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DISPERSION OF FINE PARTICLES BY WAVE-INDUCED MASS TRANSPORT
NEAR A CIRCULAR ISLAND

by

Carole A. Womeldorf

Sc.B. Mechanical Engineering, A.B. English and American Literature
Brown University, 1989

Submitted to the Massachusetts Institute of Technology
and Woods Hole Oceanographic Institution
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ABSTRACT

We begin with the linearized irrotational theory describing the diffraction of an incoming plane wave by a vertical cylinder. From the free-surface motion, we describe the boundary-layer flow field U_{0i} near that cylinder. The flow field is then used to calculate the modified convective velocities $\mathcal{U}_i$ and symmetric dispersivities $\mathcal{D}_{ij}$.

Through our calculations we have demonstrated convective flow reversal for both large and very small cylinders. In addition we have characterized four regions of waves and transport about the large cylinder relating to the different characteristics of each region: reflected region, scattered region, plane wave region, and sharp-edged shadow region. Our understanding of the transport velocities and dispersivities allows us to speculate qualitatively on the transport of fine particles near a round island in a diffracted wave field.

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In memory of Mr. Art Tenney

and

Professor George Cox

Early teachers

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CHAPTER 1-INTRODUCTION TO HEAVY PARTICLE TRANSPORT

1.1 Background

The way particles move in flowing water has been studied for many years for many reasons. In 1954 Taylor developed a theory coupling simple molecular diffusion and the complicated correlations of velocity and concentrations fluctuations into a term called the dispersivity. This insight permitted the standard diffusion equation to describe more complicated motions with simpler mathematics. As a result of his work, others were able to apply his descriptions to more complicated forms.

Taylor's work on dispersion in steady flows was extended by Aris to describe transport in an oscillating flow in a circular pipe in 1960. There have been several other extensions¹ utilizing dispersivity to describe particle transport in different types of laminar, oscillating flows. Meanwhile in 1964, Bowden developed descriptions of tidal flows using a turbulent eddy viscosity, V_e , to describe the turbulent flows in channels and tidal regions. His work on coastal flows provided the necessary footing for Holly, Harleman, & Fischer in 1970 to combine dispersivity in oscillatory flows with turbulent eddy viscosities. These works are essential for the description of neutrally buoyant pollutant transport in natural flows.

Since that time scientists and engineers have continued to try to describe specific situations in pollutant transport more exactly. Many numerical models have been developed with the dispersivities estimated by known values in uniform flow or calculated by field measurements at a few sites. Recently, Mei & Chian (1993) have applied the boundary layer theory of

¹Chatwin (1975), Dill & Brenner (1982), Watson (1983)

mass transport to the theory of dispersion of heavy particles in turbulent oscillatory flow. In this thesis a new application of their theory is described.

1.2 Introduction to the convection-diffusion equation

The flow field in the boundary layer beneath a linear wave field has been well described by Hunt & Johns (1963). The first order Stokes flow and the second order time-averaged induced-streaming velocity are known in terms of the inviscid ambient flow. With knowledge of the flow field, Mei & Chian (1993) have applied the method of multiple scales to deduce a transport equation for particle concentration, as reviewed below.

1.2.1 Basic approximations

To describe how the particles move relative to a fluid it is necessary to estimate the drag on the particle. We compare the relaxation time of a particle in water with the period of the wave. Consider a fine sand or silt grain with a small radius, $a \sim .01\text{cm}$. The particle density, ρ_s , is greater than, but on the order of the density of water, $\rho_s/\rho \sim 2.5$; the wave frequency is $\omega = 1 \sim .01\text{rad/s}$; and the molecular viscosity is $\nu = .01\text{cm}^2/\text{s}$. By Stokes formula of the drag caused by laminar flow about a sphere, a relaxation time, τ , for a particle to acquire the velocity of the surrounding fluid has been estimated by Saffman (1962)

$$\omega\tau = \frac{2}{9} \frac{\rho_s}{\rho} \frac{a^2 \omega}{\nu} \ll 1. \quad (1.1)$$

It is apparent that the period of the wave, $T = 2\pi/\omega$, is much longer than the relaxation time. Consequently, the lag time between the motion of the

particles and the motion of the water is very small, so we can assume the particles move with the fluid.

The transport of a dilute sediment cloud of fine particles with a fall velocity, w_0 , is described by the diffusion equation

$$\frac{\partial C}{\partial t} + \frac{\partial u_j C}{\partial x_j} + \frac{\partial}{\partial z} [(-w_0 + w)C] = D \left(\frac{\partial^2 C}{\partial x_j \partial x_j} + \frac{\partial^2 C}{\partial z^2} \right) \quad (1.2)$$

where the diffusion coefficient is taken as isotropic. The vertical boundary conditions of this equation are defined at the bottom of the boundary layer, on the sea floor, at $z = 0$; and at the top of the boundary layer as $z \rightarrow \infty$. At $z = 0$, we assume that the particles neither settle nor erode the sea floor. For this to be true it is necessary that the shear velocity, u_* , at the bottom is greater than but on the order of the fall velocity, w_0 , i.e. $\kappa u_* > w_0$ (Batchelor, 1968). Then the boundary condition at $z = 0$ is

$$D \frac{\partial C}{\partial z} + w_0 C = 0. \quad (1.3)$$

At the top of the boundary layer, as $z \rightarrow \infty$, the concentration goes to zero, $C \rightarrow 0$ as the particles are assumed to be heavier than water.

To analytically derive the dispersive transport equation from equation 1.2 we use two common postulates. The first of these is the equality of the momentum and mass diffusivities, $\nu_e = D$, known as the Reynolds analogy (Taylor, 1953). And the second postulate is the Boussinesq approximation, that ν_e (and by Reynold's analogy, D) is constant in time and space.

1.2.2 Vertical Length Scales

From the assumptions and equations above, three vertical characteristic length scales are suggested: d , δ , and δ_D . The first, d , is the thickness of the steady concentration layer defined by $d \sim D / w_0$. The second is $\delta \sim \sqrt{2\nu_e/\omega}$; the oscillatory boundary layer thickness due to the transient momentum diffusion. The final is the oscillatory boundary layer thickness due to transient mass diffusion, $\delta_D \sim \sqrt{2D/\omega}$.

Comparing these three length scales gives us important information. By invoking the Reynold's analogy, $D = \nu_e$, we see the Schmidt number is one and $\delta \approx \delta_D$. Kajiura (1964) has proposed to approximate the eddy viscosity in the boundary layer by $\nu_e \equiv \kappa u_* z$. As an order of magnitude estimate we may take $\nu_e \sim \kappa u_* \delta$ and again apply Reynold's analogy to get $D \sim \kappa u_* \delta$. We now divide by δ and, assuming $\delta \approx d$, we see by the definition of the steady concentration layer $D/d = w_0 \sim \kappa u_*$, which is consistent with our requirement for the bottom boundary condition. Clearly, it is reasonable then to assume $\delta \approx \delta_D \approx d$.

1.2.3 Velocity scales

We first summarize the results known for the velocity field within the boundary layer as $u_i = u_i^{(1)} + \varepsilon u_i^{(2)} + O(\varepsilon^2)$, $i = 1, 2$. and vertically, $w = w^{(1)} + \varepsilon w^{(2)} + O(\varepsilon^2)$. Here $u_i^{(1)}$ and $w^{(1)}$ describe the purely oscillatory Stokes flow at the first order, $O(kA)$ or $O(\varepsilon)$, where A is the amplitude of the inviscid flow orbital and $\varepsilon = kA$. They are given by

$$u_i^{(1)} = \Re[U_{0i} F_1(z) e^{-i\omega t}] \quad w^{(1)} = \Re\left[\frac{\delta}{(1-i)} \frac{\partial U_{0i}}{\partial x_i} F_w(z) e^{-i\omega t}\right] \quad (1.4a,b)$$

where

$$F_1(z) = 1 - e^{-(1-i)\xi} \quad F_w(z) = 1 - e^{-(1-i)\xi} - (1-i)\xi, \quad (1.5a,b)$$

$\xi = z/\delta$, and $\Re$ indicates the real part of the complex number.

The first steady contribution is from the Eulerian time-averaged induced-streaming velocity, $\bar{u}^{(2)}$, where $u^{(2)} = \bar{u}^{(2)} + \tilde{u}^{(2)}$. Since we are interested in the eventual outcome of the transport of the particles, we do not need to know what happens within a period of a wave, therefore we neglect the oscillatory part, $\tilde{u}^{(2)}$. The second order mass transport velocity is given by

$$\bar{u}^{(2)} = -\frac{1}{\omega} \Re \left(F_2 U_0 \frac{\partial U_0^*}{\partial x} + F_3 V_0 \frac{\partial U_0^*}{\partial y} + F_4 U_0 \frac{\partial V_0^*}{\partial y} \right) \quad (1.6a)$$

$$\bar{v}^{(2)} = -\frac{1}{\omega} \Re \left(F_2 V_0 \frac{\partial V_0^*}{\partial y} + F_3 U_0 \frac{\partial V_0^*}{\partial x} + F_4 V_0 \frac{\partial U_0^*}{\partial x} \right). \quad (1.6b)$$

F_2 , F_3 , and F_4 as functions of z are defined by

$$F_2 = -\frac{1}{2}(1-3i)e^{(-1+i)\xi} - \frac{i}{2}e^{-(1+i)\xi} - \frac{1+i}{4}e^{-2\xi} + \frac{1}{2}(1+i)\xi e^{(-1+i)\xi} + \frac{3}{4}(1-i) \quad (1.7a)$$

$$F_3 = \frac{i}{2}e^{(-1+i)\xi} - \frac{i}{2}e^{-(1+i)\xi} - \frac{1}{4}e^{-2\xi} + \frac{1}{4} \quad (1.7b)$$

$$F_4 = -\frac{1}{2}(1-2i)e^{(-1+i)\xi} + \frac{1+i}{2}\xi e^{(-1+i)\xi} - \frac{i}{4}e^{-2\xi} + \frac{1}{4}(2-3i) \quad (1.7c)$$

where $\xi = z/\delta$. These results were deduced originally by Hunt & Johns (1963), where i in Hunt & Johns equals $-i$ in Mei (1989) and Mei & Chian (1993).

1.2.4 Multiple scales in particle transport

In addition to the condition that $kA \ll 1$, we require the boundary layer to be thin compared to the wave length, $\beta \equiv k\delta \ll 1$. This statement is the first step toward defining the two time scales, differentiating between the

oscillatory time scale of the wave motion and the longer time scale of the particle transport. From β and the diffusive terms it is clear that the two distinct time scales are $O(\delta^2/D)$, time of vertical diffusion across the boundary layer, and $O[1/(k^2D)]$, time for the horizontal diffusion across a wave length. The two contrasting time scales are $t' = \beta^2 t$ and $t = t$. Mei & Chian assume that $\varepsilon = O(\beta)$ and therefore define a $T = \varepsilon^2 t$, so $C(x, y, t)$ is now $C(x, y, t, T)$ where the relative magnitude of each term is determined by scaling by δ , k , $A\omega$ and ω .

