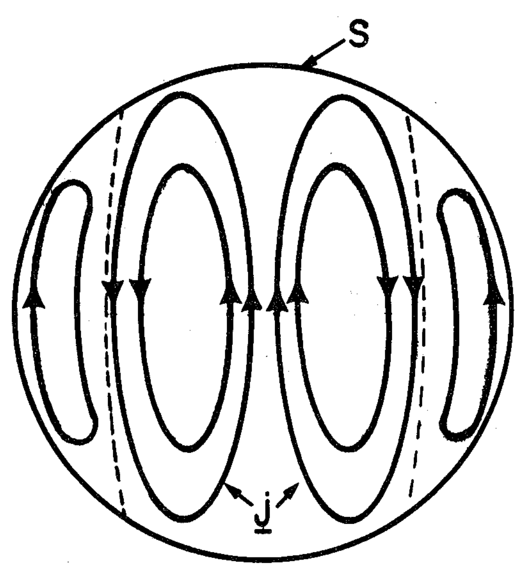

Fig. 2.1

More formal proofs of these results by different methods have been given by Backus and Chandrasekhar (1956), Cowling (1957) and Braginskiĭ (1964). They all depend on the assumption that $\underline{u}$ as well as $\underline{B}$ is axisymmetric, whereas Cowling's original proof does not. Whether an axisymmetric fluid flow can, in fact, support asymmetric fields is anybody's guess. I guess they cannot.

Similar arguments to those given in this section rule out two-dimensional dynamos, i.e., dynamos in which $\underline{B}$ does not depend on one particular Cartesian coordinate (cf. e.g. Cowling, 1957). Generalizations of this theorem, and Cowling's theorem itself, to the case in which the electrical conductivity is a function of position have recently been given by Lortz (1967).

These notes submitted by
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2A. Toroidal dynamos. The Bullard-Gellman Dynamo

2.3 Digression: Toroidal and Poloidal Vectors

When one deals with solenoidal vectors in spherical (or planar) systems, it is very useful to introduce a representation involving toroidal and poloidal vectors. An arbitrary vector field $\underline{Q}$ may be represented by three scalar fields $\underline{\Psi}$, T and S as follows:

$$\underline{Q} = -\operatorname{grad}\underline{\Psi} + \operatorname{curl} T\underline{r} + \operatorname{curl}^2 S\underline{r} \qquad (2.3.1)$$

where S and T are solutions of the second order equations

$$L^2 S = \underline{r} \cdot \underline{Q} \ , \tag{2.3.2}$$

$$L^2 T = \underline{r} \cdot \mathrm{curl}\, \underline{Q} \ , \tag{2.3.3}$$

and

$$L^2 = \sum_{\ell} \left[x_i \frac{\partial}{\partial x_i} + \left(x_i \frac{\partial}{\partial x_i} \right)^2 - x_i x_i \nabla^2 \right] \tag{2.3.4}$$

$$= - \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^3} \right] . \tag{2.3.5}$$

Here L^2 commutes with $\nabla^2, \partial/\partial\varphi, \partial/\partial r$ and any function of r . It is supposed that the mean value of S and T over any spherical shell is zero, i.e.

$$\int \left\{ \begin{matrix} S \\ T \end{matrix} \right\} \sin\theta \, d\theta \, d\varphi = 0 . \tag{2.3.6}$$

This does not involve any loss of generality since the difference between this particular choice of S and T and any other choice involves functions of r alone, and $\mathrm{curl}\, f(r)\underline{r} = \mathrm{curl}^2 f(r)\underline{r} = 0$. With this convection the operator L^{-2} necessary to solve equations (2.3.2) and (2.3.3) is well defined and unique. If, for example, we expand any scalar field, F , into spherical harmonics $Y_\alpha(\theta, \varphi)$, where (r, θ, φ) are spherical polar coordinates

$$F = \sum_{\alpha=1}^{\infty} F_\alpha(r) \, Y_\alpha(\theta, \varphi) , \tag{2.3.7}$$

we see that, since

$$L^2 f(r) Y_\alpha = \alpha(\alpha+1) f(r) Y_\alpha , \tag{2.3.8}$$

we have

$$L^{-2}F = \sum_{\alpha=1}^{\infty} \frac{F_\alpha}{\alpha(\alpha+1)} Y_\alpha(\theta,\varphi). \tag{2.3.9}$$

This series converges if the series (2.3.7) converges.

The vectors

$$\underline{T} = \mathrm{curl}\, T\underline{r} = -\underline{r} \times \mathrm{grad}\, T = \left\{0, \frac{1}{\sin\theta}\frac{\partial T}{\partial\varphi}, -\frac{\partial T}{\partial\theta}\right\}, \tag{2.3.10}$$

and

$$\begin{aligned}\underline{S} = \mathrm{curl}^2 S\underline{r} &= \mathrm{grad}\left[\frac{\partial}{\partial r}(rS)\right] - \underline{r}\nabla^2 S \\ &= \left\{\frac{L^2 S}{r}, \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial r}(rS)\right], \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left[\frac{\partial}{\partial r}(rS)\right]\right\},\end{aligned} \tag{2.3.11}$$

are called the <u>toroidal</u> and <u>poloidal</u> parts of the vector $\underline{Q}$, respectively, and T and S are called their <u>defining scalars</u>. Note that $\mathrm{curl}\,\underline{T}$ is poloidal with defining scalar T, and $\mathrm{curl}\,\underline{S}$ is toroidal with defining scalar $-\nabla^2 S$.

The toroidal and poloidal vectors are orthogonal when integrated over any spherical surface, $\mathcal{S}$, with center at the origin:

$$\int \underline{T}\cdot\underline{S}\, d\mathcal{S} = 0, \tag{2.3.12}$$

also

$$\int \underline{T}_1\cdot\underline{T}_2\, d\mathcal{S} = \int T_1 L^2 T_2\, d\mathcal{S}, \tag{2.3.13}$$

$$\int \underline{S}_1\cdot\underline{S}_2\, d\mathcal{S} = \int \nabla(rS_1)\cdot\nabla(rL^2 S_2)\frac{d\mathcal{S}}{r^2}. \tag{2.3.14}$$

The suitability of the above representation for electromagnetic problems lies in the fact that $\underline{B}_T$ is set up by $\underline{j}_S$ (defining scalar

T/μ), and $\underline{B}_S$ is set up by $\underline{j}_T$ (defining scalar $-\nabla^2 S/\mu$). Since $\hat{\underline{j}} = 0$ in $\hat{V}$, we have $\hat{T} = \nabla^2 \hat{S} = 0$. Thus, by equation (2.3.11), $\hat{\underline{B}}$ is the gradient of the single valued potential $\partial(r\hat{S})/\partial r$. The condition $\langle \underline{B} \rangle = 0$ becomes

$$T = \langle S \rangle = \langle \frac{\partial S}{\partial r} \rangle = 0 \text{ on } \mathcal{S}. \tag{2.3.15}$$

If the region $\hat{V}$ is conducting, we have in the steady state $\nabla^2 \hat{T} = \nabla^2 \hat{S} = 0$. The poloidal field is, as before, a potential field. This is bacause $\hat{\underline{j}}_T = 0$. The only currents which flow in $\hat{V}$ leak out of V, i.e. they must be poloidal.

2.4 The impossibility of a toroidal dynamo in a sphere

We intend to show that a spherical mass of fluid, of radius a, surrounded by an insulator cannot, by itself, maintain a dynamo if the velocity within it is everywhere solenoidal and toroidal (Bullard and Gellman 1954, Cowling 1957), i.e. if

$$\underline{u} \cdot \underline{r} = 0 \tag{2.4.1}$$

everywhere and not merely on $\mathcal{S}$. Here $\underline{r}$ denotes the radius vector from the center of the sphere.

We first observe that the induction equation

$$\frac{\partial \underline{B}}{\partial t} = \operatorname{curl}(\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B},$$

implies that

$$\frac{\partial(\underline{r}\cdot\underline{B})}{\partial t} = \underline{r}\cdot\mathrm{curl}\,(\underline{u}\times\underline{B}) + \eta\,\underline{r}\cdot\nabla^2\underline{B} = \mathrm{div}\,[(\underline{u}\times\underline{B})\times\underline{r}] + \eta\,\underline{r}\cdot\nabla^2\underline{B}$$
$$= \mathrm{div}[\underline{B}(\underline{r}\cdot\underline{u}) - \underline{u}\,(\underline{r}\cdot\underline{B})] + \eta[\nabla^2(\underline{r}\cdot\underline{B}) - 2\,\mathrm{div}\,\underline{B}]$$
$$= -\underline{u}\cdot\mathrm{grad}\,(\underline{r}\cdot\underline{B}) + \eta\nabla^2(\underline{r}\cdot\underline{B}), \qquad (2.4.2)$$

or

$$\frac{D}{Dt}(\underline{r}\cdot\underline{B}) = \eta\nabla^2(\underline{r}\cdot\underline{B}). \qquad (2.4.3)$$

This essentially proves that, in a frame following the fluid, $(\underline{r}\cdot\underline{B})$ diffuses away. More formally, from equation (2.4.2) we have

$$\frac{\partial}{\partial t}\int_V (\underline{r}\cdot\underline{B})^2 dV = -\int_V \underline{u}\cdot\mathrm{grad}(\underline{r}\cdot\underline{B})^2 dV + 2\eta\int_V (\underline{r}\cdot\underline{B})\nabla^2(\underline{r}\cdot\underline{B})\,dV$$
$$= -2\eta\int_V [\mathrm{grad}(\underline{r}\cdot\underline{B})]^2 dV + 2\eta\int (\underline{r}\cdot\underline{B})\,d\underline{S}\cdot\mathrm{grad}(\underline{r}\cdot\underline{B}). \qquad (2.4.4)$$

Now

$$\int (\underline{r}\cdot\underline{B})\,d\underline{S}\cdot\mathrm{grad}\,(\underline{r}\cdot\underline{B}) = -\int (\underline{r}\cdot\hat{\underline{B}})\,d\hat{\underline{S}}\cdot\mathrm{grad}\,(\underline{r}\cdot\hat{\underline{B}})$$
$$= -\int_{\hat{V}} [\mathrm{grad}\,(\underline{r}\cdot\hat{\underline{B}})]^2 d\hat{V}, \qquad (2.4.5)$$

since $\nabla^2\hat{\underline{B}} = 0$ implies that $\nabla^2(\underline{r}\cdot\hat{\underline{B}}) = 0$. Thus equation (2.4.4) gives

$$\frac{\partial}{\partial t}\int_V (\underline{r}\cdot\underline{B})^2 dV = -2\eta\int_{V+\hat{V}} [\mathrm{grad}\,(\underline{r}\cdot\underline{B})]^2 dV. \qquad (2.4.6)$$

This shows that $(\underline{r}\cdot\underline{B})$ decreases monotonically to zero. Thus, if the motion supports a field at all, that field must be toroidal, i.e. we may suppose that

$$\underline{B} = \mathrm{curl}\,L^2T\underline{r} = -\underline{r}\times\mathrm{grad}\,L^2T, \qquad (2.4.7)$$

in V, and $\hat{\underline{B}} = 0$. We have $T = 0$ on $\mathcal{S}$ and

$$\eta\int\left[\frac{1}{r}\,\mathrm{grad}(\underline{r}L^2T)\right]^2 dV = \eta\int \mathrm{curl}\,\underline{B}\cdot\mathrm{curl}^2(\underline{r}T)\,dV$$

$$= \int\left[\underline{E}+\underline{u}\times\underline{B}\right]\cdot\mathrm{curl}^2(\underline{r}T)\,dV$$

$$= \int(\mathrm{curl}\,\underline{E})\cdot\mathrm{curl}(\underline{r}T)\,dV - \int d\underline{S}\cdot\left[\underline{E}\times\mathrm{curl}(\underline{r}T)\right] +$$

$$+\int\underline{u}\cdot\left[\mathrm{curl}^2(\underline{r}T)\times(\underline{r}\times\mathrm{grad}\,L^2T)\right]dV$$

$$= -\int\mathrm{curl}\left(\underline{r}L^2\frac{\partial T}{\partial t}\right)\cdot\mathrm{curl}(\underline{r}T)\,dV +$$

$$+\int\left\{(\underline{u}\cdot\underline{r})\left[\mathrm{grad}\,L^2T\cdot\mathrm{curl}^2(\underline{r}T)\right] - (\underline{u}\cdot\mathrm{grad}\,L^2T)\left[\underline{r}\cdot\mathrm{curl}^2(\underline{r}T)\right]\right\}dV$$

$$= -\int(L^2T)\frac{\partial}{\partial t}(L^2T)\,dV - \int\underline{u}\cdot\mathrm{grad}\,\tfrac{1}{2}(L^2T)^2\,dV,$$

i.e.,

$$\frac{\partial}{\partial t}\int_V (L^2T)^2\,dV = -2\eta\int\left[\frac{1}{r}\,\mathrm{grad}(\underline{r}L^2T)\right]^2 dV \qquad (2.4.8)$$

showing that $L^2T \rightarrow 0$, i.e. that $\underline{B} \longrightarrow 0$ as $t \longrightarrow \infty$.

It may be noticed that in establishing (2.4.8), we have used, in a very essential way, the fact that the poloidal part, $\mathrm{curl}^2 S\underline{r}$ (say), of $\underline{B}$ has already disappeared. It is easily shown that, when this field is present, (2.4.8) is modified to

$$\frac{\partial}{\partial t}\int_V (L^2T)^2 dV = -2\eta\int\left[\frac{1}{r}\,\mathrm{grad}(\underline{r}L^2T)\right]^2 dV + \int\mathrm{curl}^2(\underline{r}T)\cdot\left[\underline{u}\times\mathrm{curl}^2(\underline{r}S)\right]dV. \qquad (2.4.9)$$

If $\underline{u}$ and S are large, the final integral in (2.4.9) can give rise to a tremendous initial increase in the integral on the left; essentially this is accomplished by the shearing of the poloidal lines of force by the toroidal flow onto its spherical surfaces. If $\underline{u}$ is steady, however, the final term in (2.4.9) decays exponentially with an e-folding time of $a^2/\eta\pi^2$ (cf. § 2.1). After this length of time, the energy will start decaying ohmically, no matter what the initial increase may have been.

If the laminar precessional motions discussed by Stewartson and Roberts (1963) and Roberts and Stewartson (1964) are relevant to the Earth's core, it appears from the present theorem that they are unlikely to provide the primary dynamo mechanism.

2.5 Brute force

The straightforward, and largely unsuccessful, method of solving the kinematic dynamo problem in a sphere is via the direct use of the formalism of § 2.3. Before launching into the details we might comment that so far no one has been able to give a convincing illustration of a bounded dynamo by numerical means. Also, one should note that it is quite possible to overcome the dynamo difficulties of Cowling's singular curves by adding a stray electric field in the right sense. Geophysically it might be difficult to give a satisfactory reason for such a field, but computationally it might arise incidentally from the finite difference approximation or from truncation of a representative series for the field. Thus it is especially necessary to be cautious of results obtained by purely numerical means. The hope is, however, that the numerical results will, as the degree of approximation increases, give the verisimilitude of convergence. Note that the numerical problem is necessarily three-dimensional, and that the velocity field has to have poloidal components.

We expand $\underline{u}$ and $\underline{B}$ in series of spherical harmonics of poloidal and toroidal type:

$$\underline{u} = \sum_{\alpha} (\underline{s}_{\alpha} + \underline{t}_{\alpha}),$$

$$\underline{B} = \sum_{\beta} (\underline{S}_{\beta} + \underline{T}_{\beta}),$$

where α and β are some kind of blanket label for a harmonic $Y_{\ell}^{m_s^c} = P_{\ell}^{m}(\mu)^{\cos}_{\sin} m\phi$ and $\mu = \cos\theta$. On substituting equations (2.5.1) and (2.5.2) into the induction equation

$$\frac{\partial \underline{B}}{\partial t} = R \operatorname{curl}(\underline{u} \times \underline{B}) - \operatorname{curl}^2 \underline{B}, \tag{2.5.3}$$

one obtains

$$\sum_{\gamma}\left(\frac{\partial \underline{S}_{\gamma}}{\partial t} + \frac{\partial \underline{T}_{\gamma}}{\partial t}\right) = R \sum_{\alpha\beta} \operatorname{curl}\left[(\underline{s}_{\alpha} + \underline{t}_{\alpha}) \times (\underline{S}_{\beta} + \underline{T}_{\beta})\right] - \sum_{\gamma} \operatorname{curl}^2 (\underline{S}_{\gamma} + \underline{T}_{\gamma}). \tag{2.5.4}$$

We can now make use of the orthogonality relations (2.3.12 - 14); (i) multiply scalarly equation (2.5.4) by $Y_{\gamma}\underline{r}$ and integrate over θ and ϕ , and (ii) take the curl of equation (2.5.4) and multiply scalarly by $Y_{\gamma}\underline{r}$. We obtain a doubly infinite set of equations for the S_{γ} , these being partial in t and r , or, in the case of steady dynamos, ordinary in r.

If we define ∇^2 to mean, applied to S_{γ} (or T_{γ})

$$\nabla^2 S_{\gamma} = \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{n_{\gamma}(n_{\gamma}+1)}{r^2}\right] S_{\gamma}, \tag{2.5.5}$$

these equations are of the form

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right) S_{\gamma} = R \sum_{\alpha\beta}\left[(s_{\alpha} S_{\beta} S_{\gamma}) + (t_{\alpha} S_{\beta} S_{\gamma}) + (s_{\alpha} T_{\beta} T_{\gamma})\right], \tag{2.5.6}$$

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right) T_\gamma = R \sum_{\alpha\beta} \left[(s_\alpha S_\beta T_\gamma) + (t_\alpha S_\beta T_\gamma) + (s_\alpha T_\beta T_\gamma) + (t_\alpha T_\beta T_\gamma) \right], \tag{2.5.7}$$

where

$$(s_\alpha S_\beta S_\gamma) = \frac{-\iint Y_\gamma \underline{r} \cdot \operatorname{curl} (\underline{s}_\alpha \times \underline{S}_\beta) \sin\theta \, d\theta \, d\varphi}{n_\gamma (n_\gamma + 1) \iint Y_\gamma^2 \sin\theta \, d\theta \, d\varphi}, \tag{2.5.8}$$

$$(s_\alpha S_\beta T_\gamma) = \frac{-\iint Y_\gamma \underline{r} \cdot \operatorname{curl}^2 (\underline{s}_\alpha \times \underline{S}_\beta) \sin\theta \, d\theta \, d\varphi}{n_\gamma (n_\gamma + 1) \iint Y_\gamma^2 \sin\theta \, d\theta \, d\varphi}, \tag{2.5.9}$$

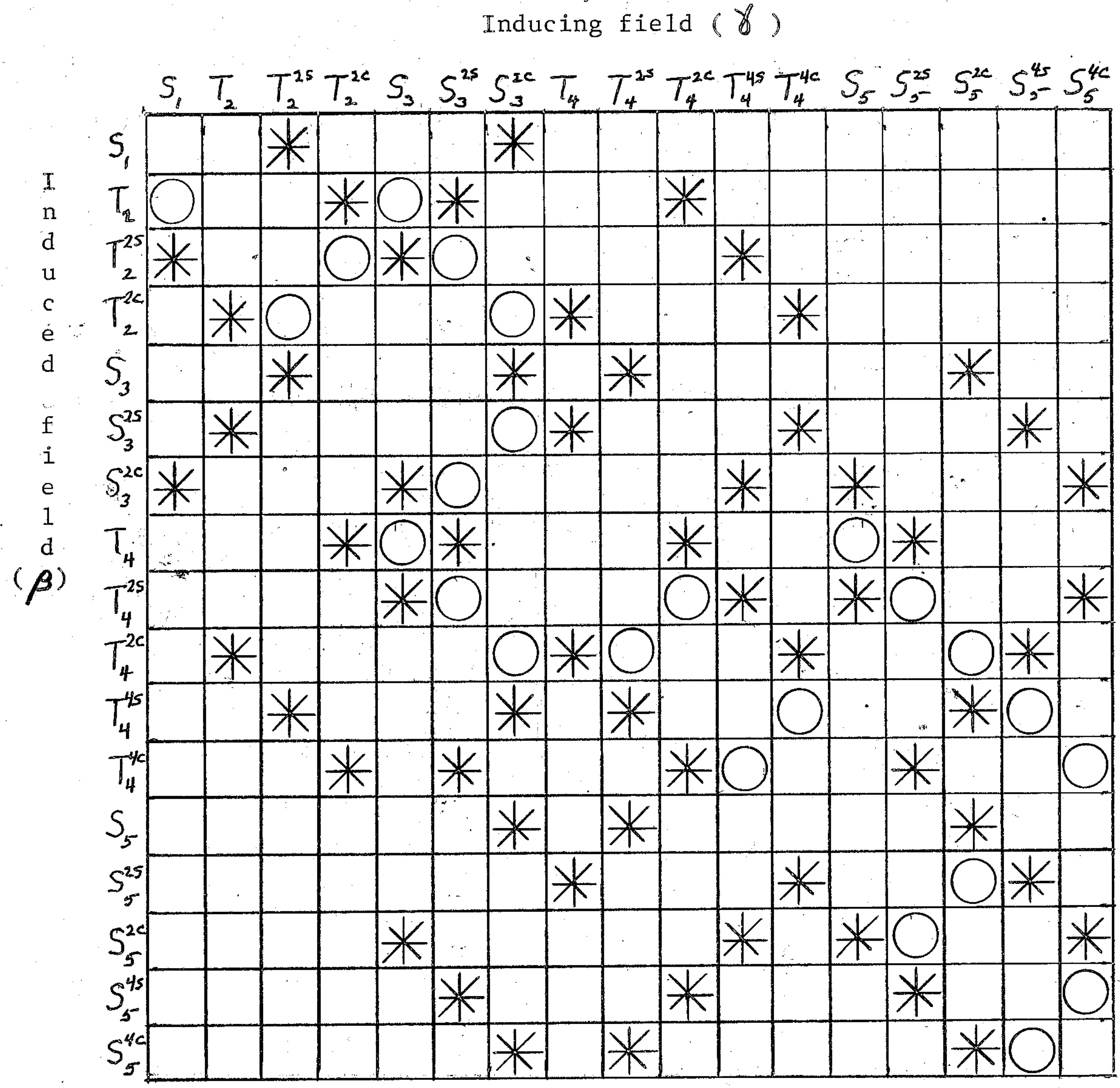
and so forth. These integrals have been evaluated by Bullard and Gellman in terms of the Gaunt and Elsasser integrals:

$$G_{\alpha\beta\gamma} = \iint Y_\alpha Y_\beta Y_\gamma \sin\theta \, d\theta \, d\varphi, \tag{2.5.10}$$

$$E_{\alpha\beta\gamma} = \iint Y_\alpha \left(\frac{\partial Y_\beta}{\partial \theta} \frac{\partial Y_\gamma}{\partial \varphi} - \frac{\partial Y_\beta}{\partial \varphi} \frac{\partial Y_\gamma}{\partial \theta} \right) \sin\theta \, d\theta \, d\varphi. \tag{2.5.11}$$

They show that, starting from a given flow and a given field harmonic, the triple products (2.5.8) and (2.5.9) excite an infinite train of other field harmonics. (An interaction diagram is given in Fig. 2.2.) Moreover, this series of successively higher excitations does not converge rapidly (at all?) because the parameter R by which they are multiplied in equation (2.5.6) and (2.5.7) is not small. The boundary conditions which must be satisfied by the solutions to equations (2.5.6) and (2.5.7) are

$$T_\gamma = \frac{\partial s_\gamma}{\partial r} + \frac{(n_\gamma + 1)}{r} S_\gamma = 0 \quad \text{at} \quad r = 1. \tag{2.5.12}$$



$*$ = s_2^{2c} interaction $\quad$ $\bigcirc$ = t_1 interaction

Figure 2.2 The interaction diagram

Many authors seem to have gained the impression that Bullard and Gellman's numerical results, obtained by taking the flow

$$\underline{u} = t_1 + \underline{s}_2^{2c}, \tag{2.5.13}$$

were convincing. Indeed, Rikitake and his co-workers have used this model extensively for examining questions of reversals and the possibility of similar magnetohydrodynamic dynamos. Dennis Gibson and I recently decided we should subject the numerical side to more scrutiny. We concentrated on one special case that Bullard and Gellman had examined, namely

$$t_1 = 5r^2(1-r), \text{ and } s_2^{2c} = r^3(1-r)^2. \tag{2.5.14}$$

As in Bullard and Gellman's case we successively truncated the number of harmonics at a higher and higher level. We went in fact as far as including all the harmonics S_1, T_2, T_2^{2s}, T_2^{2c}; S_3; S_3^{2s}, S_3^{2c}; T_4; T_4^{2s}, T_4^{2c}; T_4^{4s}, T_4^{4c}; S_5; S_5^{2s}, S_5^{2c}; S_5^{4s}, S_5^{4c}. We obtained the following:

Truncation level	Least eigenvalue
4	66.46
5	62.91
7	83.09
8	70.28
10	72.72
12	75.95
13	63.00
15	120.4
17	143.2

We tried to be very sure of the last two results although it was not practical, because of machine storage limitations, to take more than 12 terms in our representation series for each radial function. Clearly, there is no evidence that $\mathcal{R}$ is converging to a finite limit.

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3. The Herzenberg Dynamo

3.1 Induction by a rigid rotating sphere in an unbounded conductor

So far in this series, I have been completely negative. I have given theorems which exclude simple dynamos entirely, and the only dynamo I have discussed which evades these theorems (that of Bullard and Gellman) proved too complicated to provide even encouraging numerical indications of self-regeneration. This fairly accurately reflects the state of affairs ten years ago. At that time, many felt that the simple theorems were a foretaste of a much stronger anti-dynamo theorem which would rule out the mechanism entirely. Other possibilities were examined; for example, that the geomagnetic field was generated thermoelectrically from the difference in contact potentials between areas of the core surface above hot rising convection currents and areas above cold sinking fluid. All doubts were dispelled, however, in 1958, when two proofs of homogeneous dynamo action were supplied independently by Backus and Herzenberg. Since no course of lectures in the subject would be complete without it, I will discuss one of these existence theorems in some detail; namely, that of Herzenberg, who showed that two rotating spheres, embedded in a stationary conductor, can reciprocally build up each other's field.

As a preliminary, consider the field $\underline{b}$ induced outside a rigid conducting sphere of radius a which is rotating uniformly in a stationary infinite medium of the same conductivity, when there is present a constant applied magnetic field $\underline{B}_0$ generated at infinity.

Thus $\underline{B}_0$ (which is not necessarily uniform) satisfies

$$\mathrm{curl}^2\,\underline{B}_0 = \mu\,\mathrm{curl}\,\underline{j}_0 = \sigma\mu\,\mathrm{curl}\,\underline{E}_0 = 0,$$

so

$$\nabla^2 \underline{B}_o = 0 . \tag{3.1.1}$$

By definition, there are no sources of $\underline{B}_o$ in the finite plane, and so the solutions of (3.1.1) we select must have no singularities, i.e. $\underline{B}_o$ is composed of the increasing solutions $(r^n Y_n)$ of (3.1.1). The induced field $\underline{b}$ must also satisfy (3.1.1) but, since the source of this field lies in the rotor, it must die away with distance from it, i.e. $\underline{b}$ is composed of the decreasing solutions $(r^{-n-1} Y_n)$ of (3.1.1).

There is no need to require the interface between the sphere and its surroundings to be sharp; it can be thought of as containing a highly sheared film of fluid. In fact, it will even be convenient to do so initially, thinking of the angular velocity ω_o of the sphere as a function of (r, θ), where (r, θ, ϕ) are spherical polar coordinates with axis parallel to $\underline{\omega}$. On writing the total field as

$$\underline{B} = \underline{B}_o + \underline{b} , \tag{3.1.2}$$

the steady induction equation gives

$$\nabla^2 \underline{b} = -\frac{1}{\eta} \operatorname{curl} \left[\underline{u} \times (\underline{B}_o + \underline{b}) \right] , \tag{3.1.3}$$

which has the (Green's function) solution

$$\underline{b}(\underline{r}) = \frac{1}{4\pi\eta} \int \frac{\operatorname{curl}' [\underline{u}' \times (\underline{B}_o + \underline{b})']}{|\underline{r} - \underline{r}'|} \, dV' , \tag{3.1.4}$$

where the integral is over the inside of the sphere, and the primes indicate values at the dummy variable position, $\underline{r}'$. Essentially by expanding in η^{-1}, (3.1.4) may be solved by writing

$$\underline{b} = \sum_{n=1}^{\infty} \underline{b}_n, \tag{3.1.5}$$

where one iterates to obtain $\underline{b}_n$:

$$\underline{b}_{n+1}(\underline{r}) = \frac{1}{4\pi\eta}\int \frac{\mathrm{curl}'(\underline{u}'\times \underline{b}'_n)}{|\underline{r}-\underline{r}'|}\, dV', \tag{3.1.6}$$

for $n \geq 0$, where $\underline{b}_0 = \underline{B}_0$. Actually one does not need to proceed beyond $\underline{b}_1$, in either of the two main cases discussed below.

(a) Axisymmetry. Take an inducing field axisymmetric with respect to $\underline{\omega}$. The components of $\underline{B}_0$ are independent of φ although, recalling that the unit vector $\underline{1}_r$ and $\underline{1}_\varphi$ depend on φ, we have

$$\frac{\partial \underline{B}_0}{\partial \varphi} = \underline{1}_z \times \underline{B}_0 .$$

Then the induced field will also be axisymmetric, in fact, $\underline{b}$ will be completely toroidal, in the φ direction, parallel to $\underline{u}$. Therefore, $\underline{u}\times\underline{b}_n = 0$ for $n \geq 1$ and $\underline{b}_n = 0$ for $n \geq 2$, so

$$\underline{b}(r) = \frac{1}{4\pi\eta}\int \frac{\mathrm{curl}'[(\underline{\omega}'\times\underline{r}')\times\underline{B}'_0]}{|\underline{r}-\underline{r}'|}\, dV', \tag{3.1.7}$$

which may be simplified, by using $\mathrm{div}\,\underline{B}_0 = 0$ and axisymmetry, as follows

$$\mathrm{curl}\left[(\underline{\omega}\times\underline{r})\times\underline{B}_0\right] = \mathrm{curl}\left[(\underline{1}_z\times\underline{r})\times(\omega\underline{B}_0)\right]$$

$$= (\underline{1}_z\times\underline{r})\,\mathrm{div}(\omega\underline{B}_0) - (\underline{1}_z\times\underline{r})\cdot\underline{\nabla}(\omega\underline{B}_0) + \omega\underline{B}_0\cdot\underline{\nabla}(\underline{1}_z\times\underline{r})$$

$$= (\underline{1}_z\times\underline{r})(\underline{B}_0\cdot\underline{\nabla}\omega + \omega\underline{\nabla}\cdot\underline{B}_0) - \omega\left[\frac{\partial\underline{B}_0}{\partial\varphi} - \underline{1}_z\times\underline{B}_0\right] = (\underline{1}_z\times\underline{r})(\underline{B}_0\cdot\underline{\nabla}\omega), \tag{3.1.8}$$

Now (3.1.7) yields

$$\underline{b}(\underline{r}) = \frac{1}{4\pi\eta}\int \frac{(\underline{1}_z \times \underline{r}')(\underline{B}_o \cdot \underline{\nabla}\omega)'\, dV'}{|\underline{r}-\underline{r}'|} . \tag{3.1.9}$$

Now specialize $\omega(r)$ to

$$\omega = \begin{cases} \omega_o = \text{ constant, for } r < a, \\ 0, \text{ for } r > a, \end{cases} \tag{3.1.10}$$

so

$$\underline{\nabla}\omega = -\omega_o \delta(r-a)\underline{1}_r , \tag{3.1.11}$$

and (3.1.9) yields a surface integral

$$\underline{b}(\underline{r}) = -\frac{a^2\omega}{4\pi\eta}\oint \frac{(\underline{1}_z \times \underline{r}')B_{or}\sin\theta' dS'}{|\underline{r}-\underline{r}'|} ,$$

where the suffix o has been omitted from ω_o . Since $\underline{1}_z \times \underline{r} = a\underline{1}_\varphi$ on the sphere, $\underline{b}$ has only a φ component, so

$$\begin{aligned} b_\varphi = \underline{1}_\varphi \cdot \underline{b} &= -\frac{a^2\omega}{4\pi\eta}\oint \frac{(\underline{1}_\varphi \cdot \underline{1}'_\varphi)B'_{or}\sin\theta' dS'}{|\underline{r}-\underline{r}'|} \\ &= -\frac{a^2\omega}{4\pi\eta}\int \frac{B'_{or}\sin^2\theta'\cos(\varphi-\varphi')\,d\theta' d\varphi'}{|\underline{r}-\underline{r}'|} . \end{aligned} \tag{3.1.12}$$

These equations can be applied equally to the interior or exterior regions. Expand in the exterior $(r > a)$ in Legendre functions:

$$\frac{1}{|\underline{r}-\underline{r}'|} = \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n(\cos\Theta) , \tag{3.1.13}$$

where Θ is the angle between the directions of $\underline{r}$ and $\underline{r}'$, i.e.,

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'). \tag{3.1.14}$$

Using Ferrer's definition of P_n^m , one has

$$P_n(\cos\Theta) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi'), \tag{3.1.15}$$

so (3.1.13) becomes

$$\frac{1}{|r-r'|} = \sum_{n=0}^{\infty}\frac{a^n}{r^{n+1}}\left[P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi')\right], \tag{3.1.16}$$

and, performing the φ' integration of (3.1.12),

$$b_\varphi = -\left(\frac{\omega a^2}{2\eta}\right)\sum_{n=1}^{\infty}\frac{P_n^1(\cos\theta)}{n(n+1)}\left(\frac{a}{r}\right)^{n+1}\int_0^{\pi}B_n(a,\theta')P_n^1(\cos\theta')\sin^2\theta' d\theta', \tag{3.1.17}$$

since orthogonalities eliminate most terms.

As a general observation, note that the strength of the induced field increases indefinitely (linearly) as ω increases; this is not true in the case of a field with no axisymmetric part, where an electromagnetic skin effect shields against further increase (see case (b) below).

Now assume a definite form for $\underline{B}_0$, as a single harmonic:

$$\underline{B}_0 = a\,\underline{\nabla}\left[A_m\left(\frac{r}{a}\right)^m P_n(\cos\theta)\right], \tag{3.1.18}$$

where $m > 1$ and A_m is a constant. Clearly $\underline{B}_0$ is solenoidal and obeys (3.1.1); also

$$\underline{B}_{0r}(a,\theta) = m A_m P_n(\cos\theta). \tag{3.1.19}$$

Now use the identity

$$(2m+1)\sin\theta\, P_m(\cos\theta) = P_{m+1}^1(\cos\theta) - P_{m-1}^1(\cos\theta), \tag{3.1.20}$$

in equation (3.1.17) to get

$$b_\varphi = \frac{mA_m}{2m+1}\left(\frac{\omega a^2}{\eta}\right)\left[\frac{1}{(2m-1)}\left(\frac{a}{r}\right)^m P_{m-1}^1(\cos\theta) - \frac{1}{(2m+3)}\left(\frac{a}{r}\right)^{m+2}P_{m+1}^1(\cos\theta)\right], \tag{3.1.21}$$

with the convention that $P_0^1 = 0$, where the orthogonality

$$\int_0^{\pi} P_n^m(\cos\theta) P_{n'}^{m'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2m+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'}\delta_{mm'} , \qquad (3.1.22)$$

has been used.

Two special cases of (3.1.21) are relevant:

(i) $m = 1$, for which $\underline{B}_0$ is a uniform field and

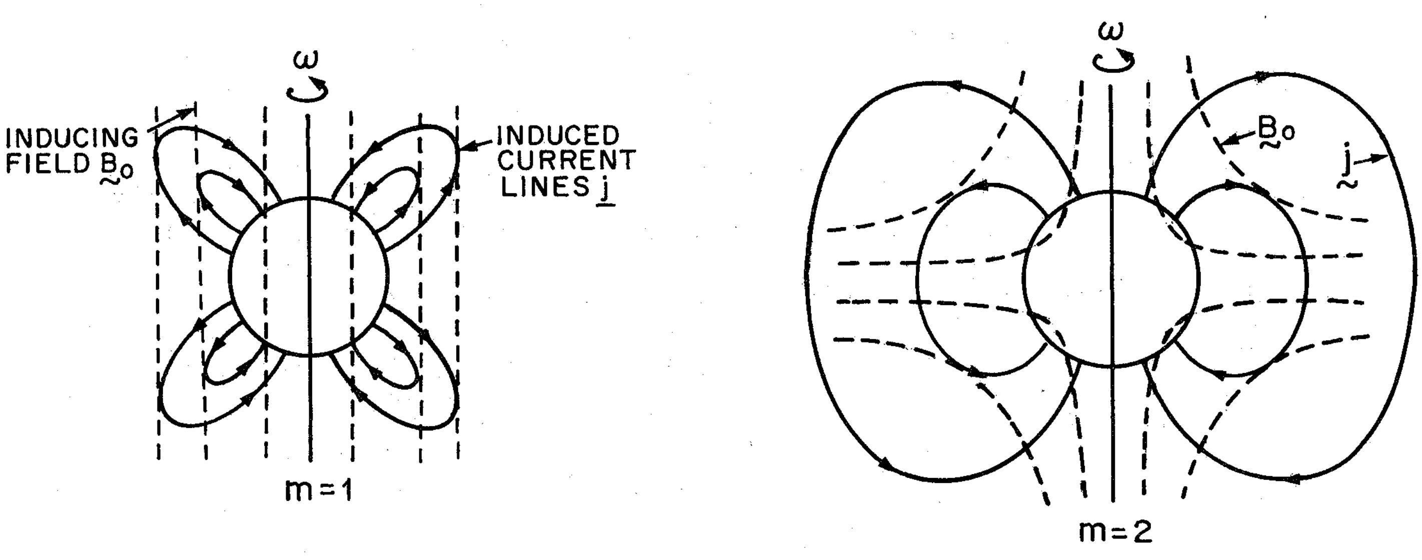
$$b_\varphi = -\frac{1}{5}A_1\left(\frac{a^2\omega}{\eta}\right)\left(\frac{a}{r}\right)^3 \sin\theta\cos\theta. \qquad (r > a) \qquad (3.1.23)$$

(ii) $m = 2$, for which $\underline{B}_0$ is linear in the coordinates

$$\left[\underline{B}_0 = A_2(-x,-y,2z)/a\right], \text{ and } b_\varphi \sim \frac{2}{15}A_2\left(\frac{a^2\omega}{\eta}\right)\left(\frac{a}{r}\right)^2 \sin\theta, \quad (r \gg a) \qquad (3.1.24)$$

is an asymptotic approximation.

Note that b_φ is proportional to ω in both (3.1.23) and (3.1.24). For all $m \geq 2$, $B_\varphi = O\left[(a/r)^m\right]$ for $r >> a$. Thus the



The currents induced for m = 1 and m = 2.

Fig. 3.1

field for $m=2$ dies off least rapidly with distance, with the uniform field $(m=1)$ and $m=3$ next. Note that the current field for $m=1$ is a quadrupole, and therefore decreases as r^{-3}, while the dipole for $m=2$ decreases as r^{-2}, for $r \longrightarrow \infty$.

Above only the poloidal axisymmetric field $\underset{\sim}{B}_0$ is considered. Clearly the toroidal axisymmetric inducing field is parallel to $\underset{\sim}{u}$ and will induce no fields.

(b) Asymmetry. Only the case in which $\underset{\sim}{B}_0$ is uniform and transverse to the axis of rotation will be considered. Form the scalar product of (3.1.6) with $\underset{\sim}{\omega}$, using

$$\underset{\sim}{\omega}\cdot \mathrm{curl}'(\underset{\sim}{u}'\times \underset{\sim}{b}_n') = \mathrm{div}'[(\underset{\sim}{u}'\times\underset{\sim}{b}_n')\times\underset{\sim}{\omega}] =$$
$$= \mathrm{div}'[\underset{\sim}{b}_n(\underset{\sim}{\omega}'\cdot\underset{\sim}{u}')-\underset{\sim}{u}'(\underset{\sim}{\omega}'\cdot\underset{\sim}{b}_n)] = -\mathrm{div}'[(\underset{\sim}{\omega}\cdot\underset{\sim}{b}_n)'\underset{\sim}{u}'] . \qquad (3.1.25)$$

Then $\underset{\sim}{\omega}\cdot\underset{\sim}{B}_0=0$ successively implies $\underset{\sim}{\omega}\cdot\underset{\sim}{b}_1=0$, $\underset{\sim}{\omega}\cdot\underset{\sim}{b}_2=0$, etc., so

$$\underset{\sim}{\omega}\cdot\underset{\sim}{B} = \underset{\sim}{\omega}\cdot\underset{\sim}{B}_0 = \underset{\sim}{\omega}\cdot\underset{\sim}{b} = 0, \text{ everywhere.} \qquad (3.1.26)$$

It is not true that $\underset{\sim}{b}_2$, $\underset{\sim}{b}_3$ etc. are zero.