Under the same assumptions of order of magnitude, we expand the concentration, $C = C^{(0)} + \varepsilon C^{(1)} + \varepsilon^2 C^{(2)} + O(\varepsilon^3)$. Through a detailed expansion and derivation of the solvability condition, Mei & Chian derive a partial differential equation for the horizontal variations of first order concentration, defined by

$$C^{(0)} = \hat{C}(x, y, T) e^{-w_0 z/D}. \quad (1.8)$$

1.2.5 The convection-diffusion equation

The evolution of $\hat{C}$ is found to be governed by the effective dispersion equation, which was derived by Mei & Chian from a perturbation expansion of the transport equation:

$$\frac{\partial \hat{C}}{\partial t} + \frac{\partial(u_i \hat{C})}{\partial x_i} = D \frac{\partial^2 \hat{C}}{\partial x_i \partial x_i} + \frac{\partial}{\partial x_i} \left(\mathcal{E}_{ij} \frac{\partial \hat{C}}{\partial x_j} \right) \quad (1.9)$$

where

$$u = \frac{1}{\omega} \Re \left(H_1 U_o \frac{\partial U_o^*}{\partial x} + H_2 V_o \frac{\partial U_o^*}{\partial y} + H_3 U_o \frac{\partial V_o^*}{\partial y} \right) \quad (1.10a)$$

$$\mathcal{V} = \frac{1}{\omega} \Re \left(H_1 V_o \frac{\partial V_o^*}{\partial y} + H_2 U_o \frac{\partial V_o^*}{\partial x} + H_3 V_o \frac{\partial U_o^*}{\partial x} \right) \quad (1.10b)$$

and

$$\mathcal{E}_{xx} = \Re \left(\frac{H_4}{\omega} |U_o|^2 \right) \quad \mathcal{E}_{xy} = \Re \left[\frac{H_4}{\omega} (U_o^* V_o) \right] \quad (1.11a,b)$$

$$\mathcal{E}_{yx} = \Re \left[\frac{H_4}{\omega} (U_o V_o^*) \right] \quad \mathcal{E}_{yy} = \Re \left(\frac{H_4}{\omega} |V_o|^2 \right) \quad (1.11c,d)$$

where H_1 , H_2 , H_3 , and H_4 are constant in space and time and T is now t .

The above coefficients can be explicitly expressed in terms of the Peclet number, $\wp$

$$\wp = w_o \delta / D = \delta / d. \quad (1.12)$$

and the Schmidt number, Σ

$$\Sigma = \nu_e / D \sim O(1). \quad (1.13)$$

By further defining the following symbols

$$N = \frac{\sqrt{8\Sigma}}{\wp} \quad (1.14)$$

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = -\frac{1}{2} (1 \mp \Gamma_1) \mp \frac{1}{2} \Gamma_2 \quad (1.15)$$

where

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \sqrt{\frac{1}{2} \left(\sqrt{1 + N^4} \pm 1 \right)}. \quad (1.16)$$

Mei & Chian then evaluate a weighted depth average of Hunt & Johns' F_2 , F_3 , and F_4 to define

$$\tilde{F}_2 = \int_0^\infty F_2 e^{-\zeta} d\zeta = \wp \left[\frac{1-3i}{2(\wp+1-i)} + \frac{i}{2(\wp+1+i)} + \frac{1+i}{4(\wp+2)} - \frac{1+i}{2(\wp+1-i)^2} \right] - \frac{3-3i}{4} \quad (1.17a,b)$$

$$\tilde{F}_3 = \int_0^\infty F_3 e^{-\zeta} d\zeta = \frac{\wp}{(\wp+1)^2 + 1} - \frac{1}{2(\wp+2)} \quad (1.18a,b)$$

$$\tilde{F}_4 = \int_0^\infty F_4 e^{-\zeta} d\zeta = \wp \left[\frac{1-2i}{2(\wp+1-i)} + \frac{i}{4(\wp+2)} - \frac{1+i}{2(\wp+1-i)^2} \right] - \frac{2-3i}{4} \quad (1.19a,b)$$

where $\zeta = z/d$, noting that $\tilde{F}_3$ is real.

In addition, the depth averages of the convection terms are defined by

$$\int_0^\infty \overline{u_1 C_1} dz = \frac{d}{\omega} \Re \left[B_1 U_0^* \left(U_0 \frac{\partial \hat{C}}{\partial x} + V_0 \frac{\partial \hat{C}}{\partial y} \right) + B_2 U_0^* \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} \right) \hat{C} \right] \quad (1.20)$$

and

$$\int_0^\infty \overline{v_1 C_1} dz = \frac{d}{\omega} \Re \left[B_1 V_0^* \left(U_0 \frac{\partial \hat{C}}{\partial x} + V_0 \frac{\partial \hat{C}}{\partial y} \right) + B_2 V_0^* \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y} \right) \hat{C} \right] \quad (1.21)$$

where

$$B_1 = \frac{\Sigma}{\wp^2} \left\{ \frac{2\wp}{A_2(1+A_2)(1+i-A_2\wp)[(1+A_1)\wp+1-i][(1+A_2)\wp+1-i]} - \frac{(1+i)\wp^3}{(\wp+1-i)[(1+A_1)\wp+1-i][(1+A_2)\wp+1-i](2+\wp)} + \frac{1+i}{(\wp+1-i)(1+A_1)(1+A_2)} \right\}$$

(1.22)

and

$$B_2 = \frac{\Sigma}{\wp^2} \left\{ \begin{aligned} & \frac{1+i}{A_2(1+i-A_2\wp)} \left[\frac{-\wp^2}{(1+A_2)[(1+A_1)\wp+1-i][(1+A_2)\wp+1-i]} \right] \\ & + \frac{1}{(1+A_2)^2(1+A_1)} \left] + \frac{(2\wp+1+i)(1+i)}{(\wp+1+i)^2(1+A_1)(1+A_2)} \right. \\ & + \frac{i}{(\wp+1-i)(\wp+2)[(1+A_1)\wp+1-i][(1+A_2)\wp+1-i]} \\ & \left. + \frac{(2+A_1+A_2)(1+i)-i(1+A_1)(1+A_2)\wp}{(\wp+1+i)(1+A_1)^2(1+A_2)^2} \right\}. \end{aligned} \right. \quad (1.23)$$

Finally, H_1 , H_2 , H_3 , and H_4 are defined by

$$H_1 = \tilde{F}_2 + B_2^* \quad H_2 = \tilde{F}_3 \quad (1.24a,b)$$

$$H_3 = \tilde{F}_4 + B_2^* \quad H_4 = -B_1 \quad (1.24c,d)$$

where $\Im H_1 = \Im H_3$ and, since $\tilde{F}_3$ is real, H_2 is also real.

It can be shown from formulas 1.7a-1.7c that

$$F_2 \equiv F_3 + F_4, \quad (1.25)$$

which implies $\tilde{F}_1 = \tilde{F}_3 + \tilde{F}_4$ and $H_1 \equiv H_2 + H_3$. This will be used to simplify the convection-diffusion equation.

CHAPTER 2 - MODIFIED CONVECTION-DIFFUSION EQUATIONS

2.1 Modified symmetric $\mathcal{D}_{ij}$

The dispersion tensor as defined is not generally symmetric, i.e. $\mathcal{E}_{ij} \neq \mathcal{E}_{ji}$, as is noted in Mei & Chian (1993). It is known that any tensor can be divided into a symmetric part and anti-symmetric part, $\mathcal{E}_{ij} = \mathcal{D}_{ij} + \mathcal{A}_{ij}$. Since the anti-symmetric part, $\mathcal{A}_{ij}$, does not contribute to dispersions, it is physically more appropriate to incorporate $\mathcal{A}_{ij}$ into the convective velocities.

From equation (1.9) we expand the original dispersivity tensor

$$\frac{\partial}{\partial x_i} \left(\mathcal{E}_{ij} \frac{\partial \hat{C}}{\partial x_j} \right) = \frac{\partial (\mathcal{D}_{ij} + \mathcal{A}_{ij})}{\partial x_i} \frac{\partial \hat{C}}{\partial x_j} + \mathcal{D}_{ij} \frac{\partial^2 \hat{C}}{\partial x_i \partial x_j} \quad (2.1)$$

where the anti-symmetric part is left out of the last term because

$$\mathcal{A}_{ij} \frac{\partial^2 \hat{C}}{\partial x_i \partial x_j} = -\mathcal{A}_{ji} \frac{\partial^2 \hat{C}}{\partial x_i \partial x_j} = 0. \quad (2.2)$$

This identity and the definition $\mathcal{A}_{ij} = -\mathcal{A}_{ji}$ permit the rearrangement of the anti-symmetric terms into the convection terms as follows:

$$\begin{aligned} \frac{\partial(u\hat{C})}{\partial x} + \frac{\partial(v\hat{C})}{\partial y} - \frac{\partial \mathcal{A}_{xy}}{\partial x} \frac{\partial \hat{C}}{\partial y} - \frac{\partial \mathcal{A}_{yx}}{\partial y} \frac{\partial \hat{C}}{\partial x} = \\ \frac{\partial}{\partial x} \left[\left(u + \frac{\partial \mathcal{A}_{xy}}{\partial y} \right) \hat{C} \right] + \frac{\partial}{\partial y} \left[\left(v + \frac{\partial \mathcal{A}_{yx}}{\partial x} \right) \hat{C} \right] \end{aligned} \quad (2.3)$$

We now expand the velocities and their complex conjugates into their real and imaginary parts

$$U_o = U_o^R + iU_o^I \quad V_o = V_o^R + iV_o^I \quad (2.4a,b)$$

$$U_o^* = U_o^R - iU_o^I \quad V_o^* = V_o^R - iV_o^I \quad (2.4c,d)$$

We then expand the dispersivities, separating the normalized symmetric and the anti-symmetric parts.

$$\mathcal{E}_{xx} = \mathcal{D}_{xx} = H_4^R (U_o^R U_o^R + U_o^I U_o^I) \quad (2.5a)$$

$$\mathcal{E}_{xy} = \mathcal{D}_{xy} + \mathcal{A}_{xy} \quad (2.5b)$$

where $\mathcal{D}_{xy} = H_4^R (U_o^R V_o^R + U_o^I V_o^I)$ and $\mathcal{A}_{xy} = H_4^I (U_o^I V_o^R + U_o^R V_o^I)$,

$$\mathcal{E}_{yx} = \mathcal{D}_{yx} + \mathcal{A}_{yx} \quad (2.5c)$$

where $\mathcal{D}_{yx} = H_4^R (U_o^R V_o^R + U_o^I V_o^I)$ and $\mathcal{A}_{yx} = -H_4^I (U_o^I V_o^R + U_o^R V_o^I)$, and

$$\mathcal{E}_{yy} = \mathcal{D}_{yy} = H_4^R (V_o^R V_o^R + V_o^I V_o^I). \quad (2.5d)$$

We can now redefine $\mathcal{U}$ and $\mathcal{V}$ as:

$$\tilde{\mathcal{U}} = \mathcal{U} + H_4^I \frac{\partial}{\partial y} (U_o^I V_o^R - U_o^R V_o^I) \quad (2.6a)$$

$$\tilde{\mathcal{V}} = \mathcal{V} - H_4^I \frac{\partial}{\partial x} (U_o^I V_o^R - U_o^R V_o^I) \quad (2.6b)$$

or, more explicitly,

$$\begin{aligned} \tilde{\mathcal{U}} = & H_1^R \left(U_o^R \frac{\partial U_o^R}{\partial x} + U_o^I \frac{\partial U_o^I}{\partial x} \right) + H_1^I \left(-U_o^I \frac{\partial U_o^R}{\partial x} + U_o^R \frac{\partial U_o^I}{\partial x} \right) \\ & + H_2^R \left(V_o^R \frac{\partial U_o^R}{\partial y} + V_o^I \frac{\partial U_o^I}{\partial y} \right) + H_2^I \left(-V_o^I \frac{\partial U_o^R}{\partial y} + V_o^R \frac{\partial U_o^I}{\partial y} \right) \\ & + H_3^R \left(U_o^R \frac{\partial V_o^R}{\partial y} + U_o^I \frac{\partial V_o^I}{\partial y} \right) + H_3^I \left(-U_o^I \frac{\partial V_o^R}{\partial y} + U_o^R \frac{\partial V_o^I}{\partial y} \right) \\ & + H_4^I \left(U_o^I \frac{\partial V_o^R}{\partial y} - U_o^R \frac{\partial V_o^I}{\partial y} + V_o^R \frac{\partial U_o^I}{\partial y} - V_o^I \frac{\partial U_o^R}{\partial y} \right). \end{aligned} \quad (2.7)$$

Clearly the last term can be incorporated in to the above equation by defining an H_5 and an H_6 and simplifying H_4 such that

$$H_5 = H_2^R + i(H_4^I) \quad H_6 = H_3^R + i(H_3^I - H_4^I) \quad (2.8a,b)$$

$$H_4 = H_4^R. \quad (2.8c)$$

These new definitions work for $\mathcal{V}$ also:

$$\begin{aligned} \tilde{\mathcal{V}} = & H_1^R \left(V_o^R \frac{\partial V_o^R}{\partial y} + V_o^I \frac{\partial V_o^I}{\partial y} \right) + H_1^I \left(-V_o^I \frac{\partial V_o^R}{\partial y} + V_o^R \frac{\partial V_o^I}{\partial y} \right) \\ & + H_2^R \left(U_o^R \frac{\partial V_o^R}{\partial x} + U_o^I \frac{\partial V_o^I}{\partial x} \right) + H_2^I \left(-U_o^I \frac{\partial V_o^R}{\partial x} + U_o^R \frac{\partial V_o^I}{\partial x} \right) \\ & + H_3^R \left(V_o^R \frac{\partial U_o^R}{\partial x} + V_o^I \frac{\partial U_o^I}{\partial x} \right) + H_3^I \left(-V_o^I \frac{\partial U_o^R}{\partial x} + V_o^R \frac{\partial U_o^I}{\partial x} \right) \\ & + H_4^I \left(V_o^I \frac{\partial U_o^R}{\partial x} - V_o^R \frac{\partial U_o^I}{\partial x} + U_o^R \frac{\partial V_o^I}{\partial x} - U_o^I \frac{\partial V_o^R}{\partial x} \right). \end{aligned} \quad (2.9)$$

The symmetric form of the convection-diffusion equation is then

$$\frac{\partial \hat{C}}{\partial t} + \frac{\partial(\tilde{u}_i \hat{C})}{\partial x_i} = D \frac{\partial^2 \hat{C}}{\partial x_i \partial x_i} + \frac{\partial}{\partial x_i} \left(\mathcal{D}_{ij} \frac{\partial \hat{C}}{\partial x_j} \right) \quad (2.10)$$

where

$$\tilde{u} = \frac{1}{\omega} \Re \left(H_1 U_o \frac{\partial U_o^*}{\partial x} + H_5 V_o \frac{\partial U_o^*}{\partial y} + H_6 U_o \frac{\partial V_o^*}{\partial y} \right) \quad (2.11a)$$

$$\tilde{\mathcal{V}} = \frac{1}{\omega} \Re \left(H_1 V_o \frac{\partial V_o^*}{\partial y} + H_5 U_o \frac{\partial V_o^*}{\partial x} + H_6 V_o \frac{\partial U_o^*}{\partial x} \right) \quad (2.11b)$$

and

$$\mathcal{D}_{xx} = \frac{H_4^R}{\omega} |U_o|^2 \quad \mathcal{D}_{xy} = \frac{H_4^R}{\omega} \Re(U_o V_o^*) \quad (2.12a,b)$$

$$\mathcal{D}_{yx} = \mathcal{D}_{xy} \quad \mathcal{D}_{yy} = \frac{H_4^R}{\omega} |V_o|^2 \quad (2.12c,d)$$

and H_1 , H_5 , H_6 and H_4^R are defined above. H_5 , H_6 and H_4^R are graphed in Figures 2-1 through 2-3.

2.2 Modified form of the convection-diffusion equation

In Chapter 1 we noted that $H_1 = H_2 + H_3$. This holds for the symmetric form also, that is

$$H_1 \equiv H_5 + H_6. \quad (2.13)$$

Accordingly, we can define modified convection velocities by

$$\tilde{U} = \frac{1}{\omega} \Re \left\{ H_5 \left[U_o \frac{\partial U_o}{\partial x} + V_o \frac{\partial U_o}{\partial y} \right] + H_6 U_o \left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y} \right) \right\} \quad (2.14a)$$

$$\tilde{V} = \frac{1}{\omega} \Re \left\{ H_5 \left[U_o \frac{\partial V_o}{\partial x} + V_o \frac{\partial V_o}{\partial y} \right] + H_6 V_o \left(\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y} \right) \right\} \quad (2.14b)$$

or, in index notation

$$\tilde{U}_i = \frac{1}{\omega} \Re \left[H_5 \left(U_{0j} \cdot \frac{\partial U_{0i}}{\partial x_j} \right) + H_6 U_{0i} \left(\frac{\partial U_{0j}}{\partial x_j} \right) \right]. \quad (2.15)$$

The first term describes a convective acceleration and the second term describes the product of U_o and the divergence of the flow field.

For the special case of a scatterer with vertical walls spanning the entire ocean depth, like our cylinder, we can define boundary layer velocity amplitudes, U_{0i} in terms of the free surface equation

FIGURE 2-1 H_5 for Schmidt Numbers 0.1, 1.0, 10.0 FIGURE 2-2 H_6 for Schmidt Numbers 0.1, 1.0, 10.0 FIGURE 2-3 H_4^R for Schmidt Numbers 0.1, 1.0, 10.0

$$U_{0i} \equiv \frac{\partial \phi}{\partial x_i} \Big|_{z=-h} = \frac{-ig}{\omega \cosh kh} \frac{\partial \eta}{\partial x_i} \quad (2.16)$$

where η satisfies the Helmholtz equation $\nabla^2 \eta + k^2 \eta = 0$.

Now substituting $H_5 + H_6$ for H_1 , a coefficient and $\partial \eta / \partial x_i$ for U_{0i} as described above; the convective velocities can be rewritten as

$$\tilde{u} = \frac{-g^2}{\omega^3 \cosh^2 kh} \Re \left\{ H_5 \left[\frac{\partial \eta}{\partial x} \frac{\partial^2 \eta^*}{\partial x^2} + \frac{\partial \eta}{\partial y} \frac{\partial^2 \eta^*}{\partial x \partial y} \right] + H_6 \frac{\partial \eta}{\partial x} \left(\frac{\partial^2 \eta^*}{\partial x^2} + \frac{\partial^2 \eta^*}{\partial y^2} \right) \right\} \quad (2.17a)$$

$$\tilde{v} = \frac{-g^2}{\omega^3 \cosh^2 kh} \Re \left\{ H_5 \left[\frac{\partial \eta}{\partial y} \frac{\partial^2 \eta^*}{\partial y^2} + \frac{\partial \eta}{\partial x} \frac{\partial^2 \eta^*}{\partial x \partial y} \right] + H_6 \frac{\partial \eta}{\partial y} \left(\frac{\partial^2 \eta^*}{\partial x^2} + \frac{\partial^2 \eta^*}{\partial y^2} \right) \right\} \quad (2.17b)$$

Finally by applying the Helmholtz relation, we arrive at a version of the modified convection-diffusion equation in terms of η is

$$\frac{\partial \hat{C}}{\partial t} + \frac{\partial}{\partial x_i} (\tilde{u}_i \hat{C}) = \frac{\partial}{\partial x_i} \left(\mathcal{D}_{ij} \frac{\partial \hat{C}}{\partial x_j} \right) + D \frac{\partial^2 \hat{C}}{\partial x_i \partial x_i} \quad (2.18a)$$

where

$$\tilde{u}_i = \frac{-g^2}{\omega^3 \cosh^2 kh} \Re \left\{ H_5 \left[\nabla \eta \cdot \nabla \left(\frac{\partial \eta^*}{\partial x_i} \right) \right] - H_6 k^2 \eta \frac{\partial \eta^*}{\partial x_i} \right\} \quad (2.18b)$$

where $\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j}$. The symmetric form of the dispersion tensor is

$$D_{ij} = \frac{-g^2 H_4^R}{\omega \cosh^2 kh} \Re \left(\frac{\partial \eta}{\partial x_i} \frac{\partial \eta}{\partial x_j} \right) \quad (2.18c)$$

where

$$H_5 = H_2 + iH_4^I = \tilde{F}_3 - i\Im(B_1) \quad (2.19a)$$

$$H_6 = H_3 - iH_4^I = \tilde{F}_4 + B_2^* - i\Im(B_1) \quad (2.19b)$$

$$H_4^R = -\Re(B_1). \quad (2.19c)$$

We note that analytically the real and imaginary parts of B_1 are difficult to separate, but can be determined numerically as a function of $\wp$ and Σ . The numerical values of $\Re(B_1)$ and $\Im(B_1)$ will then be easily determined.

CHAPTER 3-FREE SURFACE EQUATION FOR A SCATTERED WAVE

3.1 A review of the linear theory of diffraction

To apply the modified convection-diffusion equation to the case of a wave scattered by a cylinder we must first describe the boundary layer velocities, U_{0i} . These are deduced from the velocity potential $\phi(r, \theta)$, which is related to the free-surface equation $\eta(r, \theta)$ governed by the Helmholtz equation and the appropriate boundary conditions.