Henceforth, attention will be restricted to high rotation $(\omega \gg \eta/a^2)$. In the sphere's coordinates, the uniform field $\underset{\sim}{B}_0$ appears to be oscillating rapidly, and will therefore penetrate only a distance $\sqrt{(\eta/\omega)} \ll a$. Thus, the rotating sphere is almost completely shielded electromagnetically from the transverse field, i.e. in the limit $\omega \longrightarrow \infty$,

$$\underset{\sim}{B} = 0, \quad \text{for } r < a. \qquad (3.1.27)$$

To obtain (3.1.27) more formally, we may develop a "boundary layer" solution which is uniformly valid in the limit $\omega \longrightarrow \infty$. For solutions proportional to $e^{-im\phi}$, the steady induction equation for the interior of with the convention that $P_0' = 0$, where the orthogonality

$$\int_0^{\pi} P_n^m(\cos\theta) P_{n'}^{m'}(\cos\theta) \sin\theta \, d\theta = \frac{2}{2m+1} \frac{(n+m)!}{(n-m)!} \delta_{nn'} \delta_{mm'} , \tag{3.1.22}$$

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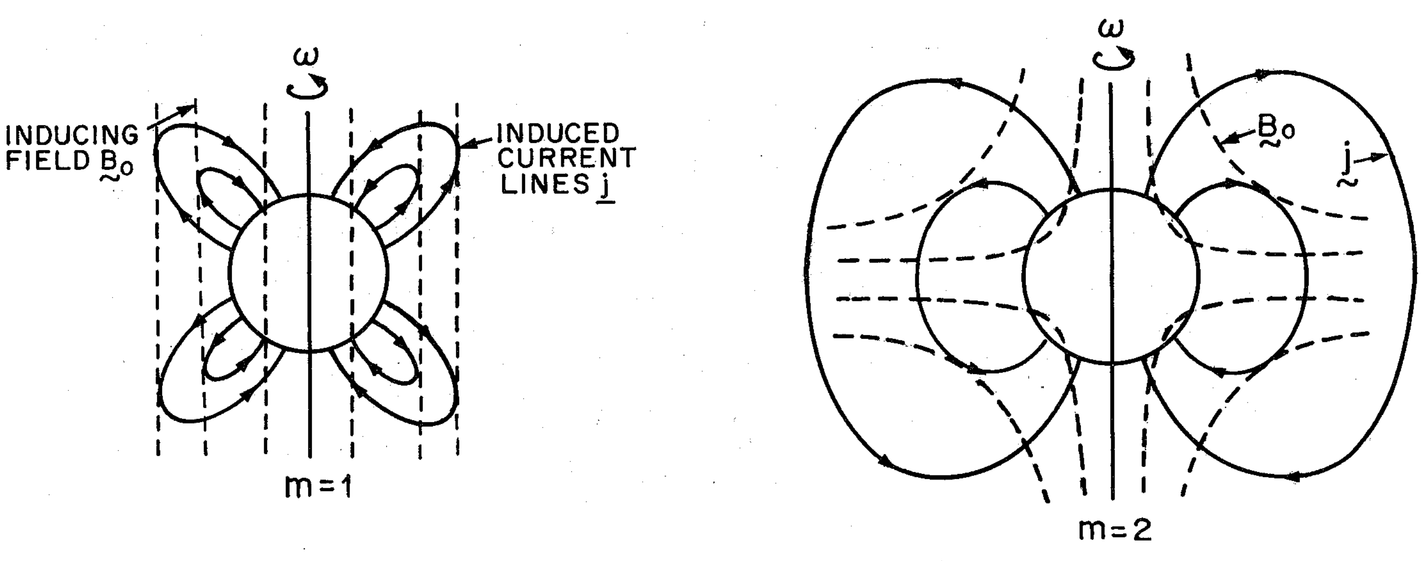

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$$\underset{\sim}{\omega}\cdot \operatorname{curl}'(\underset{\sim}{u}'\times \underset{\sim}{b}_n') = \operatorname{div}'[(\underset{\sim}{u}'\times \underset{\sim}{b}_n')\times \underset{\sim}{\omega}] =$$
$$= \operatorname{div}'[\underset{\sim}{b}_n(\underset{\sim}{\omega}'\cdot \underset{\sim}{u}') - \underset{\sim}{u}'(\underset{\sim}{\omega}'\cdot \underset{\sim}{b}_n)] = -\operatorname{div}'[(\underset{\sim}{\omega}\cdot \underset{\sim}{b}_n)\underset{\sim}{u}'] . \qquad (3.1.25)$$

Then $\underset{\sim}{\omega}\cdot \underset{\sim}{B}_0 = 0$ successively implies $\underset{\sim}{\omega}\cdot \underset{\sim}{b}_1 = 0$, $\underset{\sim}{\omega}\cdot \underset{\sim}{b}_2 = 0$, etc., so

$$\underset{\sim}{\omega}\cdot \underset{\sim}{B} = \underset{\sim}{\omega}\cdot \underset{\sim}{B}_0 = \underset{\sim}{\omega}\cdot \underset{\sim}{b} = 0, \text{ everywhere.} \qquad (3.1.26)$$

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$$\underset{\sim}{B} = 0, \text{ for } r < a. \qquad (3.1.27)$$

To obtain (3.1.27) more formally, we may develop a "boundary layer" solution which is uniformly valid in the limit $\omega \longrightarrow \infty$. For solutions proportional to $e^{-im\phi}$, the steady induction equation for the interior of the sphere becomes

$$\eta \nabla^2 \underline{B} = \omega \frac{\partial \underline{B}}{\partial \varphi} = -im\omega \underline{B}. \tag{3.1.28}$$

For the field far from the boundary layers, we have $\underline{\nabla} = O(1)$, so that, for $\omega \to \infty$, (3.1.28) reduces in leading order to (3.1.27). For the boundary layers, we may write, in the usual way,

$$\underline{B} = \underline{B}^{(int)} + \underline{B}^{(b.\ell.)}$$

where, as it happens in this case, the interior field $\underline{B}^{(int)}$ is zero. The boundary layer field $\underline{B}^{(b.\ell.)}$ is required to vanish with distance from $r = a$, and to satisfy (3.1.28). Let us introduce a boundary layer thickness, δ, which vanishes as $\omega \to \infty$, and a scaled variable

$$\zeta = (a - r)\delta,$$

to replace r; the choice of δ is determined by the postulate that

$$\frac{\partial}{\partial \zeta} = O(1), \quad \frac{\partial}{\partial \theta} = O(1), \quad \frac{\partial}{\partial \varphi} = O(1), \tag{3.1.29}$$

when operating on $\underline{B}^{(b.\ell.)}$. To leading order, (3.1.28) gives

$$\left[\frac{\partial^2}{\partial \zeta^2} + \frac{im\omega\delta^2}{\eta}\right] \underline{B}^{(b.\ell.)} = 0. \tag{3.1.30}$$

Now (3.1.29) shows that a convenient choice of δ is

$$\delta = \left(\frac{2\eta}{m\omega}\right)^{1/2}$$

which is essentially the electromagnetic skin depth. Then (3.1.30) becomes

$$\left[\frac{\partial^2}{\partial \zeta^2} + 2i\right](B_\theta, B_\varphi)^{(b.\ell.)} = 0,$$

to which the required solutions, vanishing for $\zeta \to \infty$, are

$$B_{\theta,\varphi}^{(b.\ell.)}(\zeta,\theta,\varphi) = B_{\theta,\varphi}^{(b.\ell.)}(0,\theta,\varphi)e^{-(1+i)\zeta} \tag{3.1.31}$$

The equation $\operatorname{div}\underline{B} = 0$ gives, to leading order,

$$\frac{\partial B_r^{(b.\ell.)}}{\partial \zeta} = \delta\left[-\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(B_\theta^{(b.\ell.)}\sin\theta\right) + \frac{\partial B^{(b.\ell.)}}{\partial \varphi}\right\}\right] = O\left[\delta\left|\underline{B}^{(b.\ell.)}\right|\right] \tag{3.1.32}$$

Thus, we may follow the usual boundary layer recipe: first select an exterior solution [which satisfied (3.1.1)] such that $\langle \underline{n}\cdot\underline{B}\rangle^{*}=0$, the star signifying that the jump is to be taken across the boundary layer; in other words we select an exterior solution satisfying [cf. (3.1.27)]

$$B_r^{(ext)} = 0 \text{ on } r=a. \tag{3.1.33}$$

Since this exterior field is also prescribed at infinity, it is now specified uniquely (as we will shortly show in a special case). It will not, however, satisfy $\langle \underline{n}\times\underline{B}\rangle^{*}=0$, and, since it is required that $\langle \underline{n}\times\underline{B}\rangle=0$ on S, we must have

$$B_{\theta,\varphi}^{(b.\ell.)} = -B_{\theta,\varphi}^{(ext)}, \text{ on } r=a$$

conditions which, by (3.1.31), determine the boundary layer solution uniquely. We have now satisfied all the conditions on S. The fact that, according to (3.1.32), we have introduced an error of $O(\delta)$ into $B_r^{(b.\ell.)}$ and have therefore violated $\langle \underline{n}\cdot\underline{B}\rangle=0$ to the same degree, does not matter. To dominant order, all the conditions are obeyed, and $O(\delta)$ errors could be eliminated by a correction which would introduce $O(\delta^2)$ errors, and so forth. We also see that the electromagnetic boundary layer is passive, i.e. to dominant order we can determine the interior and exterior fields uniquely [using (3.1.33)], confident that we can subsequently construct a boundary layer that will leave all the jump conditions satisfied.

Outside the sphere, $\nabla^2(rB_r)=0$ i.e., rB_r is harmonic. The applied field is uniform and directed along $\varphi=0,\ \theta=\pi/2$, so $\underline{r}\cdot\underline{B}_0 = B_0 r P_1'(\cos\theta)\cos\varphi$, so, for $r>a$, the induced field must be proportional to $r^{-2}P_1'(\cos\theta)\cos\varphi$. Since by (3.1.33) $\underline{r}\cdot\underline{B}=0$ on S,

$$B_r = B_0\left[1-\left(\frac{a}{r}\right)^3\right]\sin\theta\cos\varphi,\ (r>a), \tag{3.1.34}$$

and by (3.1.26)

$$B_\theta = B_0\left[1-\left(\frac{a}{r}\right)^3\right]\cos\theta\cos\varphi, \ (r>a), \qquad (3.1.35)$$

Combining $\operatorname{div}\underline{B}=0$ with (3.1.34) and (3.1.35) yields

$$B_\varphi = -B_0\left[1-\left(\frac{a}{r}\right)^3\right]\sin\varphi - 3B_0\left(\frac{a}{r}\right)^3\sin^2\theta\sin\varphi, \ (r>a). \qquad (3.1.36)$$

In vector form (3.1.34-36) may be written

$$\underline{B} = \underline{B}_0 + a^3 B_0 \underline{1}_z \times \underline{\nabla}\left(\frac{\sin\theta\sin\varphi}{r^2}\right), \ (r>a). \qquad (3.1.37)$$

We see that the effect of the increasing induced skin currents is canceled by the decreasing skin depth, so for large ω, the net "surface" current is nearly independent of ω. This is a general feature of fields induced by <u>any</u> field with no symmetric part.

3.2 Herzenberg dynamo: the bare bones

At first, suppose the Herzenberg dynamo consists of two spheres, $\mathcal{S}_1$ and $\mathcal{S}_2$, with centers at O_1 and O_2, both of radius a, separated* by a distance $\underline{R} = \overrightarrow{O_1 O_2}$, and imbedded in an infinite conductor of the same conductivity with which they are in perfect electrical contact.

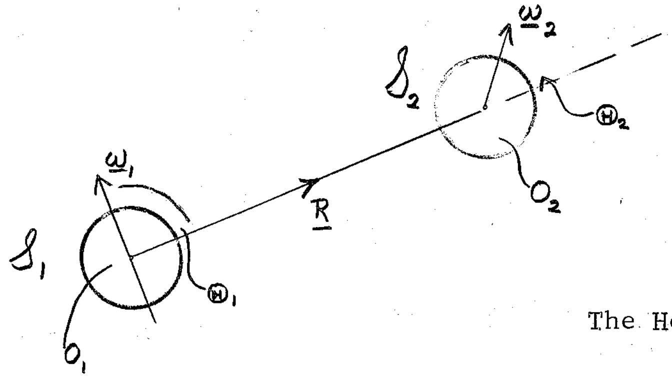

Fig. 3.2

The Herzenberg dynamo.

*Note: In this section $R(>2a)$ will denote the scalar distance O_1O_2. To avoid confusion with the magnetic Reynolds number, we will temporarily denote that quantity by R_m; see (3.2.15) below.

The angular velocities $\underline{\omega}_1$ and $\underline{\omega}_2$ of the spheres will be supposed equal in magnitude $(\omega_1 = \omega_2 = \omega)$ but different in direction. To avoid infinite velocity gradients, one may imagine that there are thin highly sheared films of fluid between the relatively moving parts. Alternatively, Gibson (1968) has shown that the entire argument to be given below is valid for onion-like rotors in which $\underline{\omega}_1$ and $\underline{\omega}_2$ can slowly vary in amplitude with radial distance within each rotor. Denote by Θ_1 the angle between $\underline{\omega}_1$ and $\underline{R}$, by Θ_2 that between $\underline{\omega}_2$ and $\underline{R}$, and by Φ that between the plane defined by $\underline{\omega}_1$ and $\underline{R}$ and the plane defined by $\underline{\omega}_2$ and $\underline{R}$. The objective is to make the field $\underline{B}_1$, created by mutual induction in $\mathcal{S}_1$, be the inducing field for $\mathcal{S}_2$ which generates, from it, the field $\underline{B}_2$ necessary to produce the initial $\underline{B}_1$. We consider only the case of small a/R and large $a^2\omega/\eta$, i.e. two well-separated spheres rotating like hell.

Consider induction in $\mathcal{S}_1$. Divide $\underline{B}_2$ into its symmetric, $\overline{\underline{B}}_2$, and antisymmetric, $\underline{B}_2'$, parts, with respect to the direction $\underline{\omega}_1$. The field $\underline{B}_1^*$ created from $\overline{\underline{B}}_2$ and the field $\underline{B}_1^+$ created from $\underline{B}_2'$ are, as previously deduced, of the orders

$$\frac{|\underline{B}_1^*|}{|\overline{\underline{B}}_2|} = O\left(\frac{a^2\omega}{\eta}\right), \quad \frac{|\underline{B}_1^+|}{|\underline{B}_2'|} = O(1). \tag{3.2.1}$$

Thus (for given $|\underline{B}_2'|/|\overline{B}_2|$) $|\underline{B}_1^+|/|\underline{B}_1^*|$ can be as small as $\eta/a^2\omega$ is set. Thus, the leading approximation need only take account of those components of $\underline{B}_2$ which are axisymmetric with respect to $\underline{\omega}_1$.

Let $\underline{r}_1$ and $\underline{r}_2$, denote the radial vectors for O_1 and O_2, the corresponding spherical polar coordinates being $(r_1, \theta_1, \varphi_1)$ and $(r_2, \theta_2, \varphi_2)$ with $\underline{\omega}_1$ and $\underline{\omega}_2$ as axes, respectively. Expand $\underline{B}_2$ about O_1 :

$$\underline{B}_2(\underline{r}_1) = \operatorname{curl}(T_2\underline{r}_1) + \operatorname{curl}^2(S_2\underline{r}_1), \tag{3.2.2}$$

and expand the defining scalars as

$$T_2 = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} b_n^{(2)m} \frac{J_{n+\frac{1}{2}}(k_m r_1)}{(k_m r_1)^{1/2}} P_n^m(\cos\theta_1)e^{im\varphi_1} + \sum_{n=1}^{\infty} B_n^{(2)}\left(\frac{r_1}{a}\right)^n P_n(\cos\theta_1), \tag{3.2.3}$$

$$S_2 = a\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} a_n^{(2)m} \frac{J_{n+\frac{1}{2}}(k_m r_1)}{(k_m r_1)^{1/2}} P_n^m(\cos\theta_1)e^{im\varphi_1} + a\sum_{n=1}^{\infty} \frac{A_n^{(2)}}{n+1}\left(\frac{r_1}{a}\right)^n P_n(\cos\theta_1), \tag{3.2.4}$$

where $k_m^2 = -i\omega m/\eta$.

It is essential in the later development to be able to extract $A_1^{(2)}$ and $A_2^{(2)}$ from a given inducing field $\underline{B}_2$. First, since $J_{n+\frac{1}{2}}(k_m r_1)/(k_m r_1)^{\frac{1}{2}} = O(r_1^n)$, as $r_1 \to 0$, only the $a_1^{\pm 1}$ and A_1 terms will enter $\underline{B}_2$ at $r_1 = 0$. But

$$\sum_i \frac{\omega_{1i}}{\omega_1} \operatorname{curl}_i^2 \left[\frac{J_{3/2}(k_m r_1)}{(k_m r_1)^{1/2}} P_1'(\cos\theta_1)e^{i\varphi_1}\underline{r}_1\right] =$$

$$= \frac{e^{i\varphi_1}}{(k_m r_1^3)^{1/2}}\left[2J_{\frac{3}{2}}(k_m r_1)\cos\theta_1 P_1'(\cos\theta_1) - \left\{k_m r_1 J_{\frac{1}{2}}(k_m r_1) - J_{\frac{3}{2}}(k_m r_1)\right\}\sin\theta_1 \frac{d P_1'(\cos\theta_1)}{d\theta_1}\right] \sim$$

$$\sim \frac{k e^{i\varphi_1}}{2^{\frac{1}{2}}\Gamma(\frac{5}{2})}\left[\cos\theta_1 P_1'(\cos\theta) - \sin\theta_1 \frac{dP_1'(\cos\theta_1)}{d\theta}\right] = 0, \text{ as } r_1 \to 0, \tag{3.2.5}$$

since $P_1'(\cos\theta) = \sin\theta$. Thus $A_1^{(2)}$ may be extracted (summation convention):

$$\frac{\omega_{1i}}{\omega_1} B_{2i}(O_1) = \frac{\underline{\omega}_1}{\omega_1}\cdot\operatorname{curl}^2\left[\frac{aA_1^{(2)}}{2}\left(\frac{r_1}{a}\right)\underline{r}_1\cos\theta_1\right] =$$

$$= \tfrac{1}{2}A_1^{(2)}\left[\cos\theta_1\, 2\cos\theta_1 + \sin\theta_1 \cdot 2\sin\theta_1\right] = A_1^{(2)},$$

so

$$A_1^{(2)} = \frac{\omega_{1i}}{\omega_1} B_{2i}(O_1). \qquad (3.2.6)$$

Like $\underline{B}_2$ itself, $\tilde{\underline{B}}_2 = \underline{\omega}_1 \cdot \underline{\nabla}\, \underline{B}_2$ is harmonic and representable in the form

$$\tilde{\underline{B}}_2 = \underline{\omega}_1 \nabla \underline{B}_2 = \mathrm{curl}\,(\tilde{T}_2 \underline{r}_1) + \mathrm{curl}^2 (\tilde{S}_2 \underline{r}_1). \qquad (3.2.7)$$

Just as in (3.2.6),

$$\tilde{A}_1^{(2)} = \frac{\omega_{1i}\, \tilde{B}_{2i}(O_1)}{\omega_1} = \frac{\omega_{1i}\, \omega_{1j}}{\omega} \frac{\partial B_{2i}(O_1)}{\partial x_{1j}}. \qquad (3.2.8)$$

Now $\tilde{S}_2 = \underline{\omega}_1 \cdot \underline{\nabla}\, S_2$, so (3.2.4) yields

$$\tilde{A}_1^{(2)} P_1(\cos\theta) = \frac{A_2^{(2)} \omega_1}{3a} \left[2\cos\theta_1 P_2(\cos\theta_1) - \sin\theta_1 \frac{\partial P_2(\cos\theta_1)}{\partial \theta_1} \right] = \frac{2A_2^{(2)} \omega_1}{a} \cos\theta_1 ,$$

giving

$$\tilde{A}_1^{(2)} = \frac{2\omega_1 A_2^{(2)}}{a} \qquad (3.2.9)$$

Combining (3.2.8) and (3.2.9) gives

$$A_2^{(2)} = \frac{a\, \omega_{1i}\, \omega_{1j}}{2\omega^2} \frac{\partial B_{2i}(O_1)}{\partial x_{1j}}. \qquad (3.2.10)$$

The argument can be extended to show

$$A_m^{(2)} = a^{m-1} O \left[\left| \frac{\partial^{m-1} \underline{B}_2}{\partial x_1^{m-1}} \right|_{O_1} \right], \quad (m \geq 1). \qquad (3.2.11)$$

The tools to attack the mutual interaction are now available. Starting from the harmonic $\underline{B}_1^{(n)}$ at $\mathcal{S}_2$, equations (3.1.23), (3.1.24), and the succeeding argument indicate that the magnitude of the field created at $\mathcal{S}_1$ is

$$O\left[\left(\frac{a^2\omega}{\eta}\right) A_n^{(1)}\right] \times \begin{Bmatrix} (a/R)^n, & \text{if} \quad n \neq 1, \\ (a/R)^3 & \text{if} \quad n = 1. \end{Bmatrix} \tag{3.2.12}$$

To dominant order, only the poloidal part of the field symmetric with respect to $\underline{\omega}_1$ need be retained, for the toriodal field is parallel to the axis and the rest is negligible by (3.2.1). So, from (3.2.6) and (3.2.10-12),

$$A_m^{(2)} = \left(\frac{a}{R}\right)^{m-1}\left(\frac{a^2\omega}{\eta}\right)\left[A_1^{(1)}\left(\frac{a}{R}\right)^3 + A_2^{(1)}\left(\frac{a}{R}\right)^2 + A_3^{(1)}\left(\frac{a}{R}\right)^3 + \cdots + A_n^{(1)}\left(\frac{a}{R}\right)^n + \cdots\right]. \tag{3.2.13}$$

Reversing the notation,

$$A_n^{(1)} = \left(\frac{a}{R}\right)^{n-1}\left(\frac{a^2\omega}{\eta}\right)\left[A_2^{(2)}\left(\frac{a}{R}\right)^3 + A_2^{(2)}\left(\frac{a}{R}\right)^2 + A_3^{(2)}\left(\frac{a}{R}\right)^3 + \cdots + A_m^{(2)}\left(\frac{a}{R}\right)^m + \cdots\right]. \tag{3.2.14}$$

To justify the retention of the symmetric part only, we used the assumption $a^2\omega/\eta >> 1$. The assumption $a/R << 1$ is now made, specialized to requiring a moderate magnetic Reynolds number, defined as

$$R_m = \left(\frac{a}{R}\right)^3\left(\frac{a^2\omega}{\eta}\right) = O(1), \tag{3.2.15}$$

so equations (3.2.13) and (3.2.14) become

$$
\begin{aligned}
A_1^{(2)} &= O\left[A_1^{(1)} + \frac{R}{a}A_2^{(1)} + A_3^{(1)} + \left(\frac{a}{R}\right)A_4^{(1)} \cdots\right], \\
\frac{R}{a}A_2^{(2)} &= O\left[A_1^{(1)} + \frac{R}{a}A_2^{(1)} + A_3^{(1)} + \left(\frac{a}{R}\right)A_4^{(1)} \cdots\right], \\
A_3^{(2)} &= O\left[\left(\frac{a}{R}\right)^2 A_1^{(1)} + \cdots\right], \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad (3.2.16) \\
A_1^{(1)} &= O\left[A_1^{(2)} + \frac{R}{a}A_2^{(2)} + A_3^{(2)} + \left(\frac{a}{R}\right)A_4^{(2)} + \cdots\right], \\
\frac{R}{a}A_2^{(1)} &= O\left[A_1^{(2)} + \frac{R}{a}A_2^{(2)} + A_3^{(2)} + \left(\frac{a}{R}\right)A_4^{(2)} + \cdots\right], \\
A_3^{(1)} &= O\left[\left(\frac{a}{R}\right)^2 A_1^{(2)} + \cdots\right], \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

These show that solutions exist in which

$$A_n = O\left[\left(\frac{a}{R}\right)^n\right], \qquad (3.2.17)$$

and, to $O\left[(a/R)^2\right]$, (3.2.16) are trimmed to

$$
\begin{aligned}
&A_1^{(2)} = O\left(A_1^{(1)} + \frac{R}{a}A_2^{(1)}\right), \quad \frac{R}{a}A_2^{(2)} = O\left(A_1^{(1)} + \frac{R}{a}A_2^{(1)}\right), \\
&A_1^{(1)} = O\left(A_1^{(2)} + \frac{R}{a}A_2^{(2)}\right), \quad \frac{R}{a}A_2^{(1)} = O\left(A_1^{(2)} + \frac{R}{a}A_2^{(2)}\right).
\end{aligned}
\qquad (3.2.18)
$$

It is seen that, to dominant order, only two harmonics from each rotor need be included, e.g., in $\underset{\sim}{B}_1^*$ we have, by (3.1.23-24),

$$
\begin{aligned}
\underset{\sim}{B}_1^* &= \left[-\frac{1}{5}A_1^{(2)}\left(\frac{a^2\omega}{\eta}\right)\left(\frac{a}{r_1}\right)^3 \sin\theta_1\cos\theta_1 + \frac{2}{15}A_2^{(2)}\left(\frac{a^2\omega}{\eta}\right)\left(\frac{a}{r_1}\right)^2 \sin\theta_1\right]\underset{\sim}{1}_{\varphi_1} = \\
&= -\frac{1}{5}A_1^{(2)}\left(\frac{a^2\omega}{\eta}\right)\left(\frac{a}{r_1}\right)^3 \frac{(\underset{\sim}{\omega}_1 \cdot \underset{\sim}{r}_1)(\underset{\sim}{\omega}_1 \times \underset{\sim}{r}_1)}{\omega^2 r_1^2} + \frac{2}{15}A_2^{(2)}\left(\frac{a^2\omega}{\eta}\right)\left(\frac{a}{r_1}\right)\frac{\underset{\sim}{\omega}_1 \times \underset{\sim}{r}_1}{\omega r},
\end{aligned}
\qquad (3.2.19)
$$

where the latter expression is in coordinate invariant form. Now $\underline{B}_1^* = \underline{B}_2$, so (3.2.6) and (3.2.10) may be used to extract $A_1^{(1)}$ and $A_2^{(1)}$:

$$\frac{a}{R}A_1^{(1)} = \left(\frac{a^5\omega}{\eta R^3}\right)\frac{\underline{R}\cdot(\underline{\omega}_1\times\underline{\omega}_2)}{5R\omega^2}\left[\frac{a}{R}\left(\frac{\underline{\omega}_1\cdot\underline{R}}{\omega R}\right)A_1^{(2)} - \frac{2}{3}A_2^{(2)}\right], \tag{3.2.20}$$

$$2A_2^{(1)} = \left(\frac{a^2\omega}{\eta R^3}\right)\frac{\underline{R}\cdot(\underline{\omega}_1\times\underline{\omega}_2)}{5R\omega^2}\left[\frac{a}{R}\left\{\frac{\underline{\omega}_1\cdot\underline{\omega}_2}{\omega^2} - \frac{5(\underline{\omega}_1\cdot\underline{R})(\underline{\omega}_2\cdot\underline{R})}{\omega^2R^2}\right\}A_1^{(2)} + \frac{2(\underline{\omega}_2\cdot\underline{R})}{\omega R}A_2^{(2)}\right]. \tag{3.2.21}$$

The reminder $\underline{R} = \overrightarrow{O_1O_2}$ may be useful.

Let

$$\Lambda = \frac{1}{5}\left(\frac{a^2\omega}{\eta}\right)\left(\frac{a}{R}\right)^3\underline{R}\cdot\frac{(\underline{\omega}_1\times\underline{\omega}_2)}{R\omega_2} = \frac{1}{5}R_m \sin\Theta_1 \sin\Theta_2 \sin\Phi, \tag{3.2.22}$$

$$p = \frac{\underline{\omega}_1\cdot\underline{\omega}_2}{\omega^2} - \frac{5(\underline{\omega}_1\cdot\underline{R})(\underline{\omega}_2\cdot\underline{R})}{\omega^2R^2} = \sin\Theta_1\sin\Theta_2\cos\Phi - 4\cos\Theta_1\cos\Theta_2, \tag{3.2.23}$$

$$F_1 = \frac{a}{R}A_1^{(1)},\quad F_2 = \frac{a}{R}A_1^{(2)},\quad G_1 = 2A_2^{(1)},\quad G_2 = 2A_2^{(2)}, \tag{3.2.24}$$

so (3.2.20) and (3.2.21) become

$$F_1 = \Lambda\left[F_2\cos\Theta_1 - \tfrac{1}{3}G_2\right],\quad G_1 = \Lambda\left[F_2 p + G_2\cos\Theta_2\right]. \tag{3.2.25}$$

Switching indices and the direction of $\underline{R}$ yields

$$F_2 = \Lambda\left[-F_1\cos\Theta_2 - \tfrac{1}{3}G_1\right],\quad G_2 = \Lambda\left[F_1 p - G_1\cos\Theta_1\right]. \tag{3.2.26}$$

These are homogeneous equations, so the condition for a non-trivial solution is that the determinant vanishes, viz:

$$\begin{vmatrix} 1 & 0 & -\Lambda\cos\Theta_1 & \frac{1}{3}\Lambda \\ 0 & 1 & -\Lambda p & -\Lambda\cos\Theta_2 \\ \Lambda\cos\Theta_2 & \frac{1}{3}\Lambda & 1 & 0 \\ -\Lambda p & \Lambda\cos\Theta_1 & 0 & 1 \end{vmatrix} = 0, \qquad (3.2.27)$$

or

$$\left[1-\tfrac{1}{3}\Lambda^2(\cos\Theta_1\cos\Theta_2-\sin\Theta_1\sin\Theta_2\cos\Phi)\right]^2=0. \qquad (3.2.28)$$

It is unfortunate that this condition has turned out to be a perfect square, for the right side is not really 0, but a term of order $(a/R)^2$. Though this is small, it might be negative, in which case the only solutions of (3.2.28) possible would have imaginary R_m, which is physically nonsensical. Thus, the Herzenberg dynamo as it stands may not work.

There are a number of ways out of this difficulty:

(i) The $O\left((a/R)^2\right)$ terms on the right of (3.2.28) can be evaluated explicitly by carrying the expansion to a higher order. Gibson (1967a) carried the expansion to $O\left((a/R)^3\right)$, but <u>again</u> got a perfect square, though a slightly different one. Thus, the Herzenberg dynamo still has not been shown to work.

(ii) The determinant remains a perfect square if the rotors are allowed different angular velocities, different radii, or even onion-like structures. [Herzenberg (1958), Gibson and Roberts (1967), Gibson (1968).] Gibson (1967b) has also considered whether a single rotor can work on its own reflected fields. He has considered cases in which the rotor lies in (and near the edge of) a wedge-shaped stationary conductor surrounded by an insulator, and another case in which the rotor lies in (and near the corner of) a stationary conductor filling the quadrant ($x > 0, y > 0, z > 0$) and surrounded by an insulator. In each case, the condition for dynamo action is a perfect square.

(iii) Instead of considering the exterior of the spheres to be unbounded, it can be cut off at some large distance $M\,(>>R)$ from the spheres, with the exterior being an insulator. Provided the cut-off is not too simple (e.g., a plane), the right side of (3.2.28) will change by $O\left((R/M)^3\right)$, which will swamp the $(a/R)^2$ term if $M << (R^5/a^2)^{1/3}$. Then, under not very restrictive conditions, the $(R/M)^3$ terms are positive, so certain modified Herzenberg dynamos can work. [See Herzenberg (1958) or Gibson (1967a).] Incidentally, to first order, no field enters the insulator.

(iv) Another way out that works in an infinite medium is to consider three rotors instead of two. (Gibson, 1967b). The condition for dynamo action is not a perfect square, in general, so all is well.

(v) Another successful way to avoid a perfect square is to suppose that the conductivity inside the spheres differs from outside. (Gibson, 1967b).

Presuming the difficulty of possible negativity has been overcome, (3.2.28) gives

$$\left(\tfrac{1}{5} R_M \sin\Theta_1 \sin\Theta_2 \sin\Phi\right)^2 = \frac{3}{(\cos\Theta_1\cos\Theta_2 - \sin\Theta_1\sin\Theta_2\cos\Phi)} . \qquad (3.2.29)$$

An R_m can be found to satisfy this if, and only if, the orientations are such that

$$\begin{gathered}\cos\Theta_1\cos\Theta_2 - \sin\Theta_1\sin\Theta_2\cos\Phi > 0\,,\\ \sin\Theta_1\sin\Theta_2\sin\Phi \neq 0\,.\end{gathered} \tag{3.2.30}$$

For example, $\Theta_1 = \pi/4$, $\Theta_2 = \pi/4$, and $\Phi = \pi/2$ satisfies (3.2.30) and allows a (dynamo) solution if

$$R_m = 10\sqrt{6} \approx 24.5\,. \tag{3.2.31}$$

It may be noted that reversing one or the other of the angular velocities of the rotors destroys the regeneration. If both are reversed, the dynamo again works. Thus, roughly half of the above dynamo models are capable of regeneration of a field.

All of the successful proofs of dynamo action have rested on being able to express the required R_m [which is $O(1)$] as the ratio of two small parameters. For example, in the above, the magnetic Reynolds number was the ratio of $(a/R)^3$ to $\eta/a^2\omega$. The proof of the Backus dynamo also rests, essentially, on the introduction of two small parameters: (1) the period of motion $(\underset{\sim}{u} \neq 0)$ divided by the period of stasis $(\underset{\sim}{u} = 0)$ and (2) the diffusion "velocity" $\eta/\mathcal{L}$ divided by the magnitude of the fluid velocity (when it occurs). The proofs of Braginskiĭ, Childress and G. O. Roberts of dynamo action also depend on similar ideas. However, it should not be thought that the small parameters are necessary physically; for example, Lowes and Wilkinson (1967) have demonstrated experimentally that the Herzenberg dynamo can work even if the parameters $(a/R)^3$ and $\eta/a^2\omega$ are not small.

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4. The Braginskiĭ Dynamo

4.1 The basic idea of the Braginskiĭ model: mean field equations

We know that, even in the axisymmetric case, the time it will take a field to decay will tend to infinity as R, the magnetic Reynolds number, tends to infinity, provided length scales are bounded below in a suitable sense. This suggests the idea that for large R this slow rate of decay could be halted by quite slight departures from asymmetry. Moreover, we could, in this limit $R \longrightarrow \infty$, avail ourselves of standard singular perturbation theory. That is, we might divide V into an interior region, in which $\underline{\nabla} = O(1)$, and in which the $R^{-1}\nabla^2 \underline{B}$ and $R^{-1}\partial \underline{B}/\partial t$ terms are small corrections to the remainder, and a boundary layer near the surface, in which $\partial/\partial n >> O(1)$, in which the interior field matches to the field $\hat{\underline{B}}$ in the exterior insulator $\hat{V}$, and in which the flow, $\underline{u}$, adjusts to the no-slip conditions.

We will denote by S the exterior surface of V , assumed axisymmetric, and by S^* the "edge of the boundary layer" on that surface. (In other words, when we specify conditions on S^* , we imply that these are to be met by the interior flow alone on the outer boundary of the fluid.)

First, consider what is the best choice for the primary axisymmetric solution (to which we will presently add the secondary asymmetric terms). If we were to start with an axisymmetric field of the Cowling type (i.e. in meridional planes), the addition of a slight degree of asymmetry would not be likely to destroy the existence of the critical singular curve: it would be more likely merely to impart a slight waviness to it. We do not expect, therefore, that the addition of slight asymmetries would allow such a dynamo to work. This is also true** of axisymmetric fields in which the meridional and azimuthal parts $\underline{B}_M$ and B_φ are comparable in magnitude. We therefore concentrate on the remaining case in which the singular curve can be terminated by an axisymmetric bounding surface, and the simplest of this class is the purely toroidal field. We are led, then, to a situation in which B_φ is large compared with the non-axisymmetric part $\underline{B}'$ of $\underline{B}$, and is also large compared with the meridional part $\underline{B}_M$. This is probably also more realistic geophysically.

First let us neglect $\underline{B}'$ entirely. The induction equation for $\underline{B}$

**See also Braginskiĭ 1964a: discussion preceding (3.24).

in this axisymmetric case may be written in dimensionless form, using the diffusion time scale $\mathcal{L}^2/\eta$,

$$\frac{\partial A_\varphi}{\partial t} + \frac{R}{\varpi}\underline{u}_M\cdot\underline{\nabla}(\varpi A_\varphi) = \Delta A_\varphi, \tag{4.1.1}$$

$$\frac{\partial B_\varphi}{\partial t} + R\varpi\underline{u}_M\cdot\nabla\left(\frac{B_\varphi}{\varpi}\right) = \Delta B_\varphi + R\left[\underline{\nabla}\left(\frac{u_\varphi}{\varpi}\right)\times\underline{\nabla}(\varpi A_\varphi)\right]_\varphi, \tag{4.1.2}$$

where $\Delta \equiv \nabla^2 - \varpi^{-2}$, and $\underline{B}$ has been written as

$$\underline{B} = \underline{B}_M + B_\varphi \underline{1}_\varphi = \underline{\nabla}\times(A_\varphi \underline{1}_\varphi) + B_\varphi \underline{1}_\varphi = \left[-\frac{\partial A_\varphi}{\partial z}, B_\varphi, \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi A_\varphi)\right]. \tag{4.1.3}$$

The forms of (4.1.1) and (4.1.2) are reminiscent of the equations (§ 2.3) through which we eliminated toroidal dynamos. A similar proof (given in full by Braginskiĭ, 1964a) shows that, almost independently of the $\underline{u}_M$ flow, A_φ decays to zero in an $\mathcal{L}^2/\eta$ time scale [i.e. 0(1) in the formalism]. The last term in (4.1.2) then vanishes, which shows that B_φ decays in a similar fashion. Before A_φ has vanished, however, B_φ is regenerated by the azimuthal shear $\underline{\nabla}\zeta$ (provided, of course, u_φ is not a solid body rotation). Here ζ is defined by

$$\zeta = \frac{u_\varphi}{\varpi}.$$

Using u_φ as the characteristic velocity defining R, we have $u_\varphi = O(1)$, and the last term of (4.1.2) in $O(RA_\varphi)$. Thus normalizing to $B_\varphi = O(1)$, we obtain a consistent picture by selecting $A_\varphi = O(R^{-1})$. Similarly, it is convenient to make the $R\underline{u}_M\cdot grad$ terms on the left of the equations of order $\partial/\partial t$. Summarizing, then, we suppose that

$$u_\varphi = O(1),\ \underline{u}_M = O(R^{-1}),\ B_\varphi = O(1),\ \underline{B}_M = O(R^{-1}),\ \partial/\partial t = O(1).$$

and, except in the boundary layers, $\underline{\nabla} = O(1)$.

Now let us introduce small deviations from axisymmetry, $\underline{u}'$ and $\underline{B}'$ into $\underline{u}$ and $\underline{B}$. The prime signifies that the φ-averages of these fields are zero. The full equation for the vector potential is

$$\frac{\partial \underline{A}}{\partial t} = -\underline{\nabla}\,\Phi + R\,\underline{u} \times \underline{B} - \underline{\nabla} \times \underline{B}, \tag{4.14}$$

and the field itself satisfies

$$\frac{\partial \underline{B}}{\partial t} = R\,\underline{\nabla} \times (\underline{u} \times \underline{B}) + \nabla^2 \underline{B}. \tag{4.1.5}$$

On taking in each case the φ-component and averaging over φ, not forgetting that the scalar potential Φ is single-valued, these equations give

$$\frac{\partial \bar{A}_\varphi}{\partial t} + \frac{R}{\varpi}\,\underline{\bar{u}}_M \cdot \underline{\nabla}(\varpi \bar{A}_\varphi) = \Delta \bar{A}_\varphi + R\,\bar{\mathcal{E}}_\varphi, \tag{4.1.6}$$

$$\frac{\partial \bar{B}_\varphi}{\partial t} + R\varpi\,\underline{\bar{u}}_M \cdot \underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right) = \Delta \bar{B}_\varphi + R\left[\underline{\nabla}\zeta \times \underline{\nabla}(\varpi \bar{A}_\varphi)\right]_\varphi + R(\underline{\nabla} \times \underline{\bar{\mathcal{E}}})_\varphi, \tag{4.1.7}$$

where $\underline{\bar{\mathcal{E}}}$ is the average e.m.f. created by the non-symmetric fields:

$$\underline{\bar{\mathcal{E}}} = \overline{\underline{u}' \times \underline{B}'}, \tag{4.1.8}$$

and where an overbar denotes an average over φ. We now see that, in order that the $\underline{\bar{\mathcal{E}}}$ terms should be of the same order as the remaining terms in (4.1.7) and (4.1.8), $(\underline{\nabla} \times \underline{\bar{\mathcal{E}}})_\varphi$ should be $O(R^{-1})$, as would be the case if $\underline{\bar{\mathcal{E}}}_M = O(R^{-1})$, but that $\bar{\mathcal{E}}_\varphi$ should be no larger than $O(R^{-2})$. This is indeed the case if we suppose that $\underline{u}' = O(R^{-1/2})$, as will now be shown.

By (4.1.4) and (4.1.5), the asymmetric part $\underline{B}'$ of $\underline{B}$ created by $\underline{u}'$ obeys

$$\frac{\partial \underline{A}'}{\partial t} = \underline{\nabla}\Phi' + R\left[\underline{\bar{u}}\times\underline{B}' + \underline{u}'\times\underline{\bar{B}} + \underline{G}'\right] - \underline{\nabla}\times\underline{B}', \tag{4.1.9}$$

$$\frac{\partial \underline{B}'}{\partial t} = R\underline{\nabla}\times\left[\underline{\bar{u}}\times\underline{B}' + \underline{u}'\times\underline{\bar{B}} + \underline{G}'\right] + \nabla^2\underline{B}', \tag{4.1.10}$$

where $\underline{G}'$ is the asymmetric e.m.f. created by the non-symmetric fields:

$$\underline{G}' = \underline{u}'\times\underline{B}' - \underline{\bar{\mathcal{E}}} = \underline{u}'\times\underline{B}' - \overline{\underline{u}'\times\underline{B}'}. \tag{4.1.11}$$

We assume that, as for the axisymmetric fields, $\partial/\partial t$ in (4.1.9) and (4.1.10) is O(1). We note that in a coordinate system rotating rapidly relative to the frame we have taken, $\underline{u}'$ will be strongly dependent on time: $\underline{u}' = \underline{u}'(\varpi, \varphi - \varphi_0(t), z, t)$, where $\varphi_0(t)$ is the relative angular displacement. Then "$d\underline{u}'/dt$" (meaning $-\dot{\varphi}_0\,\partial\underline{u}'/\partial\varphi + \partial\underline{u}'/\partial t$) would be large but $\partial\underline{u}'/\partial t$ could be small, as before. In this case we could still work out a scheme as below, using $\underline{B}'(\varpi, \varphi - \varphi_0(t), z, t)$ in place of $\underline{B}'$, and using $u_\varphi - \varpi\dot{\varphi}_0$ in place of u_φ. We shall return to this later (§ 4.4).