Boundary conditions are needed as $r \rightarrow \infty$ and at $r = a$ on the wall of the cylinder. Let the total equation of the free-surface be described by the sum of the incident and the scattered wave, $\eta = \eta^I + \eta^S$. The radiation condition states: *A locally generated sinusoidal disturbance (i.e. the scattered wave) must propagate outward to infinity.*¹

The exact solution of the incident and scattered waves is given by

$$\eta = A \sum_{n=0}^{\infty} \varepsilon_n i^n \left[J_n(kr) - H_n(kr) \frac{J'_n(ka)}{H'_n(ka)} \right] \cos n\theta \quad (3.1a)$$

where $J'_n(s) \equiv dJ_n(s)/ds$ and $H'_n(s) \equiv dH_n(s)/ds$, and the superscript on the Hankel function of the first kind has been dropped for brevity. This describes the sum of the incoming wave and its scattered wave from a cylinder of radius a .

The calculation of such a function, consisting of an infinite sum of numerous Bessel functions for an unspecified cylinder radius can only be performed on a computer. For our purposes we have used Bessel function defined in *Numerical Recipes* (1986) and the FORTRAN library functions

¹p. 53. *The Applied Dynamics of Ocean Waves..* C. C. Mei (1983).

provided on the CRAY supercomputer. These values were verified by comparing their output with Bessel function tables provided in Abramowitz & Stegun (1972). The CRAY calculations were found to correlate to the maximum accuracy provided by Abramowitz & Stegun, while the *Numerical Recipes* forms varied no more than 10^{-6} .

Once the Bessel functions are evaluated for integer values of n it is vital to determine the number of terms one must sum to minimize truncation error. Figures 3-1 through 3-6 are plots for different ka 's, kr 's, and θ 's of each term of η_n versus n where

$$\eta = \sum_{n=0}^{\infty} \eta_n = \eta_0 + \eta_1 + \eta_2 + \dots \quad (3.1b)$$

For $ka = 1.0$ and $kr = 1.0$ we see in Figure 3-1 that η_n decreases rapidly and that n needs to be no larger than 5, but for the same flow field evaluated at $kr = 3.0$, Figure 3-2, we see that n should be at least 7. For this example as for the others not only is the size of ka important but the size of the calculated domain is critical.

For larger ka or kr it is essential that one observe the lack of linear decay in the size of η_n . A simple test of the size of any single η_n will not provide sure information that this η_n is representative of the series. For this reason we have included figures for $ka = 5.0$ and $kr = 5.0, 15.0$, Figures 3-3 and 3-4; and $ka = 15$ and $kr = 15, 45$, Figures 3-5 and 3-6. These illustrate the irregularity of η_n that occurs prior to final decay. For $ka = 5.0$ and $kr = 5.0, 15.0$ a sum of no less than 22 terms are required to provide coverage for the whole domain. For $ka = 15$ and $kr = 15, 45$, 55 terms are necessary. And for higher ordered terms, like $\partial \eta^2 / \partial^2 x$, even more terms are needed.

FIGURE 3-1 Examination of Truncation Error for $ka = 1.0$, $kr = 1.0$ FIGURE 3-2 Examination of Truncation Error for $ka = 1.0$, $kr = 3.0$

FIGURE 3-3 Examination of Truncation Error for $ka = 5.0$, $kr = 5.0$ FIGURE 3-4 Examination of Truncation Error for $ka = 5.0$, $kr = 15.0$

FIGURE 3-5 Examination of Truncation Error for $ka = 15.0$, $kr = 15.0$ FIGURE 3-6 Examination of Truncation Error for $ka = 15.0$, $kr = 45.0$

3.2 Run up on the cylinder

In a polar plot of $|\eta(a, \theta)|$, where $|\eta|$ is normalized by A , we can compare the run-up on the wall of the cylinder for different ka 's. In Figure 3-7, we show overlays of run up for different ka 's, where the incoming wave is approaching from the negative x-axis. Figure 3-7 presents identical results to those in *The Applied Dynamics of Ocean Waves*, Figure 5.1, p. 314. Accordingly, this confirms our calculation of the free-surface equation and its Bessel functions.

One can see that for sufficiently large ka the run up on the negative x-side, at $\theta = \pi$, approaches the run up on a plane wall, $|\eta(a, \pi)| \rightarrow 2$. When ka is small, $ka = 0.1$ for example, the run-up is barely disturbed by the cylinder, creating a nearly uniform circle at $|\eta| = 1$ for all θ .

FIGURE 3.7 Polar plot of $|\eta(a, \theta)|$ for different ka .

3.3 The free surface amplitude

In Figures 3-8 through 3-10 we have plotted in x and y contours of the normalized $|\eta|$ for three different ka 's. There the incident wave approaches from $x = -\infty$ and the island is represented by the circle of radius ka , located at the origin. All contour plots follow the same convention: a solid contour represents a positive value, a dashed contour represents a negative value and a dotted line represents zero. To show in detail the magnitude of η along $\theta = \pi$, we have also plotted normalized $|\eta|$ versus r for increasing values of ka in Figures 3-11 through 3-18. $|\eta|$, as expected oscillates between the extremes of 0, a node, and 2, an anti-node when there is total reflection. For each line graph the incident wave approaches from the left and the cylinder is represented by a vertical line on the right at ka .

Generally for a medium to large cylinder the no-flux boundary condition, $\partial\eta(a, \theta)/\partial r = 0$, defines an absolute maximum facing the incoming wave, at $\theta = \pi$, as shown in Figures 3-11 through 3-15. For these cases we observe that $\partial^2|\eta(a, \pi)|/\partial r^2 < 0$ when $\partial|\eta(a, \pi)|/\partial r = 0$. However, for a small cylinder, $ka < 0.5$, as is shown in Figures 3-8 & 3-9 the no-flux condition on the wall defines a relative minimum, where $\partial^2|\eta(a, \pi)|/\partial r^2 > 0$ when $\partial|\eta(a, \pi)|/\partial r = 0$.

This observation can be confirmed analytically by the using the small ka approximation of η ,

$$\eta(r, \theta) = A \left\{ 1 + i \left[kr + \frac{(ka)^2}{kr} \right] \cos \theta \right\} + O[(kr)^2], \quad (3.2)$$

which is derived in Appendix A. The first derivative of $|\eta|$ for small ka shows that the no-flux boundary condition is satisfied. The slope is zero:

FIGURE 3-8 X vs. Y Contour plot of $|\eta|$ for $ka = 0.10$

FIGURE 3-9 X vs. Y Contour plot of $|\eta|$ for $ka = 1.00$

FIGURE 3-10 X vs. Y Contour plot of $|\eta|$ for $ka = 25.0$

FIGURE 3-11 r vs. $|\eta|$ at $\theta=\pi$, for $ka = 0.05$ FIGURE 3-12 r vs. $|\eta|$ at $\theta=\pi$, for $ka = 0.10$ FIGURE 3-13 r vs. $|\eta|$ at $\theta=\pi$, for $ka = 0.50$ FIGURE 3-14 r vs. $|\eta|$ at $\theta=\pi$, for $ka = 1.00$

$$\frac{\partial |\eta(r, \pi)|}{\partial r} = \frac{\left(r + \frac{a^2}{r}\right)\left(1 - \frac{a^2}{r^2}\right)}{\sqrt{1 + \left(r + \frac{a^2}{r}\right)^2}} \quad \frac{\partial |\eta(a, \pi)|}{\partial r} = \frac{\partial |\eta(-a, 0)|}{\partial x} \equiv 0, \quad (3.3a,b)$$

where $|\eta|$ is normalized by A and r and a are normalized by k . The second derivative of $|\eta|$ evaluated at the wall of the cylinder is, as observed, positive

$$\frac{\partial^2 |\eta(a, \pi)|}{\partial r^2} = 4 \left[1 + (2a)^2\right]^{-3/2} > 0. \quad (3.4)$$

This anomalous behavior of small ka will be referenced when we discuss $\mathcal{U}_i$.

Though beyond the scope of equation 3.2, which increases as kr increases, we see from the exact solution of $|\eta|$ (Figure 3-9), that the maximum $|\eta|$ is located at $kr \approx 0.9$, roughly four diameters from the wall of the cylinder.

For a medium ka , the maximum free-surface amplitude is located on the wall of the cylinder at $\theta = \pi$. When $ka = 1.0$, as shown in Figure 3-11, we see that the maximum amplitude measures about 1.7. This is significantly less than the 2.0 of the perfect reflection off of a flat wall.

For large ka the reflection located on the wall at $\theta = \pi$ very nearly approaches a flat wall reflection. This is shown in Figure 3-10. On the wall of the cylinder the maximum amplitude is ~ 2.0 , but one wavelength away it is only ~ 1.85 . It is not until $ka = 40$ that the amplitude of the second peak approaches 2.0.

3.4 The free surface pattern

To assist in visualizing the actual sea state that surrounds a cylinder for any particular ka , contour plots of the real and imaginary parts of $\eta(r, \hat{\theta})$ are provided in Figures 3-19 through 3-24. $\pm \Re[\eta(r, \theta)]$ represents a snap-shot of

FIGURE 3-19 X vs. Y Contour plot of $\Re(\eta)$ for $ka = 0.10$

FIGURE 3-20 X vs. Y Contour plot of $\Im(\eta)$ for $ka = 0.10$

FIGURE 3-21 X vs. Y Contour plot of $\Re(\eta)$ for $ka = 1.00$

FIGURE 3-22 X vs. Y Contour plot of $\Im(\eta)$ for $ka = 1.00$

FIGURE 3-23 X vs. Y Contour plot of $\Re(\eta)$ for $ka = 25.0$

FIGURE 3-24 X vs. Y Contour plot of $\Im(\eta)$ for $ka = 25.0$

the surface when $t = \pi n / \omega$ where $n = 0, 1, 2, \dots$; and similarly $\pm \Im[\eta(r, \theta)]$ represents the surface when $t = (\frac{1}{2} + n) \pi / \omega$.

As predicted by the analytical approximation for small ka , for $ka = 0.1$, $\Re[\eta(r, \theta)]$ is nearly 1.0 and $\Im[\eta(r, \theta)]$ varies about ± 0 depending on the sign of $\cos \theta$ (Figures 3-19 & 3-20). For $ka = 1.0$, (Figures 3-21 & 3-22), a similar, but significantly distorted, structure is apparent. The contour lines are no longer perfectly vertical and the effects of scattering are obvious as $\Im[\eta(a, \theta)]$ is less than -1.5.

At $ka = 25.0$ (Figures 3-23 & 3-24) we see the "sharp-edged shadow" described by Morse (1968) for sound waves scattered from large cylinders. This shadow is clearly bordered by two lines tangent to the cylinder at $\theta = \pm \pi/2$ and extending in the positive x -direction (Figure 3-25). The remaining flow field can be divided into three types; the reflected area which on a cylinder of this size is between $\pm \pi/12$ about $\theta = \pi$; the plane wave area which lies above and, by symmetry, below the sharp-edged shadow area; and the scattered area which is in the remaining area between the plane wave area and the reflected area from $\theta = \pm \pi/2$ to $\theta \approx \pm 11\pi/12$

FIGURE 3-25 Wave regions for a large ka .

CHAPTER 4-THE SPATIAL AMPLITUDES, U_0 , V_0

4.1 Derivation of U_0 , V_0 , and derivatives

Now we shall continue with the specific case of transport caused by an incident wave reflected off a vertical cylinder.

From equation (3.1) we have the equation for the surface. From the velocity potential

$$\phi = \frac{-ig}{\omega} \frac{\cosh k(z+h)}{\cosh kh} \eta(x, y) \quad (4.1)$$

we can define U_0 and V_0 at $z = -h$ to be used in equations (2.18). By applying the following transformations of $\partial/\partial x_i$

$$\alpha U_0 = \frac{\partial}{\partial x} \eta(r, \theta) = \cos \theta \frac{\partial \eta}{\partial r} - \frac{\sin \theta}{r} \frac{\partial \eta}{\partial \theta} \quad (4.2a)$$

$$\alpha V_0 = \frac{\partial}{\partial y} \eta(r, \theta) = \sin \theta \frac{\partial \eta}{\partial r} + \frac{\cos \theta}{r} \frac{\partial \eta}{\partial \theta} \quad (4.2b)$$

where $\alpha = \frac{i\omega}{g} \cosh kh$, we evaluate U_0 and V_0 .

The calculation of either $\partial\eta/\partial x$ or $\partial\eta/\partial y$, and their subsequent derivatives, involves the use of several recursion relations and trigonometric identities. For illustration purposes we will describe in detail the derivation of αU_0 . The remaining components we will state directly.

For the calculation of (4.2a) the recursion relations required are

$$\begin{aligned} \frac{d}{dr} \mathcal{G}_n(kr) &= -k \mathcal{G}_{n+1}(kr) + \frac{n}{r} \mathcal{G}_n(kr) & \mathcal{G}_{n-1}(kr) + \mathcal{G}_{n+1}(kr) &= \frac{2n}{kr} \mathcal{G}_n(kr) \\ \frac{d}{dr} \mathcal{G}_n(kr) &= k \mathcal{G}_{n-1}(kr) - \frac{n}{r} \mathcal{G}_n(kr) & \mathcal{G}_{-n}(kr) &= (-1)^n \mathcal{G}_n(kr) \end{aligned} \quad (4.3a-d)$$

where G represents J_n , Y_n , or $H_n^{(1)}$. The relevant trigonometric identities are

$$\cos \theta \cos n\theta = \frac{1}{2} [\cos(n-1)\theta + \cos(n+1)\theta] \quad (4.4a)$$

$$\sin \theta \sin n\theta = \frac{1}{2} [\cos(n-1)\theta - \cos(n+1)\theta] \quad (4.4b)$$

$$\cos \theta \sin \phi = \frac{1}{2} [\sin(\theta + \phi) - \sin(\theta - \phi)]. \quad (4.4c)$$

Introducing the free-surface equation into the polar form of the x derivative initially gives

$$\frac{\partial \eta}{\partial x} = A \sum_{n=0}^{\infty} \varepsilon_n i^n \left\{ \begin{aligned} & \left[J'_n(kr) - H'_n(kr) \frac{J'_n(ka)}{H'_n(ka)} \right] \cos \theta \cos n\theta + \\ & \left[J_n(kr) - H_n(kr) \frac{J_n(ka)}{H_n(ka)} \right] \frac{m}{r} \sin \theta \sin n\theta \end{aligned} \right\}. \quad (4.5)$$

By applying the trigonometric identities 4.4a and 4.4b we can regroup the Bessel functions and then apply the identities , 4.3a and 4.3b,

$$\frac{\partial \eta}{\partial x} = \frac{1}{2} A \sum_{n=0}^{\infty} \varepsilon_n i^n \left\{ \begin{aligned} & \cos(n-1)\theta \left[J'_n + \frac{m}{r} J_n - \left(H'_n + \frac{m}{r} H_n \frac{J'_n(ka)}{H'_n(ka)} \right) \right] \\ & + \cos(n+1)\theta \left[J'_n - \frac{m}{r} J_n - \left(H'_n - \frac{m}{r} H_n \frac{J'_n(ka)}{H'_n(ka)} \right) \right] \end{aligned} \right\}, \quad (4.6)$$

resulting in

$$\frac{\partial \eta}{\partial x} = \frac{1}{2} Ak \sum_{n=0}^{\infty} \varepsilon_n i^n \left\{ \begin{aligned} & \cos(n-1)\theta \left[J_{n-1} - H_{n-1} \frac{J'_n(ka)}{H'_n(ka)} \right] \\ & - \cos(n+1)\theta \left[J_{n+1} - H_{n+1} \frac{J'_n(ka)}{H'_n(ka)} \right] \end{aligned} \right\}. \quad (4.7)$$

Letting n equal $m-1$ for the top term and $n = m+1$ for the bottom, we get

$$\frac{\partial \eta}{\partial x} = \frac{Ak}{2} \left\{ \begin{aligned} & \sum_{m=-1}^{\infty} \varepsilon_{m+1} i^{m+1} \cos m\theta \left[J_m - H_m \frac{J'_{m+1}(ka)}{H'_{m+1}(ka)} \right] \\ & - \sum_{m=1}^{\infty} \varepsilon_{m-1} i^{m-1} \cos m\theta \left[J_m - H_m \frac{J'_{m-1}(ka)}{H'_{m-1}(ka)} \right] \end{aligned} \right\}. \quad (4.8)$$

By the expansion of the first several terms, application of 4.3d, and regrouping again, the summation above can be rewritten as

$$\frac{\partial \eta}{\partial x} = Ak \left\{ \sum_{m=0}^{\infty} \varepsilon_m i^{m+1} \cos m\theta \left[J_m - \frac{1}{2} H_m \left(\frac{J'_{m-1}(ka)}{H'_{m-1}(ka)} + \frac{J'_{m+1}(ka)}{H'_{m+1}(ka)} \right) \right] \right\}. \quad (4.9)$$

The dispersion relation $\omega^2 = gk \tanh kh$ reduces the leading coefficients of equation 4.11. Finally, the first-order velocity in the x-direction at the top of the boundary layer in dimensional form is

$$U_0(r, \theta) = \frac{A\omega}{\sinh kh} \sum_{n=0}^{\infty} \varepsilon_n i^n \cos n\theta \left\{ J_n(kr) - \frac{1}{2} H_n(kr) \left[\frac{J'_{n-1}(ka)}{H'_{n-1}(ka)} + \frac{J'_{n+1}(ka)}{H'_{n+1}(ka)} \right] \right\}. \quad (4.10)$$

4.2 Define remaining velocities exactly

By applying the following normalizations we can simplify the velocity equations.

Variable	Scaled by
η	A
$a, r, x, \text{ and } y$	$1/k$
$U_0 \text{ and } V_0$	$A\omega/\sinh kh$

Thus,

$$U_o = \sum_{n=0}^{\infty} \varepsilon_n i^n \cos n\theta \left\{ J_n(r) - \frac{1}{2} H_n(r) \left[\frac{J'_{n-1}(a)}{H'_{n-1}(a)} + \frac{J'_{n+1}(a)}{H'_{n+1}(a)} \right] \right\} \quad (4.11)$$

$$V_o = \sum_{n=0}^{\infty} i^{n+2} \sin n\theta H_n(r) \left[\frac{J'_{n-1}(a)}{H'_{n-1}(a)} - \frac{J'_{n+1}(a)}{H'_{n+1}(a)} \right] \quad (4.12)$$

where the progressive wave part in $V_0 = \partial \eta / \partial y$ as represented by $J_n(r)$ has fallen out as expected.

To fully satisfy equations 2.18a-c, in addition to 4.11 and 4.12, we will need the velocity gradients. In non-dimensional form, they are:

$$\frac{\partial U_o}{\partial x} = \sum_{n=0}^{\infty} \varepsilon_n i^{n+1} \cos n\theta \left\{ J_n(r) - \frac{1}{4} H_n(r) \left[\frac{J'_{n-2}(a)}{H'_{n-2}(a)} + 2 \frac{J'_n(a)}{H'_n(a)} + \frac{J'_{n+2}(a)}{H'_{n+2}(a)} \right] \right\} \quad (4.13)$$

$$\frac{\partial U_o}{\partial y} = \frac{\partial V_o}{\partial x} = \frac{1}{2} \sum_{n=0}^{\infty} i^{n-1} \sin n\theta H_n(r) \left[\frac{J'_{n-2}(a)}{H'_{n-2}(a)} - \frac{J'_{n+2}(a)}{H'_{n+2}(a)} \right] \quad (4.14)$$

and

$$\frac{\partial V_o}{\partial y} = \frac{1}{4} \sum_{n=0}^{\infty} i^{n+1} \cos n\theta H_n(r) \left[\frac{J'_{n-2}(a)}{H'_{n-2}(a)} - 2 \frac{J'_n(a)}{H'_n(a)} + \frac{J'_{n+2}(a)}{H'_{n+2}(a)} \right] \quad (4.15)$$

4.3 Illustrations of U_{0i}

The contour plots of the free-surface equation, η , have provided a general picture of the wave field. The following vector diagrams sketch the leading-order flow field resulting at the top of the boundary layer, U_0 and V_0 .

In Figure 4-1 where $ka = 0.1$, the small ka example demonstrates the extremely regular field that one would expect from a small cylinder or long wavelength. Comparing the vector flow field of $\Re(U_{0i})$ in Figure 4-1 with $\Re[\eta(r, \theta)]$ in Figure 3-19 is like examining snapshots of η and U_{0i} at $t = \pi n / \omega$ where $n = 0, 1, 2, \dots$. For a dynamic picture, we also examine $\Im(U_{0i})$ (Figure 4-2) alongside $\Im[\eta(r, \theta)]$ (Figure 3-20). From these two comparisons it is clear that U_0 travels strongly in the positive x-direction beneath the peaks; strongly in the negative x-direction beneath the troughs; and converges or diverges from inflection points between the peaks and troughs.

FIGURE 4-1 X vs. Y Vector plot of $\Re(U_0)$ and $\Re(V_0)$ for $ka = 0.10$

FIGURE 4-2 X vs. Y Vector plot of $\mathfrak{S}(U_0)$ and $\mathfrak{S}(V_0)$ for $ka = 0.10$

For $ka = 1.0$ the same pattern is evident (Compare Figures 4-3 & 4-4 with 3-21 & 3-22). In addition, at this size one can observe that the weak transverse flow, V_0 , within a diameter of the cylinder is directed away from the cylinder except immediately around the perimeter.

For the large ka , shown in Figures 4-5 & 4-6 for $ka = 25$, the patterns become more varied. The same four regions: sharp-edged shadow, reflected, progressive and scattered regions are again quite evident. In the reflected region by comparing the real and imaginary diagrams of η , in Figures 3-23 & 3-24, and U_{0i} , Figures 4-5 & 4-6, we see the strength of this oscillatory boundary layer flow occurs when the waves are adding destructively about $t = (\frac{1}{2} + n)\pi/\omega$, when the imaginary phase is more important.