It is clear, by (4.1.9), that to leading order,

$$\bar{u}_\varphi \underline{1}_\varphi \times \underline{B}' + \underline{u}'\times\bar{B}_\varphi \underline{1}_\varphi = R^{-1}\underline{\nabla}\Phi'. \tag{4.1.12}$$

The left-hand side has no φ-component; therefore neither does the right. Thus $\Phi' = 0$ and

$$\underline{1}_\varphi \times \left(\bar{u}_\varphi \underline{B}' - \bar{B}_\varphi \underline{u}'\right) = 0, \tag{4.1.13}$$

or

$$\underline{B}'_M = \bar{\chi}\,\underline{u}'_M\,, \tag{4.1.14}$$

where

$$\bar{\chi} = \frac{\bar{B}_\varphi}{\bar{u}_\varphi}. \tag{4.1.15}$$

From $\underline{B}'_M$, we may determine B'_φ by using the equation $\underline{\nabla}\cdot\underline{B}'=0$, i.e.,

or

$$B'_\varphi = -\varpi\,\underline{\nabla}\cdot\hat{\underline{B}}_M. \tag{4.1.16}$$

Here we have introduced the hat notation. This is the inverse of the $\partial_1/\partial\varphi$ operation, i.e., differentiation holding the unit vectors $\underline{1}_\varpi$ and $\underline{1}_\varphi$ fixed. From a vector (or scalar or tensor) field $\underline{K}'$, we define $\hat{\underline{K}}$ by

$$\frac{\partial_1}{\partial\varphi}\hat{\underline{K}} = \underline{K}', \quad \bar{\hat{\underline{K}}} = 0. \tag{4.1.17}$$

The second of these excludes axisymmetric functions of integration. For example, if $K' = k(\varpi, z)\cos m\varphi$, then $\hat{K} = m^{-1}k(\varpi, z)\sin m\varphi$ and not $m^{-1}k(\varpi, z)\sin m\varphi + \bar{h}(\varpi, z)$. We observe that, in the first approximation in which (4.1.14) holds,

$$\begin{aligned} B'_\varphi &= -\varpi\bar{\chi}\,\underline{\nabla}\cdot\hat{\underline{u}}_M - \varpi\,\hat{\underline{u}}_M\cdot\underline{\nabla}\bar{\chi} \\ &= \bar{\chi}\,u'_\varphi - \varpi\,\hat{\underline{u}}_M\cdot\underline{\nabla}\bar{\chi}. \end{aligned} \tag{4.1.18}$$

We expect that on the axis, where both $\bar{u}_\varphi$ and $\bar{B}_\varphi$ vanish,

$\bar{\chi} \longrightarrow$ limit $< \infty$. Also difficulties may arise on S^*. For example, for steady flows at large magnetic Prandtl numbers, it happens, in a wide variety of circumstances (cf. e.g. Roberts, 1967, §6.1), that the interior flow satisfies $\langle \underline{n} \times \underline{u} \rangle^* = 0, \langle \underline{n} \times \underline{B} \rangle^* \neq 0$. (The star emphasizes that the leap is to be taken <u>across</u> the boundary layer.) If applicable here, it would mean that $\bar{\chi} = \infty$ on S^*. It may also be noted that certain types of dynamos necessarily include surfaces on which $\bar{u}_\varphi$ vanishes but $\bar{B}_\varphi$ does not. For example, a Backus dynamo model suggests that the flow $\underline{t}_2 + \underline{s}_1^{1c} + \underline{t}_1^{1c}$ is possibly regenerative. From an $\underline{S}$ field, a $\underline{T}_1$ field is produced, whence $\bar{B}_\varphi$ is proportional to $\sin\theta$. But $\bar{u}_\varphi = \underline{t}_{2\varphi}$ is proportional to $\sin\theta\cos\theta$, so $\bar{\chi} \propto \sec\theta$, and is infinite everywhere on the equatorial plane. Braginskiĭ (1964a, §5) has given a method by which this type of difficulty may be overcome. It is clear that $\bar{B}_\varphi$ (and therefore $\bar{\chi}$) and $\underline{n} \cdot \underline{u}'$ vanish on S. If, as Braginskiĭ claims we should, we apply the same conditions on S^* then, in the first approximation (4.1.14), $\underline{B}_M' = 0$ on S^*, and by (4.1.18), $B_\varphi' = 0$ on S^*, i.e. $\underline{B}' = 0$ on S^* in the first approximation. This would also imply that $\underline{n} \cdot \underline{B}' = 0$ on S in the first approximation.

It is clear from (4.1.14) that if $\underline{u}' = O(R^{-1/2})$ then so is $\underline{B}'$. Moreover, using also (4.1.16),

$$\begin{aligned} \underline{u}' \times \underline{B}' &= \underline{u}_M' \times B_\varphi' \underline{1}_\varphi + u_\varphi' \underline{1}_\varphi \times \underline{B}_M' \\ &= \left[-B_\varphi' + \bar{\chi} u_\varphi'\right](\underline{1}_\varphi \times \underline{u}_M') \\ &= \left[\varpi \underline{\nabla} \cdot (\bar{\chi} \hat{\underline{u}}_M) - \varpi \bar{\chi} \underline{\nabla} \cdot \hat{\underline{u}}_M\right](\underline{1}_\varphi \times \underline{u}_M') \\ &= \varpi(\hat{\underline{u}}_M \cdot \underline{\nabla} \bar{\chi})(\underline{1}_\varphi \times \underline{u}_M'). \end{aligned} \tag{4.1.19}$$

It is clear that $\overline{\mathcal{E}}_\varphi$ vanishes to order R^{-1}, which is consistent with our hope that it is $O(R^{-2})$.

We may simplify (4.1.19) further. Since

$$\overline{A'\hat{B}} = -\overline{\hat{A}B'}, \qquad (4.1.20)$$

we have, for any arbitrary axisymmetric vector $\underline{a}$ and any vector $\underline{F}'$ with $\overline{\underline{F}'} = 0$, the result

$$\overline{(\underline{a}\cdot\underline{F}')\hat{\underline{F}}} = -\overline{(\underline{a}\cdot\hat{\underline{F}})\cdot\underline{F}'} = \tfrac{1}{2}\overline{\left[(\underline{a}\cdot\underline{F}')\hat{\underline{F}} - (\underline{a}\cdot\hat{\underline{F}})\cdot\underline{F}'\right]}$$
$$= \tfrac{1}{2}\overline{(\underline{F}'\times\hat{\underline{F}})}\times\underline{a}\,.$$

Thus, taking $\underline{F}' = \underline{u}'_M$ and $\underline{a} = \varpi\underline{\nabla}\overline{\chi}$, we have

$$\overline{\left[\varpi(\underline{u}'_M\cdot\underline{\nabla}\overline{\chi})\underline{u}'_M\right]} = \tfrac{1}{2}\varpi\overline{(\hat{\underline{u}}_M\times\underline{u}'_M)}\times\underline{\nabla}\overline{\chi},$$

i.e. (4.1.19) and (4.1.8) give

$$\overline{\underline{\mathcal{E}}} = \tfrac{1}{2}\varpi\overline{(\underline{u}'_M\times\hat{\underline{u}}_M)}_\varphi\,\underline{\nabla}\overline{\chi}, \qquad (4.1.22)$$

or, as Braginskiĭ prefers to write it,

$$\overline{\underline{\mathcal{E}}} = \overline{W}\,\overline{u}_\varphi^2\,\underline{\nabla}\overline{\chi}, \qquad (4.1.23)$$

where

$$\overline{W} = \tfrac{1}{2}\varpi\overline{(\underline{v}'_M\times\hat{\underline{v}}_M)}_\varphi, \qquad (4.1.24)$$

and

$$\underline{v}' = \underline{u}'/\overline{u}_\varphi\,. \qquad (4.1.25)$$

Clearly

$$\underline{\nabla}\times\underline{\bar{\mathcal{E}}} = \underline{\nabla}(\overline{w}\,\bar{u}_\varphi^2)\times\underline{\nabla}\bar{\chi}$$
$$= \left[\overline{w}\varpi\bar{u}_\varphi\,\underline{\nabla}\left(\frac{\bar{u}_\varphi}{\varpi}\right)+\frac{\bar{u}_\varphi}{\varpi}\underline{\nabla}(\overline{w}\varpi\bar{u}_\varphi)\right]\times\underline{\nabla}\bar{\chi}$$
$$= \underline{\nabla}\left(\frac{\bar{u}_\varphi}{\varpi}\right)\times(\overline{w}\varpi\bar{u}_\varphi\underline{\nabla}\bar{\chi})+\underline{\nabla}(\overline{w}\varpi\bar{u}_\varphi)\times\left(\frac{\bar{u}_\varphi}{\varpi}\underline{\nabla}\bar{\chi}\right)$$
$$= \underline{\nabla}\left(\frac{\bar{u}_\varphi}{\varpi}\right)\times\underline{\nabla}(\overline{w}\varpi\bar{u}_\varphi\bar{\chi})+\underline{\nabla}(\overline{w}\varpi\bar{u}_\varphi)\times\underline{\nabla}\left(\frac{\bar{u}_\varphi}{\varpi}\bar{\chi}\right)$$
$$= \underline{\nabla}\left(\frac{\bar{u}_\varphi}{\varpi}\right)\times\underline{\nabla}(\overline{w}\varpi\bar{B}_\varphi)+\underline{\nabla}(\overline{w}\varpi\bar{u}_\varphi)\times\underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right). \qquad (4.1.26)$$

Now substituting into (4.1.7) gives

$$\frac{\partial \bar{B}_\varphi}{\partial t}+R\varpi\left[\underline{\bar{u}}_M+\mathrm{curl}\,(\overline{w}\bar{u}_\varphi\underline{1}_\varphi)\right]\cdot\underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right)=$$
$$= R\left[\underline{\nabla}\zeta\times\underline{\nabla}\left\{\varpi\,(\bar{A}_\varphi+\overline{w}\,\bar{B}_\varphi)\right\}\right]_\varphi+\Delta\bar{B}_\varphi. \qquad (4.1.27)$$

The remarkable thing about this result is that, if we define new "effective quantities"

$$\bar{A}_e=\bar{A}_\varphi+\overline{w}\,\bar{B}_\varphi,\quad \underline{\bar{u}}_{eM}=\underline{\bar{u}}_M+\mathrm{curl}(\overline{w}\,\bar{u}_\varphi\underline{1}_\varphi), \qquad (4.1.28)$$

to replace $\bar{A}_\varphi$ and $\underline{\bar{u}}_M$, then

$$\frac{\partial\bar{B}_\varphi}{\partial t}+R\varpi\,\underline{\bar{u}}_{eM}\cdot\underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right)=\Delta\bar{B}_\varphi+R\left[\underline{\nabla}\zeta\times\underline{\nabla}(\varpi\bar{A}_e)\right]_\varphi , \qquad (4.1.29)$$

which is exactly the same as we would have in the axisymmetric case. The differences between the original and effective quantities are of the same order as either $(\mathrm{viz.}\ R^{-1})$. Incidentally, there is more symmetry between field and velocity in (4.1.28) than one would perhaps at first sight notice:

$$\underline{\bar{B}}_{eM}=\underline{\bar{B}}_M+\mathrm{curl}(\overline{w}\,\bar{B}_\varphi\underline{1}_\varphi),\ \underline{\bar{u}}_{eM}=\underline{\bar{u}}_M+\mathrm{curl}(\overline{w}\,\bar{u}_\varphi\underline{1}_\varphi). \qquad (4.1.30)$$

We have yet to show that $\overline{\mathcal{E}}_\varphi = O(R^{-2})$ as required for a self-consistent analysis; this will be done in the next section.

4.2 The expansion in the asymmetries

So far, we have expanded $\underline{u}$ in the form

$$\underline{u} = \bar{u}_\varphi \underline{1}_\varphi + \underline{u}' + \underline{\bar{u}}_M, \tag{4.2.1}$$

where $\underline{u}' = O(R^{-\frac{1}{2}}\bar{u}_\varphi)$, $\underline{\bar{u}}_M = O(R^{-1}\bar{u}_\varphi)$ and $R = u_0 \mathcal{L}/\eta$. On substituting (4.2.1) into the induction equation (4.1.5), we will obtain by iteration (see §4.3) a series expansion for $\underline{B}$ of the form

$$\underline{B} = \bar{B}_\varphi \underline{1}_\varphi + \underline{B}_1' + \underline{\bar{B}}_M + \underline{B}_2' + \underline{B}_3' + \ldots, \tag{4.2.2}$$

where $\underline{B}_n'$ is asymmetric$(\overline{\underline{B}_n'} = 0)$ and of order n in the asymmetries and is therefore of order $R^{-\frac{1}{2}n}$. If we cared to say that $\bar{u}_\varphi$ is independent of R, $\underline{u}'$ is proportional to $R^{-\frac{1}{2}}$, and $\underline{\bar{u}}_M$ is proportional to R^{-1}, the term $\underline{B}_n'$ would be proportional to $R^{-\frac{1}{2}n}$, and the axisymmetric terms $\bar{B}_\varphi$ and $\underline{\bar{B}}_M$ of (4.2.2) would be infinite power series in powers of $R^{-\frac{1}{2}}$ commencing with R^0 for $\bar{B}_\varphi$ and R^{-1} for $\underline{\bar{B}}_M$. Instead of supposing this proportionality of the $\underline{u}$ fields, which might be inconvenient when we later discuss the equations of motion, we may imagine that $\underline{u}'$ is multiplied by a factor ξ and $\underline{u}_M$ by a factor ξ^2, later to be set equal to 1; at the same time, R^{-1} in the induction equation is replaced by $\xi^2 R^{-1}$. The power series we seek then are in powers of ξ and, for example, although $\underline{B}_n'$ is of order $R^{-\frac{1}{2}n}$ it may also contain terms of smaller order in R. It is clearly a conceptual advantage to expand $\underline{\bar{\mathcal{E}}}$ and $\underline{G}'$ [see (4.1.8) and (4.1.11)] in powers of ξ, commencing with

$\overline{\underline{\mathcal{E}}}_2$ and $\underline{G}'_2$.

By (4.1.10) we have, taking meridional parts and using (4.1.14) for a first approximation,

$$\underline{B}'_M = \overline{X}\,\underline{u}'_M + \frac{\varpi}{\overline{u}_\varphi}\left\{\underline{\nabla}\times\left(\underline{\hat{G}} + \underline{\hat{u}}\times\underline{\overline{B}}_M\right) + \frac{1}{R}\nabla^2\underline{B}' + \underline{\hat{B}}_M\cdot\underline{\nabla}\,\underline{\overline{u}}_M - \frac{1}{R}\frac{d}{dt}\underline{\hat{B}}\right\}_M, \tag{4.2.3}$$

where d/dt is a "material" derivative based on the axisymmetric meridional flow: in these units

$$\frac{d}{dt} = \frac{\partial}{\partial t} + R\,\underline{\overline{u}}_M\cdot\underline{\nabla}\,. \tag{4.2.4}$$

The expréssion (4.2.3) can be simplified if it is recalled that, for any asymmetric vector, such as $\underline{G}'$,

$$\left(\underline{\nabla}\times\underline{\hat{G}}\right)_M = \frac{1}{\varpi}\underline{1}_\varphi\times\left[\underline{G}'_M - \underline{\nabla}\left(\varpi\hat{G}_\varphi\right)\right]. \tag{4.2.5}$$

By (4.2.3) and (4.1.8) we have

$$\begin{aligned}\overline{\mathcal{E}}_\varphi &= \left[\overline{\underline{u}'_M\times\underline{B}'_M}\right]_\varphi \\ &= \frac{1}{\overline{u}_\varphi}\left\{\underline{\nabla}\cdot\left[\overline{\varpi\left(\underline{u}'_M\times\underline{B}'_M\right)_\varphi\underline{\hat{u}}_M}\right] + \frac{1}{R}\left[\overline{\varpi\,\underline{u}'_M\times\left(\nabla^2\underline{\hat{B}}\right)_M}\right]_\varphi + \right. \\ &\quad \left. + \underline{\nabla}\cdot\left[\overline{\varpi\left(\underline{u}'_M\times\underline{\overline{B}}_M\right)_\varphi\underline{\hat{u}}_M}\right] + \varpi\left[\overline{\underline{u}'_M\times\left\{\left(\underline{\hat{B}}_M\cdot\underline{\nabla}\right)\underline{\overline{u}}_M - \frac{1}{R}\frac{d\underline{\hat{B}}_M}{dt}\right\}}\right]_\varphi\right\}.\end{aligned} \tag{4.2.6}$$

If we now introduce the expansions outlined above, we clearly have

$$\underline{G}'_n = \underline{u}'\times\underline{B}'_{n-1} - \overline{\underline{\mathcal{E}}}_n, \qquad (n\geq 2), \tag{4.2.7}$$

$$\overline{\underline{\mathcal{E}}}_n = \overline{\underline{u}'\times\underline{B}'_{n-1}}, \qquad (n\geq 2). \tag{4.2.8}$$

If we equate powers of ξ and ξ^2 in (4.2.3), we obtain

$$\underline{B}'_{M1} = \overline{X}\,\underline{u}'_M, \qquad \overline{X} = \frac{\overline{B}_\varphi}{\overline{u}_\varphi}, \tag{4.2.9}$$

as before, and

$$\underline{B}'_{M2} = \frac{\varpi}{\bar{u}_\varphi}\left[\nabla \times \hat{\underline{G}}_2\right]_M = \frac{1}{\bar{u}_\varphi}\, 1_\varphi \times \left[\underline{G}'_{M2} - \nabla\cdot(\varpi\, \hat{G}_{\varphi 2})\right] \qquad (4.2.10)$$

$$= -\varpi\, \underline{u}'_M\left(\frac{\hat{\underline{u}}_M}{\bar{u}_\varphi}\cdot \nabla \bar{\chi}\right) - \frac{1}{\bar{u}_\varphi} 1_\varphi \times \bar{\underline{\mathcal{E}}}_2 .$$

(Since $\underline{u}' \times \underline{B}'_1$ has no φ-component, $\hat{G}_{\varphi 2} = 0$.) It is clear that $\underline{n}\cdot\underline{u}' = 0$ on S . If, as Braginskiĭ claims we should, we apply the same condition on S^* , the first term of (4.2.10) has no normal component on S^* . Moreover, since $\underline{u}'_M$ and $\hat{\underline{u}}_M$ are then parallel on S^* , (4.1.24) shows that $\bar{w} = 0$ and (4.1.23) shows then that $\bar{\underline{\mathcal{E}}}_2 = 0$ on S^*. Then, like B'_{M1} , the field $\underline{B}'_{M2}$ has a zero normal component on S^*. Nevertheless, even assuming, following Braginskiĭ, that $\bar{B}_\varphi$ and $\underline{n}\cdot\underline{u}'$ vanish on S^* , a third order $(R^{-3/2})$ field will pass into $\hat{V}$. This is because (Tough, 1967)

$$B'_{M3} = \frac{\varpi}{\bar{u}_\varphi}\left[\bar{\underline{B}}_M\cdot\nabla\hat{\underline{u}} - \hat{\underline{u}}\cdot\nabla\bar{\underline{B}}_M + \bar{\chi}\,\hat{\underline{u}}_M\cdot\nabla\hat{\underline{u}}_M + \nabla\times\hat{\underline{G}}_3 + \frac{1}{R}\nabla^2\hat{\underline{B}}_1 - \frac{1}{R}\frac{d}{dt}(\bar{\chi}\,\hat{\underline{u}})\right]_M , \qquad (4.2.11)$$

and this does have a normal component.

Returning to the evaluation of $\bar{\mathcal{E}}_\varphi$, we see immediately from (4.2.8-10) that

$$\bar{\mathcal{E}}_{\varphi 2} = \overline{\left[\underline{u}'_M \times \underline{B}'_{M1}\right]_\varphi} = 0 , \qquad (4.2.12)$$

$$\bar{\mathcal{E}}_{\varphi 3} = \overline{\left[\underline{u}'_M \times \underline{B}'_{M2}\right]_\varphi} = 0 , \qquad (4.2.13)$$

since, for the latter, the first term of (4.2.10) is parallel to $\underline{u}'_M$ and therefore makes no contribution, while the second term of (4.2.10) is independent of φ .

Thus, as we had hoped, $\overline{\mathcal{E}}_\varphi$ is $O(R^{-2})$, at most. To show that, indeed, $\overline{\mathcal{E}}_{\varphi 4}$ is non-zero, we make use of (4.2.6). In evaluating the first term of this expression we must use the approximation $\underline{B}_1' + \underline{B}_2'$ for $\underline{B}'$; for the remainder it is adequate to take $\underline{B}_1'$ alone. The first, third, and fourth terms on the right of (4.2.6) may be written, respectively, as

$$\left.\begin{aligned} &-\frac{\overline{w}\,\overline{u}_\varphi^2}{\varpi}\frac{\partial\overline{\chi}}{\partial z} - \frac{1}{\overline{u}_\varphi}\frac{\partial\overline{\chi}}{\partial z}\frac{\partial}{\partial\varpi}\left(\overline{w}^2\overline{u}_\varphi^3\right) + \frac{1}{\overline{u}_\varphi}\frac{\partial\overline{\chi}}{\partial\varpi}\frac{\partial}{\partial z}\left(\overline{w}^2\overline{u}_\varphi^3\right), \\ &-\frac{\partial\overline{A}_\varphi}{\partial z}\left\{\overline{u}_\varphi\frac{\partial\overline{w}}{\partial\varpi}+2\overline{w}\frac{\partial\overline{u}_\varphi}{\partial\varpi}\right\} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\overline{A}_\varphi\right)\left\{\overline{u}_\varphi\frac{\partial\overline{w}}{\partial z}+2\overline{w}\frac{\partial\overline{u}_\varphi}{\partial z}\right\}, \\ &-\frac{1}{R}\frac{\partial}{\partial t}\left(\overline{w}\,\overline{B}_\varphi\right) - \overline{\underline{u}}_M\cdot\underline{\nabla}\left(\overline{w}\,\overline{B}_\varphi\right) - \frac{\overline{w}}{R}\frac{d\overline{B}_\varphi}{dt}. \end{aligned}\right\} \tag{4.2.14}$$

By (4.1.29) we have

$$\begin{aligned} \frac{1}{R}\frac{d\overline{B}_\varphi}{dt} &= \frac{1}{R}\Delta\overline{B}_\varphi + \left[\underline{\nabla}\left(\frac{\overline{u}_\varphi}{\varpi}\right)\times\underline{\nabla}\left\{\varpi\left(\overline{A}_\varphi+\overline{w}\,\overline{B}_\varphi\right)\right\}\right]_\varphi + \\ &\quad + \frac{\overline{B}_\varphi}{\varpi}\left[\overline{u}_\varpi - \frac{\partial}{\partial z}\left(\overline{w}\,\overline{u}_\varphi\right)\right] - \left[\underline{\nabla}\times\left(\overline{w}\,\overline{u}_\varphi\underline{1}_\varphi\right)\right]\cdot\underline{\nabla}\,\overline{B}_\varphi, \end{aligned} \tag{4.2.15}$$

which enables us to simplify the last line of (4.2.14).

Collecting those terms of (4.2.14) which involve $\overline{\underline{u}}_M$, those which involve $\overline{A}_\varphi$, and the remainder gives, respectively,

$$\left.\begin{aligned} &-\frac{1}{\varpi}\overline{\underline{u}}_M\cdot\underline{\nabla}\left(\varpi\,\overline{w}\,\overline{B}_\varphi\right), \qquad -\frac{1}{\varpi}\left[\underline{\nabla}\times\left(\overline{w}\,\overline{u}_\varphi\underline{1}_\varphi\right)\right]\cdot\underline{\nabla}\left(\varpi\overline{A}_\varphi\right), \\ &-\frac{1}{R}\frac{\partial}{\partial t}\left(\overline{w}\,\overline{B}_\varphi\right) - \frac{1}{\varpi}\left[\underline{\nabla}\times\left(\overline{w}\,\overline{u}_\varphi\underline{1}_\varphi\right)\right]\cdot\underline{\nabla}\left(\varpi\overline{w}\,\overline{B}_\varphi\right). \end{aligned}\right\} \tag{4.2.16}$$

Finally, after substituting from (4.2.9) for $\underline{B}_{M1}'$, the second term of (4.2.6) becomes

$$\overline{\mathcal{E}}_{\varphi 4} = \frac{1}{R}\Gamma_2\overline{B}_\varphi, \tag{4.2.17}$$

$$\Gamma_2 = \left[\overline{\frac{1}{\varpi \bar{u}_\varphi^2} \left(\underline{u}'_M \times \hat{\underline{u}}_M + \underline{u}'_M \times \frac{\partial_1 \bar{\underline{u}}_{M'}}{\partial \varphi} \right)_\varphi + 2 \nabla_M \left(\frac{r u'_r}{\bar{u}_\varphi} \right) \cdot \nabla_M \left(\frac{\hat{u}_z}{\bar{u}_\varphi} \right)} \right] , \qquad (4.2.18)$$

where (r, θ, φ) are spherical polar coordinates. Thus, we finally obtain the following dynamo equations [note the definitions (4.1.28), and also (4.1.29) and (4.1.6)]:

$$\frac{\partial A_e}{\partial t} + \frac{R}{\varpi} \bar{\underline{u}}_{eM} \cdot \underline{\nabla}(\varpi \bar{A}_e) = \Delta \bar{A}_e + \Gamma \bar{B}_\varphi , \qquad (4.2.19)$$

$$\frac{\partial \bar{B}_\varphi}{\partial t} + R \varpi \bar{\underline{u}}_{eM} \cdot \underline{\nabla} \left(\frac{\bar{B}_\varphi}{\varpi} \right) = \Delta \bar{B}_\varphi + R \left[\underline{\nabla} \zeta \times \underline{\nabla} (\varpi \bar{A}_e) \right]_\varphi , \qquad (4.2.20)$$

where Γ is given, in the present approximation, by (4.2.18).

The simplification resulting in (4.2.19) is absolutely remarkable and, moreover, as Tough (1967) has shown, persists in the next approximation. Tough has shown that, provided we redefine A_e and $\underline{u}_{eM}$ by

$$\bar{A}_e = \bar{A}_e + \varpi \bar{B}_\varphi + \frac{\bar{B}_\varphi}{\bar{u}_\varphi^2} \underline{\nabla} \cdot (\bar{u}_\varphi^2 \underline{\bar{W}}) - \bar{u}_\varphi \underline{\bar{W}} \cdot \underline{\nabla} \left(\frac{\bar{B}_\varphi}{\bar{u}_\varphi} \right) , \qquad (4.2.21)$$

$$\underline{u}_{eM} = \bar{\underline{u}}_M + \underline{\nabla} \times \left[\bar{w} \bar{u}_\varphi \underline{1}_\varphi + \frac{\underline{1}_\varphi}{\bar{u}_\varphi} \underline{\nabla} \cdot (\bar{u}_\varphi^2 \underline{\bar{W}}) \right] , \qquad (4.2.22)$$

where

$$\left. \begin{aligned} \bar{w} &= \tfrac{1}{2} \varpi \overline{\left[\underline{u}'_M \times \hat{\underline{u}}_M \right]_\varphi} , \\ \underline{\bar{W}} &= \tfrac{1}{3} \varpi^3 \left[\hat{\underline{u}}_M (\underline{u}'_M \times \hat{\underline{u}}_M)_\varphi \right] , \end{aligned} \right\} \qquad (4.2.23)$$

then we can still write the induction equation in the form (4.2.19) and (4.2.20), provided $\Gamma = \Gamma_2 + \Gamma_3$, where

$$\Gamma_3 = \nabla^2\left(\frac{1}{\bar{u}_\varphi}\underline{\bar{W}}\cdot\underline{\nabla}\bar{u}_\varphi\right) - \underline{\bar{W}}\cdot\nabla^2\left(\frac{1}{\bar{u}_\varphi}\underline{\nabla}\bar{u}_\varphi\right) - \frac{1}{\bar{u}_\varphi}\underline{\nabla}\bar{u}_\varphi\cdot\nabla^2\underline{\bar{W}} -$$
$$- \frac{6\varpi}{\bar{u}_\varphi}\frac{\partial}{\partial\varpi}\left(\frac{\underline{\bar{W}}}{\varpi^2}\right)\cdot\underline{\nabla}\bar{u}_\varphi - \frac{2\varpi}{\bar{u}_\varphi}\frac{\partial\bar{u}_\varphi}{\partial\varpi}\underline{\nabla}\cdot\left(\frac{\underline{\bar{W}}}{\varpi^2}\right) - 4\underline{\nabla}\cdot\left\{\varpi\frac{\partial}{\partial\varpi}\left(\frac{\underline{\bar{W}}}{\varpi^2}\right)\right\} +$$
$$+ \frac{2}{\bar{u}_\varphi\varpi^3}\frac{\partial}{\partial\varpi}\left(\varpi\bar{u}_\varphi\bar{W}_\varpi\right) + \frac{2}{\bar{u}_\varphi^2}\underline{\nabla}\cdot\left(\varpi^2\bar{u}_\varphi^2\underline{\bar{C}}\right), \qquad (4.2.24)$$

where

$$\bar{C}_\varpi = \overline{\left[u'_\varpi\underline{\nabla}\hat{u}_\varpi\cdot\underline{\nabla}\hat{u}_z + \hat{u}_\varpi\underline{\nabla}u'_\varpi\cdot\underline{\nabla}\hat{u}_z + \hat{u}_z\underline{\nabla}u'_\varpi\cdot\underline{\nabla}\hat{u}_\varpi + \right.}$$
$$\overline{\left. + \frac{1}{\varpi}u'_z\hat{u}_\varpi\frac{\partial\hat{u}_\varpi}{\partial\varpi} + \frac{1}{\varpi}u'_z\hat{u}_\varpi\frac{\partial\hat{u}_z}{\partial z}\right]}, \qquad (4.2.25_\varpi)$$

$$\bar{C}_z = -\overline{\left[\hat{u}_\varpi\underline{\nabla}u'_z\cdot\underline{\nabla}\hat{u}_z + \hat{u}_z\underline{\nabla}\hat{u}_\varpi\cdot\underline{\nabla}u'_z + u'_z\underline{\nabla}\hat{u}_\varpi\cdot\underline{\nabla}\hat{u}_z - \right.}$$
$$\overline{\left. - \frac{1}{\varpi}u'_z\hat{u}_z\frac{\partial\hat{u}_\varpi}{\partial\varpi} - \frac{1}{\varpi}u'_z\hat{u}_z\frac{\partial\hat{u}_z}{\partial z}\right]}. \qquad (4.2.25_z)$$

Suppose $\underline{u}'_M$ is expanded in the form

$$\underline{u}'_M = \bar{u}_\varphi\sum_{m=1}^{\infty}\left\{\underline{c}_m\cos m\varphi + \underline{s}_m\sin m\varphi\right\}. \qquad (4.2.26)$$

Then, by (4.2.18)

$$\Gamma_2 = \sum_{m=1}^{\infty}\frac{1}{m}\left\{\frac{1}{\varpi}(1-m^2)\left[\underline{s}_m\times\underline{c}_m\right]_\varphi + \underline{\nabla}(r s_{mr})\cdot\underline{\nabla}c_{mz} - \underline{\nabla}(r c_{mr})\cdot\underline{\nabla}s_{mz}\right\}, \qquad (4.2.27)$$

where s_{mr} is the r-component of $\underline{s}_m$, etc. It is noteworthy that, if only one of $\cos$ or $\sin$ appears for each m in (4.2.26), Γ_2 vanishes. This shows that creating a $\underline{u}'$ by tilting slightly an axisymmetric configuration won't work; this is not surprising in view of Cowling's theorem. Further, it shows that the Bullard-Gellman dynamo cannot function in the $R \to \infty$ limit (or more precisely in their notation, in the limit

$\epsilon \rightarrow \infty$, $R \rightarrow \infty$ with $\epsilon/R^{1/2} \rightarrow$ finite limit). One can show, from (4.2.24) and (4.2.25), that their dynamo also has a zero Γ_3, i.e., it fails also in the second approximation for large R. We may also observe that, by (4.2.18), Γ_2 vanishes if $\underline{u}'$ is toroidal, which agrees with the result of § 2.4.

Equations (4.2.19) and (4.2.20) were anticipated by Parker (1955) in a remarkable, largely qualitative, study. Although Parker did not recognize the need for effective quantities, nor give an explicit form for Γ in terms of $\underline{u}'$, he showed heuristically (see 4.5 below), that equations (4.2.19) and (4.2.20) might apply to a statistical geodynamo, to a good approximation.

These notes submitted by

Michael E. McIntyre

4A. Further Developments of Braginskiĭ's Theory, and Applications of It

4.3 Electromagnetic Boundary Layers

We have already observed that Braginskiĭ's method is a singular perturbation scheme. In essence we have written

$$\underline{u} = \sum_{n=0}^{\infty} \underline{u}_n R^{-\frac{1}{2}n}, \quad \underline{A} = \sum_{n=0}^{\infty} \underline{A}_n R^{-\frac{1}{2}n}, \quad \underline{B} = \sum_{n=0}^{\infty} \underline{B}_n R^{-\frac{1}{2}n}, \text{ etc.,} \tag{4.3.1}$$

in the interior region, and have chosen $\underline{u}_0$ to be in the ϕ-direction. We have substituted into (4.1.4), collected terms in $R^{-\frac{1}{2}n}$, and obtained

$$\underline{u}_0 \times \underline{B}_n = -\sum_{i=1}^{n} \underline{u}_i \times \underline{B}_{n-i} + \left[\operatorname{curl} \underline{B} + \frac{\partial \underline{A}}{\partial t} + \operatorname{grad} \Phi\right]_{n-2}. \tag{4.3.2}$$

The first of these equations was satisfied by taking $\underline{B}_0$ parallel to $\underline{u}_0$, i.e, in the φ-direction. And generally a knowledge of $\underline{B}_0$ to $\underline{B}_{n-1}$ determined the right of (4.3.2), and hence** $\underline{B}_n$. In other words, we have solved, and successively refined, the field in the "mainstream" or "interior region" far from S. We should supplement this by an analysis of the boundary layers on S. If we confine ourselves to the kinematic dynamo problem, $\underline{u}$ is specified, and does not alter as the limit $R \longrightarrow \infty$ is taken. In other words, we may regard $\underline{u}$ and $\nabla\underline{u}$ as being $O(1)$ in the induction equation (4.1.5). Thus, we must have $R^{-1}\nabla^2\underline{B} = O(B)$ in the boundary layer, whence $\partial/\partial n = O(R^{-1/2})$, where $\underline{n}$ is the normal to S. The boundary layer is, then, an electromagnetic skin of thickness $O(R^{-1/2})$ over S. Actually, the layer is associated with the asymmetric components of field only, since, by the familiar arguments of § 3, the axisymmetric fields do not change as we follow the motion in the φ-direction. Thus $\overline{\underline{B}}$ must satisfy on S^*, the edge of the boundary layer, the conditions required of it on S. Recalling that $\overline{w}$ and $\partial\overline{w}/\partial n$ vanish on S^*, we see that

$$\langle \overline{B}_\varphi \rangle = \langle \overline{A}_e \rangle = \left\langle \frac{\partial \overline{A}_e}{\partial n} \right\rangle = 0, \text{ on } S^*, \qquad (4.3.3)$$

**Lest it appears that this scheme should always work, even, for example, when $\underline{u}_n$ is axisymmetric, we should observe that, at each level of this iteration, an important consistency requirement arises. The left of (4.3.2) can have no φ-component. For $\underline{B}'_n$, this causes no problem, since the $\operatorname{grad}\Phi_{n-2}$ can be chosen specially to eliminate the φ-component on the right of (4.3.2). For $\overline{\underline{B}}_n$, however, $\underline{1}_\varphi \cdot \operatorname{grad}\Phi_{n-2}$ vanishes, and a condition arises on $\underline{B}_0$ to $\underline{B}_{n-1}$. One can, in fact, regard (4.2.19) and (4.2.20) as being equivalent to these consistency conditions. (Write $\overline{\underline{u}}_M + \underline{1}_\varphi \overline{u}_\varphi = \sum \overline{\underline{u}}_n R^{-\frac{1}{2}n}$, and equate powers of $R^{-\frac{1}{2}}$.)

conditions we will apply to the solutions of Braginskiĭ's equations constructed below.

Braginskiĭ has examined the boundary layer for $\underline{B}'$ in two main cases. The first is when $\zeta \neq 0$ on S ; he then calls these boundary layers "regions of concentrated generation", since he can show that the value of $\overline{\mathcal{E}}_\varphi$ in them is $R^{1/2}$ larger than in the interior region. Since the thickness of the boundary layer is of order $R^{-1/2}$, the sources within the layer are, in integrated effect, as important a source of e.m.f. as those outside. The second case is when $\zeta = 0$ on S ; he then calls the boundary layers "resonant surfaces". Here he finds a curiosity: that the outward-penetrating normal component of $\underline{B}'$ is of order $\underline{B}_M R^{1/6} >> B_M$. Both types of boundary layer are passive in the sense that one can develop the interior solution first without reference to them. Later, if one feels inclined, one can return to them and fit a boundary layer to match the interior field to a potential field outside. This is not surprising in view of the fact that an electromagnetic boundary layer cannot usually do more than match $\underline{B} \cdot \underline{n}$ or $\underline{B} \times \underline{n}$. If therefore we choose the magnetostatic potential field in $\hat{V}$ to match $\underline{B} \cdot \underline{n}$, we will then be able to produce a boundary layer to match $\underline{B} \times \underline{n}$. This procedure will, incidentally, introduce at the same time a new field $O(R^{-1/2} \underline{B} \times \underline{n})$ outside, which requires matching to a higher-order interior solution, and so on.

It is, of course, more realistic to regard $\underline{u}$ as being determined by the solution of the hydrodynamic equations. Although this topic more properly belongs to § 7 below, we may observe here that the Lorentz force on the boundary layer just constructed is $O(R^{1/2})$, as $R \to \infty$. A priori, this might result in such a severe modification to $\underline{u}$ that the assumption that $\underline{u}$ and $\underline{\nabla}\,\underline{u}$ is $O(1)$ as $R \to \infty$ is violated, and the analysis fails to be self-consistent. This occurs, for example, in Hartmann flow, where (when the Lorentz forces are included in a self-consistent way) the boundary layer thickness is neither of order $\eta^{1/2}$ (as above), or $\nu^{1/2}$ (as for a hydrodynamic flow at large Reynolds numbers), but of order $\sqrt{\eta\nu\mu\rho}/|\underline{B}\cdot\underline{n}|$, where (it is supposed) $\underline{B}\cdot\underline{n} \neq 0$. Since the same may be true for our dynamo models, we have not expounded Braginskiĭ's boundary layer theory here, but will return to the topic in § 7.3.

4.4 The rapidly-fluctuating dynamo

Braginskiĭ (1946b) considered a generalization of his nearly axisymmetric dynamo in which $\underline{u}'$ was not only asymmetric in φ but rapidly varying in time:

$$\underline{u}' = \sum_\ell \sum_m \underline{u}'_{\ell m}\, e^{im(\varphi-\varphi_\ell)} = \sum_\ell \underline{u}'_\ell , \tag{4.4.1}$$

where $\underline{u}'_{\ell m} = (\underline{u}'_{\ell(-m)})^*$ and $\varphi_\ell(t) = -\varphi_{-\ell}(t)$. Similarly he divided the axisymmetric parts into series of the form

$$\bar{u}_\varphi = \sum_\ell \bar{u}_{\varphi\ell}\, e^{-i\varphi_\ell}, \quad \underline{\bar{u}}_M = \sum_\ell \underline{\bar{u}}_{M\ell}\, e^{-i\varphi_\ell} .$$

He supposed that $\bar{u}_{\varphi\ell}$, $\underline{u}'_{\ell m}$, $\underline{\bar{u}}_{M\ell}$ and $\dot{\varphi}_\ell$ are slowly-varying functions of time [i.e. $\partial/\partial t = O(1)$ as in the earlier model], but not $\varphi_\ell(t)$, which then, however, has to be almost linear in t. Thus the $\partial/\partial t$ operator is small except when it operates on the phases. Analogously to the $\wedge$ operation he introduced a $\wedge t$ operation for integration over time and a $\overline{}^{t}$ operation for averaging over time. He obtained equations for the double φ and t average $\overline{\overline{}}^{t}$, by very similar methods to those explained above. Indeed, it is almost pointless to repeat his analysis. The final equations for the averaged quantities are identical with (4.2.19) and (4.2.20) above. We must, however, now define Γ as

$$\Gamma = \sum_{\ell} \Gamma_{\ell}\,(\underline{u}'_{\ell}), \tag{4.4.3}$$

where, for each phase ℓ individually, Γ_{ℓ} is given by the time average of (4.2.17). Boundary conditions, etc. are similar.

For just one wave in the φ-direction these results are not in the least surprising: our findings should be independent of an arbitrary rotation about the axis of rotation, by the invariance of the induction equation to such a transformation. What Braginskiĭ has shown is that the same results are still true when one has many waves, all propagating with different (rapid) angular velocities about the axis of symmetry, which no transformation to a rotating frame can simultaneously make slow.

4.5 Dynamo in a cylinder

In this section, and in §§ 4.7 and 5.1, we will give applications of Braginskiĭ's method. It may be wondered how this search for finite eigenvalues can be compatible with the limit $R \longrightarrow \infty$ supposed. There is, however, no contradiction. The eigenvalue sought is really not R; rather it is $\alpha^2 R$, where α measures the degree of asymmetry of $\underline{u}$ (e.g. $\alpha = R^{1/2}|\underline{u}'/\bar{u}\varphi|$ averaged over the flow). We seek solutions in which $\alpha \longrightarrow 0$, $R \rightarrow \infty$ but $\alpha^2 R \longrightarrow$ finite limit = required eigenvalue.

Tough and Gibson (1968) have solved Braginskiĭ's system in a cylinder.

Apart from the plane model to be considered in chapter 5, this is the simplest example yet studied. They chose

$$\bar{u}_\varphi = \varpi V(\varpi), \tag{4.5.1}$$

$$\underline{u}' = \left\{p(\varpi)V(\varpi)\sin\varphi, \frac{d}{d\varpi}\left[p(\varpi)V(\varpi)\right]\sin\varphi, \varpi q(\varpi)V(\varpi)\cos\varphi\right\}, \tag{4.5.2}$$

and $\underline{\bar{u}}_M = -\mathrm{curl}(\bar{w}\bar{u}_\varphi \underline{1}_\varphi)$, so that

$$\underline{\bar{u}}_{eM} = 0. \tag{4.5.3}$$

They obtained

$$\Gamma = p'q', \tag{4.5.4}$$

and basic equations

$$\left(\frac{\partial}{\partial t} - \Delta\right)\bar{A}_e = p'q'\bar{B}_\varphi, \tag{4.5.5}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\bar{B}_\varphi = -\varpi R V' \frac{\partial \bar{A}_e}{\partial z}. \tag{4.5.6}$$

Seeking solutions proportional to $exp(\sigma t + iaz)$, they obtained

$$\left(D^2 + \frac{1}{\varpi}D - \frac{1}{\varpi^2} - a^2 - \sigma\right)\bar{A}_e = -(Dp)(Dq)\bar{B}_\varphi, \tag{4.5.7}$$

$$\left(D^2 + \frac{1}{\varpi}D - \frac{1}{\varpi^2} - a^2 - \sigma\right)\bar{B}_\varphi = iaR\varpi(DV)\bar{A}_e, \tag{4.5.8}$$

where $D = \partial/\partial\varpi$. They then concentrated on the case

$$p = P(1-\varpi), \; q = Q\varpi, \; V = \tfrac{1}{3}\mathcal{U}\varpi^3, \tag{4.5.9}$$

where P, Q and $\mathcal{U}$ are constants, so that Γ is simply $-PQ$ = constant.

Introducing

$$\bar{B}_\varphi = PQ\tilde{\bar{B}}_\varphi, \; \lambda = (PQ\mathcal{V})R, \tag{4.5.10}$$

equations (4.5.7) and (4.5.8) become, after dropping the tilde,

$$\left(D^2 + \frac{1}{\varpi}D - \frac{1}{\varpi^2} - a^2 - \sigma\right)\bar{A}_e = \bar{B}_\varphi, \tag{4.5.11}$$

$$\left(D^2 + \frac{1}{\varpi}D - \frac{1}{\varpi^2} - a^2 - \sigma\right)\bar{B}_\varphi = ia\lambda\varpi^3\bar{A}_e. \tag{4.5.12}$$

Outside the conductor

$$\bar{B}_\varphi = \Delta\bar{A}_\varphi = 0; \quad \bar{A}_\varphi = O(\varpi^{-1}), \varpi \longrightarrow \infty, \tag{4.5.13}$$

showing that $\bar{A}_\varphi \propto K_1(a)$ where K denotes the modified Bessel function of the second kind. Since we have the jump conditions

$$\langle \bar{A}_\varphi \rangle = \left\langle \frac{\partial \bar{A}_\varphi}{\partial \varpi} \right\rangle = \langle \bar{B}_\varphi \rangle = 0, \text{ on } \varpi = 1, \tag{4.5.14}$$

and since $\bar{A}_\varphi = \bar{A}_e$ on $\varpi = 1$, we have

$$\frac{D\bar{A}_e}{\bar{A}_e} = \frac{aK_1'(a)}{K_1(a)}, \quad \bar{B}_\varphi = 0, \text{ on } \varpi = 1, \tag{4.5.15}$$

These two conditions, together with the condition that $\bar{A}_e$ and $\bar{B}_\varphi$ are bounded in the cylinder, pose an eigenvalue problem for σ which, in general, is complex.

Instead of specifying λ and seeking σ, Tough and Gibson set $\mathcal{R}(\sigma = 0)$, and sought λ and $\mathcal{J}(\sigma)$, so that a solution to (4.5.11), (4.5.12) and (4.5.15) exists. This, incidentally, is consistent with our remarks at the beginning of this section. In seeking λ, they imply that $R \longrightarrow \infty$, $\alpha = PQ\mathcal{V} \longrightarrow 0$ and that λ (their product) takes a finite

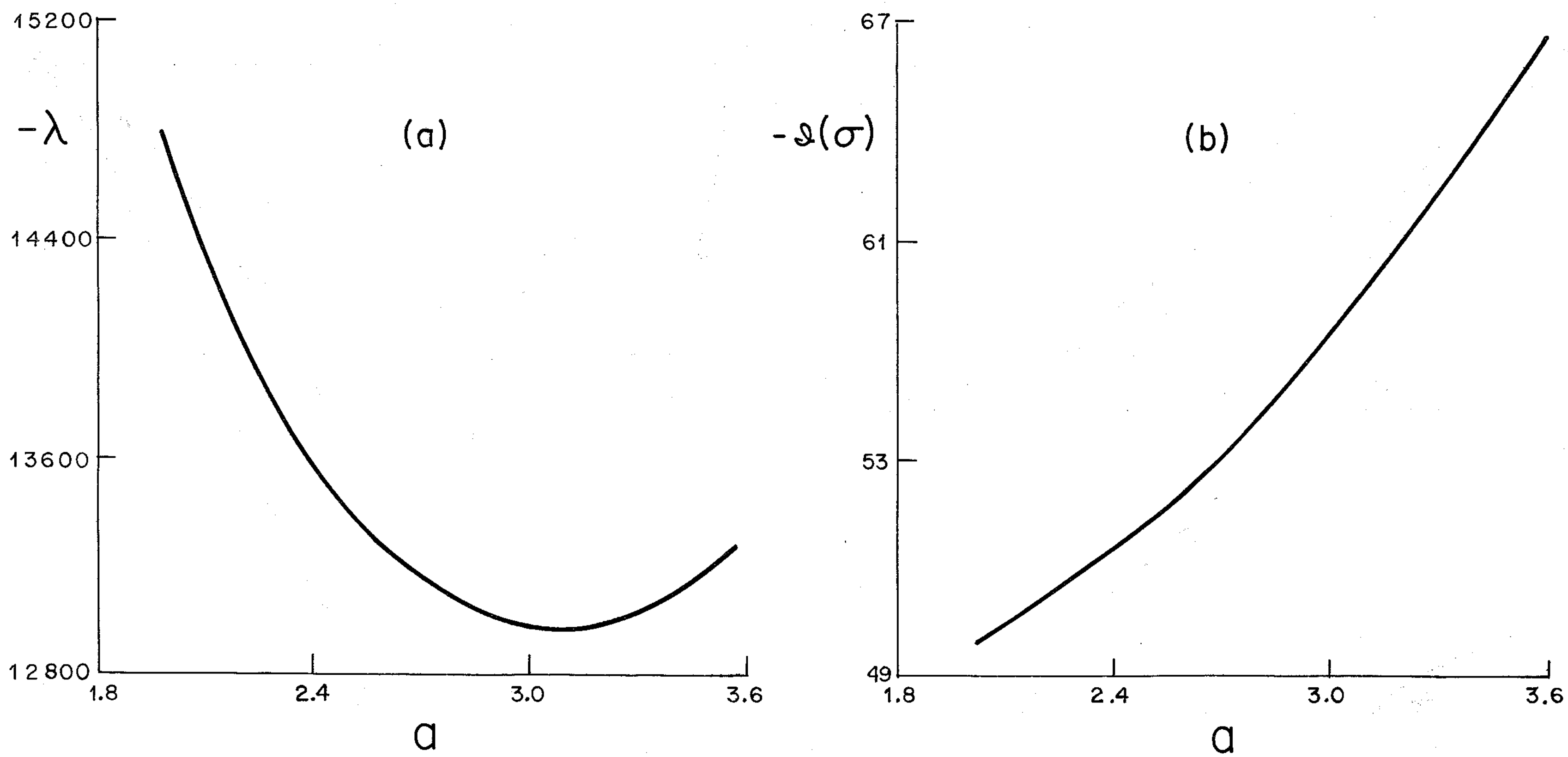
15200
14400
13600
12800
$-\lambda$
(a)
1.8
2.4
3.0
3.6
a
67
61
53
49
$-\vartheta(\sigma)$
(b)
1.8
2.4
3.0
3.6
a

Figure 4.1.

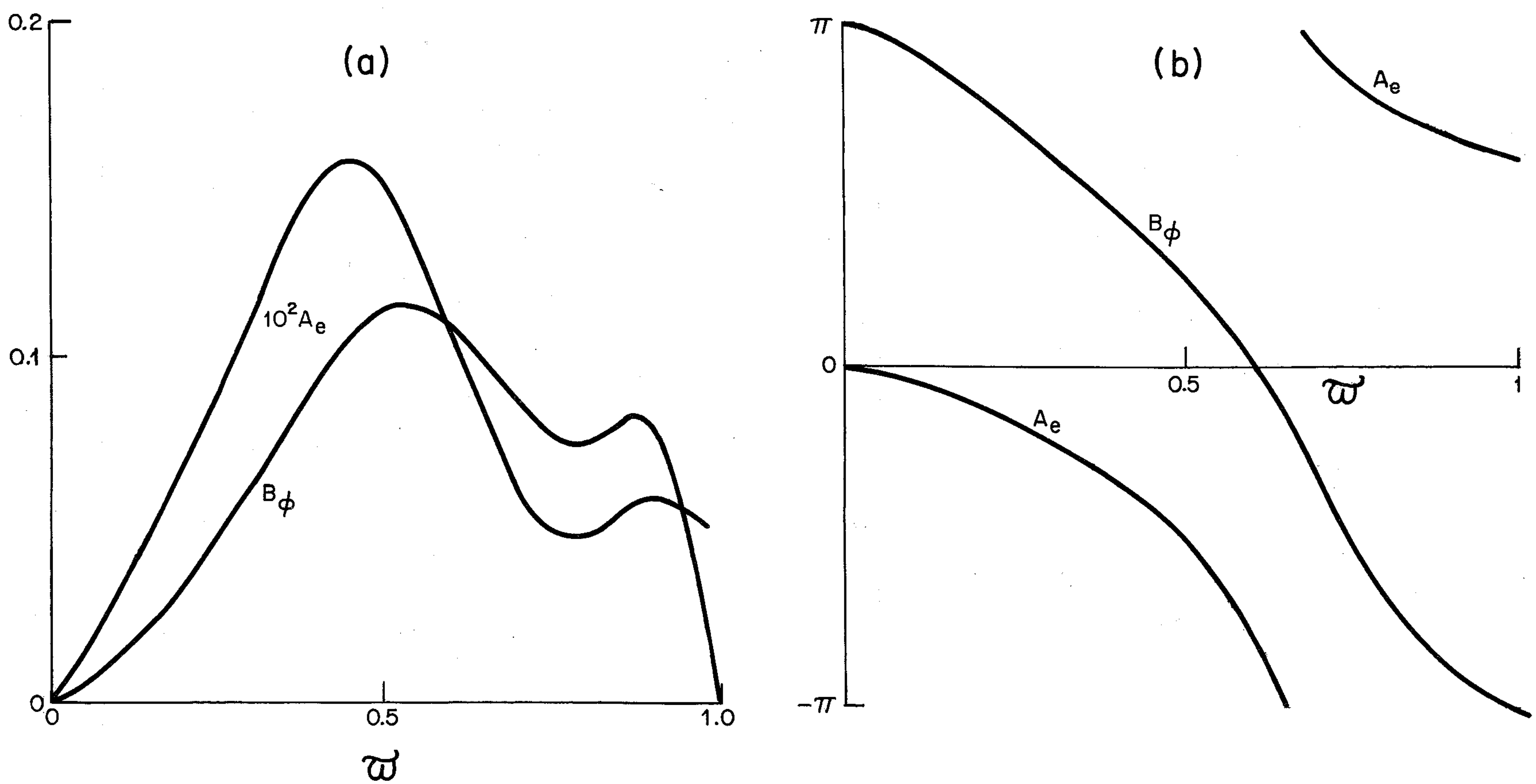

Figure 4.2.

value. The eigenvalues they obtained are shown in Figs. 4.1 and in the table below:

a	$-\lambda$	$-\wp(\sigma)$
2.0	14707.01	49.94
2.2	14071.89	51.49
2.4	13613.43	53.18
2.6	13294.28	55.05
2.8	13087.77	57.05
3.0	12973.73	59.22
3.2	12936.74	61.55
3.4	12964.41	64.05
3.6	13046.52	66.70

These eigenvalues are surprisingly large. They have a minimum at $a = 3.2$ approximately; this defines, in some sense, the dynamo most easily excited. The corresponding moduli and phases of $\bar{A}_e$ and $\bar{B}_\varphi$ are sketched in Figs. 4.2.

4.6 Parker's dynamo model

The work of Parker (1955) foreshadows that of Braginskiĭ in some of its essential ideas, although it was partially based on heuristic arguments rather than on a completely self-consistent approximation scheme. Parker was quite explicit about the details of the regenerative term $\Gamma \bar{B}_\varphi$ in (4.2.19): Consider a region in the core of the Earth, so small in extent that we may regard the field in it as uniform throughout, and introduce a local Cartesian coordinate system (ξ, η, ζ) where ξ points south, η points east and ζ points radially outwards. Hence, locally, the $\xi\zeta$-planes are meridian planes, and the toroidal field is

$$\underline{B}_\varphi = \bar{B}_\varphi \underline{1}_\varphi \equiv \bar{B}_\eta \underline{1}_\eta . \qquad (4.6.1)$$

Imagine now that there is an upward flow along the ζ-axis which also has a vorticity about that axis. Dynamically this is not too implausible; rotation will result from the Coriolis forces associated with any convergence or divergence in horizontal $(\xi\,\eta)$ planes that may be taking place - a phenomenon common enough in the atmosphere, for example. In particular, if we imagine fluid flowing into the base of a rising column somewhere in the northern hemisphere of the core, an elementary consideration of Coriolis forces suggests that the rising column will be rotating counter-clockwise. Opposite rotational forces apply to the efflux at the top of the column. It may be noted, incidentally, that Kahle, Kern and Vestine (1967) have, following a model by Roberts and Scott (1966), deduced motions near the surface of the core from observations of the geomagnetic field and its secular variation at the Earth's surface. Although the errors involved, in this process of extrapolating fields so far downwards, are so large as to make it doubtful if their results are quantitatively very reliable, it is interesting to notice that the velocity fields they obtain seem to have "whorls" of the type which might be expected on Parker's ideas*.

Let $\rho = \sqrt{(\xi + \eta^2)}$. Consider the fluid motion

$$u'_\xi = -\omega_0 \eta S(\rho), \quad u'_\eta = \omega_0 \xi S(\rho), \quad u'_\zeta = W_0 R(\rho), \tag{4.6.2}$$

where R and S vanish for large ρ and have a maximum for $\rho = 0$, and W_0 and ω_0 are constants, W_0 representing the rising or falling motion and ω_0 the vertical vorticity created from it by the Coriolis forces, as above. Thus $\omega_0 W_0$ is of one sign for all cyclones in the northern hemisphere, and of the opposite sign for all cyclones in the southern hemisphere.

*I am grateful to R. Hide for pointing this out to me. (P.H.R.)

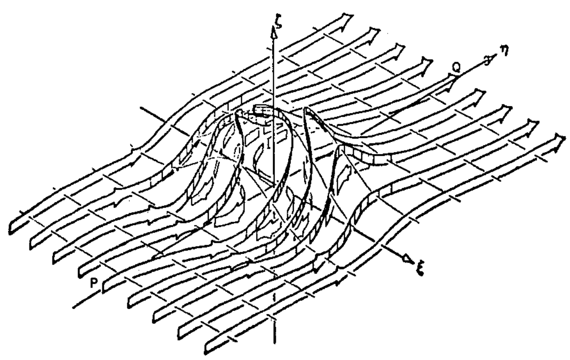

Figure 4.3.

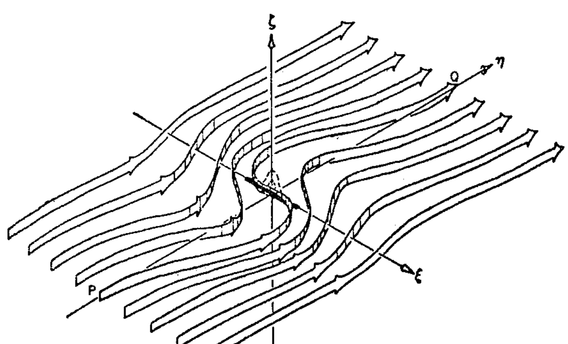

Figure 4.4.

Supposing $W_0 > 0$, the vertical u'_ζ motion of (4.6.2) raises the toroidal field lines into roughly an Ω-shape, and the u'_ξ and u'_ζ motions turn the Ω round so that it has a non-zero projection in the meridian $(\xi\,\zeta)$ plane. This is shown by the solid lines of Fig. 4.3. There is some conceptual advantage in dividing the field $\underline{B}$, created by (4.6.2) from $\underline{B}_\varphi$, into two parts:

(i) the projection, $\underline{B}_H$, of $\underline{B}$ onto the horizontal $(\xi\,\eta)$ plane. This field, shown in Fig. 4.4, is similar to that which would be created by (4.6.2) when $W_0 = 0$;

(ii) the remainder $\underline{B} - \underline{B}_H$. This can be obtained by adding to the field shown by the solid ribbons in Fig. 4.3, the field $-\underline{B}_H$, represented by the broken ribbons in that figure. The fields $\underline{B}$ and $\underline{B}_H$ are identical for large-enough ρ, so that the field $\underline{B} - \underline{B}_H$ forms closed loops. The loop of $\underline{B} - \underline{B}_H$ arising from the field line PQ (see Fig. 4.3) is shown in Fig. 4.5(a), while Fig. 4.5(b) shows its projection in the $\xi\zeta$-plane. [The arrow in Fig. 4.5(a) indicates the direction from which the loop is viewed to give Fig. 4.5(b).]

The field $\underline{B} - \underline{B}_\varphi$ induced by the cyclone (4.6.2) from the inducing field $\underline{B}_\varphi$ may be thought of as the sum of $\underline{B} - \underline{B}_H$ and $\underline{B}_H - \underline{B}_\varphi$. The $\underline{B} - \underline{B}_H$ field, pictured in Figs. 4.5, may be considered to be the sum of two loops, one in the $\xi\zeta$-plane, and the other in the $\eta\zeta$-plane (see Fig. 4.6). The corresponding line of $\underline{B}_H - \underline{B}_\varphi$ is also closed, since $\underline{B}_H$ and $\underline{B}_\varphi$ are identical at infinity, and forms two loops in the $\xi\eta$-plane having oppositely directed circulations (see Fig. 4.6).

All these arguments concern one cyclone. Now imagine many such cyclones distributed throughout the core. To satisfy continuity, we suppose that there are roughly as many with one sign of W_0 in each hemisphere as with the other. We may expect that the overall magnetic field created by these cyclones can be obtained by summing the loops shown in Fig. 4.6, allowing them to coalesce with reinforcement or cancellation, according to sense, in the way suggested by the heuristic ideas of § 1.2. Moreover, armed with the Braginskiĭ weapons (see particularly § 4.4) we might hope to be able to formalize the arguments with a fair degree of rigour.

Consider the coalescence of loops in the $\xi\eta$-, $\eta\zeta$- and $\zeta\xi$-planes separately. It is clear that the loops in the $\xi\eta$-plane will cancel on average. Indeed according to Fig. 4.6 they do so for one cyclone alone! Consider now two cyclones in the northern hemisphere (say), which are identical except that one has positive W_0 while the other has negative W_0. (Remember that $\omega_0 W_0$ is of the same sign for each!) Repeating, with the necessary minor modifications, the arguments given below (4.6.2), we see that the loop created in the $\xi\zeta$-plane by the negative-W_0 cyclone is the same as that for the positive-W_0 cyclone, while the loop created in the $\eta\zeta$-plane is in the opposite sense. Thus the former loops tend to reinforce on coalescence, while the latter tend to cancel. Furthermore, it is noteworthy that, <u>mutatis</u> <u>mutandis</u>, the dynamically analagous motions in the southern hemisphere produce loops in meridian planes that reinforce those in the northern hemisphere (i.e. give a non-zero dipole moment), <u>provided</u> $\bar{B}_\varphi$ is of the opposite sign. Thus, we seem to have a mechanism to create an average field, $\underline{\bar{B}}_M$, in meridian planes, from the initial

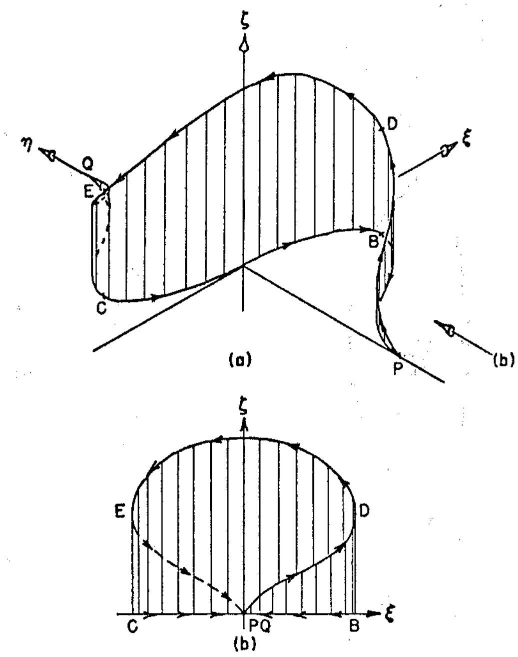

Figure 4.5.

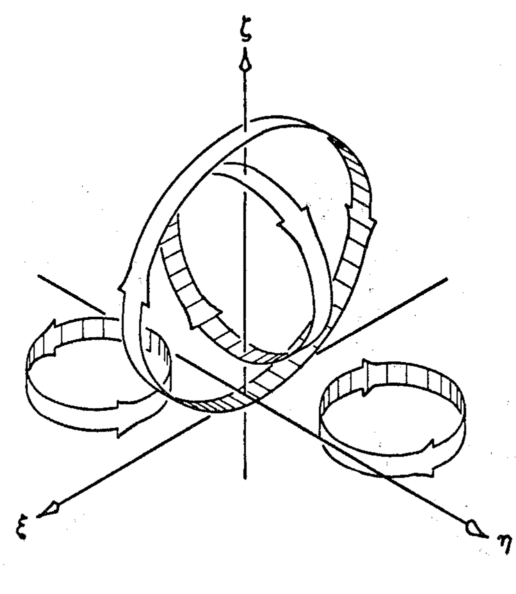

Figure 4.6.

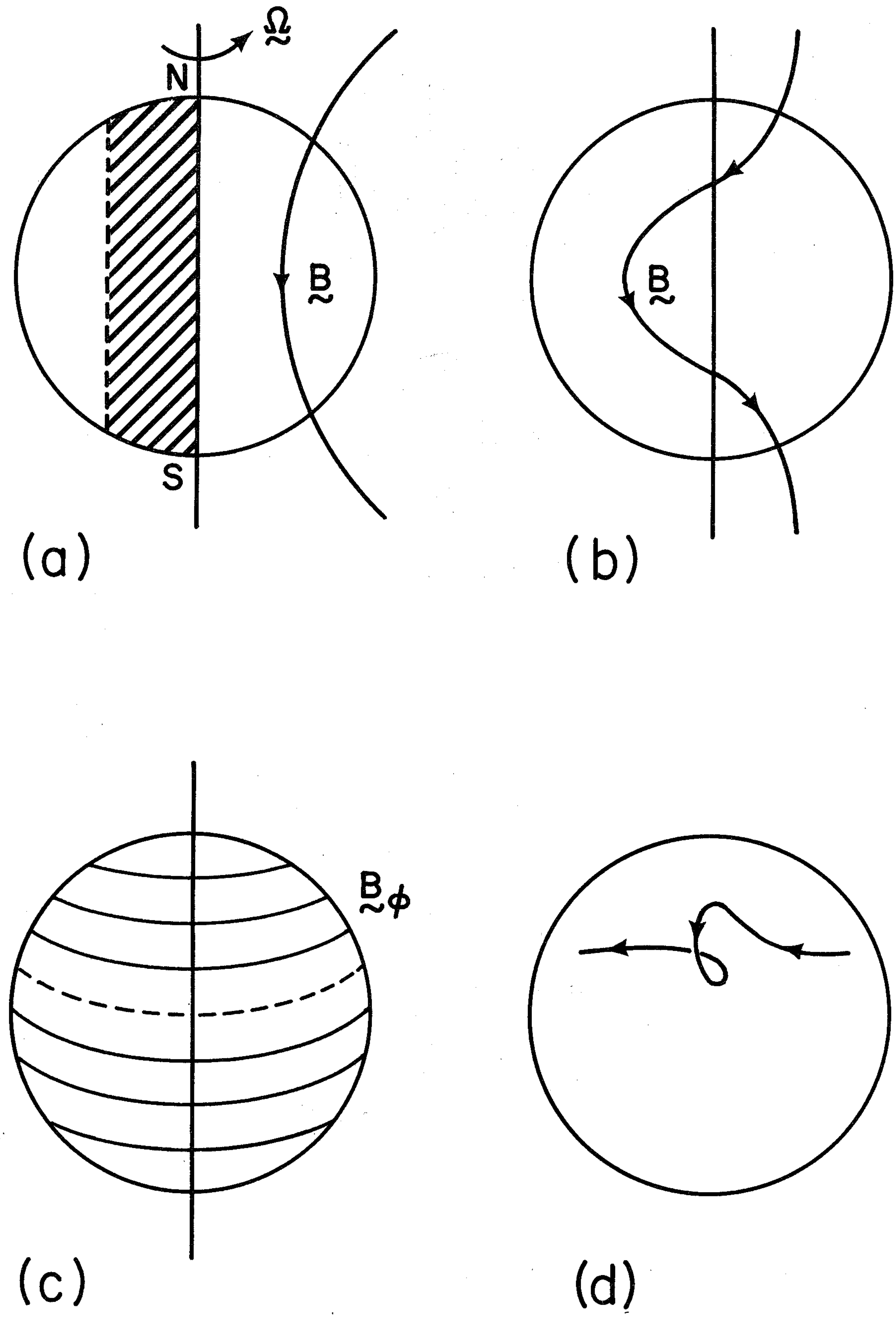

Figure 4.7.

toroidal field $(\bar{B}_\varphi)$. What is more, this is in the right sense to produce, in turn, reinforcement of the toroidal field, provided that the azimuthal component of the velocity field has, on average, the radial variation expected on the assumption that fluid elements tend to conserve their angular momentum on rising and falling through the core. To see this, consider the sequence shown in Fig. 4.7. The right half of (a) shows an initial axisymmetric poloidal field; the left half indicates a toroidal shearing motion which tends to conserve angular momentum, i.e. $\bar{u}_\varphi$ is eastwards (out of paper) in the shaded region and westwards (into paper) on the left unshaded region. This shear distorts $\underline{B}$ in the manner schematically represented in (b), the resultant field being perhaps best thought of as the sum of the original poloidal field together with the toroidal axisymmetric $\bar{B}_\varphi$ field shown in (c), which, it should be noted, has opposite signs in northern and southern hemispheres. One rising, counterclockwise (viewed from above!), cyclone in the northern hemisphere, of the type (4.6.2) with $\omega_o > 0$ and $W_o > 0$, distorts $\underline{B}_\varphi$ as shown in (d). Clearly the component of $\bar{\underline{B}}_M$ created by the cyclone is southwards, i.e. it reinforces the initial field (a).

The above simple picture is suggestive, but of course leaves many questions unanswered. Why, for instance, should the twisting of a loop obligingly cease after it has turned through one right angle? If the loops twisted much further, the regeneration would appear to be quite ineffective. (One might perhaps try to invoke the Lorentz force to limit the amount of twisting.)

The actual way in which Parker incorporates these ideas into the dynamo would seem rather unsatisfactory, but he does introduce a source term (or rather its harmonic components, denoted by $\langle\langle \gamma_{ns} \rangle\rangle$, in Parker's paper) essentially the same as Braginskiĭ's $\Gamma \bar{B}_\varphi$, to represent the creation of $\bar{A}_\varphi$ by the loops. Braginskiĭ's treatment of the dynamo models is altogether more satisfactory; nevertheless Parker's work was prophetic.

4.7 Dynamo in a sphere

According to the Braginskiĭ formalism, we require axisymmetric solutions of the induction equation with an extra source of field, $\underline{\mathcal{S}}$, added, i.e.,

$$\frac{\partial \underline{B}}{\partial t} = R \operatorname{curl} (\underline{u} \times \underline{B}) - \operatorname{curl}^2 \underline{B} + R \underline{\mathcal{S}}, \qquad (4.7.1)$$

[cf. (2.5.3)] where, in spherical polar coordinates (r, θ, φ),

$$\underline{\mathcal{S}} = \left[\frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\Gamma B_\varphi \sin\theta), -\frac{1}{r} \frac{\partial}{\partial r} (r \Gamma B_\varphi), 0 \right]. \qquad (4.7.2)$$

Here $\Gamma = \Gamma(r, \theta)$ is the generation term, given to first order by (4.2.8); and the fields $\underline{u}$ and $\underline{B}$ appearing involve the axisymmetric effective fields given, to the same accuracy, by (4.1.28).

In seeking solutions of (4.7.1) appropriate to a sphere, we may make use of the Bullard-Gellman expansion technique explained in § 2.5. From the axisymmetry of the fields involved, we see that $\underline{r} \cdot \operatorname{curl} \underline{\mathcal{S}}$ vanishes identically, where $\underline{r}$ is the radius vector. It follows that $\underline{\mathcal{S}}$ does not affect the toroidal part of $\underline{B}$ at all, and, if we expand the fields [cf. (2.5.1-2)]

$$\underline{u} = \sum_\alpha (\underline{s}_\alpha + \underline{t}_\alpha), \ \underline{B} = \sum_\alpha (\underline{S}_\alpha + \underline{T}_\alpha), \ \Gamma = \sum_\alpha \Gamma_\alpha, \qquad (4.7.3)$$

where $\Gamma_\alpha = g_\alpha(r,t)\,P_\alpha(\cos\theta)$, we obtain (2.5.7) as before:

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)T_\alpha = R\sum_{\alpha\beta}\left[(s_\alpha S_\beta T_\gamma) + (t_\alpha S_\beta T_\gamma) + (s_\alpha T_\beta T_\gamma) + (t_\alpha T_\beta T_\gamma)\right], \tag{4.7.4}$$

where $(s_\alpha S_\beta T_\gamma)$ etc. are the coupling integrals [cf. (2.5.8-9)] , and ∇^2 , in its effect on T_γ or S_γ , is given by (2.5.5).

The equation for the poloidal components of $\underline{B}$ is obtained by taking the scalar product of (4.7.1) with $Y_\gamma \underline{r}$, and integrating over θ and φ . We obtain [cf. (2.5.6)].

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)S_\gamma = R\sum_{\alpha\beta}\left[(s_\alpha S_\beta S_\gamma) + (t_\alpha S_\beta S_\gamma) + (s_\alpha T_\beta T_\gamma) + (\Gamma_\alpha T_\beta S_\gamma)\right], \tag{4.7.5}$$

where

$$(\Gamma_\alpha T_\beta S_\gamma) = \frac{\iint Y_\gamma \frac{\partial}{\partial\theta}\left[\Gamma_\alpha T_\beta \frac{\partial P_\beta}{\partial\theta}\right] d\theta\, d\varphi}{n_\gamma(n_\gamma+1)\iint Y_\gamma^2 \sin\theta\, d\theta\, d\varphi} . \tag{4.7.6}$$

It is easily seen, by integration by parts, and use of the relation $P_n^1 = -\partial P_n/\partial\theta$, that

$$\iint Y_\gamma \frac{\partial}{\partial\theta}\left[\Gamma_\alpha T_\beta \frac{\partial P_\beta}{\partial\theta}\right] d\theta\, d\varphi = -g_\alpha T_\beta \iint P_{n_\alpha} P_{n_\beta}^1 P_{n_\gamma}^1 \sin\theta\, d\theta\, d\varphi = -2 g_\alpha T_\beta\, G_{n_\alpha n_\beta n_\gamma}^{0\,1\,1} , \tag{4.7.7}$$

where $G_{n_\alpha n_\beta n_\gamma}^{0\,1\,1}$ is a Gaunt integral [cf. (2.5.10)] :

$$G_{n_\alpha n_\beta n_\gamma}^{0\,1\,1} = \pi\int_0^\pi P_{n_\alpha} P_{n_\beta}^1 P_{n_\gamma}^1 \sin\theta\, d\theta . \tag{4.7.8}$$

By the usual selection rules (cf. Bullard and Gellman, 1954, p. 229), the integral (4.7.8) vanishes unless $n_\alpha + n_\beta + n_\gamma$ is even and can form the sides of a triangle, degenerate cases (e.g. $n_\alpha = n_\beta + n_\gamma$) counting as triangles. We may now write (4.7.6) as

$$\left(\Gamma_\alpha T_\beta S_\gamma\right) = - \frac{2 g_\alpha T_\beta G^{0\ 1\ 1}_{n_\alpha n_\beta n_\gamma}}{n_\gamma (n_\gamma + 1) G^{0\ 0\ 0}_{0 n_\gamma n_\gamma}} . \qquad (4.7.9)$$

As in the Bullard-Gellman case it is inevitable that, even with simple choices such as (4.7.11-12) below, an infinite train of field harmonics is excited. Nevertheless, the presence of the coupling term $\left(\Gamma_\alpha T_\beta S_\gamma\right)$ can, on this occasion, lead to rapid convergence. The solutions of (4.7.4-5) selected must, by (4.3.3), obey the same surface conditions as those required of the Bullard-Gellman model, namely [cf. (2.5.12)]

$$T_\gamma = \frac{\partial S_\gamma}{\partial r} + \frac{(n_\gamma + 1)}{r} S_\gamma = 0, \text{ on } r = 1. \qquad (4.7.10)$$

We have here selected the radius of the sphere as unit of length. We will also use a typical azimuthal flow speed as unit of velocity.

In applying this formalism to Parker's model, we would like $\underline{B}_M$ to have a dipole component. This suggests that S might be represented by a series of odd harmonics. We wish the toroidal shear to produce a $\overline{B}_\phi$ reversing in both hemispheres. This suggests that we take t to be composed of odd harmonics, and T to be composed of even harmonics. The $\left(\Gamma_\alpha T_\beta S_\gamma\right)$ term is then non-vanishing if Γ involves odd harmonics. We set

$$t_1 = u(r) P_1(\cos\theta), \; S_2 = \frac{r v(r)}{R} P_2(\cos\theta), \qquad (4.7.11)$$

and all other components of $\underline{u}$ zero. Also we set

$$\Gamma = \frac{G(r)}{r R}\left[P_1(\cos\theta) - P_3(\cos\theta)\right], \qquad (4.7.12)$$

and all other components, g_n, of Γ zero. This conforms with Braginskiĭ's choice (Braginskiĭ, 1964c, eqs. 2.1.2-4). He states that it is easy to find examples leading to (4.7.12) and gives two: -

$$\text{(a)}\ u'_{\varpi}=0,\ u'_z=u(1-\mu^2)\left[-\frac{5G(r)}{2Rrf'(r)}\right]^{1/2}\left[kf(r)\cos\varphi-\frac{\sin\varphi}{k}\right],$$

$$\text{(b)}\ \underline{u}'_M=u(1-\mu^2)\left[-\frac{5G(r)}{2Rr}\right]^{1/2}\left[r\cos\varphi-\frac{\sqrt{1-\mu^2}\sin\varphi}{k_1k_2}\right]\left(\underline{1}_r\frac{k_1\mu}{\sqrt{r}}-\underline{1}_\theta k_2\sqrt{r}\right).$$

(Here k, k_1, k_2 are arbitrary constants, and f is an arbitrary function.) Incidentally, $\overline{W}=0$ for both these choices, so that there is no difference between an effective field and its parent field.

The introduction of the poloidal flow (4.7.11) does not destroy the symmetry argument just given, and S and T may be expanded as

$$S=\sum_{n=0}^{\infty}S_{2n+1},\quad T=\sum_{n=0}^{\infty}T_{2n+2}.$$

The poloidal flow also provides an extra degree of freedom. For, when we talk about dynamo waves (§ 5.1), we shall find that there are no steady solutions if $\underline{u}_{eM}=0$; only migratory waves are possible. Braginskiǐ argues that the same is likely to be true here, and, since it is easier, numerically, to solve for steady dynamos rather than oscillatpry ones, he decides to retain the poloidal flow $\underline{\overline{u}}_M$, and choose v in such a way that $\underline{B}$ is steady.

It may be observed that (4.7.13) only includes flows involving $\cos\varphi$ and $\sin\varphi$. The thought here is that the equatorial dipole of the Earth is its predominant non-symmetric magnetic feature. (This may be arguable if one recollects that such inferences should be made at the core surface.) On Braginskiǐ's ideas, in fact, the asymmetry of the geomagnetic field is no accidental feature but a necessary part of the generation mechanism. Indeed, if

$$\frac{|\underline{B}_3'|}{|\underline{B}_M|} = O\left(R^{-1/2}\right), \qquad (4.7.15)$$

we see that, supposing $\underline{B}_3'$ is the field of the equatorial dipole, R for the core is 20, approximately (but see below). Since the equatorial dipole does not move as rapidly as the other asymmetric harmonics in their westerly drift, we might expect that, with a $\underline{u}'$ depending on $\cos\varphi$ and $\sin\varphi$, we should choose a v in (4.7.11) in such a way that the field becomes non-oscillatory. These, it should be noted, are essentially dynamical inferences. As far as the induction equation is concerned, $\underline{u}'$ only enters in averaged form in Γ, and any other non-axisymmetric flow would do as well, provided it gave the same Γ. Nevertheless, if the field is predominantly steady and has a $\cos\varphi$ and $\sin\varphi$ dependence, it may be "sensible" to suppose that $\underline{u}'$ is steady and has the same φ-dependence also.

Braginskiĭ (1964c) seeks normal mode $(e^{i\omega t})$ solutions. He integrates (4.7.4-5) in three cases, using (4.7.11-14) and the following choices of u, v and G:

Model I (Homogeneous case)

$$u(r) = r(1-r^2)^2, \quad v(r) = -10\beta r^5(1-r)^2, \qquad (4.7.16.I)$$

$$G(r) = -\frac{96}{5}\alpha^2 r^3(1-r)^2, \quad (\alpha, \beta \text{ constants}). \qquad (4.7.17.I)$$

Model II (First case with inner core)

$$u(r) = \frac{r(1-r)^2}{(0.6)^2}, \quad v(r) = -10\beta r^3(r-0.4)(1-r)^2, \qquad (4.7.16.II,III)$$

$$G(r) = -\frac{30\alpha^2}{r^3}(r-0.4)^2(1-r)^2, \qquad (4.7.17.II)$$

these holding for $1 \geq r > r_i = 0.4$ = radius of inner core. In the solid inner core $(r < r_i)$, $\underline{u}'$, $\underline{\bar{u}}_M$, F and G are taken to be zero and $u(r)/r$ a constant angular velocity chosen so that the net electromagnetic torque on it is zero.

Model III (Second case with inner core)

This is the same as model II, except

$$G = -\frac{576}{5}\alpha^2 r y^2(1-y)^2(1-\lambda y), \; y = \frac{r-0.4}{0.6}, \qquad (4.7.17.\text{III})$$

where λ is arbitrary.

Braginskiĭ solved the ordinary differential equations (4.7.4-5) for these models by truncating after N pairs of T and S harmonics, and transforming them into integro-differential equations, using the very simple Green's functions of the ∇^2 operator [which, by (2.5.5), is of the form $d^2/dr^2 + 2r^{-1}d/dr - n_\gamma(n_\gamma+1)r^{-2}$]. He divided the range $0 \leq r \leq 1$ (or $r_i \leq r \leq 1$) into 60 grid intervals, used the trapezium rule for the integrals, and central differences for the first-order derivatives which are still present. The set is solved by iteration using, for each step, the estimate of the solution obtained at the previous step to evaluate the integrals. Values of α and β were sought for which steady solutions are possible: $\mathcal{R}(\omega) = \mathcal{I}(\omega) = 0$.

The method was found to converge well, as evinced by the successive estimate of the eigenvalues for N = 1, 2, 3 and 4 (the highest level of truncation Braginskiĭ considered). Rather surprisingly, perhaps, the results obtained depended on how the iteration was started. For model I and N = 4, for instance, he obtained

$$\alpha = 33.45 \qquad \beta = 6.167, \tag{4.7.18.I}$$

$$\alpha = 31.63, \qquad \beta = 10.484,$$

for two particular sets of starting conditions. Of course, the final eigenfunctions were also different. This suggests that (over a range of β at least) it was not, after all, necessary to introduce two parameters (α and β) to make $\mathcal{R}(\omega)$ and $\mathcal{J}(\omega)$ both vanish; there is always a value of α for which a steady dynamo exists. This is brought out by Braginskiĭ's results; for a range of values of $\rho = \beta/\alpha$, he obtained stationary dynamos at the following:

ρ	α
0.12	38.94
0.15	35.18
0.185	33.43
0.4	32.08
0.5	33.67
0.52	34.10

At either extreme of ρ , the convergence was slow, and iterations for ρ = 0.11 and ρ = 0.6 (also for ρ = 0 and $\rho < 0$) blew up. He surmised that the steady solutions only exist in this band of ρ . The weak dependence of α on ρ is noteworthy.

Integration of Models II and III proceeded similarly. For example, he obtained

$$\text{Model II:} \quad \rho = 0.5; \quad \alpha = 34.3474, \tag{4.7.18.II}$$

$$\text{Model III:} \quad \rho = 0.5, \quad \lambda = 1.8; \quad \alpha = 30.9652. \tag{4.7.18.III}$$

The value of λ in the latter of these provided the best fit with the observed poloidal field at the Earth's surface. The integrations (5.7.18) were the bases for his subsequent comparisons with the geomagnetic field.

It is important to point out again that $R^{-1/2}$ is not really our expansion parameter. Actually the degree of asymmetry is $\propto R^{-1/2}$, as may be seen from (4.7.13,17). Although, as we have seen [cf. discussion below (4.7.5)], $(|\underline{B}_M/|\underline{B}_3'|)^2$ is of the order of 20 for the Earth, this really provides an estimate of R/α^2 and not R. In all the integrations reported above, $\alpha \approx 30$, so that $R \approx 18,000$. This, Braginskiĭ asserts, is unreliable, because the constant k in the relation implied, viz.

$$\frac{|\underline{B}_3'|}{|\underline{B}_M|} \approx k\frac{\alpha}{R^{1/2}}, \tag{4.7.19}$$

is very sensitive to the precise model chosen for integration. It should be pointed out, before proceeding, that if equations (4.7.4-5) are examined in detail, the ratio of poloidal to toroidal field is not $(\alpha/R^{1/2})^2$, but

$$\frac{|\overline{\underline{B}}_M|}{|\overline{B}_\varphi|} \approx \frac{\alpha}{R}. \tag{4.7.20}$$

In order to obtain more reliable information, Braginskiĭ studied the electromagnetic core-mantle coupling. Taking the electrical of the lower mantle to be 400 mho/m (cf. Eckart, Larner and Madden, 1963), he obtained agreement with the observed time-scale ($\sim$ 10 years, he says) of the variation in the length of the day. He also applied this coupling theory to the westward drift, supposing that the relative motion of field and mantle represent a genuine fluid flow. This is in contrast to the imaginative suggestion of Hide (1966) that the westward drift is primarily a wave notion. Although Braginskiĭ seems to feel this idea is at least partially correct (cf. Braginskiĭ, 1964c, p.583), it is hard to accept in toto for the following main reason. (See also §9 of Bullard and Gellman, 1954.)

It seems likely that the dynamic balance within the Earth is primarily between Lorentz forces, Coriolis forces, and pressure gradient [cf. eq. (7.2.19) below]:

$$2\Omega\bar{u}_\varphi \approx \frac{\bar{B}_\varphi^2}{r_c\mu\rho} = \frac{R^2}{\alpha^2 r_c}\frac{\bar{B}_M^2}{\mu\rho}, \tag{4.7.21}$$

using (4.7.20). Here r_c is the core-radius, $R = \bar{u}_\varphi r_c/\eta$ is the usual magnetic Reynolds number, and ρ is the density. If V_p is the Alfvén velocity based on the poloidal axisymmetric field $\bar{\underline{B}}_M$, we may rewrite (4.7.21) as

$$R = \frac{2\Omega\eta}{V_p^2}\alpha^2. \tag{4.7.22}$$

Taking $|\bar{\underline{B}}_M| \approx 5$ gauss, we obtain $V_p \approx 4.5\ 10^{-3} m/s$. For $2\Omega = 1.5\ 10^{-4} s^{-1}$, $\eta = 1 m^2/s$ (see §1.1), and $\alpha = 30$ as before, (4.7.21) gives

$$R \approx 6300, \tag{4.7.23}$$

implying, incidentally, that

$$\bar{u}_\varphi \approx 2 mm/s, \tag{4.7.24}$$

compared with the westward drift velocity of about 0.6 mm/s.