The relation between η and U_{0i} as described above has been well examined previously for analytically described flows. For small ka , the case of a vertical cylinder in a plane wave, U_{0i} is easily derived and analyzed in the very near region around the cylinder. However, particularly for the larger ka , the exact solution provides a more detailed examination of the different phenomena that develop. This brief analysis of U_{0i} provides an introduction of the building blocks that make up the steady convection velocities $\mathcal{U}_i$ and dispersivities $\mathcal{D}_{ij}$.

FIGURE 4-3 X vs. Y Vector plot of $\Re(U_0)$ and $\Re(V_0)$ for $ka = 1.00$

FIGURE 4-4 X vs. Y Vector plot of $\Im(U_0)$ and $\Im(V_0)$ for $ka = 1.00$

FIGURE 4-5 X vs. Y Vector plot of $\Re(U_0)$ and $\Re(V_0)$ for $ka = 25.0$

FIGURE 4-6 X vs. Y Vector plot of $\Im(U_0)$ and $\Im(V_0)$ for $ka = 25.0$

CHAPTER 5-RESULTS FOR DIFFERENT ka 's

Once we have analytically defined U_{0i} and $\partial U_{0i}/\partial x_j$, we can easily calculate the modified convective velocities $\mathcal{U}_i$ and dispersion tensor $\mathcal{D}_{ij}$. By computing and plotting the normalized forms of $\mathcal{U}_i$ and $\mathcal{D}_{ij}$ for a specified domain, we can visually describe the flow fields created by a wave scattered by cylinders of various ka 's. From these we can analyze variations in the steady transport of a pollutant concentration for different sized islands and wavelengths.

5.1 The Modified Convection Velocities $\mathcal{U}_i$

For $ka = 0.1$ we see a uniform flow field beyond a diameter's length in the x-direction as shown in Figure 5-1. From the associated contour plots of $\mathcal{U}$ and $\mathcal{V}$ in Figures 5-2 & 5-3, we note $\mathcal{U} \sim 0.7$ and $\mathcal{V} \sim 0$ in the far field. Closer to the cylinder we see a more complicated flow pattern. In particular, we point out the reversals of $\mathcal{U}$ near $\theta = \pm\pi/6$ and π , visible in Figures 5-2 & 5-5.

By comparing the reversal for small cylinders at $\theta = \pi$ (Figures 5-4 & 5-5) with the slope of $|\eta|$ (Figures 3-11 & 3-12), we observe that as $r \rightarrow a$ a brief reversal in $\mathcal{U}$ occurs beneath the negative slope of $|\eta|$ in the r-direction at the wall of the cylinder. As ka increases to 0.5, we observe that $|\eta|$ no longer has a negative slope, as shown in Figure 3-13. Concurrently, the reversal vanishes in Figure 5-6. (And only reoccurs once ka has increased to larger than 10, which will be discussed below.)

On the other side of the small cylinder, near $\theta = \pm\pi/6$ the other slightly stronger reversal in $\mathcal{U}$ occurs. This does not appear to be correlated with a negative radial slope of $|\eta|$. We can make some observations about it

FIGURE 5-1 X vs. Y Vector plot of u and v for $ka = 0.10$

FIGURE 5-2 X vs. Y Contour plot of u for $ka = 0.10$

FIGURE 5-3 X vs. Y Contour plot of U for $ka = 0.10$

FIGURE 5-4 r vs. $\mathcal{U}$ at $\theta=\pi$, for $ka = 0.05$ FIGURE 5-5 r vs. $\mathcal{U}$ at $\theta=\pi$, for $ka = 0.10$ FIGURE 5-6 r vs. $\mathcal{U}$ at $\theta=\pi$, for $ka = 0.50$ FIGURE 5-7 r vs. $\mathcal{U}$ at $\theta=\pi$, for $ka = 1.00$

by closely examining the vector plot of $\mathcal{U}_i$ for $ka = 0.1$ (Figure 5-1). From $\theta = \pm 2\pi/3$ to $\theta = \pm \pi/2$, we see a very strong, nearly tangential flow, where initially $\mathcal{U}$ dominates with $\mathcal{U} \sim 2$ then following the flow to the top of the cylinder, $\mathcal{V}$ increases to $\sim \pm 1.2$ producing a flow that tends to converge on the sides of the cylinder.

This behavior for the small ka , though interesting, may have limited applications to real islands. For example, to consider a small island where $a \approx 0.5\text{km}$ and $ka \approx 0.1$, the wave length would need to be $\lambda \sim 31\text{km}$. By using a shallow depth of 10m and the dispersion relation, this gives a period of 0.9 hours. This period is too long for gravity waves without taking into account the earth's rotation. Generally the period of an appropriate gravity wave lies between 5 ~ 25 seconds, so for a depth of 10m and an island where $a \approx 0.5\text{km}$, ka varies between 85 for the shorter period waves and 12.7 for the longer.

For a moderate sized island, where $ka = 1.0$, the vector plot of $\mathcal{U}_i$, Figure 5-8, shows flow directed purely tangentially about the cylinder, not into the cylinder. Initially the flow converges on the cylinder, stalling at $\theta = \pi$. Then it weakly flows up the side, increasing in velocity until it reaches $\theta = \pi$ and as it flows down the downstream side it decays quickly toward zero near $\theta = 0$. In the far field there are some very weak transverse fluctuations.

This progression is shown quantitatively in the contour plots of $\mathcal{U}$ and $\mathcal{V}$ in Figures 5-8 & 5-9, as well as in the plot of $\mathcal{U}$ vs. r along $\theta = \pi$ in Figure 5-7. For the small ka we saw that only very near the cylinder does $\mathcal{U}$ reverse at all. For $ka = 1.0$ $\mathcal{U}$ is always positive and oscillates about the value 0.6 in the domain beyond a radius from the cylinder. Within a radius, sharp gradients in $\mathcal{U}$ are evident near 0, $\pi/2$, π and $-\pi/2$. At $\pm \pi/2$, the

FIGURE 5-8 X vs. Y Vector plot of u and v for $ka = 1.00$

FIGURE 5-9 X vs. Y Contour plot of U for $ka = 1.00$

FIGURE 5-10 X vs. Y Contour plot of U for $ka = 1.00$

convection velocity reaches a maximum value of $U \sim 1.2$, nearly twice the magnitude of the average velocity away from the cylinder, producing the strong tangential flow observed in the vector plot. And at the opposite extreme, the flow near $\theta = 0$ and $\theta = \pi$ decreases sharply to zero at the wall. An illustration of the steepness of the decay at $\theta = \pi$ is clearly visible in the line graph of Figure 5-7. The contour plot of $\mathcal{V}$ in Figure 5-10 shows a mild acceleration as the flow climbs the cylinder, reaching $\mathcal{V} \sim .5$; decaying about $\theta \sim \pm \pi/2$ and then increasing in the negative y -direction to $\mathcal{V} \sim -0.5$. As the flow descends down the back, it reaches zero again at $\theta = 0$, demonstrating a rough symmetry in y , as well as the general symmetry in x .

For the cylinder with a large ka , the vector flow field diagram, Figure 5-12 shows the variety of the complex flow at $ka = 25.0$. Again the four regions create very distinctive flow fields. The sharp-edged shadow region shows very little transport when $U < 0.25$. The fastest transport occurs on the wall of the island in the scattered region reaching a maximum value of $|U_i| \sim 1.8$. This cross-current recurs periodically off the wall within the scattered region, gradually weakening farther from the cylinder. Comparing this cross-current to that observed in Mei & Chian (1993) in bidirectional waves with an angle of incidence of $\pi/4$

FIGURE 5-11 Cross-Current U , created by bidirectional waves.

FIGURE 5-12 X vs. Y Vector plot of u and v for $ka = 25.0$

Beyond the scattered region in the plane wave region, a moderate, steady transport is observed parallel to the incident wave, with $\mathcal{U} \sim 0.5 - 0.75$ and $\mathcal{V} \sim 0$.

In the reflected region we see a spatially periodic $\mathcal{U}$, increasing and then decaying as it approaches the cylinder. Close comparison of the contour plot of $\mathcal{U}$ (Figure 5-13) and the line plot of $\mathcal{U}$ at $\theta = \pi$ (Figure 5-17) with $|\eta|$, (Figure 3-17) shows that the reflected area causes sufficient reflection of the free surface, not just at the wall but through the first peak off the wall, to produce significant reversal at $r \cong (a + 2.4)$. Here, $\mathcal{U} \approx -0.17$ from measurements of Figure 5-17. By comparing Figures 5-16 through 5-18 we see that the reversal for the large ka occurs when $ka > 10$ and as ka increases, the maximum of the reversal occurs spatially periodically at $r \cong (a + 2.4) + \pi n$.

By comparing the location of the reversal for large ka with the plots of $|\eta|$ (Figures 3-16 through 3-18), we observe that the reversed $\mathcal{U}$ occurs below the inflection point in $|\eta|$, where $\partial|\eta|/\partial r < 0$ and $\partial^2|\eta|/\partial r^2 = 0$. The spatial periodicity of the reversal for the larger cylinders on the order of $ka = 40$ (Figure 5-18) dies out as $r \rightarrow \infty$. One could speculate based on the flat wall analogy that for a very large cylinder and/or a very short wave that the reversal of $\mathcal{U}$ will eventually equal the positive part of $\mathcal{U}$.

If we want to try to generalize this observation along $\theta = \pi$, we can look back at the case for when $ka = 0.1$, and recall that the first inflection point of $|\eta|$ occurred just off the wall of the cylinder, above the reversal of $\mathcal{U}$. This appears to indicate that where $\partial|\eta|/\partial r < 0$ and $\partial^2|\eta|/\partial r^2 = 0$ there is the potential for reversal of $\mathcal{U}$. But, it is difficult to translate observations of $|\eta|$ and its derivatives into analytical information that can be applied to

FIGURE 5-13 X vs. Y Contour plot of U for $ka = 25.0$

FIGURE 5-14 X vs. Y Contour plot of U for $ka = 25.0$

equations 2.17a & 2.17b to prove that this is a necessary condition for $-\mathcal{U}$. For now, it is enough to note that there appears to be a correlation.

5.2 The Symmetric Dispersion Tensor

The normalized diffusivity is defined by

$$D' \equiv D \frac{\sinh^2 kh}{A^2 \omega}. \quad 5.1$$

The turbulent diffusivity is $D = 3 \times 10^{-3} \text{ m}^2/\text{s}$ (Gill 1983, Mei 1983). If we choose the depth $h = 10 \text{ m}$, we find for a range of gravity waves:

T (sec)	ω (1/sec)	k (1/m)	λ (m)	A (m)	D'
5	1.26	0.169	37	.2	.5
25	0.25	0.025	250	1	.0007

For the case of the Schmidt and the Peclet numbers equal to 1 for small, heavy particles, H_4^R is near maximum: $H_4^R = 0.024$ (Figure 2-3). For such particles the normalized dispersivity can be significantly larger than D' for some longer waves and cause significant spatial variation in the diffusion of a pollutant. This spatial variation depends on the wave form which in our case depends on the size of ka . Below we examine the normalized dispersivity that occurs when $ka = 0.1, 1.0$, and 25 .