Braginskiĭ uses his models to estimate the Joule heat loss in the core, and finds it difficult to believe, bearing in mind the theoretical maximum efficiency of a heat engine, that the convection driving the geodynamo (if it is convection!) is thermal in origin. He favours the notion that the stirring of the core is effected by the floating up of a light silicate to the base of the mantle.

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5. Dynamos in Unbounded Conductors

5.1 Parker's Migratory Dynamo Wave

Today we discuss dynamo waves, a subject initiated by Parker (1955). We will consider fields with a high degree of "two-dimensionality". We should, therefore, emphasize at the outset that the anti-dynamo theorem for two-dimensional dynamos is not violated. This theorem is, essentially, a form of Cowling's theorem where the independence on φ is replaced by independence on a space coordinate, for example, z . It states that fields independent of z cannot be maintained by dynamo action (cf. e.g. Cowling, 1957, §4).

First, we consider the original migratory dynamo waves of Parker (1955), but using Braginskiĭ's formalism as applied to a plane layer. This can be obtained from the theory of Sect. 4 by replacing the φ-coordinate by z and taking an appropriate $\varpi \longrightarrow \infty$ limit (to remove the curvature effects). The primary flow $\underline{u}$ and primary field $\underline{B}$ are both parallel to the direction of z and independent of z . Averages, again denoted by bar, refer to z ; primes to fields dependent on z , and hats ($\hat{\ }$) to integrations of such fields over z . The magnetic field is the sum of

$$\overline{\underline{B}} = \left[\frac{\partial \bar{A}(x,y,t)}{\partial y}, - \frac{\partial \bar{A}(x,y,t)}{\partial x}, \bar{B}(x,y,t) \right], \tag{5.1.1}$$

where $\bar{A} = O(R^{-1}\bar{B})$, and $\underline{B}' = O(R^{-1/2}\bar{B})$. The fluid velocity is the sum of

$$\overline{\underline{u}} = \left[\frac{\partial \bar{v}(x,y)}{\partial y}, - \frac{\partial \bar{v}(x,y)}{\partial x}, \bar{u}(x,y) \right], \tag{5.1.2}$$

where $\bar{v} = O(R^{-1}\bar{u})$, and $\underline{u}' = O(R^{-1/2}\bar{u})$, where $\bar{u} = O(1)$. The Braginskiĭ equations become

$$\frac{\partial \bar{A}_e}{\partial t} + R\left(\frac{\partial \bar{v}_e}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \bar{v}_e}{\partial x}\frac{\partial}{\partial y}\right)\bar{A}_e = \nabla^2 \bar{A}_e + \Gamma \bar{B}, \tag{5.1.3}$$

$$\frac{\partial B}{\partial t} + R\left(\frac{\partial \bar{v}_e}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \bar{v}_e}{\partial x}\frac{\partial}{\partial y}\right)\bar{B} = \nabla^2 \bar{B} - R\left(\frac{\partial \bar{u}}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \bar{u}}{\partial x}\frac{\partial}{\partial y}\right)\bar{A}_e, \tag{5.1.4}$$

where

$$\bar{v}_e = \bar{v} + \bar{w}\bar{u}, \; \bar{A}_e = \bar{A} + \bar{w}\,\bar{B}, \tag{5.1.5}$$

and

$$\left.\begin{aligned} w &= \frac{1}{2\bar{u}^2}\left(\overline{\underline{u}'_M \times \hat{\underline{u}}_M}\right)_z, \\ \Gamma &= \frac{1}{\bar{u}^2}\left(\overline{\underline{u}' \times \frac{\partial \underline{u}'}{\partial z}}\right)_z - 2\left[\overline{\nabla_M\left(\frac{u'_x}{\bar{u}}\right)\cdot \nabla_M\left(\frac{\hat{u}_y}{\bar{u}}\right)}\right]. \end{aligned}\right\} \tag{5.1.6}$$

Now M denotes the (x, y)-components.

Suppose

$$\bar{u} = \bar{u}(y),$$

and

$$\underline{u}' = \left[u_1(y)\cos qz, u_2(y)\sin qz, \frac{1}{q}\frac{du_2(y)}{dy}\cos qz\right], \tag{5.1.7}$$

so that by (5.1.6)

$$\left.\begin{aligned} w &= w(y) = -\frac{1}{2q\bar{u}^2}u_1 u_2, \\ \Gamma &= \Gamma(y) = q\frac{u_1 u_2}{\bar{u}^2} + \frac{1}{q}\frac{d}{dy}\left(\frac{u_1}{\bar{u}}\right)\frac{d}{dy}\left(\frac{u_2}{\bar{u}}\right) \end{aligned}\right\} \tag{5.1.8}$$

Select $\bar{v}$ to be

$$\bar{v} = \bar{v}(y) = -\bar{w}(y)\bar{u}(y), \tag{5.1.9}$$

so that

$$\bar{v}_e = 0. \tag{5.1.10}$$

Equations (5.1.3) and (5.1.4) then become

$$\frac{\partial \bar{A}_e}{\partial t} = \nabla^2 \bar{A}_e + \Gamma \bar{B}, \tag{5.1.11}$$

$$\frac{\partial \bar{B}}{\partial t} = \nabla^2 \bar{B} - R\,\frac{d\bar{u}}{dy}\,\frac{\partial \bar{A}_e}{\partial x}. \tag{5.1.12}$$

The coefficients in this equation are independent of x. We may therefore seek solutions in the form of plane waves traveling in the x-direction:

$$\bar{A}_e = A_0(y)e^{i(kx-\omega t)}, \quad \bar{B} = B_0(y)e^{i(kx-\omega t)}. \tag{5.1.13}$$

To make further progress, we specialize yet further by arranging that du/dy and Γ are constant; for example

$$\bar{u} = u_0 y \qquad u_1 = u_2 = V_0 y \sin qy, \tag{5.1.14}$$

where u_0 and V_0 are constants. Then

$$\Gamma = q\left(\frac{V_0}{u_0}\right)^2, \tag{5.1.15}$$

and (5.1.11) and (5.1.12) become

$$\frac{d^2A_0}{dy^2} - (k^2 - i\omega)A_0 = -q\left(\frac{V_0}{u_0}\right)B_0 \tag{5.1.16}$$

$$\frac{d^2B_0}{dy^2} - (k^2 - i\omega)B_0 = ikRu_0A_0. \tag{5.1.17}$$

Parker obtained solutions of these equations which are independent of y and are appropriate to an unbounded mechanism. In this case we have

$$(k^2 - i\omega)^2 = ik\left(\frac{RqV_0^2}{u_0}\right). \tag{5.1.18}$$

Suppose $qu_0 > 0$ (i.e. $\Gamma d\bar{u}/dy > 0$). If $k > 0$, (5.1.18) gives

$$k^2 - i\omega = (1-i)\left(\frac{kRqV_0^2}{2u_0}\right)^{1/2}. \tag{5.1.19}$$

For waves which neither increase nor decrease in amplitude, ω is real and, equating real and imaginary parts, we obtain

$$k^2 = \omega = \left(\frac{kR_q v_0^2}{2u_0}\right)^{1/2}, \tag{5.1.20}$$

or

$$k = \left(\frac{R_q v_0^2}{2u_0}\right)^{1/3}, \quad \omega = \left(\frac{R_q v_0^2}{2u_0}\right)^{2/3}, \quad c = \frac{\omega}{k}\left(\frac{R_q v_0^2}{2u_0}\right)^{1/3}, \tag{5.1.21}$$

where c is the phase velocity. Had we supposed that $k < 0$, we would have found that the signs of both k and ω in (5.1.21) would have reversed, but c would have remained the same. This corresponds to a progressive wave traveling in the positive x-direction. Note that by (5.1.16) and (5.1.19),

$$B_0 = e^{-\frac{1}{4}\pi i}\left(\frac{Ru_0^2}{2^{1/4}q^{1/3}v_0}\right)^{2/3} A_0, \tag{5.1.22}$$

showing a 45° phase lag often characteristic of a diffusive process.

To obtain a wave traveling in the negative x-direction one must assume the opposite sign for Γ, by reversing u_1 or u_2 for example, or the opposite sign for $d\bar{u}/dy$, by reversing u_0. In this way the sign of qu_0 is changed.

It is worth noticing that the waves necessarily progress. This appears to be the result of (5.1.10). Had we added cy/R to the definition (5.1.9) we would have (essentially) transformed to a frame moving with the progressive wave above, and this, therefore, would have appeared as stationary.

Parker has given an interesting interpretation of his waves. In Fig. 5.1 the + signs represent regions where $\bar{B} > 0$ (out of the page),

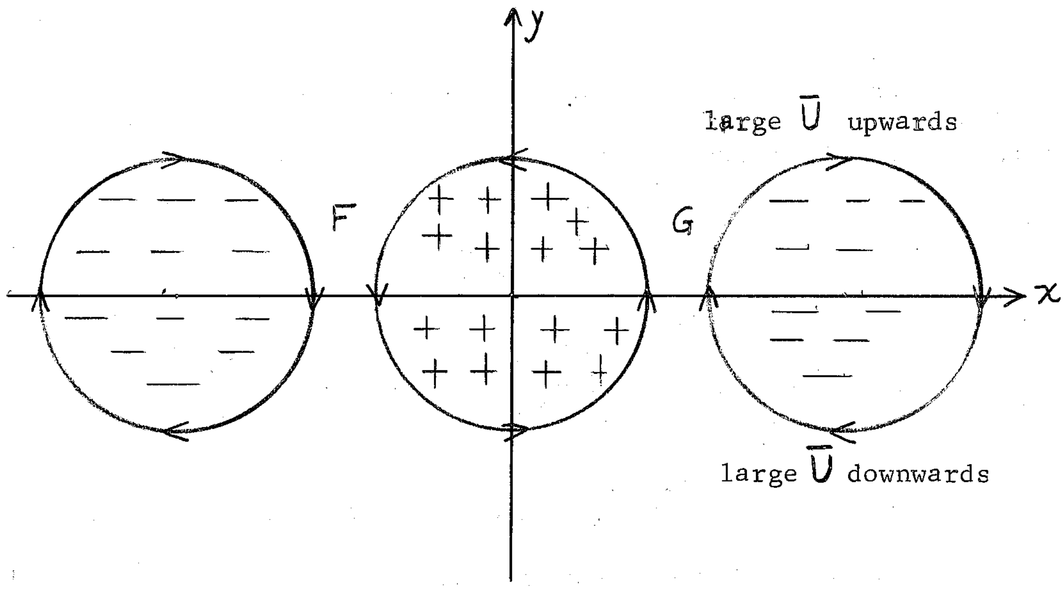


Fig. 5.1

and the - signs where $\bar{B} > 0$ (into the page). If Γ is positive, $\bar{A}_e$ grows, and this results in a secondary field in the direction shown by the arrows. The action of a positive shear (as shown) is to draw out these loops upwards (out of the paper) for the upper half plane and downwards (into the paper) for the lower half plane. Thus, in the region F, a relatively strong field is produced downwards (into the paper), i.e. in the negative z-direction; while in the region G it is produced upwards (out of the paper), i.e. in the positive x-direction. In other words the right side of each tube shown is regenerated, while that on the left side is depleted. The overall effect is a motion of the pattern to the right, i.e. in the positive x-direction.

It is interesting to note that, in the case of the Earth (x northwards, z eastwards, y upwards), the waves would tend to travel along lines of longitude. It is not inconceivable that the observed variation of the Earth's axial dipole arises from a rearrangement in latitude of the flux emanating from the core surface; and this rearrangement may be related, albeit in a rather general way, to the migratory wave described above.

One shortcoming of the model discussed here should be mentioned. It is always possible to make the waves amplify (or decay) in amplitude simply by making k smaller (or larger) than the value given in (5.1.21); this is not entirely unexpected since dissipation affects the longer (shorter) wavelengths less (more). Indeed in an infinite medium there is one wave-number which grows most rapidly

$$k=\left(\frac{R_q V_0^2}{32u_0}\right)^{1/3}, \qquad \mathcal{R}(\omega)=4\left(\frac{R_q V_0^2}{32u_0}\right)^{2/3}, \quad \mathcal{I}(\omega)=3\left(\frac{R_q V_0^2}{32u_0}\right)^{2/3}. \tag{5.1.23}$$

This feature is apparently also present in the generalization of Parker's waves to a layer bounded in the z-direction (a case studied by Braginskiĭ, 1964). For a body bounded (or cyclic) in the x-direction, however, there need be no such amplifying solutions.

5.2 Roberts's Periodic Solutions

G. O. Roberts has very recently exhibited rather conclusively the existence of two spatially periodic solutions of the dynamo equations in an infinite medium, one of which is a progressive wave and one of which is not. His work stems directly from the induction equation and does not make use of the Braginskiĭ formalism. The flows he considers are

$$\left.\begin{aligned} \underline{u}_1 &= [2U\cos ky\cos lz,\ V\sin lz,\ W\sin ky], &&\text{for model 1,}\\ \underline{u}_2 &= [U\sin(ky+lz),\ V\sin 2lz,\ W\sin 2ky], &&\text{for model 2,} \end{aligned}\right\} \tag{5.2.1}$$

where k, ℓ, U, V and W are constants. Clearly both motions satisfy the continuity equation. The magnetic field which is maintained in each case is of the form

$$\underline{B} = \underline{B}\,(y,z)\,e^{(ijx+\omega t)}, \qquad (j, \omega \text{ constants}), \tag{5.2.2}$$

where $\underline{B}\,(y,z)$ is periodic, and j, although it may be small, is non-vanishing. Rather than use the partial differential equation for $\underline{B}$, one uses the infinite set of its Fourier coefficients, revealing the dynamo equations as a linear operator equation in the space of Fourier coefficients. Dynamo action requires that $\mathcal{R}(\omega) \geq 0$, where ω is an eigenvalue of the operator involved.

It is of some interest to reproduce the heuristic arguments that led Roberts to (5.2.1) to (5.2.2). Consider the fluid motions

$$\left.\begin{array}{ll} \underline{v}_1 = (-1+\cos z, \sin z, 0), & \text{for model 1,} \\ \underline{v}_2 = (\sin z, \sin 2z, 0), & \text{for model 2.} \end{array}\right\} \tag{5.2.3}$$

Consider a line of force initially along the z-axis. Ignoring the drift of fluid through field, we may easily see that, after unit time (say), the line of force is distorted into a spiral whose projection on the (x,y)-plane is shown in Figs. 5.2(i) and 5.2(ii).

Suppose now that the initial field is $(0, 0, \sin x)$. Each field line is distorted in the same way. After the time $\pi/2$, the two oppositely-directed field-lines initially at $(0,0)$ and $(\pi, 0)$ become, in projection, the loops shown in Figs. 5.3(i) and (ii). (In these figures, the broader parts of the lines indicate nearness to the reader, i.e.

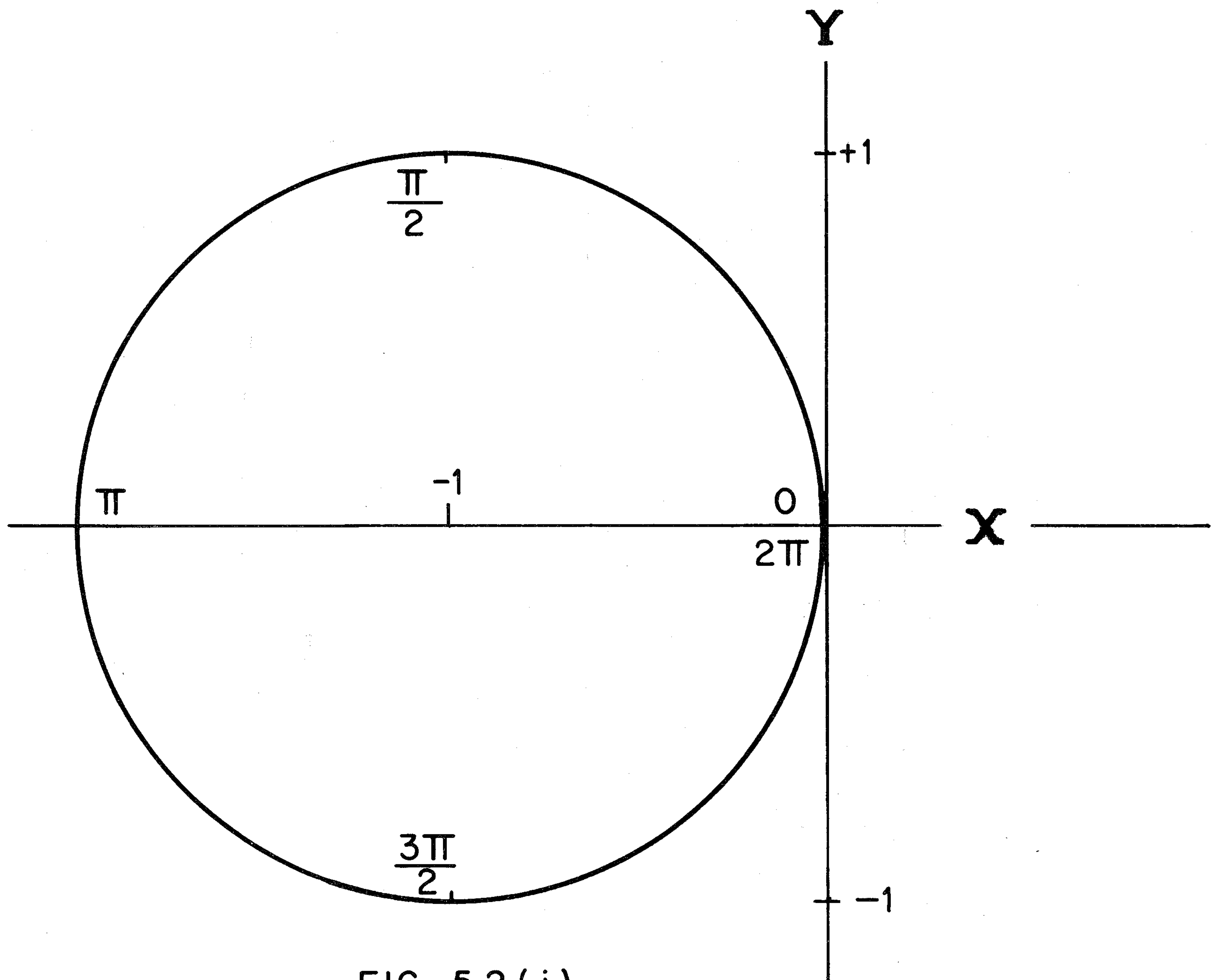
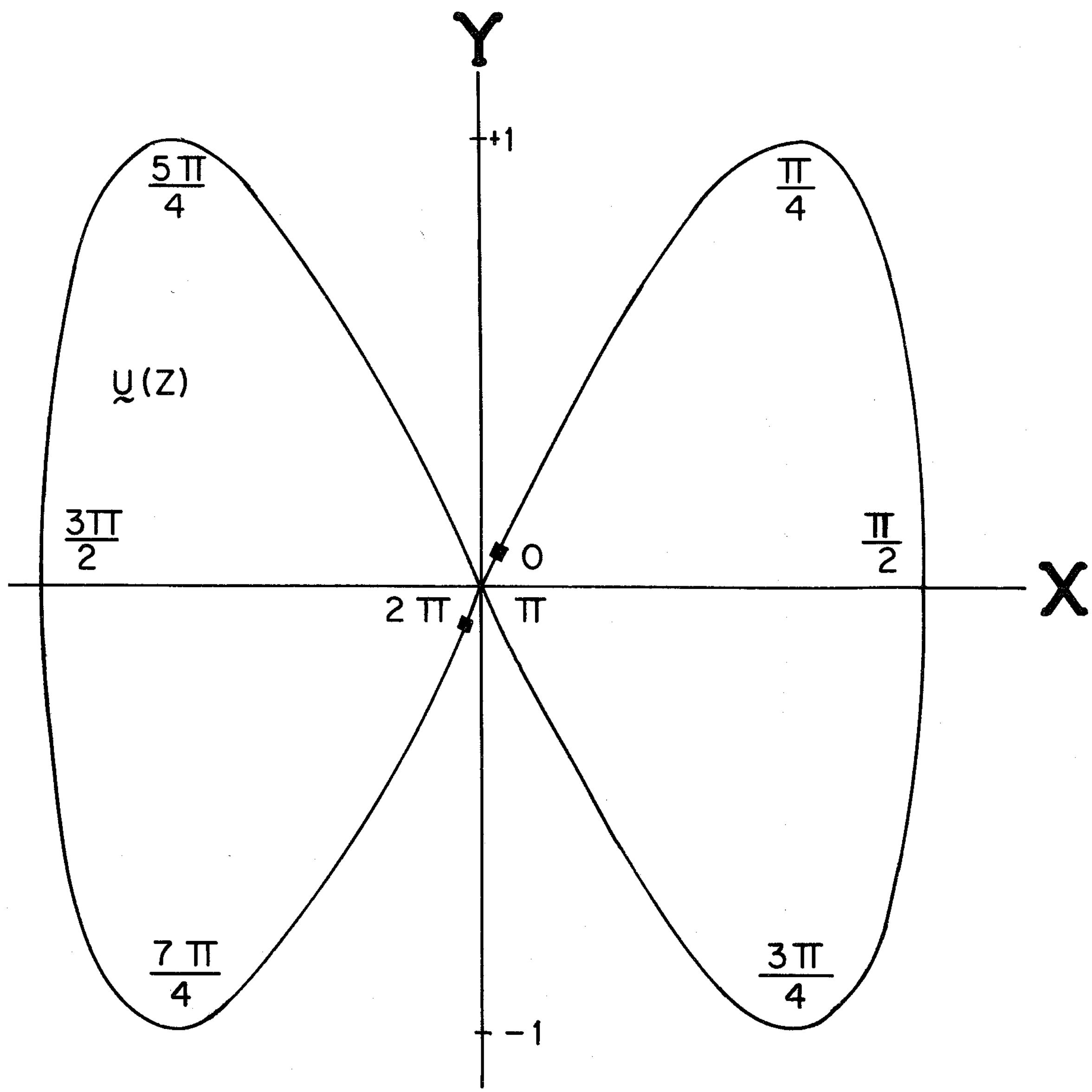


FIG. 5.2 (i)



Y
+1
X
$\frac{5\pi}{4}$
$\frac{\pi}{4}$
$\underset{\sim}{U}(Z)$
$\frac{3\pi}{2}$
$\frac{\pi}{2}$
0
2π
π
$\frac{7\pi}{4}$
$\frac{3\pi}{4}$
−1

FIG. 5.2 (ii)

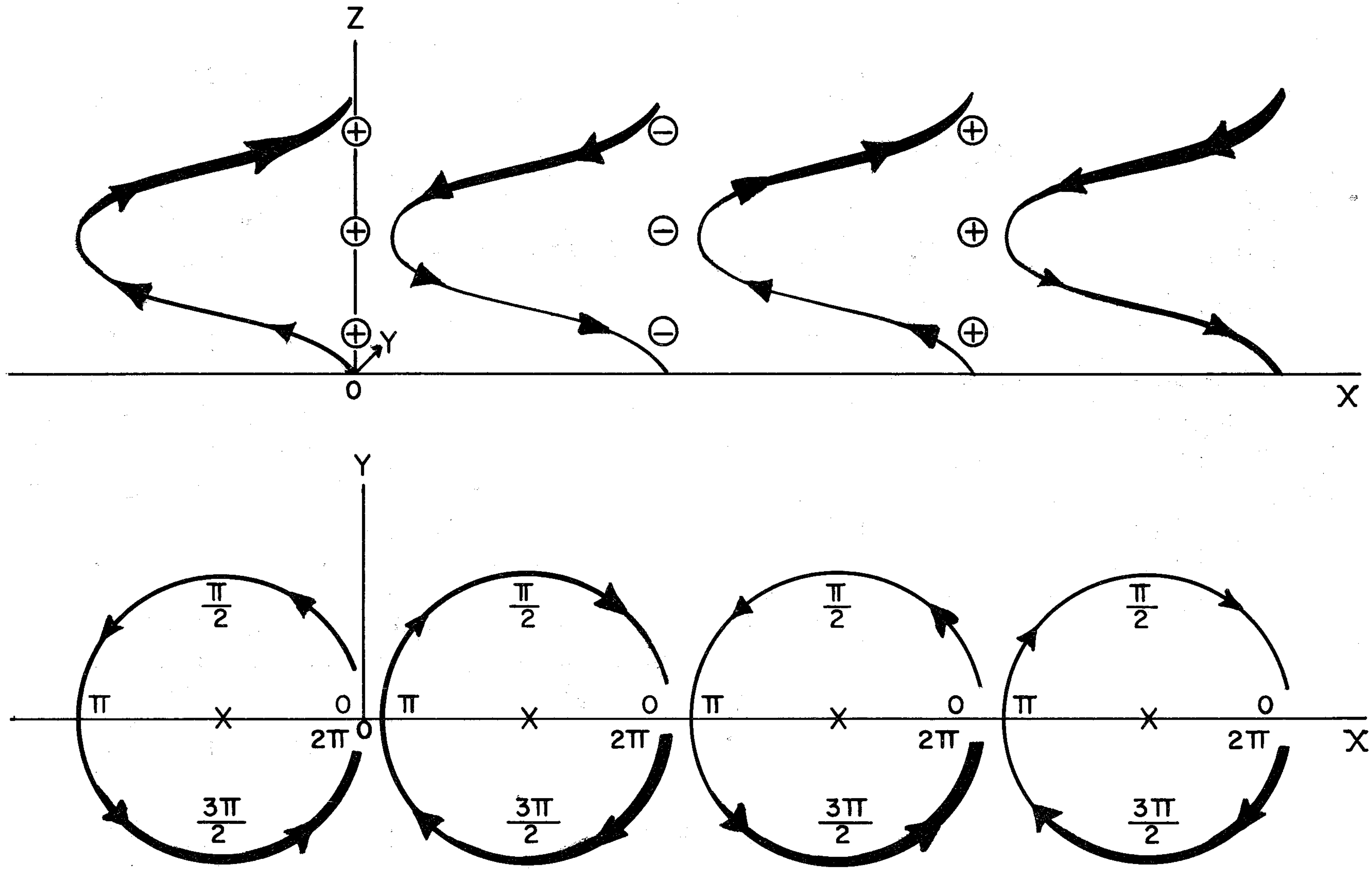


FIG. 5.3 (i)

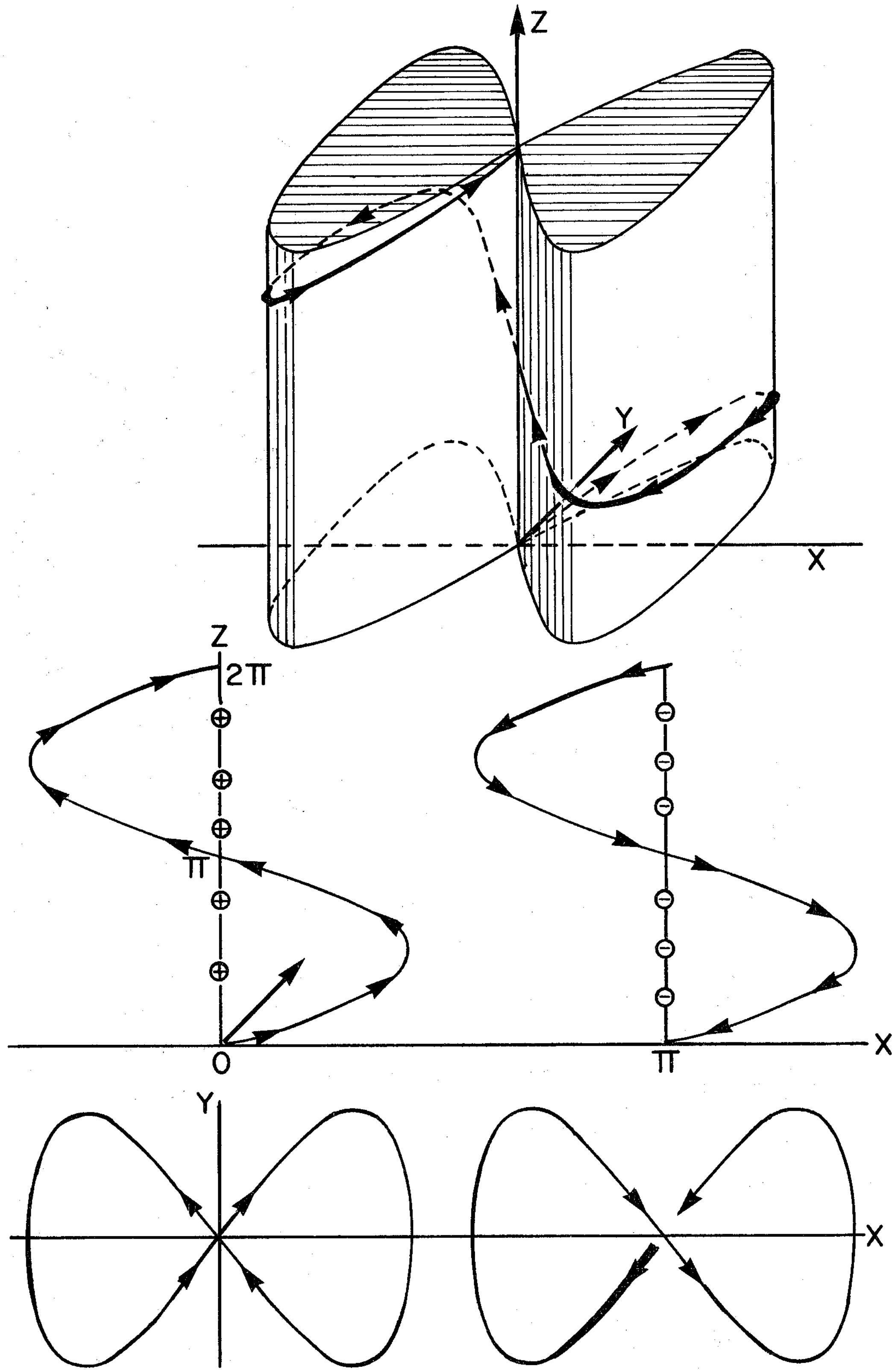


FIG. 5.3 (ii)

larger z and more negative y .)

And now, as before, the general effect is to create y-components which are particularly strong at the locations shown by the small circles in the (x,z)-planes of the figures. To regenerate the original field, we can, as before, add a constant shear in the z-direction. Unlike the case considered in § 5.1, however, the regeneration occurs at the "right place" and not displaced to right or left. In other words, we expect the solutions to be stationary, or, if we omit the constant term in V_{1x} in (5.2.3), that the first model should yield a progressive wave. We have now arrived at the flows (including constants ℓ, U, V and W)

$$\left.\begin{array}{ll} \underset{\sim}{v}_1^* = (2U\cos \ell z,\ V\sin \ell z,\ Wy), & \text{for model 1,} \\ \underset{\sim}{v}_2^* = (U\sin \ell z,\ V\sin 2\ell z,\ Wy), & \text{for model 2,} \end{array}\right\} \quad (5.2.4)$$

which, under certain circumstances, should regenerate a field of the form (5.2.2). The form (5.2.4) is not, however, very convenient since one hopes to be able to use Fourier analysis throughout. If, however, we simply replaced y by $\sin ky$ in the v_z^* components, the shear would, for half the y-cycle, be of the wrong sign for regeneration. This may be avoided by reversing the sign of the v_x^* in these regions, e.g. by including a factor $\cos ky$ in v_x^* ; and so (5.2.1) is reached for the first motion. For the second motion, a term $U\sin ky\cos \ell z$ is also added to the x-component to preserve the maximum symmetry between the y- and z-coordinate directions.

Now we look at Roberts's model from an analytic viewpoint. First we decompose $\underline{B}$ into its Fourier components:

$$\underline{B} = \sum_{m,n} \underline{B}_{mn} e^{i(jx+mky+nlz)+\omega t}, \qquad (m,n \text{ integers}), \tag{5.2.5}$$

where, since

$$\operatorname{div} \underline{B} = 0,$$

we have

$$jB_{mnx} + mkB_{mny} + nlB_{mnz} = 0. \tag{5.2.6}$$

This determines B_{mnx}, once B_{mny} and B_{mnz} are known. This is useful since the y- and z-components of the induction equations do not involve B_{mnx}. We can, therefore, solve these separately and return, if necessary, to (5.2.6) to determine B_{mnx}. We now introduce the following dimensionless quantities:

$$X = P+Q, \quad P = \frac{2\omega}{Uj}, \quad Q = \frac{2\eta j^2}{Uj}, \quad Y = \frac{Wl}{Uj}, \quad Z = \frac{Vk}{Uj}, \tag{5.2.7}$$

$$A = \frac{2\eta k^2}{Uj}, \quad B = \frac{2\eta l^2}{Uj}. \tag{5.2.8}$$

We confine ourselves to positive values of j, k, l, U, V and W. This is justified, for the first motion, by the fact that $\mathcal{R}(\omega)$ is unaffected by any change in sign of these quantities. For the second motion, the same is true except that the sign of VW should be preserved; it appears that dynamo action requires that VW must be positive. For given A, B, Y and $Z (>0)$, we seek an eigenvalue, X, of the equations (see (5.2.11) below) such that $\mathcal{R}(X) > 0$. For, once we have done that, we can always, by reducing j and increasing U (preserving Uj), make Q less than $\mathcal{R}(X)$, and then $\mathcal{R}(P) > 0$, i.e. $\mathcal{R}(\omega) > 0$, and dynamo regeneration has been established. It may be noticed that this process of reducing j may make the dynamo "nearly" two-dimensional. Let

$$\underline{h}_{mn} = (y_{mn},\ z_{mn}), \tag{5.2.9}$$

where

$$y_{mn} = \ell B_{mny};\ z_{mn} = kB_{mnz}. \tag{5.2.10}$$

The induction equation becomes a set of recurrence relations for $\underline{h}_{mn}$:

$$(X + m^2A + n^2B)\underline{h}_{mn} = [T\underline{h}]_{mn}, \tag{5.2.11}$$

where T is a linear operator defined, for the second model, by

$$[T\underline{h}]_{mn} = (\underline{h}_{m+1,n+1} - \underline{h}_{m-1,n-1}) + nY(\underline{h}_{m+2,n} - \underline{h}_{m-2,n}) + mZ(\underline{h}_{m,n+2} - \underline{h}_{m,n-2}) +$$
$$+ \{2Z(z_{m,n+2} + z_{m,n-2}),\ 2Y(y_{m+2,n} + y_{m-2,n})\}, \tag{5.2.12}$$

and similarly for the first model. The last term of this expression is a stretching term, and the other three terms on the right-hand side are convection terms. This relation is similar to the recurrence relations for Mathieu functions, and similar techniques of solution can be applied.

The analytic task of solving these equations and obtaining an eigenvalue is reduced by making the recurrence relations converge as quickly as possible. We therefore want A and/or B to be large. One starts with $\ell = 0$, $X_\ell = 0$ and then iterates as follows:

One takes first

$$\underline{h}_{00} = (0,1), \tag{5.2.13}$$

and then solves the recurrence relation

$$\underline{h}_{mn}(X_\ell) = \frac{[T\underline{h}(X_\ell)]_{mn}}{X_\ell + m^2A + n^2B}, \qquad (m,n) \neq (0,0), \tag{5.2.14}$$

under the condition that $|\underline{h}_{mn}| \longrightarrow 0$ as $(m^2+n^2) \longrightarrow \infty$, since the magnetic energy must remain finite. One then computes $\chi_{\ell+1}$ from

$$\chi_{\ell+1}(0,1) = [T\underline{h}(\chi_\ell)]_{00} , \qquad (5.2.15)$$

returns to (5.2.13), and repeats until the process converges, as it must for large A and/or B. (Note that it can be proved that the first component of the right-hand side of (5.2.15) is zero; thus the notation on the left of that equation is justified.) To solve (5.2.14) one introduces a sub-iteration, writing

$$\underline{h}_{00}^{(k)} = \begin{cases} (0,1), & k=0, \\ (0,0), & k \neq 0, \end{cases} \qquad (5.2.16)$$

and computing

$$\underline{h}_{mn}(\chi_\ell) = \sum_{k=0}^{\infty} \underline{h}_{mn}^{(k)} , \qquad (5.2.17)$$

where

$$\underline{h}_{mn}^{(k+1)} = \frac{[T\underline{h}^{(k)}]_{mn}}{\chi_\ell + m^2 A + n^2 B} . \qquad (5.2.18)$$

(A similar iteration scheme can be used to determine the smallest eigenvalue of Mathieu's equation.)

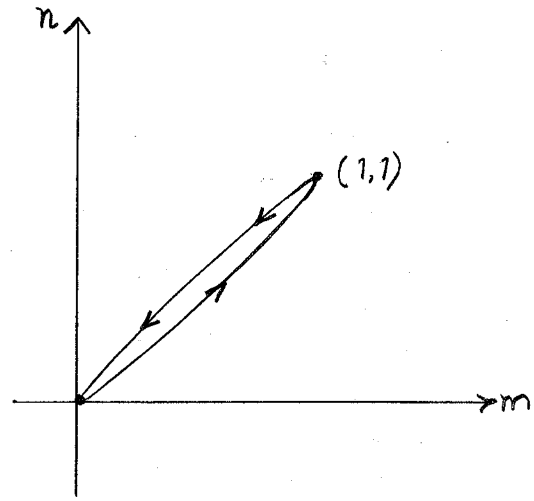
The result of this inner iteration can be expressed as a diagram expansion

$$\chi_{\ell+1}(0,1) = \sum_{k=2}^{\infty} \sum_{\substack{\text{diagrams} \\ \text{of } k \text{ sides}}} \frac{\prod_{\text{steps}, s} \alpha_s}{\prod_{\text{points}, p} \beta_p} (0,1) . \qquad (5.2.19)$$

A diagram in (m,n)-space is a series of line segments beginning and ending at the origin, and not passing through the origin at intermediate steps, e.g. those shown in Figs. 5.4 and 5.5. For each step taken the appropriate operator is applied from the left, where α_s corresponds to a term on the right of

Fig. 5.4

A two-step diagram



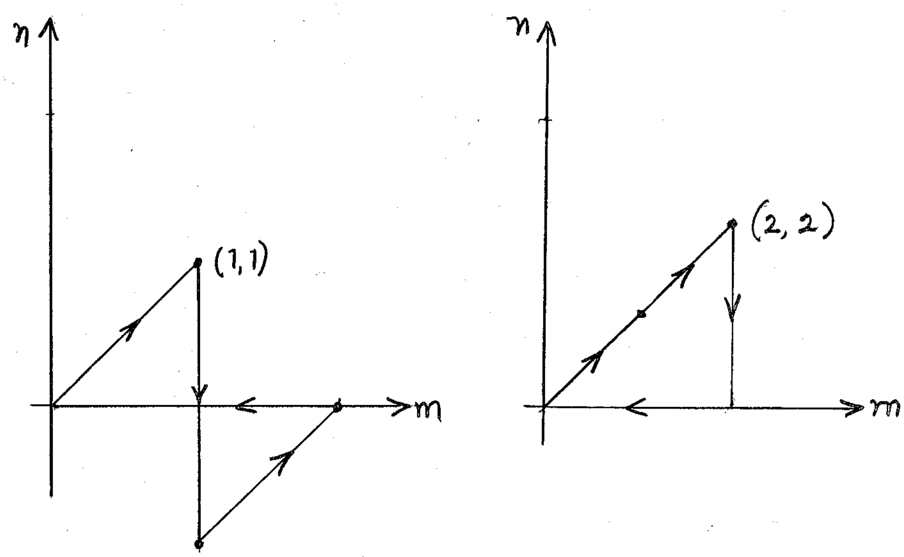


Fig. 5.5

Two four-step diagrams

(5.2.12); successively,

$$\left.\begin{array}{lll}
(m,n) \longrightarrow (m-1,n-1) & \alpha_s(y,z) = & (y,z), \\
(m,n) \longrightarrow (m+1,n+1) & \alpha_s(y,z) = & -(y,z), \\
(m,n) \longrightarrow (m,n-2) & \alpha_s(y,z) = & Z(my+2z,mz), \\
(m,n) \longrightarrow (m,n+2) & \alpha_s(y,z) = & Z(-my+2z,-mz), \\
(m,n) \longrightarrow (m-2,n) & \alpha_s(y,z) = & Y(ny,nz+2y), \\
(m,n) \longrightarrow (m+2,n) & \alpha_s(y,z) = & Y(-ny,-nz+2y).
\end{array}\right\} \quad (5.2.20)$$

For each point (m,n) of the diagram (other than the origin), an appropriate algebraic factor

$$\beta_p = X_\ell + m^2A + n^2B,$$

appears in the denominator of (5.2.19), as can be seen from (5.2.18). The series (5.2.19) starts at $k=2$, since it takes at least two steps to leave the origin and return.

For the two-step diagram shown, the successive α operations on (0,1) are (0,1) $\longrightarrow$ (0,-1) $\longrightarrow$ (0,-1) leading to a contribution to X_1, of $-1/B_{11} = -1/(A+B)$, since $X_0 = 0$. There are two diagrams of this type.

For the first of the four-step diagrams shown, the successive operations are (0,1) $\longrightarrow$ (0,1) $\longrightarrow$ Z(2,-1) $\longrightarrow$ Z(-2,1) $\longrightarrow$ YZ(0,4). Since (for $X_0 = 0$), $\beta_{11} = \beta_{1,-1} = (A+B)$ and $\beta_{20} = 4A$, the contribution to X_1 is $YZ/A(A+B)^2$. There are two diagrams of this type. For the second of the four-step diagrams, the α operations are (0,1) $\longrightarrow$ (0,-1) $\longrightarrow$ (0,1) $\longrightarrow$ Z(2,2) $\longrightarrow$ YZ(0,4). Since $\beta_{11} = (A+B)$, $\beta_{22} = 4(A+B)$ and

$B_{20} = 4A$, the contribution to X_1 is $YZ/4A(A+B)^2$. There are two diagrams of this type and two others which make the same contribution in the $B \gg A$ limit considered below. The net contribution from these types of diagrams is therefore $3YZ/AB^2$ for $B \gg A$.

The process can be made to converge very rapidly by making A or B (or both) large, X at the same time being small. It is then sufficient to consider only a few of the diagrams and only the first iterate, X_1 , for the second motion, and only the first and second iterates for the first motion. By taking

$$A = 1, \quad B = N^6, \quad Y = N^5, \quad Z = N^2, \quad Q = N^{-6}, \tag{5.2.22}$$

and considering the limit $N \to \infty$, the contribution $3YZ/AB^2 = 3N^{-5}$ the four-step diagrams shown in Fig. 5.5 dominates all others. (For example, we can see from the argument just given that the two-step diagrams contribute $-2/B = -2N^{-6}$ only.) We obtain, then,

$$P \sim 3N^{-5}, \text{ as } N \to \infty, \tag{5.2.23}$$

establishing dynamo action.

In a similar fashion, for the first model, Roberts has shown, for

$$A = N^2, \quad B = N^5, \quad Y = N^4, \quad Z = N^2, \quad Q = N^{-5}, \tag{5.2.24}$$

and

$$\mathcal{R}(P) \sim 56N^{-4}, \quad \mathcal{I}(P) \sim -8N^{-1}, \text{ as } N \to \infty. \tag{5.2.25}$$

The imaginary part of P is associated with the progression of the wave in the x-direction.

The asymptotic results just obtained depend, for their success, on constructing multiple length and velocity scales (see also § 6 below). Convincing results can, however, be obtained by numerical means even when all the length and all the velocity scales are comparable in magnitude. Indeed, Roberts work appears to be the only convincing numerical demonstration to date that the straight dynamo equations, unsullied by small or large parameter limits, have solutions. For simplicity, Roberts restricted himself to cases in which $V = W$ and $k = \ell$, so that $A = B$ and $Y = Z$. He introduced three dimensionless numbers:

$$L = \frac{k}{j}, \qquad S = \frac{U}{V}, \qquad R = \frac{V}{k\eta}, \tag{5.2.26}$$

the last being, of course, a magnetic Reynolds number. By (5.2.7-8), we have

$$Q = \frac{2}{RS^2Y}, \qquad Y = \frac{L}{S}, \qquad A = \frac{2Y}{R}. \tag{5.2.27}$$

The required eigenvalue, X, may now be considered to be a function of A and Y, or of R and Y, as convenient. It may be shown that, for given R, $X(R,Y)$ has the following properties:

$$\left.\begin{array}{l}
Y\Re(X) \rightarrow f(R) \text{ as } Y \longrightarrow \infty, \text{ for both motions,} \\
\Im(X) \rightarrow c(R) \text{ as } Y \longrightarrow \infty, \text{ for the first motion,} \\
\Im(X) = 0 \text{ for sufficiently large } Y, \text{ for the second motion,} \\
\Re(X) < 0 \text{ for sufficiently small } Y, \text{ for both motions.}
\end{array}\right\} \tag{5.2.28}$$

Roberts used two numerical methods. The first is a straightforward cut-off procedure in which one sets $\mathbf{h}_{mn} = 0$, for $|m|$ and $|n| > r$, and obtains X. One observes whether X tends to a limit as r increases, and if so what. Storage is a limitation in this method, and Roberts was confined to

$r \leq 7$. The second scheme is to divide the matrix equations into an inner (Strong) and outer (Weak) set of equations, writing (5.2.11) as

$$(X + m^2A + n^2B)\,\underline{h}_S = T_{SS}\,\underline{h}_S + T_{SW}\,\underline{h}_W, \qquad (5.2.29,\ \underline{S}trong)$$

$$(X + m^2A + n^2B)\,\underline{h}_W = T_{WW}\,\underline{h}_W + T_{WS}\,\underline{h}_S, \qquad (5.2.29,\ \underline{W}eak)$$

where $\underline{h}_S$ denotes the strong coefficients $(|m|, |n| \leq r)$, and $\underline{h}_W$ denotes the weak coefficients $(S > (|m|$ and $|n|) > r)$. By setting $\underline{h}_W = 0$ and solving (5.2.29, S), one can obtain a first estimate, X_0, of X; or it may be sufficient simply to set X_0 zero. Since m and n are large in the weak equations, the $m^2A + n^2B$ factor on the left of (5.2.29, W) dominates the X term to which it is joined. We may therefore write the solution to that equation as

$$\underline{h}_W = S_{WS}\,\underline{h}_S, \qquad (5.2.30,\ \underline{W}eak)$$

confident that the linear operator, $S_{WS}(X)$, is a slowly varying function of X well approximated by $S_{WS}(X_0)$. Now (5.2.29, S) becomes

$$(X + m^2A + n^2B)\,\underline{h}_S = T_{SS}\,\underline{h}_S + T_{SW}\,S_{WS}\,\underline{h}_S, \qquad (5.2.30,\ \underline{S}trong)$$

the linear operator $T_{SW}S_{WS}$ having, in fact, zero matrix elements except for $\underline{h}_S$ components on the edge of the rectangle $|m|, |n| < r$. It is evaluated by an iterative procedure akin to that used to obtain $\underline{h}_{mn}(X_i)$ in the analytic method already described; the rapidity of the convergence of this iteration depends on variable relaxation methods. For $r > 2$, the X_i iteration converges extremely rapidly; X_1 is in fact usually a very good approximation to the required eigenvalue.

Roberts's numerical results for $X(R,Y)$ for a range values of R and Y are in agreement with (5.2.28) above, and indicate a further property, viz.

$$Y\mathcal{R}(X)\leq f(R), \text{ all } Y. \tag{5.2.31}$$

Since $P = X - Q$, we see by (5.2.27,31) that

$$\mathcal{R}(P)\leq \frac{1}{Y}\left[f(R)-\frac{2}{RS^2}\right], \tag{5.2.32}$$

with asymptotic equality as $Y\rightarrow\infty$. Provided

$$f(R) > \frac{2}{RS^2}, \tag{5.2.33}$$

we can, by making Y sufficiently large, obtain dynamo action $[\mathcal{R}(P)\geq 0]$. In other words, provided S is chosen so that

$$\frac{1}{S}\equiv\frac{V}{U} > \sqrt{\frac{Rf(R)}{2}} = \Sigma(R), \text{ say}, \tag{5.2.34}$$

we can obtain dynamo action for a sufficiently large value of the length scale $L = SY$, or for a sufficiently large value of

$$S^2Y = \frac{U/j\eta}{V/k\eta}, \tag{5.2.35}$$

the ratio of two magnetic Reynolds numbers of the problem.

Figure 5.6(i) shows X as a function of Y for R = 2 for the first motion. It may be seen that $X > 0$ for $Y > 2$. Fig. 5.6(ii) shows X as a function of Y for R = 4 for the second motion. It may be seen that $X > 0$ for $Y > 0.63$. Thus, in both cases, the dynamo is regenerative. It is also interesting to observe that $\mathcal{R}(X)$ has a maximum in R. This, it will be recalled, was also a feature of the Parker dynamo waves (cf. (5.1.23) above). Figs. 5.7 show $\Sigma(R)$; it will be noted that, for sufficiently large R at least, Σ increases with R. For all $\Sigma > 0$,

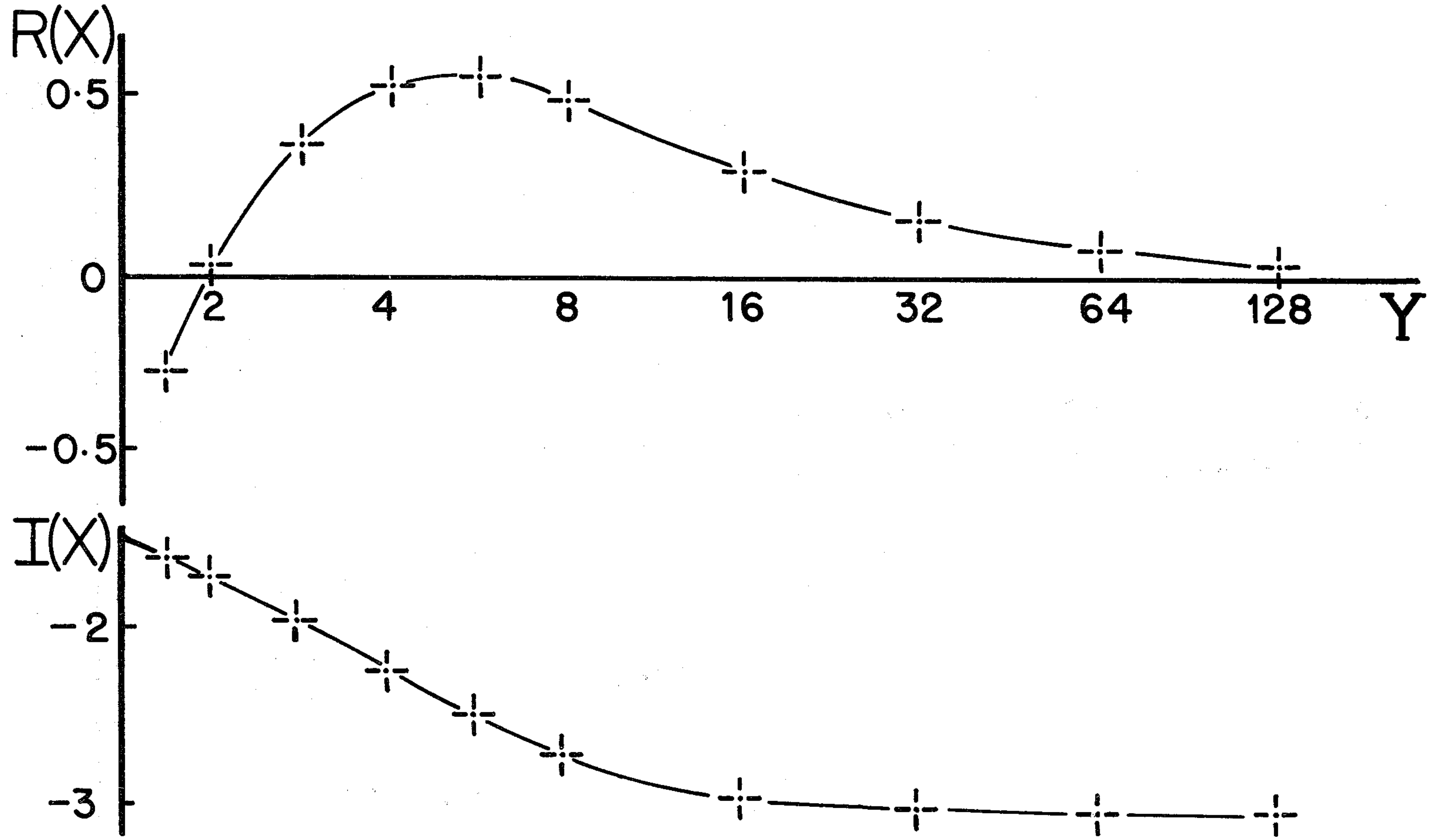


FIG. 5.6(i)

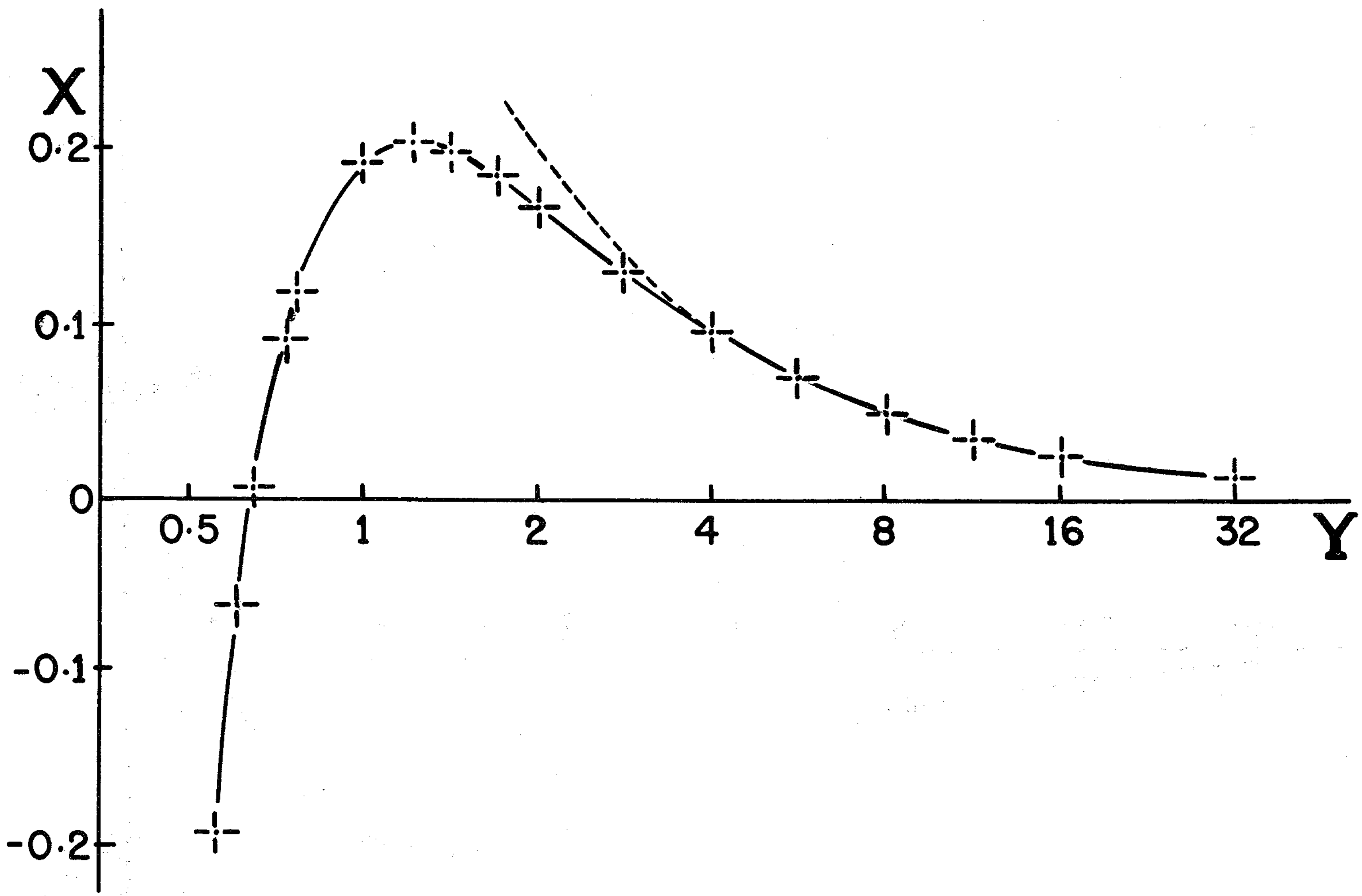


FIG. 5.6(ii)

dynamo action is possible, for small enough $\mathcal{S}$.

Roberts (private communication) has recently extended his work to other periodic motions. Preliminary results indicate that the motions (5.2.1) are atypical in the sense that they are inefficient dynamos. It appears that, for general periodic motion (not necessarily nearly two-dimensional), there is a chance slightly under ½ of dynamo action at (essentially) all magnetic Reynolds numbers, and that even if this chance is not realized, there will be dynamo action for sufficiently large Reynolds numbers. Motions for which there is no dynamo action are, it appears, exceptional.

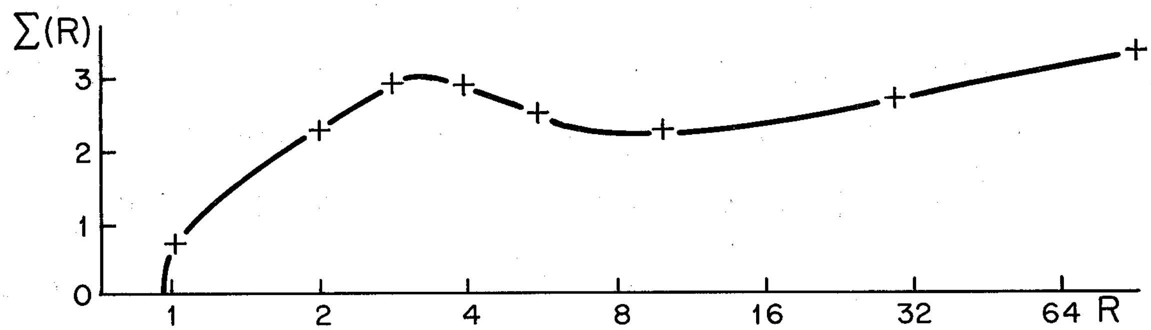


Fig. 5.7(i)

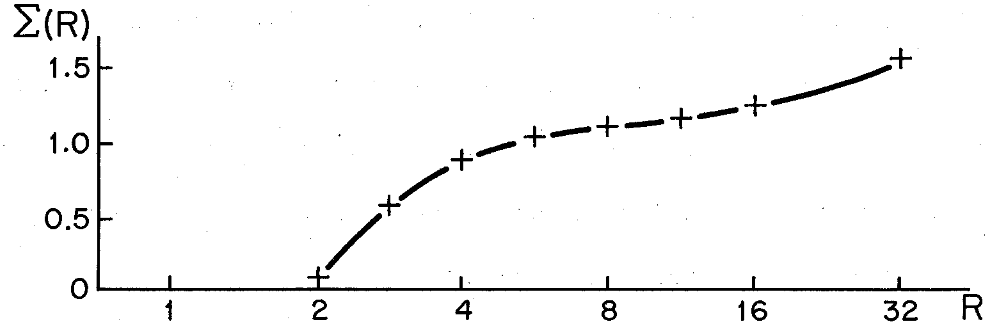


Fig. 5.7(ii)

5.3 Lortz's Family of Dynamos

While on the topic of dynamos in an unbounded conductor, we shall give an exact solution discovered very recently* by Lortz (1967). In this solution, the fields have a helical structure.

Let (ϖ, φ, z) be cylindrical polar coordinates, and let

$$\xi = m\varphi + kz. \tag{5.3.1}$$

Introduce the solenoidal vector field

$$\underline{w} = \left[0, -\frac{k\varpi}{k^2\varpi^2 + m^2}, \quad \frac{m}{k^2\varpi^2 + m^2}\right]. \tag{5.3.2}$$

---

*So recently, in fact, that when I started these lectures I did not know it existed!

This is a field orthogonal to the surfaces of constant ξ and of constant ϖ, i.e. its lines are helices of constant pitch. It may be noticed that

$$\operatorname{curl} \underline{w} = -\frac{2km}{k^2\varpi^2+m^2}\underline{w}, \tag{5.3.3}$$

i.e. $\underline{w}$ is a Beltrami (or force-free) field. We consider a class of fields which depend on ξ and ϖ alone. We may observe that, if $f = f(\xi, \varpi)$, then both $f\underline{w}$ and $\underline{w} \times \operatorname{grad} f$ are solenoidal. [To see the latter, observe that $\underline{w} \times \operatorname{grad} f = -\operatorname{curl}(f\underline{w}) + f \operatorname{curl} \underline{w}$, and the second vector on the right is solenoidal by (5.3.3).]

Thus, we may seek solutions in the form

$$\underline{B} = h\underline{w} + \underline{w} \times \operatorname{grad} H, \tag{5.3.4}$$

where h and H are both functions of ξ and ϖ above. Operating on (5.3.4) by curl, we obtain

$$\operatorname{curl} \underline{B} = J\underline{w} - \underline{w} \times \operatorname{grad} h, \tag{5.3.5}$$

where

$$J = \frac{(-2kmh + LH)}{(k^2\varpi^2+m^2)}, \tag{5.3.6}$$

and L is the linear operator

$$L = \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left[\frac{\varpi}{(k^2\varpi^2+m^2)}\frac{\partial}{\partial\varpi^2}\right] + \frac{1}{\varpi^2}\frac{\partial^2}{\partial\xi^2}. \tag{5.3.7}$$

Now introduce $\underline{u}$ as

$$\underline{u} = v\underline{w} + \underline{w} \times \operatorname{grad} V, \tag{5.3.8}$$

where v and V are both functions of ξ and ϖ alone. Then

$$\underline{u} \times \underline{B} = [-\underline{w}\cdot(h \operatorname{grad} V - v \operatorname{grad} H + \operatorname{grad} H \times \operatorname{grad} V)]\underline{w} + w^2(h \operatorname{grad} V - v \operatorname{grad} H), \tag{5.3.9}$$

and, taking components in the induction equation,

$$\eta \operatorname{curl} \underline{B} = -\operatorname{grad} \Phi + \underline{u} \times \underline{B}, \tag{5.3.10}$$

we obtain

$$-\frac{\eta}{\varpi}\frac{\partial h}{\partial \xi} = \frac{\partial \Phi}{\partial \varpi} - \frac{1}{(k^2\varpi^2+m^2)}\left(h\frac{\partial V}{\partial \varpi} - v\frac{\partial H}{\partial \varpi}\right), \tag{5.3.11}$$

$$\frac{\eta\varpi}{(k^2\varpi^2+m^2)}\frac{\partial h}{\partial \varpi} = \frac{\partial \Phi}{\partial \xi} - \frac{1}{(k^2\varpi^2+m^2)}\left(h\frac{\partial V}{\partial \xi} - v\frac{\partial H}{\partial \xi}\right), \tag{5.3.12}$$

$$\eta\varpi\left[LH - \frac{2k\ell}{(k^2\varpi^2+m^2)^2}h\right] = -\frac{1}{(k^2\varpi^2+m^2)}\left(\frac{\partial H}{\partial \varpi}\frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \varpi}\frac{\partial H}{\partial \xi}\right). \tag{5.3.13}$$

On eliminating Φ between (5.3.11) and (5.3.12), we obtain

$$\eta\varpi Lh = -\frac{1}{(k^2\varpi^2+m^2)}\left[\left(\frac{\partial h}{\partial \varpi}\frac{\partial V}{\partial \xi} - \frac{\partial h}{\partial \xi}\frac{\partial V}{\partial \varpi}\right) - \left(\frac{\partial v}{\partial \varpi}\frac{\partial H}{\partial \xi} - \frac{\partial v}{\partial \xi}\frac{\partial H}{\partial \varpi}\right)\right] + $$

$$+\frac{2k^2\varpi}{(k^2\varpi^2+m^2)^2}\left(h\frac{\partial V}{\partial \xi} - v\frac{\partial H}{\partial \xi}\right). \tag{5.3.14}$$

Equations (5.3.13) and (5.3.14) determine h and H from v and V. We now consider the particular case

$$\left.\begin{aligned} v &= v_1(\varpi)\sin\alpha\xi + v_2(\varpi)\sin 2\alpha\xi, \\ V &= V_1(\varpi)\sin\alpha\xi, \end{aligned}\right\} \tag{5.3.15}$$

seeking solutions of the form

$$\left.\begin{aligned} h &= h_0(\varpi) + h_1(\varpi)\cos\alpha\xi, \\ H &= H_0(\varpi). \end{aligned}\right\} \tag{5.3.16}$$

On equating coefficients independent of ξ and proportional to $\cos\alpha\xi$ in (5.3.13), we obtain, respectively,

$$\frac{2mk}{(k^2\varpi^2+m^2)^2}h_0 = LH_0, \tag{5.3.17}$$

$$\frac{2\eta mk\varpi}{(k^2\varpi^2+m^2)}h_1 = \alpha H_0' V_1. \tag{5.3.18}$$

On equating coefficients independent of ξ, proportional to $\cos\alpha\xi$ and proportional to $\cos 2\alpha\xi$ in (5.3.14), we obtain, respectively,

$$2\eta\varpi L h_0 = -\frac{\alpha(V_1 h_1' + h_1 V_1')}{(k^2\varpi^2+m^2)} + \frac{2k^2\varpi\alpha V_1}{(k^2\varpi^2+m^2)^2} h_1 , \tag{5.3.19}$$

$$\eta\varpi\left(L h_1 - \frac{\alpha^2}{\varpi^2} h_1\right) = -\frac{\alpha(V_1 h_0' + v_1 H_0')}{(k^2\varpi^2+m^2)} + \frac{2k^2\varpi\alpha V_1}{(k^2\varpi^2+m^2)^2} h_0 , \tag{5.3.20}$$

$$0 = -\frac{\alpha(V_1 h_1' - h_1 V_1' + 4\alpha v_2 H_0')}{(k^2\varpi^2+m^2)} + \frac{2k\varpi\alpha V_1}{(k^2\varpi^2+m^2)^2} h_1 . \tag{5.3.21}$$

The last two of these determine v_1 and v_2 when the remaining functions have been determined. We may solve for h_0, h_1, and V_1 in terms of from (5.3.17) to (5.3.19). Indeed (5.3.17) gives directly

$$h_0 = \frac{(k^2\varpi^2+m^2)^2}{2k\ell\varpi}\frac{d}{d\varpi}\left[\frac{\varpi H_0'}{k^2\varpi^2+m^2}\right] . \tag{5.3.22}$$

On writing (5.3.19) in the form

$$\alpha\frac{d}{d\varpi}\frac{h_1 V_1}{k^2\varpi^2+m^2} = -2\eta\varpi L h_0 = -2\eta\frac{d}{d\varpi}\left[\frac{\varpi h_0'}{k^2\varpi^2+m^2}\right],$$

we find

$$\alpha h_1 V_1 = -2\eta\left[\varpi h_0' + a(k^2\varpi^2+m^2)\right], \tag{5.3.23}$$

where a is a constant. By (5.3.18) we now obtain

$$h_1^2 = -\frac{(k^2\varpi^2+m^2)}{mk\varpi} H_0'\left[\varpi h_0' + a(k^2\varpi^2+m^2)\right], \tag{5.3.24}$$

$$V_1 = \frac{2\eta mk\varpi}{\alpha(k^2\varpi^2+m^2)}\frac{h_1}{H_0'} . \tag{5.3.25}$$

Substitution of these results into (5.3.20-21) and (5.3.11/12), completes the solution:

$$v_1 = -\frac{(k^2\varpi^2+m^2)}{m H_0'}\left[\eta\left(\frac{\varpi h_1'}{k^2\varpi^2+m^2}\right)' - \frac{\eta\alpha^2}{\varpi} h_1 + \alpha\left(\frac{h_0}{k^2\varpi^2+m^2}\right)' V_1\right], \tag{5.3.26}$$

$$v_2 = \frac{(k^2\varpi^2+m^2)}{4H_0'}\left[\left(\frac{h_1}{k^2\varpi^2+m^2}\right)V_1' - \left(\frac{h_1}{k^2\varpi^2+m^2}\right)' V_1\right] , \qquad (5.3.27)$$

$$\Phi = \frac{1}{(k^2\varpi^2+m^2)}\left[\left(h_0 V_1 + \frac{\eta\varpi}{\alpha}\right) h_1' \sin\alpha\xi + \frac{1}{4} h_1 V_1 \sin 2\alpha\xi\right] . \qquad (5.3.28)$$

The function H_0 is arbitrary although v_1 and v_2 are in danger of becoming infinite at a zero of H_0'. The axis $(\varpi=0)$ of the helices is, therefore, potentially dangerous. It may be shown, however, that provided $\alpha m = \pm 1$ (i.e. provided that $\xi = \pm\varphi + k'z$), a solution can be obtained in which $\underline{B}$ is uniform (and of strength, b, say) on the axis, in which $a = 0$, and in which

$$\left.\begin{aligned} &H_0' = bk\varpi - O(\varpi^3), \quad h_0 = b\ell - O(\varpi^2), \quad h_1 = O(\varpi^2), \\ &V_1 = O(\varpi), \quad v_1 = O(\varpi), \quad v_2 = O(\varpi^2); \quad \varpi \to 0. \end{aligned}\right\} \qquad (5.3.29)$$

Also one can arrange that $\underline{u}$ and $\underline{B}$ vanish at infinity, by taking

$$\left.\begin{aligned} &H_0' = c\ell\varpi^{-1} + O(\varpi^{-5}), \quad h_0 = -ck + O(\varpi^{-6}), \quad h_1 = O(\varpi^{-3}), \\ &V_1 = O(\varpi^{-3}), \quad v_1 = O(\varpi^{-1}), \quad v_2 = O(\varpi^{-6}), \quad \varpi \to \infty, \end{aligned}\right\} \qquad (5.3.30)$$

where c is a constant of the same sign as bk/ℓ, the field at great distances then being in the φ-direction and of strength c/ϖ. Apart from (5.3.29) and (5.3.30), there appears to be no restrictions on H_0', apart from the obvious one that $H_0' \neq 0$, for $0 < \varpi < \infty$. It may be noticed, however, that B_z has a non-zero component over most of the fluid even "at infinity" in z. Whether this difficulty may be overcome by "joining the ends" of the cylinder together to form a torus is not clear. Lortz's basic idea of "combining" axially-symmetric and two-dimensional fields without in any obvious way including the anti-dynamo theorems of either, is, however, very ingenious.

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6. The Multiple-scale Method

Stephen Childress

6.1 Introduction: heuristic discussion of the infinite conductor

The multiple-scale method (Childress, 1967a, b) has many points of similarity with Braginskiĭ's method explained by Roberts (see § 4 above), although a formal connection has yet to be established. In Braginskiĭ's approach, variables are divided into axially symmetric and asymmetric parts by an averaging process over φ. In the multiple-scale method, variables are supposed to consist of a spatially rapidly varying part (the microscale, ℓ) and of a spatially slowly varying part (the macroscale, L), and an averaging process over the microscale plays a central part. In Braginskiĭ's method, the variables are expanded in powers of $R^{-\frac{1}{2}}$, where R is the magnetic Reynolds number based on the scale of the conductor and the largest velocity in it. In the multiple-scale method, the variables are expanded in $\varepsilon^{\frac{1}{2}}$, where $\varepsilon = \ell/L$, the scale parameter, is small. The multiple-scale method has been developed to a much higher degree of mathematical rigour than has Braginskii's approach.

We suppose the conducting fluid is uniform, incompressible and fills all space. We write the induction equations

$$\text{curl } \underline{B} = -\text{grad } \Phi + \underline{u} \times \underline{B}, \quad \text{div } \underline{B} = 0, \tag{6.1.1}$$

in dimensionless form, by measuring length in units of L, and writing

$$B = \mathcal{B}\underline{h}, \quad \Phi = -\eta \mathcal{B}\phi, \quad \underline{u} = \frac{\eta}{L}\underline{q}, \tag{6.1.2}$$

where $\mathcal{B}$ is a typical field strength. Now $\underline{\nabla} = O(1)$ for the macroscale and is $O(\varepsilon^{-1})$ for the microscale. And, (6.1.1) becomes

$$\underline{\nabla} \times \underline{h} = \underline{\nabla}\phi + \underline{q} \times \underline{h}, \quad \underline{\nabla} \cdot \underline{h} = 0, \tag{6.1.3}$$

implying, since

$$\underline{\nabla} \cdot \underline{q} = 0, \tag{6.1.4}$$

that

$$-\nabla^2 \underline{h} = \underline{h} \cdot \underline{\nabla}\underline{q} - \underline{q} \cdot \underline{\nabla}\,\underline{h}. \tag{6.1.5}$$

We seek a bounded flow $\underline{q}$ such that (6.1.5) has a non-trivial solution vanishing at infinity.

We suppose $\underline{q}$ varies over the microscale, while $\underline{h}$ varies predominantly over the macroscale, i.e.

$$\underline{h} = \underline{h}_0(\underline{x}) + \varepsilon^{1/2}\underline{h}_1\left(\underline{x}, \frac{\underline{x}}{\varepsilon}\right) + \dots, \tag{6.1.6}$$

where $\underline{h}_0, \underline{h}_1, \dots$ are $O(1)$. We have here introduced a two-scale notation:

$f(\underline{x})$ will mean that f varies over the macroscale; $f(\underline{x}/\varepsilon)$ will mean it varies over the microscale; while $f(\underline{x}, \underline{x}/\varepsilon)$ will mean that both scales are involved. Thus $\underline{q} = \underline{q}(\underline{x}/\varepsilon)$. The three terms appearing in (6.1.5) are respectively $O(\varepsilon^{-\frac{3}{2}}\underline{h}_1)$, $O(\varepsilon^{-1}\underline{q}\,\underline{h}_0)$ and $O(\varepsilon^{-\frac{1}{2}}\underline{q}\,\underline{h}_1)$. We can therefore obtain a balance between the first and second terms by supposing $\underline{q} = O(\varepsilon^{-\frac{1}{2}})$, whereupon (6.1.5) reduces to

$$-\varepsilon^{\frac{1}{2}}\nabla^2\underline{h}_1 = \underline{h}_0\cdot\underline{\nabla}\,\underline{q}\,, \tag{6.1.7}$$

an equation which expresses a balance between the diffusion process and the stretching process, the convective term having been too small to retain to this order. This equation is solved for $\underline{h}_1$:

$$\underline{h}_1 = -\varepsilon^{-\frac{1}{2}}\nabla^{-2}(\underline{h}_0\cdot\underline{\nabla}\,\underline{q}). \tag{6.1.8}$$

If now we average (6.1.3) over the microscale we obtain

$$\underline{\nabla}\times\underline{h}_0 = \underline{\nabla}\phi_0 + \underline{\mathcal{E}}_0\,, \quad \underline{\nabla}\cdot\underline{h}_0 = 0\,, \tag{6.1.9}$$

where

$$\underline{\mathcal{E}}_0 = \varepsilon^{-\frac{1}{2}}\langle\underline{q}\times\underline{h}_1\rangle \tag{6.1.10}$$

is the effective e.m.f. of the microscale induction, and $\langle\ldots\rangle$ denotes the average over that microscale.

To give an example of this process in operation, suppose that

$$\underline{q} = \varepsilon^{-\frac{1}{2}}\left[\sin\frac{y}{\varepsilon} \pm \cos\frac{z}{\varepsilon},\ \sin\frac{z}{\varepsilon} \pm \cos\frac{x}{\varepsilon},\ \sin\frac{x}{\varepsilon} \pm \cos\frac{y}{\varepsilon}\right], \tag{6.1.11}$$

where the plus or minus signs are to be taken together. This is an example of a Beltrami field (satisfying $\underline{q}\times\mathrm{curl}\,\underline{q} = 0$). According to (6.1.8), we have

$$\underline{h}_1 = \left[h_{oy}\cos\frac{y}{\varepsilon} \mp h_{oz}\sin\frac{z}{\varepsilon},\ h_{oz}\cos\frac{z}{\varepsilon} \mp h_{ox}\sin\frac{x}{\varepsilon},\ h_{ox}\cos\frac{x}{\varepsilon} \mp h_{oy}\sin\frac{y}{\varepsilon}\right]_o \tag{6.1.12}$$

In (6.1.12) and below, the upper or lower signs conform with those chosen in (6.1.11). Now, using (6.1.11) and (6.1.12)

$$(\underline{q} \times \underline{h}_1)_x = \pm h_{ox} + \text{rapidly varying terms,} \tag{6.1.13}$$

and similarly for the y- and z-components. Thus, in this case,

$$\underline{\mathcal{E}}_o = \pm \underline{h}_o, \tag{6.1.14}$$

and (6.1.9) becomes

$$\underline{\nabla} \times \underline{h}_o = \underline{\nabla}\phi_o \pm \underline{h}_o,\quad \underline{\nabla}\cdot\underline{h}_o = 0. \tag{6.1.15}$$

These equations are readily solved using the toroidal and poloidal representation (§ 2.3). Writing $\phi_o = 0$ and

$$\underline{h}_o = \underline{\nabla} \times (\psi_o \underline{x}) \pm \underline{\nabla} \times \left[\underline{\nabla} \times (\psi_o \underline{x})\right], \tag{6.1.16}$$

equation (6.1.15) requires that

$$\nabla^2 \psi_o + \psi_o = 0, \tag{6.1.17}$$

a particular solution of which is, for example,

$$\psi_o = \frac{\sin r}{r}, \tag{6.1.18}$$

where $r = |\underline{x}|$.

The flow (6.1.11) is difficult to picture. If we ignore the terms involving z, the flow in the xy-plane is as shown in Fig. 6.1; the boundaries of the cells being nodes for both the vorticity and the z-component of $\underline{q}$. The field produced from the stretching of $\underline{h}_o$ is given by (6.1.12), with $h_{oz} = 0$. Induction now regenerates an $\underline{\mathcal{E}}$ of $\pm(h_{ox}, h_{oy}, 0)$. If we imagine three such motions superimposed in the xy-, yz- and zx-planes respectively, we obtain twice the correct inductive effect. In this sense

Fig. 6.1 represents two-thirds of the dynamo!

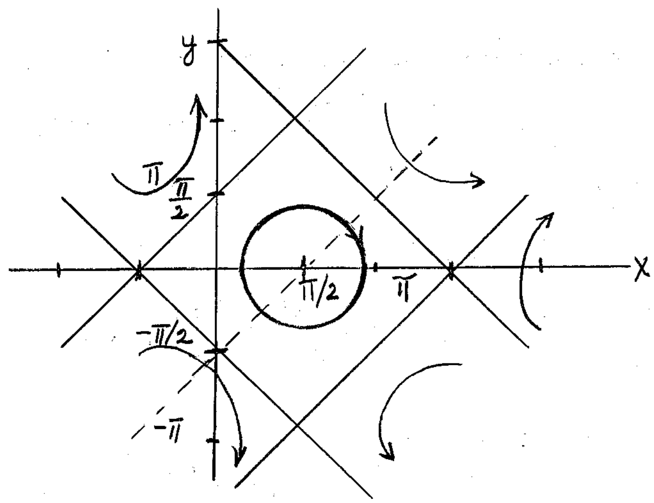


Fig. 6.1

"Two-thirds" of a Periodic Dynamo

6.2 Outline of the formal justification

We suppose that $\underline{q}$ may be expressed as a finite sum of Fourier modes:-

$$\underline{q} = \varepsilon^{\frac{1}{J}-1} \sum_{K} \underline{\mu}(\underline{k}) e^{i\underline{k}\cdot\underline{x}/\varepsilon} \quad , \tag{6.2.1}$$

where $\underline{\mu}(-\underline{k}) = \underline{\mu}^*(\underline{k})$, and K denotes a finite set of integer coordinates: J is a positive integer.

Define a projection (or averaging) operator P as follows. If a field is divided into Fourier modes of the form

$$\psi = e^{i\underline{n}\cdot\underline{x}} \cdot e^{i\underline{m}\cdot\underline{x}/\varepsilon} \quad ,$$

where $\underline{n}$ corresponds to the macroscale and $\underline{m}$ is drawn from the set of microscale vectors, then

$$\mathcal{P}\psi = \begin{cases} 0, \text{ if } \underline{m} \neq 0, \\ \psi, \text{ if } \underline{m} = 0. \end{cases} \tag{6.2.2}$$

Clearly $\mathcal{P}F$ is the slowly varying (macroscale) component of any field or operator F, while $\widetilde{F} = (I-\mathcal{P})F$ is the rapidly varying (microscale) part. Note that $\mathcal{P}\widetilde{F} = 0$. (I is the identity operator.)

We now write (6.1.5) as

$$\underline{h} = \nabla^{-2}[\underline{\nabla}\times(\underline{q}\times\underline{h})] = T\underline{h}, \textit{ say}. \tag{6.2.3}$$

Suppose

$$\underline{f} = (I-\widetilde{T})\underline{h}, \text{ i.e. } \underline{h} = (I-\widetilde{T})^{-1}\underline{f}. \tag{6.2.4}$$

Then

$$\begin{aligned} \underline{h} - T\underline{h} &= [(I-\widetilde{T})+\widetilde{T}](I-\widetilde{T})^{-1}\underline{f} - T(I-\widetilde{T})^{-1}\underline{f} \\ &= \underline{f} + (T-\mathcal{P}T)(I-\widetilde{T})^{-1}\underline{f} - T(I-\widetilde{T})^{-1}\underline{f} \\ &= \underline{f} - \mathcal{P}T(I-\widetilde{T})^{-1}\underline{f}. \end{aligned} \tag{6.2.5}$$

Thus, if

$$\underline{f} = \mathcal{P}T(I-\widetilde{T})^{-1}\underline{f}, \tag{6.2.6}$$

then (6.2.3) is satisfied, and conversely. It is clear, from the position of the $\mathcal{P}$ operator on the right-hand side of (6.2.6), that $\underline{f}$ is slowly varying, i.e.

$$\underline{f} = \sum_{\underline{n}} \underline{\Gamma}(\underline{n},\varepsilon)e^{i\underline{n}\cdot\underline{x}}. \tag{6.2.7}$$

In fact, by (6.2.4) we recognize that $\underline{f} = \mathcal{P}T\underline{h} = \mathcal{P}\underline{h}$, i.e. $\underline{f}$ is the projection of $\underline{h}$, i.e. it is the slowly varying part of $\underline{h}$.

We may write

$$(I\widetilde{T})^{-1} = I + \widetilde{T} + \widetilde{T}^2 + \cdots, \tag{6.2.8}$$

and represent the right of (6.2.6) as a sum of terms of the type $PT\tilde{T}^{j-1}\underline{f}$; the successive members (remembering that $\tilde{T}P=0$) are, for $j = 1,2,3,4 \ldots,$

$$\left.\begin{aligned} PT\underline{f} &= 0, \\ PT\tilde{T}\underline{f} &= PT(T-PT)\underline{f} = PT^2\underline{f}, \\ PT\tilde{T}^2\underline{f} &= PT(T-PT)\tilde{T}\underline{f} = PT(T^2-PT^2)\underline{f} = PT^3\underline{f}, \\ PT\tilde{T}^3\underline{f} &= PT(T-PT)(T^2-PT^2)\underline{f} = PT(T-PT)T^2\underline{f} = PT^4\underline{f} - PT^2PT^2\underline{f}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}\right\} \quad (6.2.9)$$

We can represent these terms by a cumulant expansion. The origin can be regarded as a slowly varying vector $(\underline{f})$, and the operator T can be represented as a vector joining its operand to the result of applying T to that operand. A number j of T operations is then represented by a "chain" of j such vector arcs. If P is now applied, $PT^j\underline{f}$ will vanish unless the chain has returned to the origin, since it is only in that case that $T^j\underline{f}$ is not rapidly varying. Thus we are led to a chain starting and finishing at the origin. This chain must in fact be "irreducible", i.e. it must not return to the origin except at these initial and final links [cf. the fourth of (6.2.9)]. More formally, if $\underline{f}$ is a plane wave

$$\underline{f} = \underline{\Gamma} e^{i\underline{n}\cdot\underline{x}}, \qquad (6.2.10)$$

it may be shown that (6.2.6) may be written as (cf. 6.1.9)

$$i\underline{n}\times\underline{\Gamma} = i\underline{n}\,\Phi + A\cdot\underline{\Gamma}, \quad i\underline{n}\cdot\underline{\Gamma} = 0, \qquad (6.2.11)$$

where Φ in the first of (6.2.11) is determined by the second of (6.2.11), and A is a complex matrix:

$$A \cdot \underline{\Gamma} = \sum_{j=2}^{\infty} (i)^{j-1} \varepsilon^{\frac{j}{J}-1} \sum_{(j)} \frac{\underline{\mu}_j \times \underline{m}_{j-1} \times \cdots \times \underline{m}_1 \times \underline{\mu}_1 \times \Gamma}{m_{j-1}^2 \cdots m_1^2} . \qquad (6.2.12)$$

In (6.2.12), $\underline{\mu}_j = \underline{\mu}(\underline{k}_j)$, $\underline{m} = \underline{k} + \varepsilon \underline{n}$, and the sum (j) is taken over all chains of j vectors in K which are irreducible in the above sense.

If $A'_{\alpha\beta}$ are the co-factors of A, the condition that (6.2.11) has a non-trivial solution reduces to

$$n^4 - \sum_{\alpha\beta} A'_{\alpha\beta} n_\alpha n_\beta - i n^2 \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} A_{\alpha\beta} n_\gamma = 0 . \qquad (6.2.13)$$

Thus, the components of $\underline{f}$ must be drawn from the surface (6.2.13), and must be superimposed in such a way that $\underline{h} \longrightarrow 0$ as $\underline{x} \longrightarrow \infty$.