In Figures 5-19 through 5-21 we have plotted $\mathcal{D}_{xx}$ for the range of ka . $\mathcal{D}_{xx}$ is positive for all ka , x , and y . This is generally true for $\mathcal{D}_{jj}$ for all ka , because $\mathcal{D}_{jj}$ is proportional to $|U_{0j}|^2$ and H_4^R which are both positive always.

For $ka = 0.1$ in Figure 5-19, we observe that $\mathcal{D}_{xx}$ is symmetric in both the x and y axes. This symmetry is predicted by the small ka formula for $\mathcal{D}_{xx}$ in Appendix B. We see in the far-field that $\mathcal{D}_{xx} \sim .025$, generally. Maximum

FIGURE 5-19 X vs. Y Contour plot of D_x for $ka = 0.10$

FIGURE 5-20 X vs. Y Contour plot of $\mathcal{D}_m$ for $ka = 1.00$

FIGURE 5-21 X vs. Y Contour plot of D_m for $ka = 25.0$

$\mathcal{D}_{xx}$ occurs on the wall at $\theta = \pm\pi/2$. At this location, $\mathcal{D}_{xx} = 0.08$, roughly 80 times greater than D' . At $\theta = 0, \pi$ the longitudinal dispersivity goes to zero on the cylinder wall.

For $ka = 1.0$ (Figure 5-20), the maxima and minima of $\mathcal{D}_{xx}$ are nearly the same as for $ka = 0.1$. In the far-field however, $\mathcal{D}_{xx}$ changes. For this larger cylinder, we now see spatially periodic peaks and troughs of longitudinal diffusion along $\theta = \pi$. Comparing this with the contour plot of $|\eta|$ for $ka = 1.0$, Figure 3-9 we note that along $\theta = \pi$, $\mathcal{D}_{xx}$ is maximum where $|\eta|$ is a minimum, and vice versa.

In Figure 5-21 for $ka = 25.0$ the maximum is found just off the cylinder in the reflected region. Here the maximum magnitude of $\mathcal{D}_{xx}$ is 0.080. The scattered region has an average $\mathcal{D}_{xx}$ of ~ 0.03 . The plane wave region averages about ~ 0.02 . And the sharp-edged shadow region has minimal longitudinal diffusion, $\mathcal{D}_{xx} < 0.02$.

The three contour plots of $\mathcal{D}_{yy}$, Figures 5-22 through 5-24, all possess fairly similar structures, with somewhat different emphases. For $ka = 0.1$, we see symmetry in both x and y , with maxima occurring at $\theta = \pm\pi/4$, and $\pm 3\pi/4$, where the magnitude is $\mathcal{D}_{yy} \sim 0.02$. For $ka = 1.0$, the $\mathcal{D}_{yy}$ tensor has weaker maxima at $\theta = \pm 3\pi/4$ and slightly stronger maxima at $\theta = \pm\pi/4$. While for ka , the trend is reversed. Here the stronger maxima occur at $\theta = \pm 3\pi/4$ in the scattered region and $\mathcal{D}_{yy}$ is negligible about $\theta = \pm\pi/4$.

$\mathcal{D}_{xy}$ represents the diagonal of the symmetric dispersion tensor. When $\mathcal{D}_{xy}$ is zero the dispersion tensor is already a diagonal matrix and $\mathcal{D}_{ij}$ requires no rotation to interpret the effective diffusion. However when $\mathcal{D}_{xy}$ is non-zero, it is necessary to calculate the eigenvalues and eigenvectors to deduce the effect of $\mathcal{D}_{ij}$ at that location.

FIGURE 5-22 X vs. Y Contour plot of D_n for $ka = 0.10$

FIGURE 5-23 X vs. Y Contour plot of D_n for $ka = 1.00$

For $ka = 0.1$ we see perfect inverse symmetry in the x- and y-axes in $\mathcal{D}_{xy}$, Figure 5-25. For all r , at $\theta = 0, \pm\pi/2$, and π , $\mathcal{D}_{xy}$ is zero. This along with $\mathcal{D}_{xx}$ and $\mathcal{D}_{yy}$, Figures 5-19 & 5-22, indicates that the diffusion is tangential at those points. By evaluating the eigenvalues and eigenvectors at $\theta = \pi/4$ (see Appendix C) on the cylinder, we deduce by symmetry that there is little radial diffusion anywhere. The diffusion is almost entirely tangential.

For the larger ka , Figures 5-26 & 5-27, the effect very close to the cylinder is the same: minimal radial diffusion and significant tangential. Further from the cylinder for $ka = 1.0$ the influence of $\mathcal{D}_{xy}$ is less significant because, where dispersion is significant: $\mathcal{D}_{xx} \gg \mathcal{D}_{yy}$ and $\mathcal{D}_{xy}$. However for $ka = 25$, in the scattered region we find $\mathcal{D}_{xy} \sim \mathcal{D}_{xx}$. Therefore we can expect substantial radial and/or tangential spreading in that region. In the plane wave region we can see again $\mathcal{D}_{xx} \gg \mathcal{D}_{yy}$ and $\mathcal{D}_{xy}$, so we can predict longitudinal spreading will dominate. And in the sharp-edged shadow region, not much is happening at all.

5.3 Future Numerical Tasks

To predict where these convection velocities and dispersivities will transport a concentration of pollutant one can attempt to solve numerically the partial differential equation using a scheme like the one presented in Mei & Chian (1993). Done correctly, this should provide a realistic balance between all of the competing forces involved and illustrate a progression in time of any initial concentration.

Initial attempts solve this equation using the scheme presented in Mei & Chian have provided inconclusive results. Whether this is because of an error in the method or the physics of this particular problem is not known. What has been observed numerically is the following. When a cloud of

FIGURE 5-25 X vs. Y Contour plot of D_y for $ka = 0.10$

FIGURE 5-26 X vs. Y Contour plot of D_w for $ka = 1.00$

FIGURE 5-27 X vs. Y Contour plot of $\mathcal{D}_w$ for $ka = 25.0$

contaminant is released upstream of the cylinder, it spreads into the area immediately about the cylinder. Eventually some of the contaminant is swept around the cylinder. Unfortunately our calculated concentration near the cylinder is suspiciously large. Even for a minute grid, the radically steep slope between the first and second ring of points violates the premise that the second derivative is continuous and smooth. Despite our inability to achieve conclusive numerical results for $\hat{C}$, it is still possible to make some qualitative speculations about the effective mass transport about an island for different ka 's.

CONCLUSIONS AND SPECULATIONS

For a small island the dominant trait we have observed is the convergence of $\mathcal{U}_i$ at $\theta \cong \pm\pi/2$. The reversal in $\mathcal{U}$ on the back of the cylinder and the strong flow around and into the cylinder leading toward $\theta \cong \pm\pi/2$ indicate that there should be significant piling of a heavy contaminant there. The remainder of the field should transport like the incident plane wave, spatially uniform and in the positive x-direction.

For a medium sized island, where $a \sim O(\lambda/2\pi)$ the convection velocities in the near field can be expected to spread the contaminant more evenly around the cylinder. Since there is no radially inward flow or reversal as in the small island case, a pollutant will be partially swept beyond or left behind. A contaminant cloud released upstream should be divided and spread around the island by the diverging convection velocity at $\theta = \pi$, perhaps leaving some fall out on the front of the cylinder as it passes around. Then the contaminant should be either swept along beyond the cylinder or piled near $\theta \cong \pm\pi/4$.

For a large island, where $a \gg \lambda/2\pi$, the division of the flow field into the four regions: reflected, scattered, progressive plane wave, and sharp-edged shadow; provides a reasonable way to predict transport in any region around the island. The transport in the reflected region can be expected to cause spatially periodic piling near the island. In the scattered region the spatially periodic cross-currents should sweep the majority of a small, heavy pollutant around the island avoiding most of the coast as long as it is initially released at least a wavelength away from the island. If it is released closer to the island, it is reasonable to expect that it will be swept into and tangentially along the island. Then the remainder will be deposited on the edge of the

shadow region and/or swept away in the plane wave region. In the plane wave region the transport should be spatially uniform and in the x-direction. At the edge of the plane wave region and the sharp-edged shadow region it is possible that one will find spill off from the plane wave region along the interface, because the lack of significant transport in the shadow region would tend to keep a pollutant relatively stationary.

Future work on this subject could cover many different areas. Resolution of the difficulties involved in the numerical solution of the modified effective convection-diffusion equation is most urgent. Calculations that account for the vertical variation of the eddy viscosity would also worthwhile. With a more realistic model it would be possible to take into account bottom roughness and to calibrate it with real situations, and hence quantitatively evaluate the effect of different release locations for different islands and waves. It would also be revealing to perform some laboratory experiments simulating different situations.

APPENDIX A-DERIVATION OF η FOR $ka \ll 1$

Given the equation of the surface:

$$\eta = A \sum_{n=0}^{\infty} \varepsilon_n i^n \left[J_n(kr) - H_n(kr) \frac{J'_n(ka)}{H'_n(ka)} \right] \cos n\theta \quad (\text{A.1})$$

it is possible to approximate and simplify η for the case of a cylinder of very small radius relative to the wave length, i.e. $ka \ll 1$. To do this we expand η for each n : $\eta = \eta_0 + \eta_1 + \eta_2 + \dots$ and compare the relative magnitude of each term.

By expanding each Bessel and Hankel function we can evaluate the specifics for the case of $ka \ll 1$. A Bessel function of the first kind, $J_n(x)$, is defined by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2} \right)^{2k+n} \quad (\text{A.2})$$

a Hankel function of the first kind, $H_n(x)$, is defined by $H_n(x) \equiv J_n(x) + iY_n(x)$, and a Bessel function of the second kind, $Y_n(x)$, is

$$Y_n(x) = \frac{2}{\pi} \left\{ \left(\ln \frac{x}{2} + \gamma \right) J_n(x) - \frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-n} \right. \\ \left. - \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{[\phi(k) - \phi(k+n)]}{k!(k+n)!} \left(\frac{x}{2} \right)^{2k+n} \right\} \quad (\text{A.3})$$

where $\phi(k) = 1 + \frac{1}{2} + \dots + \frac{1}{k}$ for $k \geq 1$, $\phi(0) = 0$ and γ is Euler's constant: $\gamma \equiv \lim_{n \rightarrow \infty} [\phi(n) - \ln n] = 0.5772157$.

The first term of the η expansion, for $n = 0$, is

$$\eta_0 = \left[J_0(kr) - H_0(kr) \frac{J'_0(ka)}{H'_0(ka)} \right]. \quad (\text{A.4})$$

To evaluate this for small ka , we expand $J_0(x)$, $Y_0(x)$, $J_1(x)$, and $Y_1(x)$. The derivative terms are defined generally by

$$\mathcal{G}'_n(kr) = -k\mathcal{G}_{n+1}(kr) + n\mathcal{G}_n(kr)/r \quad (\text{A.5a})$$

$$\mathcal{G}'_n(kr) = k\mathcal{G}_{n-1}(kr) - n\mathcal{G}_n(kr)/r, \quad (\text{A.5b})$$

where $\mathcal{G}_n(x) = J_n(x)$, $Y_n(x)$, and $H_n(x)$ for brevity. The expansion for $J_1(x)$, and $Y_1(x)$ is

$$J_0(kr) = 1 - \frac{(kr)^2}{4} + \frac{(kr)^4}{64} + O(kr)^6 \quad (\text{A.6a})$$

$$Y_0(kr) = \frac{2}{\pi} \left\{ \left(\ln \frac{kr}{2} - \gamma \right) J_0(kr) + \frac{kr^2}{4} - \frac{kr^4}{64} \left(\frac{3}{2} \right) + O(kr)^6 \right\} \quad (\text{A.6b})$$

$$J_1(ka) = \frac{ka}{2} - \frac{(ka)^3}{16} + O(ka)^5 \quad (\text{A.6c})$$

$$Y_1(ka) = \frac{2}{\pi} \left\{ \left(\ln \frac{ka}{2} - \gamma \right) J_1(ka) - \frac{1}{ka} + \frac{ka}{4} + \frac{ka^3}{16} \left(\frac{5}{4} \right) + O(ka)^5 \right\} \quad (\text{A.6d})$$

From (A.4) the derivative term is

$$\frac{J'_0(ka)}{H'_0(ka)} = \frac{J_1(ka)}{J_1(ka) + iY_1(ka)} \quad (\text{A.7})$$

Substituting the $J_1(x)$, and $Y_1(x)$ expansions,

$$\frac{J'_0(ka)}{H'_0(ka)} = \frac{\frac{ka}{2} - \frac{(ka)^3}{16} + \dots}{\frac{ka}{2} - \frac{(ka)^3}{16} + \dots + \frac{2i}{\pi} \left\{ \left(\ln \frac{ka}{2} + \gamma \right) \frac{ka}{2} - \frac{(ka)^3}{16} + \dots - \frac{1}{ka} - \frac{ka}{4} + \dots \right\}} \quad (\text{A.8})$$

To simplify A.7 we determine the largest term in the denominator. For $ka \ll 1$, where $ka > 0$, $\ln \frac{ka}{2} \sim O(1)$. Therefore the largest term in the denominator for $ka \ll 1$ is $\frac{2i}{\pi} \left(-\frac{1}{ka} \right) + O(ka)$. It follows that

$$\frac{J'_0(ka)}{H'_0(ka)} = \frac{i\pi(ka)^2}{4} + O(ka)^4 \quad (\text{A.9})$$

Therefore

$$\eta_0 = J_0(kr) - H_0(kr) \frac{J'_0(ka)}{H'_0(ka)} \approx A \left\{ 1 - \dots - [1 - \dots] \left[\frac{i\pi(ka)^2}{4} + \dots \right] \right\} \quad (\text{A.10})$$

or

$$\eta_0 = A + O[(kr)^2, (ka)^2] \quad (\text{A.11})$$

Using the same reasoning for larger n , we see

$$\frac{J'_1(ka)}{H'_1(ka)} = \frac{J_0(ka) - J_1(ka)/ka}{H_0(ka) - H_1(ka)/ka} \approx \frac{-i\pi(ka)^2}{4} + O(ka)^4 \quad (\text{A.12})$$

$$\frac{J'_2(ka)}{H'_2(ka)} = \frac{J_1(ka) - 2J_2(ka)/ka}{H_1(ka) - 2H_2(ka)/ka} \approx \frac{i\pi(ka)^4}{32} + O(ka)^6 \quad (\text{A.13})$$

and

$$\eta_1 = Ai \left[kr + \frac{(ka)^2}{kr} \right] \cos \theta + O[(kr)^3] \quad (\text{A.14})$$

$$\eta_2 = O[(kr)^2] \quad (\text{A.15})$$

The higher n terms are order $O[(ka)^2]$ and higher, and therefore insignificant. Finally, summing A.11 and A.14, the free-surface equation is

$$\eta(r, \theta) = A \left\{ 1 + i \left[kr + \frac{(ka)^2}{kr} \right] \cos \theta \right\} + O[(kr)^2] \quad (\text{A.16})$$

where $ka \ll 1$ and $O(kr) \approx O(ka)$.

Appendix B-Approximation of $\mathcal{U}$ and $\mathcal{V}$ for $ka \ll 1$

In Appendix A we derived

$$\eta(r, \theta) = A \left\{ 1 + i \left[kr + \frac{(ka)^2}{kr} \right] \cos \theta \right\} + O[(kr)^2] \quad \text{B.1}$$

for the small ka approximation. Using the modified form of the convection-diffusion equation we can define $\mathcal{U}$ and $\mathcal{V}$ analytically for the small ka approximation. After we transform the coordinate system by equations 4.2a & 4.2b we deduce

$$U_0 = \frac{-ig}{\omega \cosh kh} \frac{\partial \eta}{\partial x} = \frac{A\omega}{\sinh kh} \left(1 - \frac{(ka)^2}{(kr)^2} \cos 2\theta \right) \quad \text{B.2}$$

and

$$V_0 = \frac{-ig}{\omega \cosh kh} \frac{\partial \eta}{\partial y} = \frac{A\omega}{\sinh kh} \left(-\frac{(ka)^2}{(kr)^2} \sin 2\theta \right) \quad \text{B.3}$$

And the derivatives are

$$\frac{\partial U_0}{\partial x} = \frac{A\omega k}{\sinh kh} \left(2 \frac{(ka)^2}{(kr)^3} \cos 3\theta \right) \quad \frac{\partial U_0}{\partial y} = \frac{A\omega k}{\sinh kh} \left(2 \frac{(ka)^2}{(kr)^3} \sin 3\theta \right) \quad \text{B.4a,b}$$

$$\frac{\partial V_0}{\partial x} = \frac{A\omega k}{\sinh kh} \left(2 \frac{(ka)^2}{(kr)^3} \sin 3\theta \right) \quad \frac{\partial V_0}{\partial y} = \frac{A\omega k}{\sinh kh} \left(-2 \frac{(ka)^2}{(kr)^3} \cos 3\theta \right) \quad \text{B.4c,d}$$

By assembling the above parts into the modified form of the convection velocities we observe that

$$u = \frac{2A^2 \omega k}{\sinh^2 kh} \left\{ \begin{aligned} &\Re(H_1) \left[\frac{(ka)^2}{(kr)^3} \cos 3\theta - \frac{(ka)^4}{(kr)^5} \cos 2\theta \cos 3\theta \right] \\ &+ \Re(H_5) \left[-\frac{(ka)^4}{(kr)^5} \sin 2\theta \sin 3\theta \right] \\ &+ \Re(H_6) \left[\frac{-(ka)^2}{(kr)^3} \cos 3\theta + \frac{(ka)^4}{(kr)^5} \cos 2\theta \cos 3\theta \right] \end{aligned} \right\} \quad \text{B.5}$$

simplifies to

$$u = \frac{2A^2 \omega k}{\sinh^2 kh} \left\{ \Re(H_5) \left[\frac{(ka)^2}{(kr)^3} \cos 3\theta - \frac{(ka)^4}{(kr)^5} \cos 5\theta \right] \right\}, \quad \text{B.6}$$

after applying the following two trigonometric identities

$$\cos m\theta \cos n\theta = \frac{1}{2} [\cos(m-n)\theta + \cos(m+n)\theta] \quad \text{B.7a}$$

$$\sin m\theta \sin n\theta = \frac{1}{2} [\cos(m-n)\theta - \cos(m+n)\theta]. \quad \text{B.7b}$$

Similarly

$$v = \frac{2A^2 \omega k}{\sinh^2 kh} \left\{ \Re(H_2) \left[\frac{(ka)^2}{(kr)^3} \sin 3\theta - \frac{(ka)^4}{(kr)^5} \sin 5\theta \right] \right\}, \quad \text{B.8}$$

using

$$\sin m\theta \cos n\theta = \frac{1}{2} [\sin(m-n)\theta + \sin(m+n)\theta]. \quad \text{B.9}$$

In normalized form:

$$u = 2\Re(H_5) \left[\frac{a^2}{r^3} \cos 3\theta - \frac{a^4}{r^5} \cos 5\theta \right] \quad \text{B.10}$$

$$\mathcal{V} = 2\Re(H_2) \left[\frac{a^2}{r^3} \sin 3\theta - \frac{a^4}{r^5} \sin 5\theta \right] \quad \text{B.11}$$

And the normalized dispersivities are

$$\mathcal{D}_{xx} = H_4^R \left[1 - 2 \frac{a^2}{r^2} \cos 2\theta + \frac{a^4}{2r^4} (1 + \cos 4\theta) \right] \quad \text{B.12a}$$

$$\mathcal{D}_{yy} = H_4^R \left[\frac{a^4}{2r^4} (1 - \cos 4\theta) \right] \quad \text{B.12b}$$

$$\mathcal{D}_{xy} = H_4^R \left[\frac{a^4}{2r^4} \sin 4\theta - \frac{a^2}{r^2} \sin 2\theta \right]. \quad \text{B.12c}$$

Appendix C - A Sample Calculation of the Rotation of $\mathcal{D}_{ij}$

Evaluating $\mathcal{D}_{ij}\left(a, \frac{\pi}{4}\right)$ on Figures 5-18, 5-21, and 5-24, we find

$$\mathcal{D}_{xx} = 0.025$$

$$\mathcal{D}_{xy} = \mathcal{D}_{yx} = 0.025$$

$$\mathcal{D}_{yy} \approx 0.027$$

By diagonalizing the above tensor, we find the eigenvalues are

$$\lambda_n = 0.026 \pm 0.025$$

$$\lambda_1 = 0.001$$

$$\lambda_2 = 0.051$$

And then the eigenvectors are

$$x_1 = \frac{1}{\sqrt{2}}$$

$$x_2 = \frac{-1}{\sqrt{2}}$$

$$y_1 = \frac{1}{\sqrt{2}}$$

$$y_2 = \frac{1}{\sqrt{2}}.$$

The transformation of the above eigenvectors results in a rotation of $\phi = \pi/4$. And it is clear that $\lambda_1 \equiv \mathcal{D}_{rr}$ and $\lambda_2 \equiv \mathcal{D}_{\theta\theta}$. Since $\lambda_2 \gg \lambda_1$, we find purely tangential diffusion at this location.

[illegible]

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