The fact that the sum (6.2.12) starts from $j = 2$, reflects the fact (cf. the first of 6.2.9) that the first term in the expansion of (6.2.6) is zero. Let J be the smallest value of j for which $PT\tilde{T}^{j-1}$ is non-zero (in general, $J = 2$, as indicated by the example of § 6.1). Then the equations (6.2.11) have a non-trivial limiting form as $\varepsilon \rightarrow 0$. It can be shown that

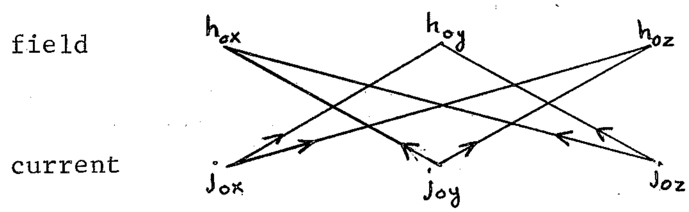
$$A_0 = \lim_{\varepsilon \rightarrow 0} A$$

is independent of $\underline{n}$ and is a non-zero symmetric (J even) or antisymmetric (J odd) matrix operator. From (6.2.13) we now see that, if $\underline{q}$ is asymptotically regenerative (i.e. regenerative in the limit $\varepsilon \longrightarrow 0$), J is even and A'_0 is positive definite. Moreover, if in addition, all the odd order cumulants $PT\tilde{T}^{j-1}$ for all odd $j \geq 3$ vanish, these conditions are sufficient for $\underline{q}$ to be a dynamo.

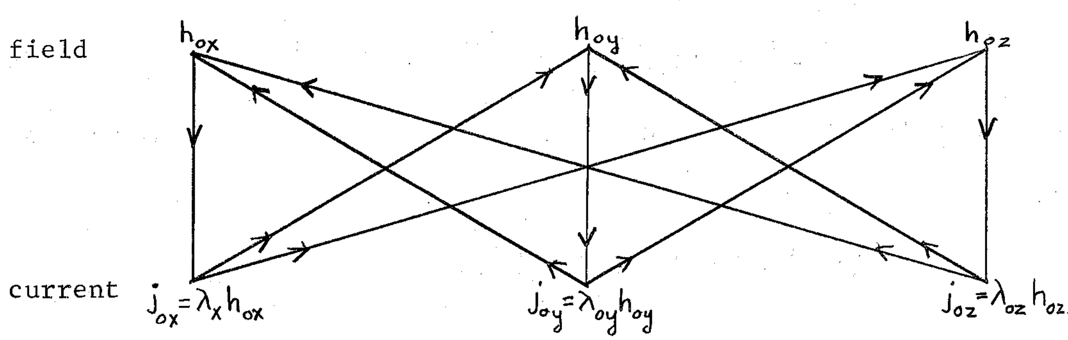
If, for a rough physical picture, we suppose that $A_{0\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, the field $\underline{h}_0 = (h_{0x}, h_{0y}, h_{0z})$ regenerates the current $\underline{j}_0 = (\lambda_x h_{0x}, y_y h_{0y}, \lambda_z h_{0z})$ or diagrammatically,

$$\begin{array}{llll}
\text{field} & h_{ox} & h_{oy} & h_{oz} \\
 & \downarrow & \downarrow & \downarrow \\
\text{current} & \lambda_x h_{ox} & \lambda_y h_{oy} & \lambda_z h_{oz}
\end{array}$$

On the other hand, if λ_x, λ_y and λ_z all have the same sign, Ampère's law $\dot{\underline{j}}_o = \underline{\nabla} \times \underline{h}_o$ may be represented diagrammatically by



Combining these two diagrams we obtain a composite picture showing the three distinct closed cycles which represent the regenerative effect of the three orthogonal families of cells (cf. Fig. 6.1).



6.3 <u>Fitting the solutions into a sphere</u>

The solutions obtained in §§ 6.1 and 6.2 for unbounded flows may be tailored to suit a bounded conductor, such as the sphere $r = a$. We divide $r \leq a$ into a "core" (not to be confused with the word as applied to the Earth!), $0 \leq r \leq a - \varepsilon$, and a surrounding shell $a - \varepsilon \leq r \leq a$. Outside both is an insulator. We introduce a "cut-off" function $\omega(r)$;

(see Fig. 6.2):

$$\omega = \begin{cases} 1, \text{ in core,} \\ \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{p} e^{-\zeta^2}\, d\zeta, \text{ where } p(r) = \dfrac{(a-\frac{1}{2}\varepsilon - r)}{(r-a-\varepsilon)(a-r)}, \text{ in shell.} \end{cases} \tag{6.3.1}$$

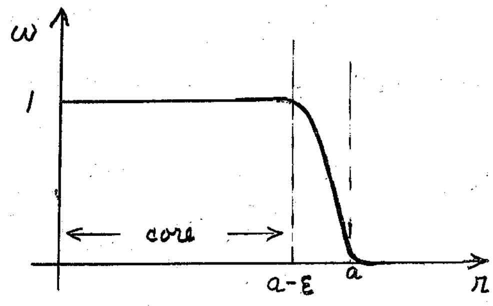


Fig. 6.2

Now, for the microscale velocity we may in place of the $\underline{q}$ of (6.2.1) use, in essence, $\omega\underline{q}$. More generally, we may include in $\underline{q}$ a slowly varying part (macroscale $L=a$):

$$\underline{q} = \underline{W}(\underline{x}) - \varepsilon^{1/2}\underline{\nabla}\times\left[\omega\underline{v}\left(\frac{\underline{x}}{\varepsilon}\right)\right], \qquad r \leq a, \tag{6.3.2}$$

where $\underline{\nabla}\cdot\underline{W}=0$. For $\underline{v}$ we take (6.1.11), so that (6.3.2) may also be written

$$\underline{q} = \underline{W}(r) \pm \omega\varepsilon^{-1/2}\underline{v} + \underline{\varepsilon}^{1/2}\underline{v} \tag{6.3.2}$$

If ε and W_{max} are sufficiently small, we can find a solution to the dynamo equations provided a certain comparison problem can be solved. This essentially is that of solving the induction equation (6.1.15) for a bounded region. To see this, it is convenient first to abandon the scaling (6.1.2) writing, instead of (6.1.3),

$$\underline{\nabla}\times\underline{h} = \underline{\nabla}\phi + R\underline{q}\times\underline{h}, \quad \underline{\nabla}\cdot\underline{h} = 0, \tag{6.3.3}$$

where $\mathcal{R}$ is a magnetic Reynolds number based on the macroscales a and w_{max}. We may note that $\varepsilon^{\frac{1}{2}}\mathcal{R}$ is the magnetic Reynolds number based on the microscales ℓ and $\varepsilon^{-\frac{1}{2}}\underline{v}$, while $\varepsilon^{-\frac{1}{2}}\mathcal{R}$ is the magnetic Reynolds number based on the largest length (a) and largest fluid velocity $(\varepsilon^{-\frac{1}{2}}\underline{v})$.

Again, the effect of inductive interaction between a field $\underline{h}$ and the motion $\underline{g}$ can be represented by an operator T_s (say), (6.3.3) being written [cf. (6.2.3)]

$$\mathcal{R}^{-1}\underline{h} = T_s\underline{h}. \tag{6.3.4}$$

(The subscript s emphasizes that the inductor is now a sphere.) By applying (6.3.4) twice, we see that the eigenvalues of (6.3.4) provide eigenvalues, $\lambda = \mathcal{R}^{-2}$, of the equation

$$T_s^2\underline{h} = \lambda\underline{h}. \tag{6.3.5}$$

To appreciate the technical advantages of this formulation over (6.3.4), suppose that ε and w_{max} are small, so that far from the boundaries the system resembles the infinite conductor. We know from § 6.1-2 that, even though $\underline{h}_0$ may be predominantly slowly varying, $\underline{h}_1 = T_s\underline{h}_0$ may fluctuate rapidly. Nevertheless $\underline{g} = T_s\underline{h}_1 = T_s^2\underline{h}_0$ should again be predominantly slowly varying and should satisfy (cf. 6.1.13)

$$\underline{\nabla}\times\underline{g} = \underline{\nabla}\phi_0 \pm \mathcal{R}^2\underline{h}_0, \quad \underline{\nabla}\cdot\underline{g} = 0, \tag{6.3.6}$$

over the interior of the sphere for some ϕ_0. This suggests that, if we set $\underline{g} = \underline{h}_0$ in (6.3.6) and impose the usual boundary and matching conditions, the resulting (very simple) problem, which may be written operationally as

$$\pm\mathcal{R}^{-2}\underline{h} = H\underline{h}, \tag{6.3.7}$$

should be "close" to the real system (6.3.5), and should provide a comparison problem for it. The basic idea, then, is that for small ε and w_{max}, the operator $T_s^2 - H$ is, in some appropriate sense, a "small" perturbation of H. We should observe that, if we take for $\underline{v}$ and $\underline{q}$ of (6.1.11) with the plus sign, we look for a real eigenvalue of T_s near to a square root of one of the <u>positive</u> eigenvalues of H. If the $\underline{q}$ has opposite parity, the desired real eigenvalues are obtained by perturbing a <u>negative</u> real eigenvalue of H. We may, however, restrict our attention to dynamos of positive parity, since it easily follows from the exact dynamo problem in a sphere that $-\underline{q}(-\underline{x})$ is a dynamo (with the same eigenvalues) if $\underline{q}(\underline{x})$ is a dynamo.

In fact, it can be shown that in a sphere there is a symmetric infinite sequence of real eigenvalues of H having odd multiplicity. Because the multiplicity is odd, the perturbation always leaves a real eigenvalue in addition to complex pairs. It should be observed that in no sense is T_s itself assumed to be "close" to H. Indeed, the existence of positive and negative eigenvalues of H suggests that the point spectrum of T_s probably lies close to the axis in the complex λ-plane when ε is small, so that T_s behaves something like the "square root" of a self-adjoint, but not necessarily positive, operator. This point deserves, perhaps, further elucidation, since it would be interesting to know for what other choices of $\underline{q}$ an even iterate of T_s would have similar properties.

The proofs of the above assertions are rather involved owing to the technical problem of extracting good estimates on the operator T_s as $\varepsilon \to 0$. The principal difficulty is the selection of a suitable measure of "smallness" of $T_s^2 - H$, i.e., the choice of a norm and an associated function space, when the kernels of integral operators oscillate rapidly. It is natural to introduce a norm involving derivatives, and it turns out that a particularly useful family is

$$\|\underline{h}\|_\gamma = |\underline{h}|_{max} + \varepsilon^\gamma |\underline{\nabla}\, \underline{h}|_{max}, \quad \tfrac{1}{2} \le \gamma < 1,$$

where the maximum is taken over the core. These norms determine a certain two-parameter (ε, γ) family of Banach spaces in which it is possible to construct eigenfunctions of T_s associated with real eigenvalues. For $\gamma = \frac{1}{2}$, the norm of T_s (defined in the usual way by its action on the corresponding Banach space) is uniformly bounded as $\varepsilon \to 0$, even though T_s is <u>not</u> bounded in ε on the Hilbert space of magnetic fields with finite core energy. It is interesting to note in this connection that Herzenberg's proof of steady dynamo maintenance (outlined in § 3) also avoids the use of Hilbert space norms.

The smallest comparison eigenvalues correspond to dipole vacuum fields, with $R^2 = 4.5$. Thus the corresponding magnetic Reynolds number based upon the maximum velocity and length scales is approximately $2.1\, \varepsilon^{-\frac{1}{2}}$, while the local effective value of R on the microscale is $2.1\, \varepsilon^{\frac{1}{2}}$. We may conclude that, in the multiple-scale dynamo theory, regeneration is basically a low-R phenomenon, despite the possibility of large velocities and despite the presence of such macroscopic high-R effects as distortion of poloidal field by differential rotation.

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7. The Magnetohydrodynamic Dynamo Problem

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7.1 Taylor's necessary and sufficient condition

The immediate aim of dynamo theory is to solve the equations of motion and of induction simultaneously. For example, it is only by solving this non-linear problem that we can, in the case of the Earth, hope to understand the magnitude of the fields in the core. The Bullard-Gellman type of approach, so inadequate in the kinematic dynamo problem, is clearly doubly unsatisfactory here, as the work of Rikitake (1959) and Stevenson and Wolfson (1966) has confirmed. The dynamic Herzenberg dynamo can be fairly satisfactorily treated, but it can hardly be said to be geophysically realistic! At the present time, the most promising avenues of approach seem to lie in extending Braginskiĭ's ideas of nearly symmetric dynamos, and in generalizing Childress's multiple-scale method.

The geomagnetohydrodynamic dynamo problem may not be as severe as the general case. For the motions in the Earth's core are probably dominated by a balance between Coriolis and Lorentz forces, and this seems to imply some simplifications analagous to the Proudman-Taylor theorem for rotating non-magnetic fluids, as we shall see. One example of this, due to

Taylor (1963), will now be considered.

The equations of motion for the fluid in the core (supposed a cylindrically symmetric cavity) is

$$\rho\left[\frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\underline{\nabla}\underline{u}+2\underline{\Omega}\times\underline{u}\right]=-\underline{\nabla}p+\underline{j}\times\underline{B}+\rho\nu\nabla^2\underline{u}+\rho\underline{F}. \qquad (7.1.1)$$

We make the following postulates:-

(i) $\underline{F}$ has no φ-component, as would be the case, for example, if it is a buoyancy force,

(ii) The motion is "slow", by which we mean the term $\underline{u}\cdot\underline{\nabla}\,\underline{u}$ is negligible,

(iii) The motion is "nearly steady", by which we mean that the $\partial\underline{u}/\partial t$ term can be ignored, and

(iv) The viscosity is "small" by which we mean that we are dealing with small Ekman numbers, $E=\nu/2\Omega\mathcal{L}^2$, $\mathcal{L}$ being a typical length, and that the flow may be considered to be of the boundary layer type. Then, provided we are in the interior region far from these boundary layers, we may omit the $\rho\nu\nabla^2\underline{u}$ term in (7.1.1), in the first approximation.

Under assumptions (i) to (iv) above, (7.1.1) reduces to

$$2\rho\underline{\Omega}\times\underline{u}=-\underline{\nabla}p+\underline{j}\times\underline{B}+\rho\underline{F}, \qquad (7.1.2)$$

where

$$F_\varphi=0. \qquad (7.1.3)$$

Let $\mathcal{C}$ be the section of the cylinder $\varpi=\varpi_0$ lying within the mainstream (not including the boundary layers). Let C^*, the end caps of this cylinder, be part of S^*, the edge of the boundary layer on $\mathcal{S}$, the container surface. Let $\mathcal{V}$ be the interior volume and $\hat{\mathcal{V}}$ the exterior region (assumed insulating); see Fig. 7.1. If

$$\int_{C^*} \rho \underline{u} \cdot \underline{dS} = 0 \tag{7.1.4}$$

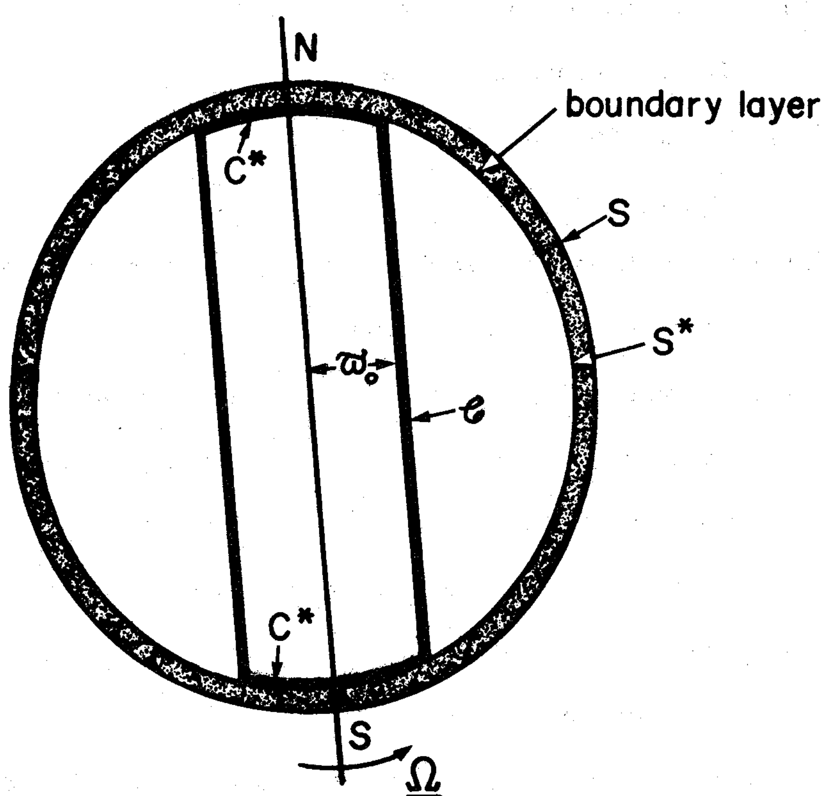

Fig. 7.1

we have, by the equation of continuity** (for slow flows if density

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**We are going to suppose that effects of density stratification, over and above the adiabatic, are negligible. This may be dubious, but the theory is difficult enough as it is! As a consequence, it will be a good first step to suppose the core is uniform and incompressible.

variations are allowed) $\operatorname{div}\rho\underline{u} = 0$,

$$\int_{\mathscr{C}} \rho\underline{u}\cdot d\underline{S} = \int_{\mathscr{C}+S^*} \rho\underline{u}\cdot d\underline{S} = \int_V \operatorname{div}(\rho\underline{u})\, dV = 0,$$

i.e.,

$$\int_{\mathscr{C}} 2\rho(\underline{\Omega}\times\underline{u})_\varphi\, dS = 0. \tag{7.1.5}$$

Thus, taking the φ-component of (7.1.2) and using (7.1.3) and (7.1.5), we have

$$\int_{\mathscr{C}} (\underline{j}\times\underline{B})_\varphi\, dS = 0, \tag{7.1.6}$$

which is Taylor's result.

Stephen Childress (1967) has found an interesting consequence of (7.1.6). Suppose $\underline{B}$ is axisymmetric so that, in cylindrical polar coordinates, (ϖ, φ, z),

$$\underline{B} = \left[-\frac{\partial A}{\partial z},\ B,\ \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi A)\right]. \tag{7.1.7}$$

The integral appearing in (7.1.6) is then

$$\begin{aligned}\mu\int_{\mathscr{C}} (\underline{j}\times\underline{B})_\varphi\, dS &= -2\pi\int_{-Z}^{Z}\left[\frac{\partial A}{\partial z}\frac{\partial}{\partial\varpi}(\varpi B) - \frac{\partial B}{\partial z}\frac{\partial}{\partial\varpi}(\varpi A)\right] dz \\ &= -2\pi\int_{-Z}^{Z}\left[\frac{\partial A}{\partial z}\frac{\partial}{\partial\varpi}(\varpi B) + B\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial A}{\partial z}\right)\right] dz + 2\pi\left[B\frac{\partial}{\partial\varpi}(\varpi A)\right]_{-Z}^{Z} \\ &= -\left(\frac{\partial}{\partial\varpi} + \frac{2}{\varpi}\right)\int_{\mathscr{C}} B\frac{\partial A}{\partial z}\, dS + 2\pi\left[B\frac{\partial}{\partial\varpi}(\varpi A)\right]_{-Z}^{Z},\end{aligned} \tag{7.1.8}$$

where $Z = +\sqrt{(1-\varpi^2)}$. Rearranging (7.1.8) gives

$$\left(\frac{\partial}{\partial\varpi} + \frac{2}{\varpi}\right)\int_{\mathscr{C}} B\frac{\partial A}{\partial z}\, dS = -\mu\int_{\mathscr{C}} (\underline{j}\times\underline{B})_\varphi\, dS + 2\pi\left[B\frac{\partial}{\partial\varpi}(\varpi A)\right]_{-Z}^{Z}. \tag{7.1.9}$$

If

$$B = 0 \text{ on } S^*, \tag{7.1.10}$$

and if Taylor's result (7.1.6) is true, (7.1.9) gives

$$\int_{\mathscr{C}} B \frac{\partial A}{\partial z} \, dS = \frac{C}{\varpi^2}, \tag{7.1.11}$$

where C is a constant; but this implies that the fields are singular on the axis of symmetry (which is impossible unless the inner core is cylindrical!). It follows that $C = 0$, and

$$\int_{\mathscr{C}} B \frac{\partial A}{\partial z} \, dS = 0. \tag{7.1.12}$$

Taylor (1963) has shown that, provided the tangent plane to S is perpendicular to $\underline{\Omega}$ only at isolated points (e.g. the poles) and not over areas (as would be the case if the core were cylindrical), we can construct, from any field $\underline{B}$ satisfying (7.1.6), a unique flow $\underline{u}$ satisfying continuity and the equation of motion (7.1.2). We will not give an account of this development, since we may already take issue with the statement (7.1.4). It is true that $\underline{n} \cdot \underline{u}$ must vanish on S, but this does not imply the same is true on S^* for "between the two" there is a boundary layer which must adjust the interior flow to the no-slip conditions on S, and must adjust the interior magnetic field to the potential field for $\hat{V}$. For the former, in the absence of magnetic effects, the Ekman layer exists and, depending on the local vorticity on S^*, there is a boundary layer suction; i.e. $\underline{n} \cdot \underline{u}$ does not vanish on S^*. Even without working out the details of the magnetohydrodynamic Ekman layer relevant here, we may expect that it has the same property, and that (7.1.2) will not in general be obeyed. We may expect that the primary azimuthal flow $\bar{u}_\varphi$ will be associated with a secondary axisymmetric poloidal flow $E^{\frac{1}{2}}\bar{u}_\varphi$

over $\mathcal{C}^*$, which will be compensated for by a net $\bar{\underline{u}}_M$ flow out of $\mathcal{C}$. We will return to this in § 7.3. Meanwhile we may also note that, while $\mathcal{B}$ vanishes on $\mathcal{S}$, it need not vanish on $\mathcal{S}^*$. And since $\partial A/\partial z$ may be large in the boundary layer, it is not obvious that $\mathcal{C}$ can be extended through the boundary layer to $\mathcal{S}$, in order to be able to apply (7.1.10) on $\mathcal{S}$ instead of $\mathcal{S}^*$. Thus, even were (7.1.6) true, (7.1.12) need not follow from it.

Finally we should point out that, although we have taken issue with Taylor's assumptions in one important respect, his overall theme, that the integral appearing in (7.1.6) is a fundamental quantity in the dynamics of the core, is hard to dispute.

7.2 <u>The magnetohydrodynamic dynamo a la Braginskiĭ</u>

We will examine the possibility of extending Braginskiĭ's formalism to include the Navier-Stokes equations on the assumption that $\underline{F}$ is given, satisfying (7.1.3), and that $\underline{u}$ and $\underline{B}$ are nearly axisymmetric. It is clear that such a flow cannot be set up unless the non-axisymmetric part, $\underline{F}'$, of $\underline{F}$ is non-zero. Moreover, as (7.1.3) makes clear, $\underline{F}$ does not drive $\bar{u}_\varphi$ directly, though it may drive $\bar{\underline{u}}_M$. When expanding the solution in the usual Braginskiĭ fashion we omit the $\bar{F}_\varphi$ term, the dominant part of $\underline{F}$ being assumed to be $\underline{F}'$. We write

$$\left.\begin{aligned}
\underline{u} &= \bar{u}_\varphi \underline{1}_\varphi + \underline{u}' + \bar{\underline{u}}_M, \\
\underline{B} &= \bar{B}_\varphi \underline{1}_\varphi + \underline{B}' + \bar{\underline{B}}_M, \\
\underline{F} &= \underline{F}' + \bar{\underline{F}}_M,
\end{aligned}\right\} \qquad (7.2.1)$$

and so forth. We suppose that $\mathcal{F}$ is a typical value of $\underline{F}'$ and $\mathcal{L}$ a typical dimension of the body. We then measure $\underline{F}$ in units of $\mathcal{F}$, $\bar{u}_\varphi$ in units of $\mathcal{U} = \mathcal{L}\mathcal{F}^2/4\Omega^2\eta$, and $\bar{B}_\varphi$ in units of $\mathcal{B} = \mathcal{L}\mathcal{F}(\mu\rho/2\Omega\eta)^{1/2}$. We suppose, in the usual way, that $\underline{u}' = O(R^{-1/2}\mathcal{U}), \bar{\underline{u}}_M = O(R^{-1}\mathcal{U}), \underline{B}' = O(R^{-1/2}\mathcal{B})$, and $\bar{\underline{B}}_M = O(R^{-1}\mathcal{B})$, where

$$R = \frac{\mathcal{U}\mathcal{L}}{\eta} = \left(\frac{\mathcal{L}\mathcal{F}}{2\Omega\eta}\right)^2, \tag{7.2.2}$$

is the magnetic Reynolds number. It must be realized that the functions $\bar{u}_\varphi, \underline{u}', \bar{\underline{u}}_M, \bar{B}_\varphi, \underline{B}'$ and $\bar{\underline{B}}_M$ are themselves power series in $R^{-1/2}$, the dominant terms of which are given by the order of magnitude statements above (7.2.2). There is no significant loss of generality in supposing that $\underline{F}'$ is pro-proportional to $R^{-1/2}$ without an expansion in $R^{-1/2}$ attached.

On writing

$$\underline{u} = \mathcal{U}\underline{u}^*, \quad \underline{B} = \mathcal{B}\,\underline{B}^*, \quad \underline{F} = \mathcal{F}\underline{F}^*, \quad \underline{x} = \mathcal{L}\underline{x}^*, \quad t = \mathcal{L}^2 t^*/\eta, \tag{7.2.3}$$

and dropping asterisks, the basic equations assume the form

$$\underline{1}_z \times \underline{u} = -\underline{\nabla} p + \underline{j} \times \underline{B} + E\nabla^2\underline{u} + R^{-1/2}\underline{F}, \tag{7.2.4}$$

$$\frac{\partial \underline{B}}{\partial t} + R\underline{u}\cdot\underline{\nabla}\,\underline{B} = R\,\underline{B}\cdot\underline{\nabla}\,\underline{u} + \nabla^2\underline{B}, \tag{7.2.5}$$

$$\underline{\nabla}\cdot\underline{u} = \underline{\nabla}\cdot\underline{B} = 0, \quad \underline{j} = \underline{\nabla}\times\underline{B}, \tag{7.2.6}$$

where

$$E = \frac{\nu}{2\Omega\mathcal{L}^2} \tag{7.2.7}$$

is the Ekman number. On the left of (7.2.4) we have omitted the term

$$R_d\left[\frac{\partial \underline{u}}{\partial t} + R\underline{u}\cdot\underline{\nabla}\,\underline{u}\right], \tag{7.2.8}$$

where R_d is a Rossby number based on the drift velocity of flux tubes

$$R_d = \frac{(\eta/\mathcal{L})}{2\Omega\mathcal{L}} = \frac{1}{R}\left(\frac{\mathcal{U}}{2\Omega\mathcal{L}}\right). \qquad (7.2.9)$$

We may observe that, in the case of the Earth, $R_d \approx 10^{-8}$, so it is plausible that we can omit the terms (7.2.8). This is equivalent to the assumptions that the flow is slow and nearly steady (cf. (ii) and (iii) of § 7.1). It is also probably true that E is small in the core, but to omit this term from (7.2.4) would lead to difficulties with Taylor's results of § 7.1.

We now examine the interior $[\underline{\nabla} = O(1)]$ magnetohydrodynamic flow in the double limit $\nu \to 0, \eta \to 0$, by the Braginskiĭ method. Averaging (7.2.4) over φ gives directly

$$-\bar{u}_\varphi \underline{1}_\varpi = -\underline{\nabla}\left[\bar{p} + \tfrac{1}{2}\bar{\underline{B}}^2\right] + \bar{\underline{B}}_M \cdot \underline{\nabla}\,\bar{\underline{B}}_M - \frac{\bar{B}_\varphi^2}{\varpi}\underline{1}_\varpi + \bar{\underline{S}}_M + \bar{\underline{F}}_M + E\left(\overline{\nabla^2 \underline{u}}\right)_M , \qquad (7.2.10)$$

$$\bar{u}_\varpi = \frac{1}{\varpi}\,\bar{\underline{B}}_M \cdot \underline{\nabla}\left(\varpi \bar{B}_\varphi\right) + \bar{S}_\varphi + E\left(\overline{\nabla^2 \underline{u}}\right)_\varphi , \qquad (7.2.11)$$

where

$$\bar{\underline{S}} = \overline{\text{curl}\, \underline{B}' \times \underline{B}'} \qquad (7.2.12)$$

is the axisymmetric Lorentz force arising from the asymmetric field. In components, (7.2.12) may be written as

$$\bar{\underline{S}}_M = \begin{cases} \bar{S}_\varpi = \overline{\left[B_z'\left(\frac{\partial B_\varpi'}{\partial z} - \frac{\partial B_z'}{\partial \varpi}\right) - B_\varpi' \underline{\nabla}\cdot\underline{B}_M' - \frac{1}{2\varpi^2}\frac{\partial}{\partial \varpi}\left\{\varpi^4\left(\underline{\nabla}\cdot\hat{\underline{B}}_M\right)^2\right\}\right]}, & (7.2.13) \\ \bar{S}_z = \overline{\left[-B_\varpi'\left(\frac{\partial B_\varpi'}{\partial z} - \frac{\partial B_z'}{\partial \varpi}\right) + B_z' \underline{\nabla}\cdot\underline{B}_M' - \frac{1}{2\varpi^2}\frac{\partial}{\partial z}\left\{\varpi^4\left(\underline{\nabla}\cdot\hat{\underline{B}}_M\right)^2\right\}\right]}, & (7.2.14) \end{cases}$$

$$\bar{S}_\varphi = \frac{1}{\varpi}\overline{\left[\hat{\underline{B}}_M \cdot \underline{\nabla}\left(\varpi^2 \underline{\nabla}\cdot\underline{B}_M'\right)\right]} . \qquad (7.2.15)$$

The non-axisymmetric part of (7.2.4) is

$$\underline{1}_z \times \underline{u}' = -\underline{\nabla} p' + (\underline{\nabla}\times\underline{B}')\times\bar{\underline{B}} + (\underline{\nabla}\times\bar{\underline{B}})\times\underline{B}' + \underline{T}' + \underline{F}' + E\nabla^2\underline{u}', \qquad (7.2.16)$$

where $\underline{T}'$ is the asymmetric Lorentz forces of the asymmetric fields:

$$\underline{T}' = (\text{curl}\ \underline{B}') \times \underline{B}' - \overline{\underline{S}}. \tag{7.2.17}$$

We now proceed, as in § 4, to compute the leading terms in $\overline{\underline{S}}_M$ and $\overline{S}_\varphi$.

In the first approximation, $O(R^\circ)$, equation (7.2.10) gives

$$\bar{u}_\varphi = \frac{\partial}{\partial \varpi}\left[\bar{p} + \tfrac{1}{2}\overline{\underline{B}}^2\right] + \frac{\overline{B}_\varphi^2}{\varpi}, \qquad 0 = \frac{\partial}{\partial z}\left[\bar{p} + \frac{1}{2}\overline{\underline{B}}^2\right], \tag{7.2.18}$$

from which we obtain the well-known integral

$$\bar{u}_\varphi = \frac{\overline{B}_\varphi^2}{\varpi} + \varpi g(\varpi), \tag{7.2.19}$$

where $g(\varpi)$ is arbitrary and is to be determined from the boundary layer equations. (It may also depend on t .) Next we use (7.2.19) and Braginskii's equations (4.1.14) for the first approximation

$$\overline{B}_\varphi = \overline{\chi}\,\bar{u}_\varphi, \qquad \underline{B}'_M = \overline{\chi}\,\underline{u}'_M, \tag{7.2.20}$$

to simplify (7.2.16). After elimination of p' , Graham Tough (1967) has found, after considerable reductions, equations determining $\underline{u}'$ from $\underline{F}'$, $\bar{u}_\varphi$ and g :

$$\left(\frac{\partial^2}{\partial\varphi^2} + 1\right)u'_\varpi + \frac{1}{\varpi}\frac{\partial G'}{\partial \varpi} - \frac{1}{\varpi^2}\frac{\partial H'}{\partial\varpi} + \varpi g\left(\varpi\frac{\partial}{\partial\varpi} - \frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{u'_\varpi}{\bar{u}_\varphi}\right) = \frac{\partial F'_\varpi}{\partial\varphi}, \tag{7.2.21}$$

$$\frac{\partial^2 u'_z}{\partial\varphi^2} + \frac{1}{\varpi}\frac{\partial G'}{\partial z} - \frac{1}{\varpi^2}\frac{\partial H'}{\partial z} + \varpi g\left(\varpi\frac{\partial}{\partial z} - \frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{u'_z}{\bar{u}_\varphi}\right) = \frac{\partial F'_z}{\partial\varphi}, \tag{7.2.22}$$

where

$$G' = \varpi^3\bar{u}_\varphi\,\underline{\nabla}\cdot\left(\frac{\underline{u}'_M}{\bar{u}_\varphi}\right), \qquad H' = \varpi^5 g\,\underline{\nabla}\cdot\left(\frac{\underline{u}'_M}{\bar{u}_\varphi}\right). \tag{7.2.23}$$

These equations have some nice properties we will not dwell on here. They can be used to simplify $\overline{S}_\varphi$ considerably. From (7.2.15) and (7.2.20), Graham Tough (1967) has shown that

$$\bar{S}_\varphi = \frac{1}{\varpi}\left[\underline{\nabla}\times(\bar{w}\bar{B}_\varphi \underline{1}_\varphi)\right]\cdot\underline{\nabla}(\varpi\bar{B}_\varphi) + \frac{1}{\varpi^2\bar{u}_\varphi}\overline{\left[\hat{\underline{u}}_M\cdot\underline{\nabla}\left\{\varpi^4\bar{B}_\varphi^2\underline{\nabla}\cdot\left(\frac{\underline{u}'_M}{\varpi\bar{u}_\varphi}\right)\right\}\right]},$$

which, using (7.2.19) and (7.2.21) - (7.2.23), eventually leads to

$$\bar{S}_\varphi = \frac{1}{\varpi}\left[\underline{\nabla}\times(\bar{w}\bar{B}_\varphi \underline{1}_\varphi)\right]\cdot\underline{\nabla}(\varpi\bar{B}_\varphi) + \frac{\partial}{\partial z}(\bar{w}\bar{u}_\varphi) + \frac{1}{\bar{u}_\varphi}\overline{(\underline{u}'_M\cdot\underline{F}')}. \tag{7.2.24}$$

Substituting in (7.2.11) results in yet another remarkable simplification of the Braginskiĭ type: the effective fields once more make an appearance! We obtain

$$\bar{u}_{e\varpi} = \frac{1}{\varpi}\underline{\bar{B}}_{eM}\cdot\underline{\nabla}(\varpi\bar{B}_\varphi) + \Lambda, \tag{7.2.25}$$

where

$$\Lambda = \frac{1}{\bar{u}_\varphi}\overline{\left[\underline{u}'_M\cdot\underline{F}'\right]}, \tag{7.2.26}$$

is, in essence, the average rate of working of the applied body forces, and $\underline{B}_{eM}$ is given, as usual, by

$$\underline{B}_{eM} = \mathrm{curl}\,\bar{A}_e\underline{1}_\varphi = \mathrm{curl}\,(\bar{A}_\varphi + \bar{w}\bar{B}_\varphi)\underline{1}_\varphi, \tag{7.2.27}$$

[cf. eq. (4.1.28)]. We could, if we wished, return to (7.2.10), (7.2.13) and (7.2.14) to find an equation for the first, $O(R^{-1})$, corrections to $\bar{u}_\varphi$. Provided Λ is even in μ, (7.2.25) agrees with the parity of the Braginskiĭ solutions of § 4.7.

According to (7.2.12),

$$\overline{\underline{j}\times\underline{B}} = \underline{\bar{j}}\times\underline{\bar{B}} + \overline{\underline{j}'\times\underline{B}'} = \underline{\bar{j}}\times\underline{\bar{B}} + \underline{\bar{S}},$$

giving, by (7.2.24),

$$(\overline{\underline{j}\times\underline{B}})_\varphi = \frac{1}{\varpi}\underline{\bar{B}}_{eM}\cdot\underline{\nabla}(\varpi\bar{B}_\varphi) + \frac{\partial}{\partial z}(\bar{w}\bar{u}_\varphi) + \Lambda. \tag{7.2.28}$$

If we integrate this equation over the Taylor cylinder $\mathcal{C}$ of § 7.1 we find, in place of (7.1.9),

$$\left(\frac{\partial}{\partial\varpi}+\frac{2}{\varpi}\right)\int_{\mathcal{C}}\bar{B}_\varphi\frac{\partial\bar{A}_e}{\partial z}dS=-\int_{\mathcal{C}}(\overline{\underline{j}\times\underline{B}})_\varphi dS+\int_{\mathcal{C}}\Lambda dS+2\pi\left[\bar{B}_\varphi\frac{\partial}{\partial\varpi}(\varpi\bar{A}_e)+\varpi\bar{w}\bar{u}_\varphi\right]_{-Z}^{Z}. \tag{7.2.29}$$

Next, extending an argument due to Stephen Childress (1967), multiply (4.2.19) by $\bar{B}_\varphi/\varpi^2$ and integrate over the interior region. After some reductions we obtain

$$\frac{\partial}{\partial t}\int\frac{1}{2}\left(\frac{\bar{B}_\varphi}{\varpi}\right)^2 dV=-\int\left[\underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right)\right]^2 dV-\int_0^1\frac{g(\varpi)d\varpi}{\varpi}\int_{\mathcal{C}}\bar{B}_\varphi\frac{\partial\bar{A}_e}{\partial z}dS+Q\,, \tag{7.2.30}$$

where Q is the contribution from the integrated parts:

$$Q=\int_{S^*}\left[-\tfrac{1}{2}R\left(\frac{\bar{B}_\varphi}{\varpi}\right)^2\underline{u}_{eM}+\frac{\bar{B}_\varphi}{\varpi^2}\,\underline{\nabla}\bar{B}_\varphi\right]\cdot\underline{dS}-\int_{S^*}\frac{1}{\varpi}\bar{A}_e\bar{B}_\varphi\left[\underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right)^2\times\underline{dS}\right]_\varphi. \tag{7.2.31}$$

If $\Lambda=0$, and if we apply to S^* the same boundary conditions as apply at S, we see [cf. eq. (7.1.6)] that the right of (7.2.29) is zero, showing, as in §7.1, that

$$\int_{\mathcal{C}}\bar{B}_\varphi\frac{\partial\bar{A}_e}{\partial z}dS=0. \tag{7.2.32}$$

Also $Q=0$; so the second and third terms on the right of (7.2.30) are zero, and

$$\frac{\partial}{\partial t}\int\frac{1}{2}\left(\frac{\bar{B}_\varphi}{\varpi}\right)^2 dV=-\int\left[\underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right)\right]^2 dV. \tag{7.2.33}$$

Thus $\bar{B}_\varphi$ declines monotonically to zero, and dynamo action is impossible even though Γ may be non-zero!

The result just proved again highlights the essential rôle played by the boundary layer on S .

For ease of reference we collect together the basic equations for the interior flow, as found in this section and in §4:

$$\zeta \equiv \frac{\bar{u}_\varphi}{\varpi} = \left(\frac{\bar{B}_\varphi}{\varpi}\right)^2 + g(\varpi), \tag{7.2.34}$$

$$\bar{u}_{e\varpi} = \frac{1}{\varpi}\,\underline{\bar{B}}_{eM}\cdot\underline{\nabla}\left(\varpi\,\bar{B}_\varphi\right) + \Lambda, \tag{7.2.35}$$

$$\underline{\nabla}\cdot\underline{\bar{u}}_e = 0, \tag{7.2.36}$$

$$\frac{\partial\bar{A}_e}{\partial t} + \frac{R}{\varpi}\,\underline{\bar{u}}_{eM}\cdot\underline{\nabla}\left(\varpi\bar{A}_e\right) = \Delta\bar{A}_e + \Gamma\,\bar{B}_\varphi, \tag{7.2.37}$$

$$\frac{\partial\bar{B}_\varphi}{\partial t} + R\varpi\;\underline{\bar{u}}_{eM}\cdot\underline{\nabla}\left(\frac{\bar{B}_\varphi}{\varpi}\right) = \Delta\bar{B}_\varphi + R\left[\underline{\nabla}\,\zeta\times\underline{\nabla}\left(\varpi\bar{A}_e\right)\right]_\varphi, \tag{7.2.38}$$

the sources Λ and Γ being given, in this level of approximation, by

$$\Lambda = \frac{1}{\bar{u}_\varphi}\,\overline{\left(\underline{u}'\cdot\underline{F}'\right)}, \tag{7.2.39}$$

$$\Gamma = \frac{1}{\varpi\,\bar{u}_\varphi^2}\overline{\left(\underline{u}'_M\times\hat{\underline{u}}_M + \underline{u}'_M\times\frac{\partial_1\underline{u}'_M}{\partial\varphi}\right)}_\varphi + 2\,\overline{\underline{\nabla}_M\left(\frac{r\,u'_r}{\bar{u}_\varphi}\right)\cdot\underline{\nabla}_M\left(\frac{\hat{u}_z}{\bar{u}_\varphi}\right)}. \tag{7.2.40}$$

And to determine these quantities for given $\bar{u}_\varphi$ and g (also given $\underline{F}'$), we solve (7.2.21 - 23). Thus, Λ and Γ are complicated functionals of $\bar{u}_\varphi$ and g . Equations (7.2.34 - 38) determine $\bar{u}_\varphi, \underline{\bar{u}}_M, \bar{B}_\varphi$ and $\underline{\bar{B}}_M$ for given g , Λ and Γ. Thus they provide a complicated functional relationship between $\bar{u}_\varphi$ and g . Once g is determined from the boundary layer equations, $\bar{u}_\varphi$ and hence all other quantities are known (in principle!).

7.3 The boundary layers

The basic flow

$$\underline{u} = \bar{u}_\varphi\,\underline{1}_\varphi + \underline{u}', \qquad \underline{B} = \bar{B}_\varphi\,\underline{1}_\varphi + \underline{B}', \tag{7.3.1}$$

of the Braginskiĭ dynamo is aligned $\left[\underline{u}\times\underline{B} = O(R^{-1}),\ \text{cf. (7.2.20)}\right]$. It obeys the equations of motion for a steady, highly rotating, <u>inviscid</u>, <u>perfectly-conducting</u>, fluid. It does not, in general, satisfy the boundary conditions on S, viz. the no-slip conditions on $\underline{u}$, and the continuity of $\underline{B}$ to a suitably constructed potential field in the external insulator, $\hat{V}$. It is the boundary layer, $\mathcal{B}$, on S which effects the adjustment of the interior flow (7.3.1) to these conditions.

We have already commented on the similarity of $\mathcal{B}$ to the Ekman layer familiar in the hydrodynamics of rotating systems. Its thickness is $O(E^{\frac{1}{2}})$. The role of the magnetic field is more equivocal. We are familiar in magnetohydrodynamics with boundary layers of two main types. The first of these is the layer of the Hartmann type associated particularly with crossed flows $(\underline{u}\times\underline{B}\neq 0)$ in ducts. In this $\underline{n}\cdot\underline{B}$ is non-zero (except usually at isolated points on S) as the limit $\nu\to 0$, $\eta\to 0$ is taken. Like the Ekman layer, the Hartmann layer is <u>local</u> in the sense that the jump conditions across it are determined by the adjacent interior flow and not upon the previous "history" of the flow. Also like the Ekman layer, the Hartmann layer <u>controls</u> the interior flow since it selects which, of an infinite sequence of possible flows satisfying the interior equations, is actually realized. The second type of magnetohydrodynamic boundary layer is the magnetic and viscous diffusion layer associated (mainly) with aligned flows: the magnetohydrodynamic Blasius flow is a good example (cf. e.g. Roberts, 1967, § 6.3). This layer is <u>historical</u>, in the sense that the local conditions in the interior are not sufficient to determine its structure; the conditions previously encountered by the flow enters the boundary layer description. It is also <u>passive</u>, in the sense that the interior flow can be determined uniquely as a first step, and the boundary layer may be matched to it later. In the present situation, it is not clear which case (if either!) we are dealing with. For, when ν and η are finite, $\underline{n}\cdot\underline{B}(=\underline{n}\cdot\underline{B}')$ is non-zero; but, as ν and η tend to zero, so does $\underline{n}\cdot\underline{B}$. In some sense, we are dealing with a triple limiting process, viz. $\nu\to 0$, $\eta\to 0$, $\underline{n}\cdot\underline{B}\to 0$!

Let $\underline{n}$ be the inward normal at a point, P, of S and let φ and θ be tangential coordinates at P. Use n, φ, and θ to distinguish components of vectors and also of vector equations. By (7.2.4,n), the total pressure P is constant across the boundary layer in the first approximation; also we note that (7.2.5,n) is essentially a consequence of (7.2.5,φ, θ) and the continuity equation $\operatorname{div}\underline{B}=0$. We may concentrate on (7.2.4,φ, θ) and (7.2.5, φ, θ). We observe that the $B_n\partial/\partial$ part of the $\underline{B}\cdot\underline{\nabla}$ operator in these equations is, at most, $O(R^{-\frac{1}{2}}\bar{B}_\varphi)$ times its operand, since $\underline{n}\cdot\underline{B}=0$ to leading order. In fact, if the boundary layer is of Hartmann character, the $B_n\partial/\partial n$ terms in (7.2.4, φ, θ) and (7.2.5, φ, θ) must be of the same order as the diffusion terms, requiring

$$\frac{B_n}{\delta}\bar{B}_\varphi \approx \frac{E}{\delta^2}\bar{u}_\varphi, \qquad \frac{B_n}{\delta}\bar{u}_\varphi \approx \frac{1}{R\delta^2}\bar{B}_\varphi, \tag{7.3.2}$$

where δ is the boundary layer thickness. This gives

$$\delta = \left(\frac{E}{RB_n^2}\right)^{\frac{1}{2}}, \tag{7.3.3}$$

i.e. if $B_n=O(R^{-\frac{1}{2}})$, then $\delta=O(E^{\frac{1}{2}})$, and is greater otherwise. Now comparison of the $\underline{1}_z\times\underline{u}$ and $E\nabla^2\underline{u}$ terms of (7.2.4) shows that the flow adjusts to the no-slip conditions on S in an Ekman distance of $E^{\frac{1}{2}}$.

Thus, if $B_n = O(R^{-\frac{1}{2}})$, the boundary layer just discussed must really be a composite Ekman-Hartmann layer. If, however, $B_n = O(R^{-1})$, for example, there must be a second (inner) boundary layer of Ekman character within the (outer) boundary layer just described.

The radial flow in or out of the Ekman layer is $O(\bar{u}_\varphi \delta) = O(E^{\frac{1}{2}} \bar{u}_\varphi)$, which will, in general, have an axially symmetric part. In the Braginskiĭ theory as we have developed it, however, the largest radial flow is $O(R^{-1} \bar{u}_\varphi)$. It follows that**

$$\frac{1}{RE^{1/2}} \longrightarrow \text{limit } (\lambda, \text{say}) \neq \infty, \tag{7.3.4}$$

as $E \longrightarrow 0$ and $R \longrightarrow \infty$. This goes against the grain: one usually expects to be allowed to let $\nu \longrightarrow 0$ and $\eta \longrightarrow 0$ in any order, and quite independently. This does not seem to be the case here. One must conclude that if, for example, $ER \longrightarrow$ limit as $E \longrightarrow 0$ and $R \longrightarrow \infty$, the resulting dynamo, if one exists, will not be of the Braginskiĭ type.

Suppose $B_n' = O(R^{-\frac{1}{2}})$. By (7.3.2), the jump in B_φ across $\mathcal{B}$ is $O(R^{-\frac{1}{2}} u_\varphi)$. In other words, $\bar{B}_\varphi$ must in the interior flow satisfy on S^*, the boundary condition required on S, viz. $\bar{B}_\varphi = 0$. Thus $\bar{\chi} = 0$ on S^*, where, it is recalled (cf. 7.2.20)

$$\bar{B}_\varphi = \bar{\chi}\, \bar{u}_\varphi, \quad \underline{B}_M' = \bar{\chi}\, \underline{u}_M'. \tag{7.3.5}$$

We have, however, already seen that $\underline{n} \cdot \underline{u} = O(R^{-1})$ on S^*, i.e. $\underline{n} \cdot \underline{u}'$ vanishes, despite the fact that $\underline{n} \cdot \underline{B}'$ does not. In other words, the second of

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**If we take $R = 10^4$ and $\lambda = 1$, this gives $\nu = 1.8\ 10^3\ m^2/s$ for the Earth's core ($L \approx 3.5.10^6 m$, $2\Omega = 1.4\ 10^{-4} s^{-1}$). This is rather large by usual geophysical reckoning, though well within the generous uncertainties familiarly quoted; it may imply, if taken seriously, that ν is a <u>turbulent</u> viscosity.

(7.3.5) requires that $\bar{\chi} \to \infty$ on S^*. This contradiction shows that the supposition that $\underline{n} \cdot \underline{B} = O(R^{-\frac{1}{2}})$ is incorrect: $\underline{n} \cdot \underline{B}$ vanishes to a higher order. In fact, we may now go further. The boundary layer has the double structure mentioned below (7.3.3), the inner layer being, to the first approximation, an axisymmetric Ekman layer. The radial influx or efflux created by $\bar{u}_\varphi$ is therefore axisymmetric also. Thus, the first non-vanishing component of $\underline{u}'$ on S^* will be $\underline{u}_3'$, at most. This supports Braginskiĭ's contention that $\underline{B}_3'$ is the largest asymmetric field escaping the fluid.

Summarizing our findings so far: there is a double boundary layer on S consisting of (i) an inner layer, of thickness $E^{\frac{1}{2}}$, which is to first approximation an axisymmetric Ekman layer. It is local and controls the flow in the outer boundary layer and interior region. The no-slip conditions are satisfied on S, its outer edge, but not on its inner edge. There is also (ii) a magnetic diffusion layer in which the aligned flow in the interior region adjusts to the potential field in the surrounding insulator. In this layer, which is passive and historical, the flow and field become, to some extent at least, crossed in direction. If we examine (7.2.5, φ), however, we soon convince ourselves that there is no such boundary layer for $\bar{B}_\varphi$; for the notion that the layer thickens by diffusion in the direction of field and flow (as in the mangetohydrodynamic Blasius situation) is inconsistent with its necessary periodicity in φ. The outer boundary layer is, then, associated with $\underline{B}'$ only. It is given presumably by a theory akin to that developed by Braginskiĭ [1964, p.733, see circa eq. (4.18)].

Since the inner boundary layer is an Ekman layer, we may apply the usual jump conditions on $\underline{n}\cdot\underline{u}$ across it. Consider a point P on S at which the angle between $\underline{\Omega}$ and the outward normal to S is θ. Let n be a coordinate drawn inwards along $\underline{n}$ from P (see Fig. 7.2). Let $r_m(\theta)$ and $r_\ell(\theta)$ be the radii of curvature of the meridian section of S and the circle of latitude at P. Denoting by a superimposed tilde the boundary layer variables and by a superfix zero the corresponding values in the outer boundary layer and interior region, we have, in the usual way,

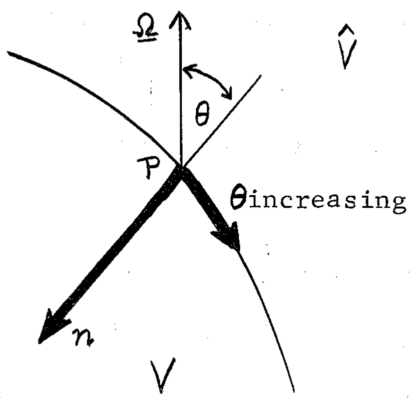

Fig. 7.2

$$\frac{\partial^2}{\partial n^2}(\tilde{u}_\varphi + i\tilde{u}_\theta) = -i\cos\theta(\tilde{u}_\varphi + i\tilde{u}_\theta). \tag{7.3.6}$$

Defining $\lambda = \lambda(\theta)$ by

$$\lambda = -\lambda_1 + i\lambda_2 = \begin{cases} (-1+i)\sqrt{\dfrac{\cos\theta}{2}}, & 0<\theta<\tfrac{1}{2}\pi; \\ (-1-i)\sqrt{\dfrac{-\cos\theta}{2}}, & \tfrac{1}{2}\pi<\theta<\pi; \end{cases} \tag{7.3.7}$$

we find, from (7.3.6) that

$$\tilde{u}_\varphi + i\tilde{u}_\theta = -(u^0_\varphi + i u^0_\theta)e^{\lambda n}, \tag{7.3.8}$$

From the equation of continuity

$$\frac{1}{E^{1/2}}\frac{\partial\tilde{u}_n}{\partial n} = -\frac{1}{r_\ell r_m}\left[\frac{\partial}{\partial\theta}(r_\ell\tilde{u}_\theta) + \frac{\partial}{\partial\varphi}(r_m\tilde{u}_\varphi)\right]$$

$$= \frac{1}{r_\ell r_m}\left[\frac{\partial}{\partial\theta}\left\{r_\ell \mathcal{J}\,(u^\circ_\varphi + iu^\circ_\theta)e^{\lambda n}\right\} + r_m\frac{\partial}{\partial\varphi}\left\{\mathcal{R}(u^\circ_\varphi + iu^\circ_\theta)e^{\lambda n}\right\}\right]. \qquad (7.3.9)$$

Integrating this equation from $n = 0$ to $n = \infty$, recalling that $\tilde{u}_n$ must vanish as $n \to \infty$, we obtain

$$\tilde{u}_n(0) = \frac{E^{1/2}}{r_\ell}\left\{\frac{1}{r_m}\frac{\partial}{\partial\theta}\left[r_\ell\,\mathcal{J}\left(\frac{u^\circ_\varphi + iu^\circ_\theta}{\lambda}\right)\right\} \pm \frac{\partial}{\partial\varphi}\left\{\mathcal{R}\left(\frac{u^\circ_\varphi + iu^\circ_\theta}{\lambda}\right)\right\}\right]. \qquad (7.3.10)$$

Since $u^\circ_n = -\tilde{u}_n$ on S, we have

$$u^\circ_n(S^*) = \frac{E^{1/2}}{r_\ell}\left\{\pm\frac{1}{r_m}\frac{\partial}{\partial\theta}\left[\frac{r_\ell}{\sqrt{\pm 2\cos\theta}}\left(u^\circ_\varphi \pm u^\circ_\theta\right)\right] + \frac{\partial}{\partial\varphi}\left[\frac{u^\circ_\varphi \mp u^\circ_\theta}{\sqrt{(\pm 2\cos\theta)}}\right]\right\}, \qquad (7.3.11)$$

where here (and below) the upper sign is to be taken for $0 < \theta < \frac{1}{2}\pi$, and the lower sign for $\frac{1}{2}\pi < \theta < \pi$.

In our application, we have $u^\circ_\theta = 0$, and

$$u^\circ_\varphi = \bar{u}_\varphi = \varpi g(\varpi), \text{ on } S^*, \qquad (7.3.12)$$

[cf. (7.3.35)]. Thus, (7.3.11) gives

$$u^\circ_n(S^*) = \pm\frac{1}{\lambda R r_\ell r_m}\frac{\partial}{\partial\theta}\left[\frac{r_\ell\varpi g(\varpi)}{\sqrt{(\pm 2\cos\theta)}}\right]. \qquad (7.3.13)$$

For the particular case of a unit sphere $\varpi = r_\ell = \sin\theta,\ r_m = 1$, so that (7.3.13) becomes

$$\bar{u}_{en}(S^*) = \mp\frac{1}{\lambda R\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\sin^2\theta\, g(\sin\theta)}{\sqrt{(\pm 2\cos\theta)}}\right]. \qquad (7.3.14)$$

(This result, and the boundary layer theory which gave rise to it, break down near the equator. One may expect, however, that the boundary layer singularities in these regions will be passive.) Since $\overline{W} = 0$ on S^*, there is no difference between $\underline{\bar{u}}_M$ and $\underline{\bar{u}}_{eM}$ on S^*. We have therefore used $\bar{u}_{en}$ rather than $\bar{u}_n$ in (7.3.14).

The flow (7.3.14) is either inwards at each end of the cylinder $\mathcal{C}$ (see Fig. 7.1), or outwards at each end. This again agrees with the parity of the Braginskiĭ solution of §4.7. The net inward flux into $\mathcal{C}$ through its ends by (7.3.14) is compensated by a net outward flux through its curved faces ϖ_0 and $\varpi_0 + d\varpi$. Thus (7.3.14) and (7.2.35) provide the required equation determining g. It is not, however, a straightforward matter to find it, since the flow $\underline{u}_{em}$ itself is a functional of g, as we pointed out below (7.2.40). At least, however, we now have a closed problem. Graham Tough and I are looking for simple solutions, without cheating! By this I mean that, by working backwards there is little doubt that we could find a Λ and Γ such that the equations have a solution. From these it would probably be easy to find a sufficiently artificial $\underline{F}'$ to do the trick. The same is true of the kinematic dynamo; once one has a working model, he can substitute the $\underline{u}$ into the Navier-Stokes equation and define the body force as that force required to satisfy that equation. But one would hardly feel one had furthered the cause by doing so!

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8. Reversals

8.1 Introduction: some facts about reversals

It is now plainly established by palaeomagnetism that the Earth's field has reversed its polarity many times in the past. The reversals occur irregularly, the length of time between them ranging from 10^5 to 10^7 years. The time that elapses as the field actually undergoes one reversal is of the order of 10^4 years. Many of the facts about reversals are not known in detail. For example, does the axial dipole reverse sign, the equatorial dipole remaining more or less fixed in magnitude? Or does the whole dipole moment of the Earth swing through 180^o? Again, is the reversal one which affects the entire geomagnetic field, i.e. do both poloidal and toroidal components reverse? Bullard and Gellman (1954) (see also Runcorn, 1955) have suggested that quite small changes in the nature of the poloidal velocity could change the poloidal field quite appreciably. The idea being, essentially, that the dipole field and its reversals are mere perturbations of a much larger toroidal field in the Earth. This view is not easy to accept in the Bragniskiĭ model where regeneration requires that $\Gamma \, \mathrm{grad}\, \zeta$ remains "of one sign". It is true that quite small changes in $\underline{u}'$ will reverse Γ , but the dynamo would then become degenerative unless grad ζ also reversed sign, and it is not clear how this would come about. It is clear, nevertheless, that the time-scale between reversals rules out a purely electromagnetic theory of the phenomena. It is virtually certain that the reversals are a magnetohydrodynamic phenomenon, i.e. a process in which both the electromagnetic induction and mechanical processes play a part. Moreover, the irregularity and amplitude of the oscillation suggests that non-linear effects play an important role (cf. the van der Pol oscillator). This raises the order of difficulty of the problem substantially. For this reason, the only attempts that have been made to understand the process have been extremely qualitative and have been based on the crudest of models.

8.2 The Bullard single rotor model

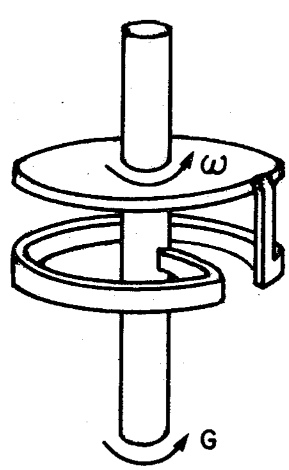

Fig. 8.1

A disk rotates about its axis in a field parallel to its axis, and the current (drawn from two brushes, one on the periphery and one on the axle of the disk) pass through a coil to produce the magnetic field. The behavior of this homopolar dynamo is governed by a pair of differential equations:

$$L\frac{dI'}{dt'}+RI'=M\omega' I', \tag{8.2.1}$$

$$C\frac{d\omega'}{dt'}=G-MI'^2. \tag{8.2.2}$$

Here I' and ω' are the current flowing in the circuit and the angular velocity of the disk; L is the self-inductive and R the resistance of the current; M is the mutual inductive between the coil and disk; C is the moment of inertia of the disk, and G is the torque driving it.

Introduce non-dimensionless variables by

$$I'=\left(\frac{G}{M}\right)^{1/2}I,\quad \omega'=\left(\frac{GL}{CM}\right)^{1/2}\omega,\quad t'=\left(\frac{CL}{GM}\right)^{1/2}t. \tag{8.2.3}$$

Then (8.2.1) and (8.2.2) reduce to

$$\frac{dI}{dt}+\mu I=\omega I, \tag{8.2.4}$$

$$\frac{d\omega}{dt}=1-I^2, \tag{8.2.5}$$

where

$$\mu=\left(\frac{CR}{GM}\right)^{1/2}\Big/\left(\frac{L}{R}\right)^{1/2}. \tag{8.2.6}$$

The critical points (or equilibrium states) of (8.2.4) and (8.2.5) are given by

$$I^2=1,\qquad I(\omega-\mu)=0,$$

i.e.

$$\omega=\mu,\quad I=\pm 1. \tag{8.2.7}$$

It is easy to perform a partial integration of (8.2.4) and (8.2.5). We see from these equations that

$$\frac{d\omega}{dt} = \frac{d\omega/dt}{dI/dt} = \frac{1-I^2}{I(\omega-\mu)},$$

or

$$(\omega-\mu)\,d\omega = \left(\frac{1}{I} - I\right)dI,$$

giving

$$\tfrac{1}{2}(\omega-\mu)^2 = \log I - \tfrac{1}{2}I^2 + \tfrac{1}{4}A. \tag{8.2.8}$$

where A = constant. In the (ω, I) phase plane these solutions are represented by closed curves about the equilibrium solutions A = 2, showing these to be stable (see Fig. 8.2). When A is large, the large currents only occur in short bursts which serve to decelerate the disk,

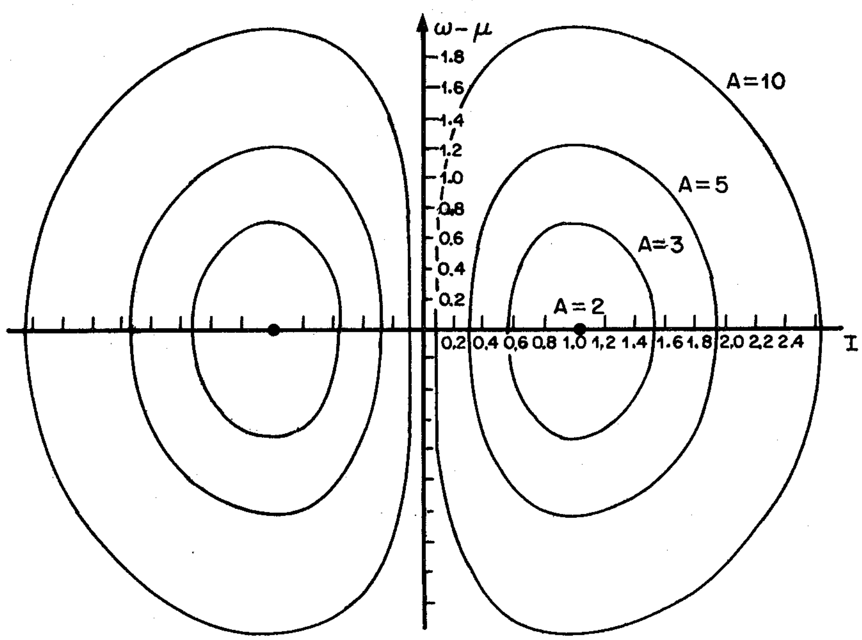

Fig. 8.2

the angular speed of which, up to the time of a burst, grows more or less linearly (see Fig. 8.3). Indeed the burst reverses the direction of the disk's motion. Since no curve in Fig. 8.2 crosses $I = 0$, no reversal can occur. For A close to 2, the oscillations are sinusoidal with periods $\pi\sqrt{2}$. For large A, the oscillations are far from sinusoidal (as we have seen) and the period is* of order $A^{\frac{1}{2}}$.

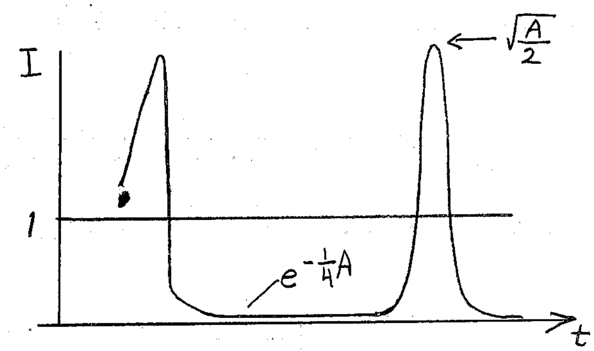

Fig. 8.3

Returning to the physical variables, the equilibria (8.2.7) are

$$\omega' = \mu\left(\frac{GL}{CM}\right)^{\frac{1}{2}} = \frac{R}{M}, \qquad I' = \pm\left(\frac{G}{M}\right)^{\frac{1}{2}}. \tag{8.2.9}$$

From a state of rest in the absence of a magnetic field, the disk would attain this angular velocity under the action of G in a time of order

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*If $\ell = \log I$, it is easily shown from (8.2.4) and (8.2.5) that $d^2\ell/dt^2 = 1 - e^{2\ell}$ so that, if $x = d\ell/dt$, $x\,dx/d\ell = 1 - e^{2\ell}$. Thus $(dx/dt)^2 = 2\left[\ell - \frac{1}{2}e^{2\ell} + \frac{1}{4}A\right]$, giving Period $= \sqrt{2}\oint \frac{d\ell}{[A + 4\ell - 2e^{2\ell}]^{1/2}}$. The greatest contribution to this integral is in the region where $I \ll 1$. This gives Period $\approx 2\sqrt{2}\int_0^{-\frac{1}{4}A} \frac{d\ell}{\sqrt{A+4\ell}} = \sqrt{2A}$. The rest gives a contribution which is $O(A^{-\frac{1}{2}})$.

$$\tau_M = \frac{CR}{GM}, \tag{8.2.10}$$

which we might term 'the mechanical response time'. If the disk were stopped with a magnetic field present, that field would decay according to (8.2.1) in a time of order

$$\tau_E = \frac{L}{R}, \tag{8.2.11}$$

which we might term 'the electromagnetic diffusion time'. It is clear that

$$\mu^2 = \frac{\text{mechanical response time}}{\text{electromagnetic diffusion time}} \tag{8.2.12}$$

Moreover, the ratio of stored mechanical energy to stored electromagnetic energy is

$$\frac{\frac{1}{2}C\omega'^2}{\frac{1}{2}CI'^2},$$

which, for solution (8.2.9) is simply CR^2/LGM, i.e.

$$\mu^2 = \frac{\text{stored mechanical energy}}{\text{stored electromagnetic energy}} \tag{8.2.13}$$

giving a second interpretation of μ.

The time-scale (8.2.3) is seen to be

$$t' = (\tau_M \tau_E)^{1/2} t, \tag{8.2.14}$$

the geometric mean of the two characteristic times. This determines the oscillation period for small A, while for large A it is $(\tau_M \tau_E A)^{1/2}$, essentially.

It is very difficult to know what the parameter μ will be for the Earth. If we used (8.2.12) taking for (8.2.14), the time-scale of slow Alfvén waves across the core, we might guess that

$$\tau_M = \frac{\mathcal{L}}{V_A^2}\left[V_A^2 + (\mathcal{L}\Omega)^2\right]^{1/2} \approx \frac{\mathcal{L}^2\Omega}{V_A^2} = \frac{(3\cdot 10^6)6\cdot 10^{-5}}{2\cdot 5\ 10^{-5}} = 2\cdot 10^{13}s = 10^6 \text{ years},$$

taking $\mathcal{L} = 3\cdot 10^6$m and a poloidal field of 5 gauss as appropriate for V_A. And, if $\tau_E = 10^4$ years, we would find $\mu = 10$. On the other hand, we might use (9.2.13) and, following Bullard and Gellman (1954, p.272), guess that $\mu^2 = (3\cdot 10^{-3}\text{erg/cm}^3)/(2\cdot 10^3\text{erg/cm}^3) = 10^{-6}$, leading to $\mu = 10^{-3}$. Both Lowes and Allan have argued that $\mu \approx 3\cdot 10^{-3}$ is plausible, although Rikitake (1966) does not seem to feel the value convincing. At any rate, values of μ in the range $10^{-3} < \mu < 10$ seem appropriate.

We may ask what would happen to the model if the coil were wound in the opposite sense, or if the dynamo was driven by a reversed couple. The system would not act as an anti-dynamo, and would destroy flux more quickly than ohmic diffusion would unaided. The sign of M in (9.2.1) would be reversed (or ω' and G would be reversed, which has the same effect). Equations (8.2.4) and (8.2.5) would be replaced by

$$\frac{dI}{dt} + \mu I = -\omega I, \tag{8.2.15}$$

$$\frac{d\omega}{dt} = 1 + I^2. \tag{8.2.16}$$

There are no equilibria and the (ω, I) phase plane becomes that shown in Fig. 8.4. As $t \longrightarrow \infty$ the system tends to a state in which $(\omega + \mu) \longrightarrow +\infty$ and $I \longrightarrow 0$.

Reverting to the regenerating case, if a small viscosity is added to the model, the equilibria ($A = 2$) are slightly displaced, and the solutions spiral down onto them in time.

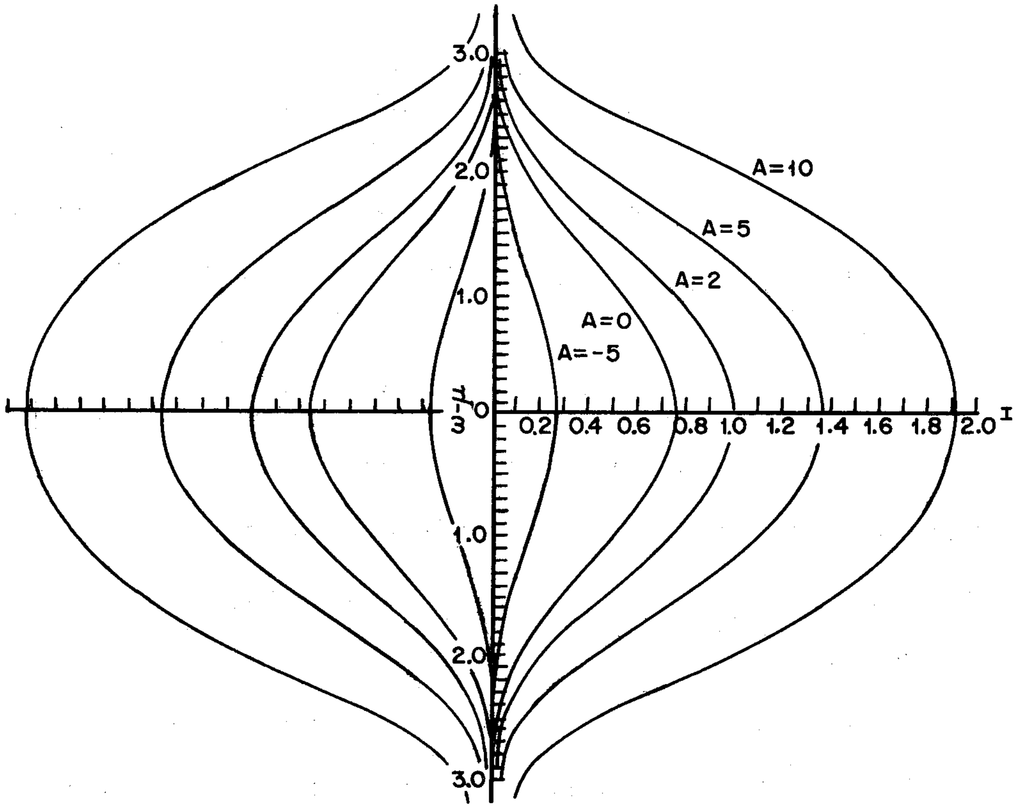

Fig. 8.4

8.3 The Rikitake two-rotor model

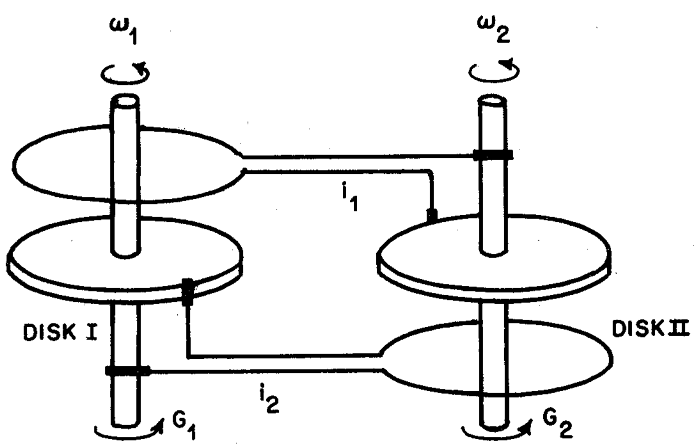

Fig. 8.5

Rikitake (1958) was the first to consider a system of two disk dynamos coupled in such a way that the current from each feeds the field coil of the other. This is an attempt to picture the dynamics of two large eddies in the Earth's core and the energy exchange between them. Crude though it is, the model is surprisingly suggestive.

The governing equations are

$$L_1 \frac{dI_1'}{dt'} + R_1 I_1' = M_{21} \omega_1' I_2' , \tag{8.3.1}$$

$$L_2 \frac{dI_2'}{dt'} + R_2 I_2' = M_{12} \omega_2' I_1' , \tag{8.3.2}$$

$$C_1 \frac{d\omega_1'}{dt'} = G_1 - M_{21} I_1' I_2' , \tag{8.3.3}$$

$$C_2 \frac{d\omega_2'}{dt'} = G_2 - M_{12} I_2' I_1' , \tag{8.3.4}$$

where now M_{ij} is the mutual inductance of the dynamos defined as the e.m.f. induced in dynamo j when the current in dynamo i is unity and $\omega_j' = 1$. It may be noted that, unless

$$\frac{G_1}{G_2} = \frac{M_{21}}{M_{12}} \quad \left(= I_1' I_2'\right), \tag{8.3.5}$$

there is no possibility of obtaining stationary solutions at all! Some of the more interesting results are obtained, however, by making the rotors and applied couples identical. Then, on introducing the scaling (8.2.3) and (8.2.6) we have

$$\frac{dI_1}{dt} + \mu I_1 = \omega_1 I_2 , \tag{8.3.6}$$

$$\frac{dI_2}{dt} + \mu I_2 = \omega_2 I_1 , \tag{8.3.7}$$

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = 1 - I_1 I_2 . \tag{8.3.8}$$

It is clear from (8.3.8), that the difference in the angular velocity of the two rotors is always constant:

$$\omega_1 - \omega_2 = a = \text{constant}. \tag{8.3.9}$$

The steady states of (8.3.6) to (8.3.8) form a one-parameter family

$$I_1 = \pm k, \quad I_2 = \pm \frac{1}{k}, \quad \omega_1 = \mu k^2, \quad \omega_2 = \frac{\mu}{k^2}, \tag{8.3.10}$$

where k is arbitrary, and is related to a by

$$\mu\left(k^2 - k^{-2}\right) = a, \tag{8.3.11}$$

$$k = \pm\left[\frac{a}{2\mu} + \sqrt{\left[1 + \left(\frac{a}{2\mu}\right)^2\right]}\right]^{1/2}. \tag{8.3.12}$$

Thus one value of a determines two equal and opposite values of k, and one value of k determines one value of a. Of course, the initially assigned values of ω_1 and ω_2 determine the relevant values of $|k|$ and a.

The normal modes $\left(e^{st}\right)$ of infinitesimal perturbation about the state $\pm k$ can be shown to be

$$s = 0, \quad s = -2\mu, \quad s = \pm i\left(k^2 + k^{-2}\right)^{1/2}. \tag{8.3.13}$$

(There is a slight error in Rikitake's work here, as was pointed out first by Gjellestad, and later by Barr and Allan independently.) The $s = 0$ in (8.3.13) reflects the existence of the neighboring steady states (8.3.10) of slightly different k, and, in fact, this normal mode does not conserve $\omega_1 - \omega_2$. The other roots do. These show that each equilibrium (8.3.10) is stable to infinitesimal perturbations, but this, of course, does not settle the question of stability to finite amplitude disturbances. The numerical results, to be described presently, suggest a measure of stability.

Equations (8.3.6) to (8.3.8) have some nice properties. If we consider the trajectories in $(I_1, I_2, \omega_1, \omega_2)$ space as describing a fluid, its specific volume,

$$J = \frac{\partial(I_1, I_2, \omega_1, \omega_2)}{\partial(I_{10}, I_{20}, \omega_{10}, \omega_{20})}, \tag{8.3.14}$$

where the suffix zero denotes initial values, satisfies the equation of continuity

$$\frac{DJ}{Dt} = J \operatorname{div} \underline{u}, \tag{8.3.15}$$

where $\underline{u}$ denotes "Eulerian velocity" in $(I_1, I_2, \omega_1, \omega_2)$ space. According to (8.3.6) to (8.3.8), written as

$$\frac{dI_1}{dt} = -\mu I_1 + \omega_1 I_2, \quad \frac{dI_2}{dt} = -\mu I_2 + \omega_2 I_1, \quad \frac{d\omega_1}{dt} = 1 - I_1 I_2, \quad \frac{d\omega_2}{dt} = 1 - I_1 I_2, \tag{8.3.16}$$

this Eulerian velocity is

$$\underline{u} = (-\mu I_1 + \omega_1 I_2, -\mu I_2 + \omega_2 I_1, 1 - I_1 I_2), \tag{8.3.17}$$

from which it is apparent that

$$\operatorname{div} \underline{u} = -\mu - \mu + 0 + 0 = -2\mu. \tag{8.3.18}$$

So, by (8.3.15), we have

$$J = J_0 e^{-2\mu t}. \tag{8.3.19}$$

Thus, if we take a blob of solutions $\delta I_1\, \delta I_2\, \delta\omega_1\, \delta\omega_2$ at $t = 0$, its volume in this four-dimensional space will decrease exponentially in, essentially, at the beginning of any numerical integration, we must expect a transient "shake-down period" lasting a time of under τ_E in which transients are important.

The case $\mu = a = 0$ may be integrated exactly. First suppose only that $a = 0$ (i.e. $|k| = 1$). Then $\omega_1 = \omega_2 = \omega$, (say), and (8.3.6) to (8.3.8) gives

$$\frac{dI_1}{dt} + \mu I_1 = \omega I_2, \quad \frac{dI_2}{dt} + \mu I_2 = \omega I_1, \quad \frac{d\omega}{dt} = 1 - I_1 I_2. \tag{8.3.20}$$

Put

$$u = \frac{I_1 + I_2}{2}, \qquad v = \frac{I_1 - I_2}{2}, \tag{8.3.21}$$

Then (8.3.20) gives

$$\frac{du}{dt} = -(\mu - \omega)u, \quad \frac{dv}{dt} = -(\mu + \omega)v, \quad \frac{d\omega}{dt} = 1 - u^2 + v^2. \tag{8.3.22}$$

It is clear that

$$\frac{d}{dt}(uv) = \left[-(\mu - \omega) - (\mu + \omega)\right] uv = -2\mu uv,$$

i.e.

$$uv = \frac{1}{4}A^2 e^{-2\mu t}, \tag{8.3.23}$$

where A is a constant. This is essentially the integral (8.3.19). It shows that, for $t >> 1/\mu$, either $u \approx 0$ or $v \approx 0$. If $v \approx 0$, we essentially return to the regenerative case of (8.2.4) and (8.2.5). If $u \approx 0$, we are dealing with the degenerative case of (8.2.15) and (8.2.16). The solutions do not, however, necessarily tend to remain on one or other branch. There is the possibility of a switch-over from one kind of trajectory to the other, and back. And this kind of reversal does, in fact, seem to occur.

By (8.3.23), we may write

$$u = \frac{1}{2}Ae^{\phi - \mu t}, \quad v = \frac{1}{2}Ae^{-\phi - \mu t}, \tag{8.3.24}$$

and (8.3.22) then gives

$$\frac{d\phi}{dt} = \omega, \quad \frac{d\omega}{dt} = 1 - \frac{1}{2}A^2 \sinh 2\phi\, e^{-2\mu t}. \tag{8.3.25}$$

These cannot be integrated in closed form except in the case $\mu = 0$, for which

$$\frac{1}{2}\omega^2 = B + \phi - \frac{1}{4}A^2 \cosh 2\phi,$$

giving solutions in closed trajectories about the point

$$\left[\frac{1}{2}\sinh^{-1}\left(\frac{2}{A^2}\right), 0\right],$$

in the (ϕ, ω) plane. For small ω, it is possible to solve the motion in terms of a slow drift for one of these trajectories to another.

On the numerical side most attention has been focussed on the case of large μ in which the initial phase (in which the initial J_0 dies to zero) is over rapidly*. The oscillations occur about one or other of the two states (8.3.10). Let us, for definiteness, suppose that $k^2 > 1$. (If this is not the case, interchange the suffice 1 and 2 in the following discussion.) In a large amplitude oscillation I_1 tends to resemble the solutions in the Bullard single rotor model, i.e. its graph against time has flat bottom, and sharp peaks. During these oscillations I_2 shows a similar behavior, but may be reversed in sign from (8.3.10) during an inward excursion. The velocity oscillations tend to a saw-toothed form in which the accelerations are practically constant while the currents are small, rapid decelerations occurring during current peaks. One or both velocities may become negative on the inward excursions.

As the amplitude of oscillation increases, and inward excursion of I_1 comes nearer to $I_1 = 0$ and eventually it crosses this axis. When this happens,

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*Rikitake (1958) who considered the case $\mu = 1$, $k^2 = 4$ did not, however, follow the system past this initial period. The results he obtained were not, therefore, really typical of the late behavior.

I_2 is already reversed, and both currents are now amplified in their new sense, and an oscillation about the other steady state commences; the velocities oscillate about the same positive value as before. The reversed sign of I_1 may persist for one, two or more oscillations, increasing in violence, until there is another instability. The nearer any inward excursion to $I_1 = 0$ comes, without crossing the axis, the more certainly it crosses the axis on the next oscillation, and the smaller the initial oscillation about the reversed state. This means that it stays in the neighborhood of that state longer than average before reversing again. If, on the other hand, I_1 only "just" crosses its axis on one excursion, it may oscillate only once about the reversed state before returning. There is, in any case, no regularity or repetition in the pattern, and there is no evidence that a particular sequence ever repeats. Fig. 8.6 shows a reversal occurring (Allan, 1962).

Although, as we have already remarked, systems of disk dynamos appear to model the Earth's core extremely indirectly, if at all, the very fact that they, too, suffer irregular reversals is an encouraging sign which should not be underestimated. Also they are rather simple to treat analytically and numerically, and further work, particularly that of examining the statistics of their reversals (as deduced from many integrations at each μ and α) might be revealing without being too tedious. It might, for example, help to decide on the most plausible value of μ for the Earth (cf. eq. 8.2.6). Also generalizations of the coupled dynamos might be considered; for example, the effects of self-inductance (Matthews) or of time-delays (Lowes) between the two rotors could be examined. The former is

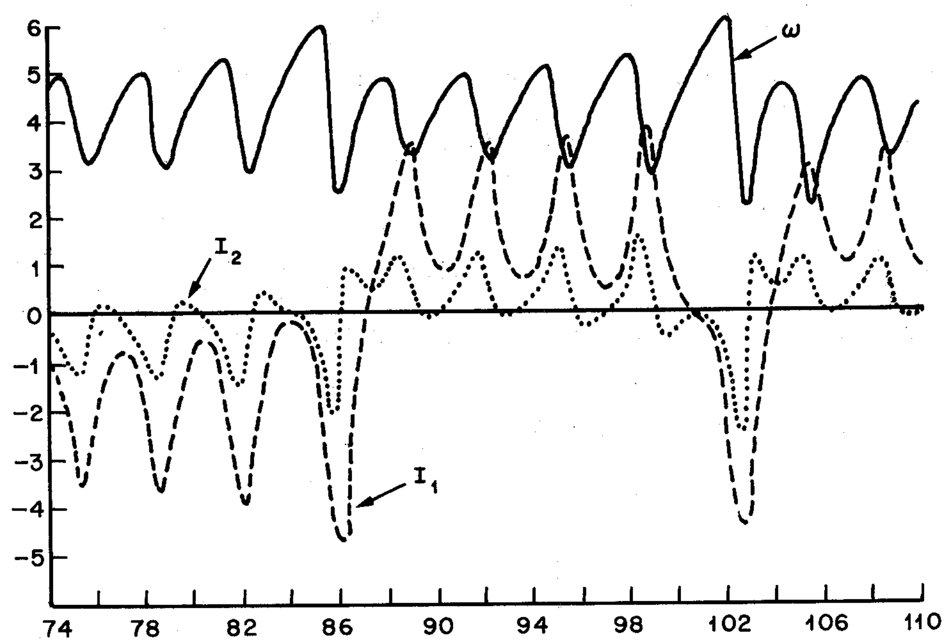

Fig. 8.6

obviously present in the two eddies in the case which the stars are meant to imitate, and the latter might simulate the time taken for electromagnetic energy to pass from one eddy to the other. Also, the generalization to more than two rotors might be rewarding. In this connection, we should note that Lebovitz (1960) has shown that a chain of N identical disk dynamos, each linked to its two neighbours electromagnetically is linearly unstable if $N>2$.

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